DOMAINS OF PROPER DISCONTINUITY ON THE BOUNDARY OF OUTER SPACE

ILYA KAPOVICH AND MARTIN LUSTIG

Abstract. Motivated by the work of McCarthy and Papadopoulos for subgroups of mapping class groups, we construct domains of proper discontinuity in the compactified Outer space and in the projectivized space of geodesic currents for any "sufficiently large" subgroup of $\mathrm{Out}(F_N)$ (that is, a subgroup containing a hyperbolic iwip).

As a corollary we prove that for $N \geq 3$ the action of $\mathrm{Out}(F_N)$ on the subset of $\mathbb{P}Curr(F_N)$ consisting of all projectivized currents with full support is properly discontinuous.

1. Introduction

One of several important recent events in the study of mapping class groups and Teichmüller spaces is the introduction and development of the theory of convex-cocompact subgroups of mapping class groups through the work of Farb and Mosher [15], Hamenstadt [23], and by Kent and Leininger [39, 40, 41]. This theory is inspired by the classical notion of convex-cocompactness for Kleinian groups and is motivated, in part, by looking for new examples of word-hyperbolic extensions of surface groups by non-elementary word-hyperbolic groups. A key component of the theory of convex-cocompactness in the mapping class group context is the construction of domains of discontinuity for subgroups of mapping class groups in the Thurston boundary of the Teichmüller space. This construction of domains of discontinuity was first put forward by Masur for the handle-body group [47] and for arbitrary subgroups of mapping class groups by McCarthy and Papadopoulos [49], the main case being that of "sufficiently large" subgroups of mapping class groups. More detail about the work of McCarthy and Papadopoulos will be given below.

For automorphisms of free groups many concepts and results proved for mapping classes have been successfully "translated" in the past 20 years, sometimes giving rise to new interesting variations of the mapping class group ideas, and occasionally even to a deeper understanding of them. In some rare cases the innovative impulse has even gone in the converse direction.

A satisfying translation of the subgroup theory into the $\mathrm{Out}(F_N)$ world, however, is still far from being achieved. In particular, it would be much desirable if some analogues of the above quoted results for convex-cocompact subgroups could be established. The aim of this paper is to provide a first step into this direction. In order to state our result we first provide some terminology; more detail will be given later.

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Teichmüller space and its Thurston boundary admit two independent translations into the $\mathrm{Out}(F_N)$ world: One candidate is Culler-Vogtmann's (compactified) Outer space $\overline{\mathrm{CV}}_N$, and the other one is the projectivized space of currents on F_N, which is also compact but (contrary to $\overline{\mathrm{CV}}_N$) infinite dimensional. The group $\mathrm{Out}(F_N)$ acts on both spaces, and it is known that *hyperbolic iwip* automorphisms (= strong analogues of pseudo-Anosov mapping classes, see Definition 3.1 below) act on both spaces with North-South dynamics. For precise references and more details see section 2.

Theorem 1.1. *Let $G \le Out(F_N)$ be a subgroup which contains at least one hyperbolic iwip. Then there exists canonical non-empty open G-invariant subsets $\Delta_G^{cv} \subseteq \overline{CV}_N$ and $\Delta_G^{curr} \subseteq \mathbb{P}Curr(F_N)$ on which the action of G is properly discontinuous.*

The precise definitions of the sets $\Delta_G^{cv} \subset \overline{\mathrm{CV}}_N$ and $\Delta_G^{curr} \subset \mathbb{P}Curr(F_N)$ are given in Section 4 and we also revisit their construction in the discussion below.

In order to motivate and explain Theorem 1.1 properly, we need to recall some details of the above mentioned construction of McCarthy and Papadopoulos [49]. Let S be a closed hyperbolic surface, let $Mod(S)$ be the mapping class group of S and let $G \le Mod(S)$ be a "sufficiently large subgroup", i.e. G contains two independent pseudo-Anosov elements. For such a G, there is a well-defined *limit set* $\Lambda_G \subseteq \mathcal{PML}(S)$ which is the unique smallest G-invariant closed subset of $\mathcal{PML}(S)$. Here $\mathcal{PML}(S)$ is the space of projective measured laminations on S. One then defines the *zero locus* Z_G of G as the set of all $[\lambda] \in \mathcal{PML}(S)$ such that there exists $[\lambda'] \in \Lambda_G$ satisfying $i(\lambda, \lambda') = 0$. Then Z_G is a closed G-invariant set containing Λ_G. Put $\Delta_G = \mathcal{PML}(S) - Z_G$. McCarthy and Papadopoulos prove [49] that G acts properly discontinuously on Δ_G. Moreover, G acts properly discontinuously on $\mathcal{T}(S) \cup \Delta_G$, where $\mathcal{T}(S)$ is the Teichmüller space of S [40].

In the free group context, the most frequently used analogue of Teichmüller space is given by the above mentioned Outer space. Let F_N be a free group of finite rank $N \ge 2$. The (unprojectivized) Outer space cv_N consists of all minimal free and discrete isometric actions of F_N on $\mathbb{R}$-trees. The projectivized Outer space $\mathrm{CV}_N = \mathbb{P}\mathrm{cv}_N$ consists of equivalence classes of points from cv_N up to homothety. The closure $\overline{\mathrm{cv}}_N$ of cv_N in the equivariant Gromov-Hausdorff convergence topology consists precisely of all the *very small* minimal isometric actions of F_N on $\mathbb{R}$-trees, considered up to F_N-equivariant isometry. The projectivization $\overline{\mathrm{CV}}_N = \mathbb{P}\overline{\mathrm{cv}}_N$ of $\overline{\mathrm{cv}}_N$ is compact and contains CV_N as a dense subset. The space $\overline{\mathrm{CV}}_N$ is an analog of the Thurston compactification of the Teichmüller space. The difference $\partial\mathrm{CV}_N := \overline{\mathrm{CV}}_N - \mathrm{CV}_N$ is called the *boundary of the Outer space* CV_N. All of the above spaces come equipped with natural $\mathrm{Out}(F_N)$-actions, see Section 2 for more details and further references.

There is a companion space for cv_N consisting of *geodesic currents* on F_N. A geodesic current on F_N is a positive Radon measure μ on $\partial^2 F_N = \partial F_N \times \partial F_N - diag$ that is invariant under the natural F_N-translation action and under the "flip" map interchanging the two coordinates of $\partial^2 F_N$. Motivated by the work of Bonahon about geodesic currents on hyperbolic surfaces, geodesic currents on free groups have been introduced in a 1995 dissertation of Reiner Martin [46]. The notion was recently re-introduced in the work of Kapovich [28, 29] and the theory of geodesic currents on free groups has been developed in the work of Kapovich and

Lustig [32, 33, 34, 35], Kapovich [30], Kapovich and Nagnibeda [38], Francaviglia [16], Coulbois, Hilion and Lustig [13] and others. The space $Curr(F_N)$ of all geodesic currents is locally compact and comes equipped with a natural continuous action of $\mathrm{Out}(F_N)$ by linear transformations. There is a projectivization $\mathbb{P}Curr(F_N)$ of $Curr(F_N)$ that consists of projective classes $[\mu]$ of nonzero geodesic currents μ, where two such currents are in the same projective class if they are positive scalar multiples of each other. The space $\mathbb{P}Curr(F_N)$ is compact and inherits a natural $\mathrm{Out}(F_N)$-action. A crucial tool in this theory is the notion of a continuous *geometric intersection form* $\langle\ .\ ,\ .\ \rangle : \overline{\mathrm{cv}}_N \times Curr(F_N) \to \mathbb{R}$ that was constructed by Kapovich and Lustig in [33]. This intersection form has some key properties in common with Bonahon's notion of a geometric intersection number between two geodesic currents on a surface, see Proposition 2.1 below for a precise formulation.

In order to construct domains of discontinuity for subgroups of $\mathrm{Out}(F_N)$ in $\partial\mathrm{CV}_N$ it turns out to be necessary to "undualize" the picture and to play the spaces $\overline{\mathrm{CV}}_N$ and $\mathbb{P}Curr(F_N)$ off each other. We say that a subgroup $G \le \mathrm{Out}(F_N)$ is *dynamically large* if it contains an atoroidal iwip (irreducible with irreducible powers) element, see Definition 3.1 for a precise definition of iwips. Being an atoroidal iwip element of $\mathrm{Out}(F_N)$ is the strongest free group analog of being a pseudo-Anosov mapping class. Atoroidal iwips act with "North-South" dynamics both on $\overline{\mathrm{CV}}_N$ and $\mathbb{P}Curr(F_N)$ (see Section 3 for precise statements). This fact allows us to define in Section 4, for a dynamically large $G \le \mathrm{Out}(F_N)$, its *limit sets* $\Lambda_G^{cv} \subseteq \overline{\mathrm{CV}}_N$ and $\Lambda_G^{curr} \subseteq \mathbb{P}Curr(F_N)$. When G is not virtually cyclic, these limit sets are exactly the unique minimal closed G-invariant subsets of $\overline{\mathrm{CV}}_N$ and $\mathbb{P}Curr(F_N)$ accordingly. We then define the *zero sets* of G:

$$Z_G^{cv} = \{[T] \in \overline{\mathrm{CV}}_N : \langle T, \mu\rangle = 0 \text{ for some } [\mu] \in \Lambda_G^{curr}\} \subseteq \overline{\mathrm{CV}}_N$$

and

$$Z_G^{curr} = \{[\mu] \in \mathbb{P}Curr(F_N) : \langle T, \mu\rangle = 0 \text{ for some } [T] \in \Lambda_G^{cv}\} \subseteq \overline{\mathrm{CV}}_N.$$

The zero sets are closed, G-invariant and contain the corresponding limit sets. We put $\Delta_G^{cv} = \overline{\mathrm{CV}}_N - Z_G^{cv}$ and $\Delta_G^{curr} = \mathbb{P}Curr(F_N) - Z_G^{curr}$, so that Δ_G^{cv} and Δ_G^{curr} are open G-invariant sets.

Thus Theorem 1.1 can be viewed as a free group analogue of the result of McCarthy and Papadopoulos [49] for subgroups of $Mod(S)$, mentioned above. It is easy to see that we always have $\mathrm{CV}_N \subseteq \Delta_G^{cv}$ because for any $[T] \in \mathrm{CV}_N$ and any $[\mu] \in \mathbb{P}Curr(F_N)$ we have $\langle T, \mu\rangle > 0$. The main result of [34] implies a similar property for currents with full support: if $\mu \in Curr(F_N)$ is such a current then for any $[T] \in \overline{\mathrm{CV}}_N$ we have $\langle T, \mu\rangle > 0$. As a consequence, we obtain the following application of Theorem 1.1:

Corollary 1.2. *Let $N \ge 3$ and denote by $\mathbb{P}Curr_+(F_N)$ the set of all $[\mu] \in \mathbb{P}Curr(F_N)$ such that μ has full support. Then the action of $Out(F_N)$ on $\mathbb{P}Curr_+(F_N)$ is properly discontinuous.*

Examples of currents with full support include the *Patterson-Sullivan currents* corresponding to points of cv_N, see [38] for details.

The domains of discontinuity provided by Theorem 1.1 are not, in general, maximal possible. In [21] Guirardel constructed a nonempty open $\mathrm{Out}(F_N)$-invariant subset $\mathcal{O}_n \subseteq \partial\mathrm{CV}_N$ such that $\mathrm{Out}(F_N)$ acts properly discontinuously on $\mathcal{O}_n$. As explained in Remark 6.6 below, there are points in $\mathcal{O}_n$ which do not belong to our discontinuity domain $\Delta^{cv}_{Out(F_N)}$.

However, there is a natural class of examples where our domains of discontinuity are likely to be maximal possible. Namely, suppose $\varphi, \psi \in \mathrm{Out}(F_N)$ are atoroidal iwips such that the subgroup $\langle \varphi, \psi \rangle$ is not virtually cyclic. Then, as we proved in [35], there exist $n, m \geq 1$ such that $G = \langle \varphi^n, \psi^m \rangle$ is free of rank two and such that every nontrivial element of G is again an atoroidal iwip and such that $F_N \rtimes \langle \Phi^n, \Psi^m \rangle$ is word-hyperbolic (where $\Phi, \Psi \in Aut(F_N)$ are representatives of φ, ψ). Based on the mapping class group analogy (where similar statements are known as a general part of the theory of convex-cocompact subgroups of mapping class groups, see [15]), we believe that in this case it should be true that $\Lambda^{cv}_G = Z^{cv}_G$. This would imply that $\Delta^{cv}_G \subset \overline{\mathrm{CV}}_N$ is the maximal domain of discontinuity for G in this case. Moreover, again by analogy with the known mapping class group results for convex-cocompact subgroups [15], we expect that in this situation there is a natural homeomorphism between the hyperbolic boundary of the free group $\langle \varphi^n, \psi^m \rangle$ and the limit set Λ^{cv}_G and that, moreover, every $[T] \in \Lambda^{cv}_G$ is uniquely ergodic, that is, there is a unique $[\mu] \in \mathbb{P}Curr(F_N)$ such that $\langle T, \mu \rangle = 0$. If true, these statements would indicate that the "Schottky type" groups $G = \langle \varphi^n, \psi^m \rangle$ as above are good candidates for being examples of "convex-cocompact" subgroups of $\mathrm{Out}(F_N)$.

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2. Outer space and the space of geodesic currents

We give here only a brief overview of basic facts related to Outer space and the space of geodesic currents. We refer the reader to [14, 29] for more detailed background information.

2.1. **Outer space.** Let $N \geq 2$. The *unprojectivized Outer space* cv_N consists of all minimal free and discrete isometric actions on F_N on $\mathbb{R}$-trees (where two such actions are considered equal if there exists an F_N-equivariant isometry between the corresponding trees). There are several different topologies on cv_N that are known to coincide, in particular the equivariant Gromov-Hausdorff convergence topology and the so-called *axis* or *length function* topology. Every $T \in \mathrm{cv}_N$ is uniquely determined by its *translation length function* $||.||_T : F_N \to \mathbb{R}$, where $||g||_T$ is the translation length of g on T. Two trees $T_1, T_2 \in \mathrm{cv}_N$ are close if the functions $||.||_{T_1}$ and $||.||_{T_1}$ are close pointwise on a large ball in F_N. The closure $\overline{\mathrm{cv}}_N$ of cv_N in either of these two topologies is well-understood and known to consists precisely of all the so-called *very small* minimal isometric actions of F_N on $\mathbb{R}$-trees, see [3] and [10]. The outer automorphism group $\mathrm{Out}(F_N)$ has a natural continuous right action on $\overline{\mathrm{cv}}_N$ (that leaves cv_N invariant) given at the level of length functions as follows: for $T \in \mathrm{cv}_N$ and $\varphi \in \mathrm{Out}(F_N)$ we have $||g||_{T\varphi} = ||\varphi(g)||_T$, where $g \in F_N$. In terms of tree actions, $T\varphi$ is equal to T as a metric space, but the action of F_N is modified as: $g \underset{T\varphi}{\cdot} x = \widehat{\varphi}(g) \underset{T}{\cdot} x$ where $x \in T$, $g \in F_N$ are arbitrary and where $\widehat{\varphi} \in Aut(F_N)$ is some fixed representative of the outer automorphism φ is $Aut(F_N)$. The *projectivized Outer space* $\overline{\mathrm{CV}}_N = \mathbb{P}\overline{\mathrm{cv}}_N$ is defined as the quotient $\mathrm{cv}_N/\sim$ where for $T_1 \sim T_2$ whenever $T_2 = cT_1$ for some $c > 0$. One similarly defines the projectivization $\overline{\mathrm{CV}}_N = \mathbb{P}\overline{\mathrm{cv}}_N$ of $\overline{cv}(F_N$ as $\overline{cv}(F_N/\sim$ where $\sim$ is the same as above. The space $\overline{\mathrm{CV}}_N$ is compact and contains CV_N as a dense $\mathrm{Out}(F_N)$-invariant subset. The compactification $\overline{\mathrm{CV}}_N$ of CV_N is a free group analog of the Thurston compactification of the Teichmüller space. For $T \in \overline{\mathrm{cv}}_N$ its $\sim$-equivalence class is denoted by $[T]$, so that $[T]$ is the image of T in $\overline{\mathrm{CV}}_N$.

2.2. **Geodesic currents.** Let $\partial^2 F_N := \{(x,y) | x, y \in \partial F_N, x \neq y\}$. The action of F_N by translations on its hyperbolic boundary ∂F_N defines a natural diagonal action of F_N on $\partial^2 F_N$. A *geodesic current* on F_N is a positive Radon measure on $\partial^2 F_N$ that is F_N-invariant and is also invariant under the "flip" map $\partial^2 F_N \to \partial^2 F_N$, $(x,y) \mapsto (y,x)$. The space $Curr(F_N)$ of all geodesic currents on F_N has a natural $\mathbb{R}_{\geq 0}$-linear structure and is equipped with the weak-* topology of pointwise convergence on continuous functions. Every point $T \in \mathrm{cv}_N$ defines a *simplicial chart* on $Curr(F_N)$ which allows one to think about geodesic currents as systems of non-negative weights satisfying certain Kirchhoff-type equations; see [29] for details. We briefly recall the simplicial chart construction for the case where $T_A \in \mathrm{cv}_N$ is the Cayley tree corresponding to a free basis A of F_N. For a nondegenerate geodesic segment $\gamma = [p,q]$ in T_A the *two-sided cylinder* $Cyl_A(\gamma) \subseteq \partial^2 F_N$ consists of all $(x,y) \in \partial^2 F_N$ such that the geodesic from x to y in T_A passes through $\gamma = [p,q]$. Given a nontrivial freely reduced word $v \in F(A) = F_N$ and a current $\mu \in Curr(F_N)$, the "weight" $\langle v, \mu \rangle_A$ is defined as $\mu(Cyl_A(\gamma))$ where γ is any segment in the Cayley graph T_A labelled by v (the fact that the measure μ is F_N-invariant implies that a particular choice of γ does not matter). A current μ is uniquely determined by a family of weights $\big(\langle v, \mu \rangle_A\big)_{v \in F_N - \{1\}}$. The weak-* topology on $Curr(F_N)$ corresponds to pointwise convergence of the weights for every $v \in F_N, v \neq 1$.

There is a natural left action of $\mathrm{Out}(F_N)$ on $Curr(F_N)$ by continuous linear transformations. Specifically, let $\mu \in Curr(F_N)$, $\varphi \in \mathrm{Out}(F_N)$ and let $\Phi \in Aut(F_N)$ be a representative of φ in $Aut(F_N)$. Since Φ is a quasi-isometry of F_N, it extends to a homeomorphism of ∂F_N and, diagonally, defines a homeomorphism of $\partial^2 F_N$. The measure $\varphi\mu$ on $\partial^2 F_N$ is defined as follows. For a Borel subset $S \subseteq \partial^2 F_N$ we have $(\varphi\mu)(S) := \mu(\Phi^{-1}(S))$. One then checks that $\varphi\mu$ is a current and that it does not depend on the choice of a representative Φ of φ.

For every $g \in F_N, g \neq 1$ there is an associated *counting current* $\eta_g \in Curr(F_N)$. If A is a free basis of F_N and the conjugacy class $[g]$ of g is realized by a "cyclic word" W (that is a cyclically reduced word in $F(A)$ written on a circle with no specified base-vertex), then for every nontrivial freely reduced word $v \in F(A) = F_N$ the weight $\langle v, \eta_g \rangle_A$ is equal to the total number of occurrences of $v^{\pm 1}$ in W (where an occurrence of v in W is a vertex on W such that we can read v in W clockwise without going off the circle). We refer the reader to [29] for a detailed exposition on the topic. By construction the counting current η_g depends only on the conjugacy class $[g]$ of $[g]$ and it also satisfies $\eta_g = \eta_{g^{-1}}$. One can check [29] that for $\varphi \in \mathrm{Out}(F_N)$ and $g \in F_N, g \neq 1$ we have $\varphi\eta_g = \eta_{\varphi(g)}$. Scalar multiples $c\eta_g \in Curr(F_N)$, where $c \geq 0$, $g \in F_N, g \neq 1$ are called *rational currents*. A key fact about $Curr(F_N)$ states that the set of all rational currents is dense in $Curr(F_N)$.

The *space of projectivized geodesic currents* is defined as $\mathbb{P}Curr(F_N) = Curr(F_N) - \{0\}/\sim$ where $\mu_1 \sim \mu_2$ whenever there exists $c > 0$ such that $\mu_2 = c\mu_1$. The $\sim$-equivalence class of $\mu \in Curr(F_N) - \{0\}$ is denoted by $[\mu]$. The action of $\mathrm{Out}(F_N)$ on $Curr(F_N)$ descends to a continuous action of $\mathrm{Out}(F_N)$ on $\mathbb{P}Curr(F_N)$. The space $\mathbb{P}Curr(F_N)$ is compact and the set $\{[\eta_g\ g \in F_N, g \neq 1\}$ is a dense subset of it.

2.3. **Intersection form.** In [33] we constructed a natural geometric *intersection form* which pairs trees and currents:

Proposition 2.1. [33] *Let $N \geq 2$. There exists a unique continuous map $\langle , \rangle : \overline{cv}_N \times Curr(F_N) \to \mathbb{R}_{\geq 0}$ with the following properties:*

1. *We have $\langle T, c_1\mu_1 + c_2\mu_2 \rangle = c_1\langle T, \mu_1\rangle + c_2\langle T, \mu_2\rangle$ for any $T \in \overline{cv}_N$, $\mu_1, \mu_2 \in Curr(F_N)$, $c_1, c_2 \geq 0$.*
2. *We have $\langle cT, \mu\rangle = c\langle T, \mu\rangle$ for any $T \in \overline{cv}_N$, $\mu \in Curr(F_N)$ and $c \geq 0$.*
3. *We have $\langle T\varphi, \mu\rangle = \langle T, \varphi\mu\rangle$ for any $T \in \overline{cv}_N$, $\mu \in Curr(F_N)$ and $\varphi \in Out(F_N)$.*
4. *We have $\langle T, \eta_g\rangle = ||g||_T$ for any $T \in \overline{cv}_N$ and $g \in F_N, g \neq 1$.*

3. North-South dynamics for atoroidal iwips

Definition 3.1. An element $\varphi \in \mathrm{Out}(F_N)$ is *reducible* if there exists a free product decomposition $F_N = C_1 * \dots C_k * F'$, where $k \geq 1$ and $C_i \neq \{1\}$, such that φ permutes the conjugacy classes of subgroups $C_1, \dots, C_k$ in F_N. An element $\varphi \in \mathrm{Out}(F_N)$ is called *irreducible* if it is not reducible. An element $\varphi \in \mathrm{Out}(F_N)$ is said to be *irreducible with irreducible powers* or an *iwip* for short, if for every $n \geq 1$ φ^n is irreducible (sometimes such automorphisms are also called *fully irreducible*).

Thus it is easy to see that $\varphi \in \mathrm{Out}(F_N)$ is an iwip if and only if no positive power of φ preserves the conjugacy class of a proper free factor of F_N. Recall also that $\varphi \in \mathrm{Out}(F_N)$ is called *atoroidal* if it has no periodic conjugacy classes, that is, if there do not exist $n \geq 1$ and $g \in F_N - \{1\}$ such that φ^n fixes the conjugacy class $[g]$ of g in F_N.

The following result is due to Reiner Martin [46]:

Proposition 3.2. *Let $\varphi \in Out(F_N)$ be a atoroidal iwip. Then there exist unique $[\mu_+], [\mu_-] \in \mathbb{P}Curr(F_N)$ with the following properties:*

1. *The elements $[\mu_+], [\mu_-] \in \mathbb{P}Curr(F_N)$ are the only fixed points of φ in $\mathbb{P}Curr(F_N)$.*
2. *For any $[\mu] \neq [\mu_-]$ we have $\lim_{n\to\infty} \varphi^n[\mu] = [\mu_+]$ and for any $[\mu] \neq [\mu_+]$ we have $\lim_{n\to\infty} \varphi^{-n}[\mu] = [\mu_-]$.*
3. *We have $\varphi\mu_+ = \lambda_+\mu_+$ and $\varphi^{-1}\mu_- = \lambda_-\mu_-$ where $\lambda_+ > 1$ and $\lambda_- > 1$. Moreover λ_+ is the Perron-Frobenius eigenvalue of any train-track representative of φ and λ_- is the Perron-Frobenius eigenvalue of any train-track representative of φ^{-1}.*

A similar statement is known for $\overline{\mathrm{CV}}_N$ by a result of Levitt and Lustig [44]:

Proposition 3.3. *Let $\varphi \in Out(F_N)$ be an iwip. Then there exist unique $[T_+], [T_-] \in \overline{CV}_N$ with the following properties:*

1. *The elements $[T_+], [T_-] \in \overline{CV}_N$ are the only fixed points of φ in $\overline{CV}_N$.*
2. *For any $[T] \in \overline{CV}_N$, $[T] \neq [T_-]$ we have $\lim_{n\to\infty}[T\varphi^n] = [T_+]$ and for any $[T] \in \overline{CV}_N$, $[T] \neq [T_+]$ we have $\lim_{n\to\infty}[T\varphi^{-n}] = [T_-]$.*

(3) *We have* $T_+\varphi = \lambda_+ T$ *and* $T_-\varphi^{-1} = \lambda_- T_-$ *where* $\lambda_+ > 1$ *and* $\lambda_- > 1$. *Moreover* λ_+ *is the Perron-Frobenius eigenvalue of any train-track representative of* φ *and* λ_- *is the Perron-Frobenius eigenvalue of any train-track representative of* φ^{-1}.

Recall that as proved in [34, 35] we have:

Proposition 3.4. *Let* $\varphi \in Out(F_N)$ *be an iwip and let* $[T_+], [T_-] \in \overline{CV}_N$ *and* $[\mu_+], [\mu_-] \in \mathbb{P}Curr(F_N)$ *be as in Proposition 3.3 and Proposition 3.2 accordingly. Then the following hold:*

(1) *For* $T \in \overline{cv}_N$ *we have* $\langle T, \mu_+\rangle = 0$ *if and only if* $[T] = [T_-]$ *and we have* $\langle T, \mu_-\rangle = 0$ *if and only if* $[T] = [T_+]$.
(2) *For* $\mu \in Curr(F_N)$, $\mu \neq 0$ *we have* $\langle T_+, \mu\rangle = 0$ *if and only if* $[\mu] = [\mu_-]$ *and we have* $\langle T_-, \mu\rangle = 0$ *if and only if* $[\mu] = [\mu_+]$.
(3) *We have* $\langle T_+, \mu_+\rangle > 0$ *and* $\langle T_-, \mu_-\rangle > 0$.
(4) *We have*

$$Stab_{Out(F_N)}([T_+]) = Stab_{Out(F_N)}([T_-]) =$$
$$= Stab_{Out(F_N)}([\mu_+]) = Stab_{Out(F_N)}([\mu_-])$$

and this stabilizer is virtually cyclic.

We also need the following fact [35, 36]:

Proposition 3.5. *Let* $G \leq Out(F_N)$ *be a subgroup and such that there exist an atoroidal iwip* $\varphi \in G$. *Let* $[T_+(\varphi)], [T_-(\varphi)] \in \overline{CV}_N$, $[\mu_+(\varphi)], [\mu_-(\varphi)] \in \mathbb{P}Curr(F_N)$ *be the attracting and repelling fixed points of* φ *in* $\overline{CV}_N$ *and* $\mathbb{P}Curr(F_N)$ *accordingly. Then exactly one of the following occurs:*

(1) *The group* G *is virtually cyclic and preserves the sets* $\{[T_+(\varphi)], [T_-(\varphi)]\} \subseteq \overline{CV}_N$, $\{[\mu_+(\varphi)], [\mu_-(\varphi)]\} \subseteq \mathbb{P}Curr(F_N)$.
(2) *The group* G *contains an atoroidal iwip* $\psi = g\varphi g^{-1}$ *for some* $g \in G$ *such that* $\{[T_+(\varphi)], [T_-(\varphi)]\} \cap \{[T_+(\psi)], [T_-(\psi)]\} = \emptyset$ *and* $\{[\mu_+(\varphi)], [\mu_-(\varphi)]\} \cap \{[\mu_+(\psi)], [\mu_-(\psi)]\} = \emptyset$. *In this case there are some* $m, n \geq 1$ *such that* $\langle \varphi^m, \psi^n\rangle \leq Out(F_N)$ *is free of rank two and, moreover, every nontrivial element of* $\langle \varphi^m, \psi^n\rangle$ *is an atoroidal iwip.*

4. Dynamically large subgroups and their associated invariant sets

Definition 4.1 (Dynamically large)**.** We say that a subgroup $G \leq \mathrm{Out}(F_N)$ is *dynamically large* if there exist atoroidal iwip $\varphi \in G$. We will say that a dynamically large subgroup $G \leq \mathrm{Out}(F_N)$ is *elementary* if it is virtually cyclic and that it is *non-elementary* otherwise. Thus by Proposition 3.5 a non-elementary dynamically large subgroup contains a free subgroup of rank two.

Proposition-Definition 4.2 (Limit set for a nonelementary dynamically large subgroup)**.** Let $G \leq \mathrm{Out}(F_N)$ be a non-elementary dynamically large subgroup. Then the following hold:

(1) There exists a unique minimal nonempty closed G-invariant subset Λ_G^{cv} of $\overline{\mathrm{CV}}_N$. Moreover, $\Lambda_G^{cv} \subseteq \partial\mathrm{CV}_N$ and every G-orbit of a point of Λ_G^{cv} is dense in Λ_G^{cv}. We call Λ_G^{cv} the *limit set* of G in $\overline{\mathrm{CV}}_N$.

(2) There exists a unique minimal nonempty closed G-invariant subset Λ_G^{curr} of $\mathbb{P}Curr(F_N)$. Moreover, every G-orbit of a point of Λ_G^{curr} is dense in Λ_G^{curr}. We call Λ_G^{curr} the *limit set* of G in $\mathbb{P}Curr(F_N)$.

Proof. We will prove part (1) of the proposition as the proof of (2) is similar.

By Proposition 3.5 and Proposition 3.4 there exist atoriodal iwips $\varphi, \psi \in G$, such that $[T_\pm(\varphi)], [T_\pm(\psi)]$ are four distinct points and $[\mu_\pm(\varphi)], [\mu_\pm(\psi)]$ are four distinct points.

Put Λ_G^{cv} to be the closure in $\overline{\mathrm{CV}}_N$ of the orbit $G[T_+(\varphi)]$. Thus Λ_G^{cv} is a closed G-invariant subset. Let $X \subseteq \overline{\mathrm{CV}}_N$ be a nonempty closed invariant subset. Note that X must contain a point $[T] \neq [T_\pm(\varphi)]$ since G is non-elementary and $\psi \in G$ does not leave invariant a nonempty subset of $\{[T_\pm(\varphi)]\}$. Then $\lim_{n\to\infty} \varphi^n[T] = [T_+(\varphi)]$ and hence $[T_+\varphi] \in X$. Since X is closed and G-invariant, it follows that $\Lambda_G^{cv} = \overline{G[T_+(\varphi)]} \subseteq X$. Thus $\Lambda_G^{cv} = \overline{G[T_+(\varphi)]}$ is the unique minimal nonempty closed G-invariant subset of $\overline{\mathrm{CV}}_N$. It follows that the G-orbit of every point of Λ_G^{cv} is dense in Λ_G^{cv}, since the closure of such an orbit is a closed G-invariant set and thus must contain Λ_G^{cv}. □

Definition 4.3 (Limit set of an elementary dynamically large subgroup)**.** Let $G \le \mathrm{Out}(F_N)$ be an elementary dynamically large subgroup and let $\varphi \in G$ be an atoroidal iwip. We put $\Lambda_G^{cv} := \{[T_+(\varphi)], [T_-(\varphi)]\}$ and call it the *limit set* of G in $\overline{\mathrm{CV}}_N$. Similarly, we put $\Lambda_G^{curr} := \{[\mu_+(\varphi)], [\mu_-(\varphi)]\}$ and call it the *limit set* of G in $\mathbb{P}Curr(F_N)$.

Definition 4.4 (Zero sets)**.** Let $G \le \mathrm{Out}(F_N)$ be a dynamically large subgroup.

(1) Put
$$Z_G^{cv} = \{[T] \in \overline{\mathrm{CV}}_N : \langle T, \mu\rangle = 0 \text{ for some } [\mu] \in \Lambda_G^{curr}\}.$$
(2) Put
$$Z_G^{curr} = \{[\mu] \in \mathbb{P}Curr(F_N) : \langle T, \mu\rangle = 0 \text{ for some } [T] \in \Lambda_G^{cv}\}.$$

The following is an immediate corollary of the definitions and of Proposition 3.4:

Proposition 4.5. *Let $G \le Out(F_N)$ be an elementary dynamically large subgroup and let $\varphi \in G$ be an atoroidal iwip. Then $Z_G^{cv} = \Lambda_G^{cv} = \{[T_\pm(\varphi)]\}$ and $Z_G^{curr} = \Lambda_G^{curr} = \{[\mu_\pm(\varphi)]\}$.*

The proof of the following proposition is straightforward, particularly in view of continuity of the intersection form, and we leave the details to the reader.

Proposition 4.6. *Let $G \le Out(F_N)$. Then $Z_G^{cv} \subseteq \overline{CV}_N$ and $Z_G^{curr} \subseteq \mathbb{P}Curr(F_N)$ are closed G-invariant subsets and $\Lambda_G^{cv} \subseteq \mathbb{Z}_G^{cv}$ and $\Lambda_G^{curr} \subseteq Z_G^{curr}$.*

Remark 4.7. Let $G \le \mathrm{Out}(F_N)$ and let $\varphi \in G$ be an atoroidal iwip. Then $[T_\pm(\varphi)] \in \Lambda_G^{cv}$ and $[\mu_\pm(\varphi)] \in \Lambda_G^{curr}$. Indeed, if G is elementary, this follows from the definitions. If G is nonelementary, then there exists $[T] \in \Lambda_G^{cv}$ such that $[T] \neq [T_\pm(\varphi)]$. Then $\lim_{n\to\pm\infty}[T]\varphi^n = [T_\pm] \in \Lambda_G^{cv}$. The argument for $[\mu_\pm]$ is the same.

Definition 4.8. Let $G \le \mathrm{Out}(F_N)$ be a dynamically large subgroup. Put
$$\Delta_G^{cv} := \overline{\mathrm{CV}}_N - Z_G^{cv}$$
and
$$\Delta_G^{curr} := \mathbb{P}Curr(F_N) - Z_G^{curr}$$

Note that by construction Δ_G^{cv} and Δ_G^{curr} are open G-invariant subsets of $\overline{\mathrm{CV}}_N$ and of $\mathbb{P}Curr(F_N)$ accordingly.

5. Dichotomy

Convention 5.1. Through this section, unless specified otherwise, let $G \le \mathrm{Out}(F_N)$, where $N \ge 3$, be a dynamically large subgroup. Let $g \in G$ be an atoroidal iwip. Let $[T_+], [T_-] \in \overline{\mathrm{CV}}_N$, $[\mu_+], [\mu_-] \in \mathbb{P}Curr(F_N)$ be the attracting and repelling fixed points of g in $\overline{\mathrm{CV}}_N$ and $\mathbb{P}Curr(F_N)$ accordingly. For a current $\mu \in Curr(F)$ denote $\langle T_\pm, \mu\rangle := \max\{\langle T_-, \mu\rangle, \langle T_+, \mu\rangle\}$.

5.1. **Basic dichotomy for trees.**

Lemma 5.2. *Let $T \in cv(F)$ and let $g_n \in Out(F_N)$ be an infinite sequence of distinct elements. Let $T_\infty \in \overline{cv}_N$, $c_n \ge 0$ be such that $\lim_{n\to\infty} c_n T g_n \to T_\infty$ in $\overline{cv}_N$. Then $\lim_{n\to\infty} c_n = 0$.*

Proof. Recall that $\mathrm{CV}_N = \mathbb{P}\mathrm{cv}_N$ and $\overline{\mathrm{CV}}_N = \mathbb{P}\overline{\mathrm{cv}}_N$. Since $\lim_{n\to\infty}[T]g_n = [T_\infty]$ in $\overline{\mathrm{CV}}_N$ and since the action of $\mathrm{Out}(F_N)$ on CV_N is properly discontinuous, it follows that $[T_\infty] \in \partial\mathrm{CV}_N = \overline{\mathrm{CV}}_N - \mathrm{CV}_N$. Therefore F_N has nontrivial elements of arbitrarily small translation length with respect to the action on T_∞. On the other hand, the action of F_N on T is free and discrete and therefore there exists $C > 0$ such that for every $w \in F_N, w \ne 1$ and for every $n \ge 1$ we have $||w||_{Tg_n} = ||g_n(w)||_T \ge C$. The statement of the lemma now follows from point-wise convergence of the translation length functions of Tg_n to that of T_∞. □

Corollary 5.3. *Let $g_n \in Out(F_N)$ be an infinite sequence of distinct elements. Then there exists a conjugacy class $[w]$ in F_N such that the set of conjugacy classes $g_n[w]$ is infinite.*

Proof. Let $T \in \mathrm{cv}_N$ be arbitrary. Choose a limit point $[T_\infty]$ of $[T]g_n$ in $\overline{\mathrm{CV}}_N$. Then, after passing to a subsequence, we have $\lim_{n\to\infty} c_n T g_n = T_\infty$. Choose $w \in F_N$ such that $||w||_{T_\infty} > 0$. Thus $\lim_{n\to\infty} c_n ||g_n(w)||_T = ||w||_{T_\infty} > 0$. Since by Lemma 5.2 we have $\lim_{n\to\infty} c_n = 0$, it follows that $\lim_{n\to\infty} ||g_n(w)||_T = \infty$, so that the sequence of conjugacy classes $g_n[w]$ contains infinitely many distinct elements. □

Lemma 5.4. *Let $T \in cv_N$ and $\mu \in Curr(F_N)$. Let $g_n \in Out(F_N)$ be an infinite sequence of distinct elements. Then one of the following holds:*

1. *The sequence $\langle T, g_n\mu\rangle$ is unbounded.*
2. *For every limit point $[T_\infty]$ of $[T]g_n$ we have $\langle T_\infty, \mu\rangle = 0$.*

Proof. Suppose that (2) fails and there exists a limit point $[T_\infty]$ of $[T]g_n$ such that $\langle T_\infty, \mu\rangle > 0$. After passing to a subsequence g_{n_i} of g_n we have $T_\infty = \lim_{i\to\infty} c_i T g_{n_i}$ for some $c_i \ge 0$. Then

$$\langle T_\infty, \mu\rangle = \lim_{i\to\infty} \langle c_i T g_{n_i}, \mu\rangle = \lim_{i\to\infty} c_i \langle T, g_{n_i}\mu\rangle.$$

Since by Lemma 5.2 we have $\lim_{i\to\infty} c_i = 0$ and since $\langle T_\infty, \mu\rangle > 0$, it follows that $\lim_{i\to\infty}\langle T, g_{n_i}\mu\rangle = \infty$. Therefore the sequence $\langle T, g_n\mu\rangle$ is unbounded, as required. □

Recall that $g \in G$ is an atoroidal iwip and that $[T_\pm]$, $[\mu_\pm]$ are its fixed points in $\overline{\mathrm{CV}}_N$ and $\mathbb{P}Curr(F_N)$.

The following fact is established in [34]:

Proposition 5.5. *Let $T_0 \in cv_N$ be arbitrary. Then the functions $||.||_{T_0}$ and $||.||_{T_+} + ||.||_{T_-}$ on F_N are bi-Lipschitz equivalent.*

Lemma 5.6. *Let $g_n \in Out(F_N)$ be an infinite sequence of distinct elements. Then there exist a subsequence $g'_n = g_{s_n}$, $s_n \to \infty$, and a tree $T_* \in \{T_+, T_-\}$ such that the conclusion of Lemma 5.2 holds for T_* and g'_n:*

That is, if, after passing to a further subsequence g'_{n_i}, we have $\lim_{i\to\infty} c_i T_* g'_{n_i} = T_\infty$ in $\overline{cv}_N$ for some $c_i \geq 0$ then $\lim_{i\to\infty} c_i = 0$.

Proof. By Proposition 5.5, for any $T_0 \in \mathrm{cv}_N$ there exists $C > 0$ such that

$$||.||_{T+} + ||.||_{T_-} \geq C||.||_{T_0} \text{ on } F_N.$$

Corollary 5.3 implies that there exists a nontrivial conjugacy class $[w]$ in F_N such that the set of conjugacy classes $g_n[w]$ is infinite, so that $\lim_{n\to\infty} ||g_n(w)||_{T_0} = \infty$. Hence, after passing to a subsequence g'_n of g_n, by the above inequality we conclude that for some $T_* \in \{T_+, T_-\}$, we have $\lim_{n\to\infty} ||g'_n(w)||_{T_*} = \infty$.

Suppose now that $\lim_{i\to\infty} c_i T_* g'_{n_i} = T_\infty$. Since there exists a finite limit

$$||w||_{T_\infty} = \lim_{i\to\infty} c_i ||w||_{T_* g'_{n_i}} = \lim_{i\to\infty} c_i ||g'_{n_i}(w)||_{T_*},$$

it follows that $\lim_{i\to\infty} c_i = 0$, as required. $\square$

Convention 5.7. For the remainder of this section, unless specified otherwise, we assume that $g_n \in G$ is a (fixed) infinite sequence of distinct elements of a dynamically large subgroup $G \leq \mathrm{Out}(F_N)$ and that g'_n and $T_* \in \{T_+, T_-\}$ are provided by Lemma 5.6.

Corollary 5.8. *For any $[\mu] \in \Delta_G^{curr}$, we have $\lim_{n\to\infty} \langle T_*, g'_n \mu\rangle = \infty$. In particular, the sequence $\langle T_*, g_n \mu\rangle$ is unbounded.*

Proof. Suppose that for some $[\mu] \in \Delta_G^{curr}$, we have $\lim_{n\to\infty} \langle T_*, g'_n\mu\rangle \neq \infty$. Then there exists a subsequence g''_n of g'_n such that the sequence $\langle T_*, g''_n \mu\rangle$ is bounded. Since $\overline{\mathrm{CV}}_N$ is compact, after passing to a further subsequence g_{n_i} of g''_n, there exist $c_i \geq 0$ and $T_\infty \in \overline{\mathrm{cv}}_N$ such that $\lim_{i\to\infty} c_i T_* g_{n_i} = T_\infty$. By Lemma 5.6 we have $\lim_{i\to\infty} c_i = 0$. Therefore, as in the proof of Lemma 5.4, either the sequence $\langle T_*, g_{n_i}\mu\rangle$ is unbounded or $\langle T_\infty, \mu\rangle = 0$. The former is impossible since by assumption the sequence $\langle T_*, g''_n\mu\rangle$ is bounded. Thus $\langle T_\infty, \mu\rangle = 0$. However, $[T_\pm] \in \Lambda_G^{cv}$ by Remark 4.7 and hence $[T_\infty] \in \Lambda_G^{cv}$. Therefore the possibility that $\langle T_\infty, \mu\rangle = 0$ is ruled out by the assumption that $[\mu] \in \Delta_G^{curr}$, yielding a contradiction. $\square$

5.2. **Basic dichotomy for currents.** We now want to dualize the previous arguments for the case of geodesic currents. The difficulty here is that currents analogues of Lemma 5.2 and Lemma 5.6 are not readily available. We get around this problem by using the results of the previous steps to obtain such analogues first.

Lemma 5.9. *Let $[\mu] \in \Delta_G^{curr}$. Suppose that for a subsequence g_{n_i} of g'_n there are $c_n \geq 0$ and $\mu_\infty \in Curr(F_N)$ be such that $\lim_{i\to\infty} c_i g_{n_i}\mu = \mu_\infty$. Then $\lim_{i\to\infty} c_i = 0$.*

Proof. We have

$$\langle T_*, \mu_\infty\rangle = \lim_{i\to\infty} \langle T_*, c_i g_{n_i}\mu\rangle = \lim_{i\to\infty} c_i \langle T_*, g_{n_i}\mu\rangle.$$

By Corollary 5.8 we have $\lim_{i\to\infty}\langle T_*, g_{n_i}\mu\rangle = \infty$. Hence, from the above equation, we conclude that $\lim_{i\to\infty} c_i = 0$, as required. $\square$

We now obtain an analogue of Lemma 5.4 for currents, with the important difference however, that, as specified in Convention 5.7, here we do need to assume that $g_n \in G$ while the proof of Lemma 5.4 works for an arbitrary infinite sequence $g_n \in \mathrm{Out}(F_N)$. Similarly, we had to assume that $g_n \in G$ in Lemma 5.9 while this assumption was not needed in Lemma 5.2. Recall that $g_n \in G$, as well as g'_n and T_* are as in Convention 5.7.

Lemma 5.10. *Let $T \in \overline{cv}_N$ be arbitrary and let $[\mu] \in \Delta_G^{curr}$. Then one of the following holds:*

(1) *The sequence $\langle Tg'_n, \mu\rangle$ is unbounded.*
(2) *We have $\langle T, \mu_\infty\rangle = 0$ for any limit point $[\mu_\infty]$ of the sequence $[g'_n\mu]$ in $\mathbb{P}Curr(F_N)$.*

Proof. Suppose that (2) fails and there exist $c_i \geq 0$ and $\mu_\infty \in Curr(F_N)$ be such that, after possibly passing to a subsequence g_{n_i} of g'_n, we have $\lim_{i\to\infty} c_i g_{n_i}\mu = \mu_\infty$ and such that $\langle T, \mu_\infty\rangle > 0$. Then

$$\langle T, \mu_\infty\rangle = \lim_{i\to\infty} \langle T, c_i g_{n_i}\mu\rangle = \lim_{i\to\infty} c_i\langle Tg_{n_i}, \mu\rangle.$$

Since by Lemma 5.9 we have $\lim_{i\to\infty} c_i = 0$, and since by assumption $\langle T, \mu_\infty\rangle > 0$, it follows that the sequence $\langle Tg'_n, \mu\rangle$ is unbounded. □

Lemma 5.11. *There exists $\mu \in \{\mu_\pm\}$ with the following property. Whenever, after passing to a further subsequence g_{n_i} of g'_n, we have $c_i g_{n_i}\mu \to \mu_\infty \neq 0$ for some $c_i \geq 0$, then $\lim_{i\to\infty} c_i = 0$.*

Proof. Suppose there exist $c_i, g_{n'_i}$ and μ_∞ as above such that $c_i g_{n_i}\mu \to \mu_\infty \neq 0$ but $\lim_{i\to\infty} c_i \neq 0$. Then after passing to a further subsequence, we may assume that $c_{n_i} \geq c > 0$ for all $i \geq 1$.

Let $T \in \mathrm{cv}_N$. First, we claim that either the sequence $\langle T, g_{n_i}\mu_+\rangle$ or the sequence $\langle T, g_{n_i}\mu_-\rangle$ is unbounded. Suppose not, that is, that they are both bounded. Choose a limit point $[T_\infty]$ of $[Tg_{n_i}]$. Then by Lemma 5.4 we have $\langle T_\infty, \mu_+\rangle = \langle T_\infty, \mu_-\rangle = 0$. However, this is impossible by Proposition 3.4. Thus there is $\mu \in \{\mu\pm\}$ such that the sequence $\langle T, g_{n_i}\mu\rangle$ is unbounded. After taking a further subsequence of g_{n_i}, we may thus assume that $\lim_{i\to\infty}\langle T, g_{n_i}\mu\rangle = \infty$. Recall that $c_i \geq c > 0$ for every $i \geq 1$. Now,

$$\langle T, \mu_\infty\rangle = \lim_{i\to\infty} \langle T, c_i g_{n_i}\mu\rangle = \lim_{i\to\infty} c_i\langle T, g_{n_i}\mu\rangle$$

which yields a contradiction since $\lim_{i\to\infty}\langle T, g_{n_i}\mu\rangle = \infty$ and $c_i \geq c > 0$ for every $i \geq 1$. □

Corollary 5.12. *Let $\mu \in \{\mu_\pm\}$ be provided by Lemma 5.11. Let $[T] \in \Delta_G^{cv}$. Then the sequence $\langle Tg_n, \mu\rangle$ is unbounded.*

Proof. We first pass to the subsequence g'_n of g_n provided by Lemma 5.6 so that μ has the properties guaranteed by Lemma 5.11. Let $[\mu_\infty]$ be a limit point of $[g'_n\mu]$. Thus, for a subsequence g_{n_i} of g'_n and for some $c_i \geq 0$ we have $\lim_{n\to\infty} c_i g_{n_i}\mu = \mu_\infty$. By Lemma 5.11 we have $\lim_{i\to\infty} c_i = 0$. Note also that $\mu \in \{\mu_\pm\}$ which implies that $[\mu] \in \Lambda_G^{curr}$ and hence $[\mu_\infty] \in \Lambda_G^{curr}$.

Now let $[T] \in \Delta_G^{cv}$. Then

$$\langle T, \mu_\infty\rangle = \lim_{i\to\infty} c_i\langle T, g_{n_i}\mu\rangle = \lim_{i\to\infty} c_i\langle Tg_{n_i}, \mu\rangle.$$

Note that $\langle T, \mu_\infty\rangle > 0$ since $[\mu_\infty] \in \Lambda_G^{curr}$ and $[T] \in \Delta_G^{cv}$. By Lemma 5.11 we have $\lim_{i\to\infty} c_i = 0$ which now implies that $\lim_{i\to\infty}\langle Tg_{n_i}, \mu\rangle = \infty$. Thus the sequence $\langle Tg_n, \mu\rangle$ is unbounded, as claimed. □

Corollary 5.13. *Let $[T] \in \Delta_G^{cv}$ and let $c_i \ge 0$, and let g_{n_i} be a subsequence of g'_n such that $\lim_{i\to\infty} c_i T g_{n_i} = T_\infty \in \overline{cv}_N$. Then $\lim_{i\to\infty} c_i = 0$.*

Proof. Let $\mu \in \{\mu_\pm\}$ have the properties guaranteed by Lemma 5.11. Suppose that $\lim_{i\to\infty} c_i \ne 0$. Then, after passing to a further subsequence, we have $c_i \ge c > 0$ for all $i \ge 1$. After passing to a further subsequence, we may assume that $c'_i g_{n_i}\mu \to \mu_\infty$. We have $\lim_{i\to\infty} c'_i = 0$. The proof of Corollary 5.12 shows that in this case $\lim_{i\to\infty}\langle Tg_{n_i}, \mu\rangle = \infty$ since

$$\langle T, \mu_\infty\rangle = \lim_{i\to\infty} c'_i \langle T, g_{n_i}\mu\rangle = \lim_{i\to\infty} c'_i\langle Tg_{n_i}, \mu\rangle.$$

Then

$$\langle T_\infty, \mu\rangle = \lim_{i\to\infty} c_i\langle Tg_{n_i}, \mu\rangle = \infty,$$

since $c_i \ge c > 0$ and since $\lim_{i\to\infty}\langle Tg_{n_i}, \mu\rangle = \infty$. This yield a contradiction since the value of intersection form is always finite. □

Proposition 5.14. *Let $G \le Out(F_N)$ be a dynamically large subgroup. Then the following hold:*

1. *Let $[T] \in \Delta_G^{cv}$ and let $[T_\infty]$ be a limit point of $G[T]$. Then $[T_\infty] \in Z_G^{cv}$.*
2. *Let $[\mu] \in \Delta_G^{curr}$ and let $[\mu_\infty]$ be a limit point of $G[\mu]$. Then $[\mu_\infty] \in Z_G^{curr}$.*

Proof. Let $[T] \in \Delta_G^{cv}$ and let $[T_\infty]$ be a limit point of $G[T]$. Then there is an infinite sequence of distinct elements $g_n \in G$ such that $\lim_{n\to\infty}[T]g_n = [T_\infty]$. Let g'_n be the subsequence of g_n provided by Lemma 5.6. Then for some subsequence g_{n_i} of g'_n and for some $c_i \ge 0$ we have $\lim_{i\to\infty} c_i T g_{n_i} = T_\infty$. By Corollary 5.13 we have $\lim_{i\to\infty} c_i = 0$. We now apply Lemma 5.11 to the sequence $h_i = g_{n_i}^{-1}$. Lemma 5.11 implies that after passing to a further subsequence for some $\mu \in \{\mu_\pm\}$ we have $\lim_{i\to\infty} c'_i g_{n_i}^{-1}\mu = \mu_\infty \ne 0$ and $\lim_{i\to\infty} c'_i = 0$.

Therefore

$$\langle T_\infty, \mu_\infty\rangle = \lim_{i\to\infty} c_i c'_i\langle Tg_{n_i}, g_{n_i}^{-1}\mu\rangle = \lim_{i\to\infty} c_i c'_i\langle T, \mu\rangle = 0.$$

By construction we have $[\mu] = [\mu_\pm] \in \Lambda_G^{curr}$ and hence $[\mu_\infty] = \lim_{i\to\infty} g_{n_i}^{-1}[\mu] \in \Lambda_G^{curr}$. Therefore by definition of the zero-set we have $[T_\infty] \in Z_G^{cv}$, as required.

The proof of part (2) of the proposition is similar. The use of Corollary 5.13 and Lemma 5.11 has to be replaced by the use of Lemma 5.9 and Lemma 5.6. We leave the details to the reader. □

6. Domains of discontinuity

Convention 6.1. For the remainder of this section let $G \le \mathrm{Out}(F_N)$ be a dynamically large subgroup, let $\varphi \in G$ be an atoroidal iwip and let $[T_\pm]$, $[\mu_\pm]$ be the fixed points of φ in $\overline{\mathrm{CV}}_N$ and in $\mathbb{P}Curr(F_N)$ accordingly. Recall that for $T \in \overline{\mathrm{cv}}_N$ we denote $\langle T, \mu_\pm\rangle = \langle T, \mu_+\rangle + \langle T, \mu_-\rangle$. Similarly, for $\mu \in Curr(F_N)$ we denote $\langle T_\pm, \mu\rangle = \langle T_+, \mu\rangle + \langle T_-, \mu\rangle$.

Notation 6.2. Denote

$$D_G^{curr} = \{[\mu] \in \Delta_G^{curr} : \langle T_\pm, \mu\rangle \le \langle T_\pm, g\mu\rangle \text{ for every } g \in G\}$$

and

$$D_G^{cv} = \{[T] \in \Delta_G^{cv} : \langle T, \mu_\pm\rangle \le \langle gT, \mu_\pm\rangle \text{ for every } g \in G\}$$

Lemma 6.3. *We have $GD_G^{curr} = \Delta_G^{curr}$ and $GD_G^{cv} = \Delta_G^{cv}$.*

Proof. Let $[\mu] \in \Delta_G^{curr}$. Then for any $C \ge 1$ the set

$$\{g \in G : \langle g\mu, T_\pm\rangle \le C\}$$

is finite by Corollary 5.8. Hence there exists $g_0 \in G$ with $\langle g_0\mu, T_\pm\rangle = \min_{g\in G}\langle g\mu, T_\pm\rangle$, that is, $g_0[\mu] \in D_G^{curr}$. Therefore $GD_G^{curr} = \Delta_G^{curr}$, as required. The proof that $GD_G^{cv} = \Delta_G^{cv}$ is similar, with the use of Corollary 5.8 replaced by Corollary 5.12. □

Lemma 6.4. *For any compact set $K' \subseteq \Delta_G^{curr}$ and for any compact set $K \subseteq \Delta_G^{cv}$ we have*

$$\{g \in G : K' \cap gD_G^{curr} \ne \emptyset\} \textit{ is finite}$$

and

$$\{g \in G : K \cap gD_G^{cv} \ne \emptyset\} \textit{ is finite.}$$

Proof. Suppose that there exists an infinite sequence of distinct elements $g_n \in G$ such that $K \cap g_nD_G^{cv} \ne \emptyset$ for all $n \ge 1$. Then there is also a sequence $[T_n] \in K \cap g_nD_G^{cv}$. After passing to a subsequence, we may assume that $[T_n] \to [T_\infty]$ in $\overline{\mathrm{CV}}_N$. Moreover, after choosing the projective representatives of $[T_n]$ appropriately, we may assume that $T_n \to T_\infty$ as $n \to \infty$. Note that $[T_\infty] \in K$ since K is compact.

By Lemma 5.11, after passing to a further subsequence, there exist $\mu \in \{\mu_\pm\}$, $c_n \ge 0$ and $0 \ne \mu_\infty \in Curr(F_N)$ such that $\lim_{n\to\infty} c_n g_n\mu = \mu_\infty$ and such that $\lim_{n\to\infty} c_n = 0$. We have

$$\begin{aligned}\langle T_\infty, \mu_\infty\rangle &= \lim_{n\to\infty} c_n\langle T_n, g_n\mu\rangle \le \lim_{n\to\infty} c_n\langle T_n, g_n\mu_\pm\rangle = \\ &= \lim_{n\to\infty} c_n\langle g_n^{-1}T_n, \mu_\pm\rangle \le \qquad \text{since } [T_n] \in g_nD_G^{cv} \\ &\le \lim_{n\to\infty} c_n\langle T_n, \mu_\pm\rangle = \lim_{n\to\infty} c_n \lim_{n\to\infty}\langle T_n, \mu_\pm\rangle = 0\cdot\langle T_\infty, \mu_\pm\rangle = 0.\end{aligned}$$

Thus $\langle T_\infty, \mu_\infty\rangle = 0$. Recall that $[T_\infty] \in K \subseteq \Delta_G^{cv}$. Also, $[\mu] \in \{[\mu_\pm]\}$ and hence $[\mu] \in \Lambda_G^{curr}$ and therefore $[\mu_\infty] \in \Lambda_G^{curr}$. The fact that $\langle T_\infty, \mu_\infty\rangle = 0$ now gives a contradiction with the definition of Δ_G^{cv}.

The proof that $\{g \in G : K' \cap gD_G^{curr} \ne \emptyset\}$ is finite is similar. The only difference is that in the argument the use of Lemma 5.11 has to be replaced by the use of Lemma 5.6. □

Theorem 6.5. *Let $G \le Out(F_N)$ be a dynamically large subgroup. Then the actions of G on Δ_G^{cv} and Δ_G^{curr} are properly discontinuous.*

Proof. We will show that the action of G on Δ_G^{curr} is properly discontinuous. The argument for Δ_G^{cv} is similar.

Let $K' \subseteq \Delta_G^{curr}$ be a compact subset and suppose that $\{g \in G : K' \cap gK' \ne \emptyset\}$ is infinite. Let $g_n \in G$ be an infinite sequence of distinct elements such that $K' \cap g_nK' \ne \emptyset$ for every $n \ge 1$.

By Lemma 6.4 there exists a finite collection $h_1, \dots, h_t \in G$ such that

$$\{h_1, \dots, h_t\} = \{g \in G : K' \cap gD_G^{curr} \ne \emptyset\}.$$

Moreover, since by Lemma 6.3 $GD_G^{curr} = \Delta_G^{curr}$, it follows that $K' \subseteq \cup_{i=1}^t h_i D_G^{curr}$.

For every $n \geq 1$ there exists $[\mu_n] \in K' \cap g_n K'$ and hence $[\mu_n] \in K' \cap \cup_{i=1}^t g_n h_i D_G^{curr}$. Therefore for every $n \geq 1$ there exists $i_n, 1 \leq i_n \leq t$ such that $K' \cap g_n h_{i_n} D_G^{curr} \neq \emptyset$. It follows that

$$\{g_n h_{i_n} : n \geq 1\} \subseteq \{h_1, \dots, h_t\}$$

which yields a contradiction since by assumption the sequence $g_n, n \geq 1$ consists of infinitely many distinct elements of G. □

Remark 6.6. The domains of discontinuity constructed in Theorem 6.5 are not necessarily maximal. Thus for the case of $G = \mathrm{Out}(F_N)$ Guirardel [21] constructed an open G-invariant subset $\mathcal{O}_N \subset \overline{\mathrm{CV}}_N$ such that $\mathrm{Out}(F_N)$ acts on $\mathcal{O}_N$ properly discontinuously. In Guirardel's construction, $\mathrm{CV}_N \subsetneq \mathcal{O}_N$. One can see that for every $[T] \in \mathcal{O}_N - \mathrm{CV}_N$ we have $[T] \in Z_{Out(F_N)}^{cv}$ and hence $[T] \notin \Delta_G^{cv}$. The reason for this is that every such T corresponds to a (non-free) simplicial action of F_N with trivial edge stabilizers (plus some additional conditions on the quotient graph that are not relevant here) and therefore there exists a primitive element a of F_N such that a fixes a vertex in T. For every such a we have $[\eta_a] \in \Lambda_{Out(F_N)}^{curr}$ (see [32]) and $\langle T, \eta_a \rangle = ||a||_T = 0$, so that indeed $[T] \in Z_{Out(F_N)}^{cv}$, as claimed.

Let $Curr_+(F_N)$ be the set of all $\mu \in Curr(F_N)$ with full support and let $\mathbb{P}Curr_+(F_N) = \{[\mu] : \mu \in Curr_+(F_N)\}$. Note that $\mathbb{P}Curr_+(F_N)$ is an open $\mathrm{Out}(F_N)$-invariant subset of $\mathbb{P}Curr(F_N)$.

Corollary 6.7. *Let $N \geq 3$. Then the action of Out(F_N) on $\mathbb{P}Curr_+(F_N)$ is properly discontinuous.*

Proof. Note that for $N \geq 3$ the group $G = \mathrm{Out}(F_N)$ is dynamically large. By the main result of [34] we have $\langle T, \mu \rangle > 0$ for any $T \in \overline{\mathrm{cv}}_N$ and any $\mu \in Curr_+(F_N)$. Therefore, by Definition 4.8, we have $\mathbb{P}Curr_+(F_N) \subseteq \Delta_{Out(F_N)}^{curr}$. Hence the action of $\mathrm{Out}(F_N)$ on $\Delta_{Out(F_N)}^{curr}$ is properly discontinuous by Theorem 6.5 and therefore the action of $\mathrm{Out}(F_N)$ on $\mathbb{P}Curr_+(F_N)$ is properly discontinuous as well. □

Note, however, that in the proof of Corollary 6.7 the containment $\mathbb{P}Curr_+(F_N) \subsetneq \Delta_{Out(F_N)}^{curr}$ is proper. Recall that a current $\mu \in Curr(F_N)$ is called *filling* if for every $T \in \overline{\mathrm{cv}}_N$ we have $\langle T, \mu \rangle > 0$. The same argument as in the proof of Corollary 6.7 shows that if $\mu \in Curr(F_N)$ is filling then $[\mu] \in \Delta_{Out(F_N)}^{curr}$. It was proved in [34] that there exist filling rational currents $\eta_g \in Curr(F_N)$ and in fact, the property of being filling for a rational current is, in a sense, "generic". However, a rational current never has full support, so if η_g is filling, then $\eta_g \in \Delta_{Out(F_N)}^{curr} - \mathbb{P}Curr_+(F_N)$.

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Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 West Green Street, Urbana, IL 61801, USA
http://www.math.uiuc.edu/~kapovich/

E-mail address: kapovich@math.uiuc.edu

Mathématiques (LATP), Université Paul Cézanne -Aix Marseille III,, av. Escadrille Normandie-Niémen, 13397 Marseille 20, France

E-mail address: Martin.Lustig@univ-cezanne.fr