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EQUIVALENCE OF THE BROWNIAN AND ENERGY REPRESENTATIONS

Abstract. We consider two unitary representations of the infinite-dimensional groups of smooth paths with values in a compact Lie group. The first representation is induced by quasi-invariance of the Wiener measure, and the second representation is the energy representation. We define these representations and their basic properties, and then we prove that these representations are unitarily equivalent.

§1. Introduction

The main subject of this paper is a study of two unitary representations of the group $H(G)$ of smooth paths in a compact Lie group G. The first representation is on the Hilbert space $L^2(W(G), \mu)$, where $W(G)$ is the Wiener space of continuous path in G and μ is the corresponding Wiener measure. This representation is induced by the quasi-invariance of the Wiener measure μ with respect to the left (right) multiplication on $W(G)$ by elements in $H(G)$. The necessary preliminaries from stochastic analysis are introduced in Section 2. We define the corresponding Brownian representations in Section 4. One of the questions mentioned in the previous works such as [1] is whether the constant function $\mathbb{1}$ is the cyclic vector for these representations. This is what we prove in Section 3.

Another representation of the the group $H(G)$ is the energy representation. The representation space in this case is $L^2(W(\mathfrak{g}), \nu)$, where $\mathfrak{g}$ is the Lie algebra of G, and ν is the standard Gaussian measure on $W(\mathfrak{g})$. Our main result in Section 5 is the (unitary) equivalence of the Brownian and energy representations.

Key words and phrases: quasi-invariance; stochastic differential equations; Lie groups; representations of infinite-dimensional groups.

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These representations have been studied previously in a number of articles including [1–4, 9, 11, 12, 27, 28]. We will not attempt to give a comprehensive review of the mathematical literature on the subject, but rather explain the choice of this particular topic for this volume.

Acknowledgment. Even though M.I. had no publications in this field, the combination of representation theory, stochastic analysis and von Neumann algebras appealed to him. Moreover, he introduced MG to the latter subject which resulted in [13].

§2. Notation

Let G be a compact connected Lie group, $e \in G$ denote the identity of G, $\mathfrak{g}$ be its Lie algebra, and $d = \dim_{\mathbb{R}} \mathfrak{g}$ be the dimension of G and $\mathfrak{g}$. Without loss of generality we may and do assume that G is a Lie subgroup of $\mathrm{GL}_n(\mathbb{R})$. By identifying G with a matrix group, we are able to minimize the differential geometric notation required of the reader. We assume that the Lie algebra $\mathfrak{g}$ of G is identified with the tangent space at e, and $\mathfrak{g}$ is equipped with an Ad_G-invariant inner product $\langle \cdot, \cdot \rangle$, which we could take to be the negative of the Killing form if $\mathfrak{g}$ is semi-simple. Associated to the Ad_G-invariant inner product is the Laplace operator described below.

2.1. Heat kernels. This section reviews some basic facts about heat kernels on unimodular Lie groups. Let dx denote a bi-invariant Haar measure on G which is unique up to normalization. For $A \in \mathfrak{g}$, let $\widetilde{A}(\widehat{A})$ denote the unique left (right) invariant vector field on G which agrees with A at $e \in G$. Let $\mathfrak{g}_0 \subset \mathfrak{g}$ be an orthonormal basis for $\mathfrak{g}$. The left and right invariant Laplacian is then given $\Delta := \sum_{A \in \mathfrak{g}_0} \widetilde{A}^2$ and $\Delta' := \sum_{A \in \mathfrak{g}_0} \widehat{A}^2$ respectively. Since G is unimodular, it is easy to check the formal adjoint, relative to $\mathrm{L}^2(G, dx)$, of $\widetilde{A}$ $(\widehat{A})$ is $-\widetilde{A}$ $(-\widehat{A})$. Hence, $\Delta/2$ and $\Delta'/2$ are symmetric operators on the smooth functions with compact support on G. It is well known, see for example Robinson [22, Theorem 2.1, p. 152], that $\Delta/2$ and $\Delta'/2$ are essentially self-adjoint and the closures of $\Delta/2$ and $\Delta'/2$ generate strongly continuous, self-adjoint contraction semigroups $e^{t\Delta/2}$ and $e^{t\Delta'/2}$ on $\mathrm{L}^2(G, dx)$. Let $p_t = e^{t\Delta/2}\delta_e$, $t \geqslant 0$, be the fundamental solution, i.e.,

$$\partial p_t / \partial t = \frac{1}{2} \Delta p_t \text{ with } \lim_{t \to 0} p_t = \delta_e. \tag{2.1}$$

For a proof of the following theorem see Robinson [22, Theorem 2.1, p.257].

Theorem 2.1. *Assuming the above notation, let p_t denote the fundamental solution to the left heat equation* (2.1)*. Then $p_t(x) = p_t(x^{-1})$ for all $x \in G$ and*

$$e^{t\Delta/2}f(x) = \int_G p_t(x^{-1}h)f(h)dh = \int_G p_t(h^{-1}x)f(h)dh.$$

Example 2.2. *In the case we take G to be $\mathfrak{g}$ thought of as a Lie group with its additive structure, we recover the standard convolution heat kernel relative to the Lebesgue measure given by*

$$p_t(x) = \left(\frac{1}{2\pi t}\right)^{d/2} \exp\left(-\frac{1}{2t}|x|_{\mathfrak{g}}^2\right).$$

2.2. Wiener Measures. The reader is referred to [24, p. 502], [20, Theorem 1.4], [6, 7] and perhaps also in [8] for more details on the summary presented here.

Notation 2.3. *Suppose $0 < T < \infty$. Let us introduce the Wiener and Cameron-Martin (finite energy) spaces, and the corresponding probability measures.*

(1) ***Wiener space*** *will refer to the continuous path space*

$$W(G) = W([0,T], G) = \{\gamma \in C([0,T], G) : \gamma_0 = e\},$$

where we equip $W(G)$ with the uniform metric

$$d_\infty(\alpha, \beta) := \max_{t\in[0,T]} d(\alpha_t, \beta_t).$$

Here d is the left invariant metric on G associated to the left invariant Riemannian metric on G induced from the Ad_G*–invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$.* [*In fact, these metrics are bi-invariant, i.e., both left and right invariant.*] *Let $g_t : W(G) \to G$ (for $0 \leqslant t \leqslant T$) be the projection maps defined by*

$$g_t(\gamma) := \gamma_t, \text{ for all } \gamma \in W(G).$$

We further make $W(G)$ into a group using pointwise multiplication by $(hk)_t := h_t k_t$ for all $h, k \in W(G)$) and $\Theta : W(G) \to W(G)$ be the group inversion defined by

$$\Theta(\gamma) = \gamma^{-1} \text{ for all } \gamma \in W(G).$$

(2) *Given* $h \in W(G)$, *let*

$$\|h\|_{H,T}^2 = \begin{cases} \infty, & \text{if } h \text{ is not absolutely continuous,} \\ \int_0^T |h(s)^{-1}h'(s)|^2 ds, & \text{if } h \text{ is absolutely continuous.} \end{cases}$$

Here $|\cdot|$ *is the norm induced by the inner product* $\langle \cdot, \cdot \rangle$ *on the Lie algebra* $\mathfrak{g}$.

(3) *The* ***Cameron–Martin*** (***finite energy***) *subgroup*, $H(G) \subset W(G)$, *is defined by*

$$H(G) = \{h \in W(G) : \|h\|_{H,T} < \infty\}.$$

(4) *The corresponding spaces of paths with values in the Lie algebra* $\mathfrak{g}$ *and starting at* 0 *are denoted by* $W(\mathfrak{g})$, *and* $H(\mathfrak{g})$, *and the Wiener measure on* $W(\mathfrak{g})$ *is denoted by* ν.

Theorem 2.4 (Wiener measures)**.** *Let* $\mathcal{B}$ *be the Borel* σ*–algebra on* $W(G)$. *There is a probability measure* μ *on* $(W(G), \mathcal{B})$ *uniquely determined by specifying its finite dimensional distributions as follows. For all* $k \in \mathbb{N}$, *partitions* $0 = s_0 < s_1 < s_2 < \ldots < s_{k-1} < s_k = T$ *of* $[0, T]$, *and for all bounded measurable functions* $f : G^k \to \mathbb{R}$

$$\mu(f(g_{s_1}, \ldots, g_{s_k})) = \int_{G^k} f(x_1, \ldots, x_k) \prod_{i=1}^k p_{\Delta s_i}(x_{i-1}^{-1}x_i) dx_1 \cdots dx_k, \quad (2.2)$$

where $x_0 := e$, $\Delta s_i \equiv s_i - s_{i-1}$, $p_t(x)$ *is the convolution heat kernel described in Theorem* 2.1.

The process, $\{g_t\}_{0 \leqslant t \leqslant T}$, *is a* G*-valued Brownian motion with respect to the filtered probability space* $(W(G), \{\mathcal{B}_t\}, \mathcal{B}, \mu)$. *In more detail,* $\{g_t\}_{0 \leqslant t \leqslant T}$ *is a diffusion process on* G *with generator* $\frac{1}{2}\Delta$ *such that* $g_0 = e$ *a.s. As usual, this process has the following martingale property: for all* $f \in (C^\infty(G))$ *the process*

$$M_t^f := f(g_t) - f(g_0) - \frac{1}{2}\int_0^t \Delta f(g_\tau) d\tau \quad (2.3)$$

is a local martingale. In differential form this can be written as

$$df(g) \overset{m}{=} \frac{1}{2}(\Delta f)(g)\, dt, \quad (2.4)$$

where $da \overset{m}{=} db$ *if* $a - b$ *is a local martingale.*

Proof. Equation (2.3) is well known from the theory of Markov processes, see [25]. Indeed, using the Markovian property of μ one computes for $s > t$, F a bounded $\mathcal{B}_t$-measurable function, and $f \in C^\infty(G)$

$$\begin{aligned}\frac{d}{ds}\mu(f(g_s)F) &= \frac{d}{ds}\mu((e^{\frac{s-t}{2}\Delta}f)(g_t)F) \\ &= \frac{1}{2}\mu(e^{\frac{s-t}{2}\Delta}\Delta f)(g_t)F) = \mu(\frac{1}{2}\Delta f(g_s)F).\end{aligned}$$

Integrating the last expression from t to s shows that

$$\mu([M_t^f - M_s^f]F) = \mu\left(\left\{f(g_t) - f(g_s) - \int_s^t \frac{1}{2}\Delta f(g_\tau)d\tau\right\}F\right) = 0,$$

which shows that M^f is a martingale. □

Remark 2.5. Note that the martingale property (2.2) can be extended to vector-valued function. In particular, this applies to G-valued functions since G is assumed to be a matrix-valued Lie group.

2.3. Left and right Brownian motions.

Theorem 2.6 (Quadratic variations)**.** *If u and v are smooth functions on G then*

$$\begin{aligned}d\left[u\left(g_t\right)\right]\cdot d\left[v\left(g_t\right)\right] &= dM_t^u dM_t^v \\ &= \left(\nabla u\left(g_t\right)\cdot\nabla v\left(g_t\right)\right)dt = \sum_{A\in\mathfrak{g}_0}\left(\widetilde{A}u\right)\left(g_t\right)\widetilde{A}v\left(g_t\right)dt.\end{aligned}$$

In particular,

$$dg_t \otimes dg_t = g_t \otimes g_t C dt,$$

where $C := \sum_{A\in\mathfrak{g}_0} A\otimes A$.

Proof. On one hand,

$$d\left[uv(g)\right] \overset{m}{=} \frac{1}{2}\Delta\left(uv\right)(g)\,dt = \frac{1}{2}\left(\Delta uv + u\Delta v + 2\nabla u\cdot\nabla v\right)\left(g_t\right)dt$$

while on the other by Itô's formula,

$$\begin{aligned}d\left[u\left(g\right)v\left(g\right)\right] =& du\left(g\right)\cdot v\left(g\right) + u\left(g\right)\cdot dv\left(g\right) + du\left(g\right)dv\left(g\right) \\ \overset{m}{=}& \frac{1}{2}\left(\Delta uv + u\Delta v\right)\left(g_t dt + dM^u dM^v\right.\end{aligned}$$

Comparing these two equations shows

$$dM^u dM^v \overset{m}{=} (\nabla u \cdot \nabla v)(g_t) dt$$

which gives the first result. More generally, suppose that u and v are vector valued, then

$$\begin{aligned} d[u \otimes v](g) &\overset{m}{=} \frac{1}{2}\Delta(u \otimes v)(g)\, dt \\ &= \frac{1}{2}\left(\Delta u \otimes v + u \otimes \Delta v + 2\widetilde{A}u \otimes \widetilde{A}v\right)(g_t)\, dt \end{aligned}$$

while on the other hand by Itô's formula,

$$\begin{aligned} d[u(g) \otimes v(g)] &= d[u(g)] \otimes v(g) + u(g) \otimes d[v(g)] + d[u(g)] \otimes d[v(g)] \\ &\overset{m}{=} \frac{1}{2}(\Delta u \otimes v + u \otimes \Delta v)(g_t)\, dt + dM^u \otimes dM^v \end{aligned}$$

Comparing these two equations shows

$$dM^u \otimes dM^v \overset{m}{=} \sum_{A \in \mathfrak{g}_0} \left(\widetilde{A}u \otimes \widetilde{A}v\right)(g_t)\, dt.$$

By Remark 2.5 we can take $u(g) = g$ and $v(g) = g$ to see that

$$dg_t \otimes dg_t = \sum_{A \in \mathfrak{g}_0} gA \otimes gA dt \tag{2.5}$$

and $dg = dM + \frac{1}{2} gC dt$, where $C = \sum_{A \in \mathfrak{g}_0} A^2$. □

Remark 2.7. Note that C is independent of the choice of the orthonormal basis of $\mathfrak{g}$ as was pointed out in [14, Lemma 3.1].

Definition 2.8 (Left and right Brownian motions)**.** The process $\{g_t\}_{0 \leqslant t \leqslant T}$ is a semi-martingale and therefore we may define two $\mathfrak{g}$–valued processes by

$$B_t^L := \int_0^t g_\tau^{-1} \delta g_\tau \text{ and } B_t^R := \int_0^t \delta g_\tau g_\tau^{-1}.$$

We refer to B^L (B^R) as the left (right) Brownian motion associated to $\{g_t\}_{0 \leqslant t \leqslant T}$. The terminology will be justified by the next theorem.

Theorem 2.9. *$B_t^L := \int_0^t g_\tau^{-1} \delta g_\tau$ and $B_t^R := \int_0^t \delta g_\tau g_\tau^{-1}$ are standard $\mathfrak{g}$-valued Brownian motions with covariances determined by $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$.*

Proof. Let $b_t := B_t^L := \int_0^t g_\tau^{-1}\delta g_\tau$ temporarily. Then

$$\begin{aligned} db = g^{-1}\delta g &= g^{-1}dg + \frac{1}{2}d\left[g^{-1}\right]dg \\ &= g^{-1}dM + \frac{1}{2}Cdt - \frac{1}{2}\,dbg^{-1}dg \\ &= g^{-1}dM + \frac{1}{2}Cdt - \frac{1}{2}\,dbdb \end{aligned}$$

but $dbdb = \left(g^{-1}dg\right)\left(g^{-1}dg\right) = Cdt$ from (2.5). This shows db is a martingale and that

$$\begin{aligned} db \otimes db &= \left(g^{-1}\otimes g^{-1}\right) dg \otimes dg \\ &= \left(g^{-1}\otimes g^{-1}\right) \sum_{A\in\mathfrak{g}_0} gA \otimes gA\,dt = \sum_{A\in\mathfrak{g}_0} A\otimes A\,dt, \end{aligned}$$

and so by Lévy's criterion b is a standard $\mathfrak{g}$-valued Brownian motion. We now call $b = B^L$. □

Theorem 2.10. *Let $\varphi \in H(G)$. The processes $\left\{B_t^L\right\}_{0\leqslant t\leqslant T}$ and $\left\{B_t^R\right\}_{0\leqslant t\leqslant T}$ are $\mathfrak{g}$-valued Brownian motions satisfying the following properties*

(1) $dB_t^R = \mathrm{Ad}_{g_t}\, dB_t^L = \mathrm{Ad}_{g_t}\,\delta B_t^L$,
(2) $dB_t^L = \mathrm{Ad}_{g_t^{-1}}\, dB_t^R = \mathrm{Ad}_{g_t^{-1}}\,\delta B_t^R$,
(3) $B_t^L\left(\varphi^{-1}g\right) = B_t^L - \int_0^t \mathrm{Ad}_{g^{-1}}\left(\delta\varphi\varphi^{-1}\right)$,
(4) $B_t^L\left(g\varphi\right) = \int_0^t \mathrm{Ad}_{\varphi^{-1}}\, dB^L + \int_0^t \varphi^{-1}\delta\varphi$
(5) $B_t^R\left(\varphi^{-1}g\right) = -\int_0^t \varphi^{-1}\delta\varphi + \int_0^t \mathrm{Ad}_{\varphi^{-1}}\,\delta B^R$
(6) $B_t^R\left(g\varphi\right) = B_t^R + \int_0^t \mathrm{Ad}_g\left(\delta\varphi\varphi^{-1}\right)$.

Proof. For (1)

$$\begin{aligned} \delta B_t^R = \delta g_t g_t^{-1} &= g_t g_t^{-1}\delta g_t g_t^{-1} = \mathrm{Ad}_{g_t}\,\delta B_t^L \\ &= \mathrm{Ad}_{g_t}\, dB_t^L + \frac{1}{2}\left(d\left[\mathrm{Ad}_{g_t}\right]\right) dB_t^L \\ &= \mathrm{Ad}_{g_t}\, dB_t^L + \frac{1}{2}\,\mathrm{Ad}_g\, ad_{dB^L} dB_t^L = \mathrm{Ad}_{g_t}\, dB_t^L, \end{aligned}$$

where we have used the fact that $\delta g = g\delta B^L$ implies $\delta \operatorname{Ad}_g = \operatorname{Ad}_g ad_{\delta B^L}$, and $ad_{dB^L} dB_t^L = 0$. For (2)

$$dB_t^L = g_t^{-1}\delta g_t = g_t^{-1}\left[\delta g_t g_t^{-1}\right] g_t = \operatorname{Ad}_{g_t^{-1}} \delta B_t^R. \tag{2.6}$$

Since $\delta g = \delta B^R g$ implies that $\delta g^{-1} = -g^{-1}\delta B^R$, and therefore $\delta \operatorname{Ad}_{g^{-1}} = -\operatorname{Ad}_{g^{-1}} ad_{\delta B^R}$, and so the Itô form of (2.6) is

$$\begin{aligned} dB_t^L &= \operatorname{Ad}_{g_t^{-1}} dB_t^R + \frac{1}{2}\left(d\left[\operatorname{Ad}_{g_t^{-1}}\right]\right) dB_t^R \\ &= \operatorname{Ad}_{g_t^{-1}} dB_t^R - \frac{1}{2}\operatorname{Ad}_{g^{-1}} ad_{\delta B^R} dB_t^R = \operatorname{Ad}_{g_t^{-1}} dB_t^R. \end{aligned}$$

The remaining items, (3–6), follow from simple computations in Itô's calculus

$$\begin{aligned} B_t^L\left(\varphi^{-1}g\right) &= \int_0^t \left(\varphi^{-1}g\right)^{-1}\delta\left(\varphi^{-1}g\right) = \int_0^t g^{-1}\varphi\left(-\varphi^{-1}\delta\varphi\varphi^{-1}g + \varphi^{-1}\delta g\right) \\ &= B_t^L - \int_0^t \operatorname{Ad}_{g^{-1}}\left(\delta\varphi\varphi^{-1}\right), \\ B_t^L\left(g\varphi\right) &= \int_0^t \left(g\varphi\right)^{-1}\delta\left(g\varphi\right) = \int_0^t \varphi^{-1}g^{-1}\left(\delta g\varphi + g\delta\varphi\right) \\ &= \int_0^t \operatorname{Ad}_{\varphi^{-1}} dB^L + \int_0^t \varphi^{-1}\delta\varphi, \\ B_t^R\left(\varphi^{-1}g\right) &= \int_0^t \delta\left(\varphi^{-1}g\right)\left(\varphi^{-1}g\right)^{-1} = \int_0^t \left(-\varphi^{-1}\delta\varphi\varphi^{-1}g + \varphi^{-1}\delta g\right) g^{-1}\varphi \\ &= -\int_0^t \varphi^{-1}\delta\varphi + \int_0^t \operatorname{Ad}_{\varphi^{-1}}\delta B^R, \text{ and} \\ B_t^R\left(g\varphi\right) &= \int_0^t \delta\left(g\varphi\right)\left(g\varphi\right)^{-1} = \int_0^t \left(\delta g\varphi + g\delta\varphi\right)\varphi^{-1}g^{-1} \end{aligned}$$

$$= B_t^R + \int_0^t \mathrm{Ad}_g \left[\delta\varphi\varphi^{-1}\right]. \qquad \square$$

Before introducing Itô maps, recall some standard definitions.

Notation 2.11. *Suppose $(X, \mathcal{B}, \mu)$ is a measurable space with a σ-finite Borel measure μ, and R is a measurable bijection on X. Then the push-forward of μ is defined by*

$$(R_* \mu)(A) := \left(\mu \circ R^{-1}\right)(A) = \mu\left(R^{-1}(A)\right), \ A \in \mathcal{B}.$$

If the pushforward measure $R_*\mu$ is equivalent to μ, we will denote the Radon–Nikodym derivative as usual by

$$\frac{dR_*\mu}{d\mu}(x), \ x \in X.$$

In particular, for any $A \in \mathcal{B}(X)$ we have

$$\int_X \mathbb{1}_A(x)\, dR_*\mu = \int_X \mathbb{1}_{R^{-1}(A)}(x)\, d\mu = \int_X \mathbb{1}_A(R(x))\, d\mu.$$

Notation 2.12. *Let $(X, \mathcal{Q}_1)$, $(Y, \mathcal{Q}_2)$ be two measurable spaces, and let $I : X \to Y$ be a measurable map. Then for any measurable function $f : Y \to \mathbb{R}$ we denote by*

$$(I^* f)(x) := f(I(x))$$

the induced map on the set of measurable functions on X.

Proposition 2.13. *The maps $B^L, B^R : (W(G), \mu) \to (W(\mathfrak{g}), \nu)$ are μ-a.e. defined maps such that $B_*^L\mu = \nu = B_*^R\mu$. In fact, these maps are measure-preserving isomorphisms from $(W(G), \mu)$ to $(W(\mathfrak{g}), \nu)$ with the inverse maps given by solving the SDEs*

$$\delta w = w\delta B^L \text{ or } \delta w = \delta B^R w \text{ with } w_0 = e$$

for w. Moreover, we have the identities

$$B^L \circ \Theta = -B^R \text{ a.e. and } B^R \circ \Theta = -B^L \text{ a.e.}, \tag{2.7}$$

where the inversion map Θ is defined in Notation 2.3.

Proof. Since

$$\delta g = \delta B^R g \implies \delta g^{-1} = -g^{-1}\delta B^R$$

and hence

$$\begin{aligned} B^L \circ \Theta = B^L \circ \Theta(g) &= \int_0^{\cdot} \left(g^{-1}\right)^{-1} \delta g^{-1} \\ &= \int_0^{\cdot} g\left(-g^{-1}\delta B^R\right) = \int_0^{\cdot} -\delta B^R = -B^R. \end{aligned}$$

Similarly one shows $B^R \circ \Theta = -B^L$ a.e. □

Note that the maps B^L and B^R induce maps on measurable functions from $(W(G), \mu)$ to $(W(\mathfrak{g}), \nu)$ as described in Notation 2.12.

2.4. Quasi-invariance. Our goal in this section is to understand the quasi-invariance properties of μ under left and right translations by $\varphi \in H(G)$.

Theorem 2.14. *For $\varphi \in H(G)$ let*

$$Z_T^R(\varphi) := \exp\left(-\int_0^T \left\langle \varphi' \varphi^{-1}, \delta B^L \right\rangle - \frac{1}{2}\int_0^T \left|\varphi'\varphi^{-1}\right|^2 dt\right)$$

and

$$Z_T^L(\varphi) := \exp\left(\int_0^T \left\langle \varphi' \varphi^{-1}, \delta B^R \right\rangle - \frac{1}{2}\int_0^T \left|\varphi'\varphi^{-1}\right|^2 dt\right)$$

then

$$\mathrm{Law}_{Z_T^R \cdot \mu}(g\varphi) = \mathrm{Law}_\mu(g) = \mathrm{Law}_{Z_T^L \cdot \mu}\left(\varphi^{-1} g\right).$$

That is, for every bounded and measurable function F on $W(G)$

$$\int_{W(G)} F(g\varphi)\, Z_T^R(\varphi)\, d\mu = \int_{W(G)} F d\mu = \int_{W(G)} F\left(\varphi^{-1} g\right) Z_T^L(\varphi)\, d\mu.$$

Proof. We will only prove the assertion involving the right translation here as the second case is proved similarly. To simplify notation let $b := B^L$,

$$M_t := -\int_0^t \left\langle \varphi' \varphi^{-1}, \delta b \right\rangle = -\int_0^t \left\langle \varphi' \varphi^{-1}, db \right\rangle$$

and let Z solve

$$dZ = Z dM = -Z \left\langle \varphi' \varphi^{-1}, db \right\rangle \text{ with } Z_0 = 1, \tag{2.8}$$

i.e.,

$$Z_t := \exp\left(-\int_0^t \left\langle \varphi' \varphi^{-1}, \delta b \right\rangle - \frac{1}{2} \int_0^t \left| \varphi' \varphi^{-1} \right|^2 dt \right) = Z_t^R(\varphi).$$

By (4) of Theorem 2.10

$$(g\varphi)^{-1} \delta (g\varphi) = \mathrm{Ad}_{\varphi^{-1}} \delta b + \varphi^{-1} d\varphi.$$

So given a smooth function, $f : G \to \mathbb{R}$, we have by Itô's lemma that

$$\delta (f(g\varphi)) = f'(g\varphi) \left(\mathrm{Ad}_{\varphi^{-1}} \delta b + \varphi^{-1} d\varphi \right), \tag{2.9}$$

where for $A, B \in \mathfrak{g}$

$$f'(g) A = \widetilde{A} f(g) := \left. \frac{d}{dt} \right|_0 f\left(g e^{tA} \right) \text{ and}$$

$$f''(g)[A \otimes B] := \left(\widetilde{A} \widetilde{B} f \right)(g) = \left. \frac{d}{dt} \right|_0 \left. \frac{d}{ds} \right|_0 f\left(g e^{tA} e^{sB} \right).$$

Note that

$$\begin{aligned} f'(g\varphi) \mathrm{Ad}_{\varphi^{-1}} \delta b &= f'(g\varphi) \mathrm{Ad}_{\varphi^{-1}} db + \frac{1}{2} d[f'(g\varphi)] \mathrm{Ad}_{\varphi^{-1}} db \\ &= f'(g\varphi) \mathrm{Ad}_{\varphi^{-1}} db + \frac{1}{2} [f''(g\varphi)] \left[\mathrm{Ad}_{\varphi^{-1}} db \otimes \mathrm{Ad}_{\varphi^{-1}} db \right] \\ &= f'(g\varphi) \mathrm{Ad}_{\varphi^{-1}} db + \frac{1}{2} \Delta f(g\varphi) dt. \end{aligned}$$

Now we can use the fact that

$$\int_0 \mathrm{Ad}_{\varphi^{-1}} db \tag{2.10}$$

is a $\mathfrak{g}$-valued Brownian motion by Lévy's criterion and due to the Ad-invariance of the inner product on $\mathfrak{g}$. Then the Itô form of (2.9) is

$$d\left[f\left(g\varphi\right)\right]=f'\left(g\varphi\right)\mathrm{Ad}_{\varphi^{-1}}\,db+\left[f'\left(g\varphi\right)\varphi^{-1}\varphi'+\frac{1}{2}\Delta f\left(g\varphi\right)\right]dt.$$

So if we define

$$N_t=N_t^f:=f\left(g_t\varphi_t\right)-\frac{1}{2}\int_0^t\Delta f\left(g_\tau\varphi_\tau\right)d\tau,$$

then

$$dN=f'\left(g\varphi\right)\mathrm{Ad}_{\varphi^{-1}}\,db+f'\left(g\varphi\right)\varphi^{-1}\varphi'dt.$$

Observe that using the orthonormal basis $\mathfrak{g}_0$ of the Lie algebra $\mathfrak{g}$ we have (using $db\otimes db=\sum_{A\in\mathfrak{g}_0}A\otimes Adt$) that

$$\begin{aligned}\left(\mathrm{Ad}_{\varphi^{-1}}\,db\right)\left\langle\varphi'\varphi^{-1},db\right\rangle&=\sum_{A\in\mathfrak{g}_0}\left(\mathrm{Ad}_{\varphi^{-1}}\,A\right)\left\langle\varphi'\varphi^{-1},A\right\rangle dt\\&=\mathrm{Ad}_{\varphi^{-1}}\left(\varphi'\varphi^{-1}\right)dt=\varphi^{-1}\varphi'dt.\end{aligned}$$

Another application of Itô's lemma then implies

$$\begin{aligned}d\left[NZ\right]&=dNZ+NdZ+dNdZ\\&\overset{m}{=}Z\left[f'\left(g\varphi\right)\varphi^{-1}\varphi'dt\right]-\left(f'\left(g\varphi\right)\mathrm{Ad}_{\varphi^{-1}}\,db\right)\cdot Z\left\langle\varphi'\varphi^{-1},db\right\rangle\\&=Z\left[f'\left(g\varphi\right)\varphi^{-1}\varphi'dt\right]-Z\left(f'\left(g\varphi\right)\mathrm{Ad}_{\varphi^{-1}}\,\varphi'\varphi^{-1}\right)dt=0,\end{aligned}$$

where as in (2.4) we write $dX\overset{m}{=}dY$ if X and Y are two processes such that $Y-X$ is a martingale. The previous computations show NZ is martingale and so

$$\mathbb{E}\left[\left(N_t-N_s\right)FZ_T\right]=0$$

for all bounded $\mathcal{B}_s$–measurable functions F. Therefore $\left\{N_t^f\right\}_{0\leqslant t\leqslant T}$ is a $Z_T\cdot\mu$–martingale for all smooth f. Thus it follows from uniqueness to the martingale problems that $\mathrm{Law}_{Z_T\cdot\mu}\left(g\varphi\right)=\mathrm{Law}_\mu\left(g\right)$. □

Theorem 2.14 can be interpreted also using Notation 2.11. Namely, for $X=W\left(G\right)$ and a measurable bijection R on $W\left(G\right)$ we have that for any Borel measurable f on $W\left(G\right)$

$$\mathbb{E}_{R_*\mu}f\left(g\right)=\mathbb{E}_{R_\mu}f\left(R\left(g\right)\right).$$

Let L_φ, R_φ be the left and right multiplication on $W(G)$ defined by

$$\begin{aligned} L_\varphi g &:= \varphi^{-1} g, \\ R_\varphi g &:= g\varphi, \end{aligned} \tag{2.11}$$

where $\varphi \in H(G)$, and $g \in W(G)$, together with their counterparts on functions on $W(G)$ denoted by $L_{\varphi *}$ and $R_{\varphi *}$ according to Notation 2.12. In addition, taking inverses in $(W(G), \mu)$ induces a map on the set of measurable functions on $(W(G), \mu)$ by

$$(Jf)(\gamma) := f \circ \Theta(\gamma) = f\left(\gamma^{-1}\right). \tag{2.12}$$

Note that by Proposition 2.13 the map J is a unitary involution on $L^2(W(G), \mu)$.

Then Theorem 2.10 can be re-written as follows. For any $\varphi \in H(G)$ and $g \in W(G)$ we have

$$\begin{aligned} B^L(L_\varphi g) &= B^L(g) - \int_0^\cdot \mathrm{Ad}_{g^{-1}}\left(d\varphi\varphi^{-1}\right), \\ B^L(R_\varphi g) &= \int_0^\cdot \varphi^{-1} d\varphi + \int_0^\cdot \mathrm{Ad}_{\varphi^{-1}}\left(\delta B^L\right), \\ B^R(L_\varphi g) &= -\int_0^\cdot \varphi^{-1} d\varphi + \int_0^\cdot \mathrm{Ad}_{\varphi^{-1}}\left(\delta B^R\right), \\ B^R(R_\varphi g) &= B^R(g) + \int_0^\cdot \mathrm{Ad}_g\left(d\varphi\varphi^{-1}\right), \end{aligned} \tag{2.13}$$

where we use $d\varphi$ to indicate that it is the usual differential since φ is smooth.

Then the right Radon-Nikodym density $Z^R(\varphi)$ for $R_{\varphi *}\mu$ with respect to μ is in $L^1(W(G), \mu)$ is described in Theorem 2.14. Similarly the Wiener measure μ is quasi-invariant under the left multiplication by elements in $H(G)$, and the left Radon-Nikodym density for μ is in $L^1(W(G), \mu)$ as well.

Proposition 2.15. *The left and right Radon-Nikodym densities for μ satisfy*

$$Z_\varphi^R = JZ_\varphi^L = Z_\varphi^L \circ \Theta$$

for μ-almost every g. Here J is the map defined by (2.12).

Proof. First proof. By Proposition 2.13 μ is invariant under the taking group inverses, that is, for any bounded measurable f

$$\int_{W(G)} f(g^{-1})d\mu(g) = \int_{W(G)} f(g)d\mu(g).$$

Then

$$\int_{W(G)} f(g\varphi)d\mu(g) = \int_{W(G)} f(g^{-1}\varphi)d\mu(g) = \int_{W(G)} f\left(\left(\varphi^{-1}g^{-1}\right)^{-1}\right)d\mu(g)$$

$$= \int_{W(G)} f\left(g^{-1}\right)Z_{\varphi}^{L}(g)d\mu(g) \int_{W(G)} f\left(g\right)Z_{\varphi}^{L}(g^{-1})d\mu(g). \square$$

§3. Cyclicity

Cyclicty is one of the basic properties of representations of $H(G)$ we consider later. Note that the main result of this section, Theorem 3.1, follows from Corollary 14 in [17]. In that paper B. Hall and A. Sengupta used the Segal–Bargmann transform to prove the cyclicity of $\mathbb{1}$, and also that the Radon–Nikodym densities are coherent states as Theorem 10 in [17] states. We give a more direct proof using the inverse Itô map B^L and ideas of L. Gross in [15].

Theorem 3.1 (Cyclicity of $\mathbb{1}$). *Suppose that G is a compact connected Lie group, then*

$$\mathcal{H}_G := \operatorname{Span}\left\{\left(Z_{\varphi}^{R}(g)\right)^{1/2}, \varphi \in H(G)\right\}$$

is dense in $L^2(W(G), \mu)$.

Proof. Note that $\left(B^L\right)^*\left(Z_{\varphi}^{R}\right)^{1/2}$ is a function on $W(\mathfrak{g})$ since B^L is a measure space isomorphism, so we can reduce the problem to the Lie algebra level. Namely, let $0 = t_0 < t_1 < ... < t_{n-1} < t_n = T$, $\xi_0 = 0, \xi_1, ..., \xi_n \in \mathfrak{g}$. We assume that

$$|\xi_j||t_j - t_{j-1}| = 1, \text{ for any } j = 1, 2, ..., n, \tag{3.1}$$

unless $\xi_j = 0$. It is known that the linear span of multidimensional Hermite polynomials in $\langle \xi_j, w(t_j) - w(t_{j-1})\rangle$ is dense in $L^2(W(\mathfrak{g}), \nu)$ (e.g. [21]).

This means that it is enough to show that the linear span of cylinder Hermite polynomials is contained in the $L^2(W(\mathfrak{g}),\nu)$-closure of $\left(B^L\right)^*(\mathcal{H}_G)$.

First we observe that $\mathcal{H}_G$, and therefore $\left(B^L\right)^*(\mathcal{H}_G)$, contains all constant functions. Let $0=t_0<t_1<...<t_{n-1}<t_n=T$, $\xi_0=0,\xi_1,...,\xi_n\in\mathfrak{g}$. We define a function $\varphi=\varphi_{\xi_1,\dots,\xi_n}(s)$ recursively for $j=1,2,...,n$ by

$$\varphi(t_0)=\varphi(0)=e,\ \varphi(s)=e^{-(s-t_{j-1})\xi_j}\varphi(t_{j-1}),\ s\in[t_{j-1},t_j). \tag{3.2}$$

Then

$$\varphi'(s)\varphi(s)^{-1}=-\xi_j,\ s\in[t_{j-1},t_j),$$

therefore $\varphi\in H(G)$ and

$$\begin{aligned}&\left(B^L\right)^*\left(Z_\varphi^R\right)^{1/2}(w_t)\\&\quad=\prod_{j=1}^n\exp\left(\frac{1}{2}\langle\xi_j,w(t_j)-w(t_{j-1})\rangle-\frac{1}{4}|\xi_j|^2(t_j-t_{j-1})^2\right).\end{aligned}$$

Suppose $x_1,\dots,x_n\in\mathbb{R}$ and define $\varphi_{\vec{x}}(s):=\varphi_{x_1\xi_1,\dots,x_n\xi_n}(s)$, then $\varphi'_{\vec{x}}(s)\varphi_{\vec{x}}(s)^{-1}=x_j\xi_j$. Now let a function F on $\mathbb{R}^n$ be defined as $F(\vec{x}):=\left(B^L\right)^*\left(Z_{\varphi_{\vec{x}}}^R\right)^{1/2}$ then

$$\frac{\partial F}{\partial x_j}(0)=\frac{1}{2}\langle\xi_j,w(t_j)-w(t_{j-1})\rangle,$$

Note that for any $\vec{x}\in\mathbb{R}^n$ we have $F(\vec{x})\in\left(B^L\right)^*(\mathcal{H}_G)$. Therefore $\frac{\partial F}{\partial x_j}(0)$ as well as all other partial derivatives of F at 0 are in $\overline{\left(B^L\right)^*(\mathcal{H}_G)}$, the L^2-closure of $\left(B^L\right)^*(\mathcal{H}_G)$. Indeed, this follows from the simple observation that $F(0)=1\in\left(B^L\right)^*(\mathcal{H}_G)$ and

$$\frac{\partial F}{\partial x_j}(0)=\lim_{x_j\to 0}\frac{F((0,...,x_j,0,...,0))-1}{x_j}.$$

Now we would like to describe the functions we can get by taking partial derivatives of F. First we observe that we can write F as

$$F(\vec{x})=\prod_{j=1}^n e^{a_jx_j-b_j^2x_j^2},a_j=\frac{\langle\xi_j,w(t_j)-w(t_{j-1})\rangle}{2},b_j=\frac{|\xi_j||t_j-t_{j-1}|}{2}=\frac{1}{2}$$

by assumption (3.1). Using [5, Lemma 1.3.2 (part (iii))] we can take partial derivatives of F of all orders to see that all multidimensional Hermite polynomials in $\langle \xi_j, w(t_j) - w(t_{j-1})\rangle$ are in $\overline{(B^L)^* (\mathcal{H}_G)}$. □

§4. Brownian measure representation

4.1. Definitions and notation. The unitary representations of $H(G)$ on the Hilbert space $L^2(W(G), \mu)$ we define in this section are induced by quasi-invariance of the Wiener measure μ. Recall that L_φ and R_φ are left and right multiplication on $W(G)$ by elements $H(G)$ as defined in (2.11), i.e., $R_\varphi \gamma = \gamma\varphi$ and $L_\varphi \gamma = \varphi^{-1}\gamma$.

Definition 4.1. Let $W(G)$ and $H(G)$ be as before.

(1) The **right Brownian measure representation** U^R of $H(G)$ on $L^2(W(G), \mu)$ is defined as

$$\left(U^R_\varphi f\right)(g) := \left(Z^R_\varphi(g)\right)^{1/2} f(R_\varphi g)$$

for any $f \in L^2(W(G), \mu)$, $\varphi \in H(G)$, $g \in W(G)$;

(2) the **left Brownian measure representation** U^L on $L^2(W(G), \mu)$ is defined as

$$\left(U^L_\varphi f\right)(g) := \left(Z^L_\varphi(g)\right)^{1/2} f(L_\varphi g)$$

for any $f \in L^2(W(G), \mu)$, $\varphi \in H(G)$, $g \in W(G)$.

Recall that by Proposition 2.15 we have $Z^R_\varphi = JZ^L_\varphi$, where J a unitary involution on $L^2(W(G), \mu)$ defined by (2.12). In addition, the functions $\left(Z^R_\varphi\right)^{1/2}$ and $\left(Z^L_\varphi\right)^{1/2}$ have the norm 1 in $L^2(W(G), \mu)$ for any $\varphi, \psi \in H(G)$, which is a consequence of the next Proposition.

Proposition 4.2. *For any* $\varphi, \psi \in H(G)$

$$\langle \left(Z^R_\varphi\right)^{1/2}, \left(Z^R_\psi\right)^{1/2} \rangle =$$

$$\exp\left(-\frac{\|\varphi\|^2_{H,T} + \|\psi\|^2_{H,T}}{8}\right) \exp\left(\frac{1}{4}\int_0^T \langle \left(\varphi^{-1}\varphi'\right)(t), \left(\psi^{-1}\psi'\right)(t)\rangle dt\right).$$

Proof. This follows from Theorem 2.14. □

Proposition 4.3. *For any $\varphi, \psi \in H(I, G)$ we have*

$$Z_{\varphi}^{R}(\cdot) = Z_{\psi}^{R}(\cdot) \text{ if and only if } \varphi = \psi,$$

and similarly

$$Z_{\varphi}^{L}(\cdot) = Z_{\psi}^{L}(\cdot) \text{ if and only if } \varphi = \psi,$$

where $Z^{R}(\varphi)(\cdot)$ and $Z^{L}(\varphi)(\cdot)$ are viewed as random variables, and the equalities hold for μ-a.e. $g, t \in [0, T]$.

Proof. If

$$Z_{\varphi}^{R}(\cdot) = Z_{\psi}^{R}(\cdot),$$

then for any $t \in [0, T]$,

$$\mathbb{E}\left(Z_{\varphi}^{R}(\cdot) | \mathcal{F}_t\right) = \mathbb{E}\left(Z_{\psi}^{R}(\cdot) | \mathcal{F}_t\right)$$

and therefore

$$\int_0^t \langle \psi^{-1}\psi'(s) - \varphi^{-1}\varphi'(s), dB_s^L \rangle = \frac{1}{2}\int_0^t \left(|\varphi^{-1}\varphi'|^2 - |\psi^{-1}\psi'|^2\right) ds.$$

Taking expectations of this equation then shows

$$0 = \frac{1}{2}\int_0^t \left(|\varphi^{-1}\varphi'|^2 - |\psi^{-1}\psi'|^2\right) ds \text{ for all } t$$

and therefore $|\varphi^{-1}\varphi'|^2 = |\psi^{-1}\psi'|^2$ a.e. In particular, we then have

$$\begin{aligned} 0 &= \mathbb{E}\left[\left(\int_0^t \langle \psi^{-1}\psi'(s) - \varphi^{-1}\varphi'(s), dB_s^L \rangle\right)^2\right] \\ &= \int_0^t \left|\psi^{-1}\psi'(s) - \varphi^{-1}\varphi'(s)\right|^2 ds \end{aligned}$$

from which it follows $\psi^{-1}\psi'(t) - \varphi^{-1}\varphi'(t) = 0$ for any $t \in [0, T]$. Finally, we see that for any $t \in [0, T]$

$$\left(\varphi\psi^{-1}\right)'(t) = \varphi'\psi^{-1}(t) - \varphi\psi^{-1}\psi'\psi^{-1}(t) = \varphi'\psi^{-1}(t) - \varphi\varphi^{-1}\varphi'\psi^{-1}(t) = 0$$

and therefore $\varphi^{-1}\psi \equiv e$. □

Proposition 4.4. *For any* $\varphi, \psi, \varphi_1, ..., \varphi_n, \psi_1, .., \psi_n \in H(G)$, $f \in L^2(W(G), \mu)$

$$\begin{aligned}
&\left(U^R_{\varphi_1} ... U^R_{\varphi_n}\right) f(g) = \left(Z^R_{\varphi_n \cdots \varphi_1}\right)^{1/2}(g) f\left(R_{\varphi_1 \cdots \varphi_n} g\right), \\
&\left(U^L_{\psi_1} ... U^L_{\psi_n}\right) f(g) = \left(Z^L_{\psi_n \cdots \psi_1}\right)^{1/2}(g) f\left(L_{\psi_1 \cdots \psi_n} g\right), \\
&\left(U^R_{\varphi}\right)^{-1} = \left(U^R_{\varphi}\right)^* = U^R_{\varphi^{-1}}, \\
&\left(U^L_{\psi}\right)^{-1} = \left(U^L_{\psi}\right)^* = U^L_{\psi^{-1}}.
\end{aligned}$$

In particular, this implies that U^R_φ, U^L_ψ *are unitary operators on* $L^2(W(G), \mu)$.

Proof. For any $f, h \in L^2(W(G), \mu)$, $\varphi, \varphi_1, \varphi_2 \in H(G)$ we have

$$\begin{aligned}
\left(U^R_{\varphi_1} U^R_{\varphi_2} f\right)(g) &= \left(Z^R_{\varphi_1}(g) Z^R_{\varphi_2}(g\varphi_1)\right)^{1/2} f(g\varphi_1\varphi_2) \\
&= \left(Z^R_{\varphi_2\varphi_1}\right)^{1/2}(g) f(g\varphi_1\varphi_2)
\end{aligned}$$

by the properties of the Radon–Nikodym densities, and

$$\begin{aligned}
&\langle \left(U^R_\varphi\right)^* f, h\rangle_{L^2(W(G),\mu)} = \langle f, U^R_\varphi h\rangle_{L^2(W(G),\mu)} \\
&\quad = \int_{W(G)} f(g) h(g\varphi) h_\varphi(g) d\mu(g) \\
&\quad = \int_{W(G)} f(g\varphi^{-1}) h(g) \left(Z^R_\varphi\right)^{1/2}(g\varphi^{-1}) Z^R_{\varphi^{-1}}(g)\, d\mu(g) \\
&\quad = \int_{W(G)} f(g\varphi^{-1}) \left(Z^R_{\varphi^{-1}}\right)^{1/2}(g) h(g) d\mu(g) = \langle U^R_{\varphi^{-1}} f, h\rangle_{L^2(W(G),\mu)}.
\end{aligned}$$

The case of U^L can be checked similarly. □

4.2. Properties of the Brownian representations.

Notation 4.5. *We denote by*

$$\begin{aligned}
\mathcal{M}^R &:= \left(U^R_\varphi, \varphi \in H(G)\right)'' \\
\mathcal{M}^L &:= \left(U^L_\varphi, \varphi \in H(G)\right)''
\end{aligned}$$

the von Neumann algebras generated by the operators U^R_φ, U^L_φ *respectively.*

Theorem 4.6 collects some basic facts about the left and right Brownian representations. Most of these properties are what one expects from the classical case of regular representations of locally compact groups. But some of the proofs are fundamentally different. For example, the fact that the von Neumann algebras generated by the left and right representations are commutants of each other has been originally proved by I. Segal in [23] for the regular representation of a unimodular locally compact Lie group with a bi-invariant Haar measure. One of the major facts he used was existence of an approximating identity and the one-to-one correspondence between unitary representation of the group G and the non-degenerate $*$-representations of the group algebra $L^1(G)$ (e.g. [10, Section 3.2]). These fundamental constructions are not available in our case. Theorem 4.6 does not answer the question whether $\mathcal{M}^L$ and $\mathcal{M}^R$ are commutants of each other, which will be addressed in another article.

Theorem 4.6.
(1) *the unitary operators U_φ^R and U_ψ^L commute for any $\varphi, \psi \in H(G)$, and so $\left(\mathcal{M}^R\right)' \subseteq \mathcal{M}^L$ and $\left(\mathcal{M}^L\right)' \subseteq \mathcal{M}^R$. The representations U^L and U^R are unitarily equivalent, and the intertwining operator is the unitary involution J defined by* (2.12)*;*
(2) *$\Omega = \mathbb{1}$ is a separating cyclic vector of norm 1 for both $\mathcal{M}^R$ and $\mathcal{M}^L$ in $L^2(W(G), \mu)$. If G is abelian, then the corresponding von Neumann algebra $\mathcal{M}^R = \mathcal{M}^L$ is maximal abelian in $B\left(L^2\left(W(G); \mu\right)\right)$.*
(3) *For any $T \in \mathcal{M}^R$ the map $T \longmapsto T\mathbb{1}$ is injective.*
(4) *The vacuum vector $\Omega = \mathbb{1}$ defines a faithful normal weight τ on $\mathcal{M}^R$ (and similarly on $\mathcal{M}^L$) by*

$$\tau(m) := \langle m\Omega, \Omega \rangle_{L^2(W(G),\mu)} = \int_{W(G)} m(\mathbb{1})(g)\, d\mu(g) \tag{4.1}$$

for any $m \in \mathcal{M}^R$. In addition, $\tau(\mathrm{I})$ is finite, and so τ is a faithful normal state.

Proof. 1. First we observe that U_φ^L and U_ψ^R commute. Indeed, for any $\varphi, \psi \in H(G)$, $f \in L^2(W(G), \mu)$ we have

$$\left(U_\psi^L U_\varphi^R f\right)(g) = \left(\frac{d\mu\left(\psi^{-1} g\right)}{d\mu(g)}\right)^{1/2} \left(\frac{d\mu\left(\psi^{-1} g \varphi\right)}{d\mu\left(\psi^{-1} g\right)}\right)^{1/2} f\left(\psi^{-1} g \varphi\right)$$

$$= \left(\frac{d\mu\left(\psi^{-1}g\varphi\right)}{d\mu\left(g\right)}\right)^{1/2} f\left(\psi^{-1}g\varphi\right) = \left(U_{\varphi}^{R}U_{\psi}^{L}f\right)(g).$$

To see that U^L and U^R are unitarily equivalent we use Proposition 2.15, and the following simple observation. Using Notation 2.12 for the left and right multiplication operators on $W(G)$, we see that

$$JR_{\varphi\,*} = L_{\varphi\,*}J.$$

Then by Proposition 2.15 for any $f \in L^2(W(G), \mu)$

$$\begin{aligned}\left(JU_{\varphi}^{R}f\right)(g) &= J\left(Z_{\varphi}^{R}(g)\left(R_{\varphi\,*}f\right)(g)\right) = Z_{\varphi}^{L}(g)\,J\left(R_{\varphi\,*}f(g)\right)\\ &= Z_{\varphi}^{L}(g)\left(L_{\varphi\,*}Jf(g)\right) = \left(U_{\varphi}^{L}Jf\right)(g).\end{aligned}$$

2. Theorem 3.1 shows that $\mathbb{1}$ is cyclic for $\mathcal{M}^R$, and similarly one can show that it is cyclic for $\mathcal{M}^L$.

Now suppose that G is abelian. It is clear that in this case $\mathcal{M} = \mathcal{M}^R = \mathcal{M}^L$ is abelian, and therefore $\mathcal{M}' = \mathcal{M}$ which implies that it is maximal abelian. Note that another explanation for $\mathcal{M}$ being maximal abelian is that as we know it has a cyclic vector. Then by [19, Corollary 7.2.16] $\mathcal{M}$ is maximal abelian as an abelian subalgebra with a cyclic vector.

3. This is a standard fact from the Tomita-Takesaki theory, but in this case it is easy to verify and we include the argument for completeness. Let $T \in \mathcal{M}^R$ be such that $T\mathbb{1} = 0$. Then T commutes with all operators in $\mathcal{M}^L$, and therefore

$$U_{\psi^{-1}}^{L}TU_{\psi}^{L}\mathbb{1} = T\mathbb{1} = 0,$$

and so

$$TU_{\psi}^{L}\mathbb{1} = 0$$

for all $\psi \in H(G)$. Since $\mathbb{1}$ is cyclic for both left and right representations, we see that $T = 0$.

4. The first part of this statement is a standard fact following from the GNS construction (e.g. [26]). To see that τ is a state, we note that the identity operator I in $\mathcal{M}^R$ can be represented as U_e^R, where $e(t) \equiv e$ for $t \in [0, T]$. Thus

$$\tau(\mathrm{I}) = \tau\left(U_e^R\right) = 1.$$

The same holds for $\mathcal{M}^L$. □

Proposition 4.7 (τ is not a trace)**.** *For any* $\varphi, \psi \in H^T(G)$,

$$\tau\left(U_\varphi^R U_\psi^R\right) = \tau\left(U_\psi^R U_\varphi^R\right)$$

if and only if

$$\int_0^T \langle \varphi^{-1}\varphi', \psi'\psi^{-1}\rangle ds = \int_0^T \langle \varphi'\varphi^{-1}, \psi^{-1}\psi'\rangle ds. \tag{4.2}$$

Proof. By definition of τ and Propositions 4.2 and 4.4 we see that

$$\begin{aligned}
\tau\left(U_\varphi^R U_\psi^R\right) &= E_\mu Z_{\psi\varphi}^R(g) = \exp\frac{-\|\psi\varphi\|_{H,T}^2}{8} \\
&= \exp\frac{-\|\varphi\|_{H,T}^2 - \|\psi\|_{H,T}^2}{8} \exp\frac{1}{4}\int_0^T \langle \operatorname{Ad}_\varphi \varphi'\varphi^{-1}, \psi'\psi^{-1}\rangle\, dt \\
&= \exp\frac{-\|\varphi\|_{H,T}^2 - \|\psi\|_{H,T}^2}{8} \exp\frac{1}{4}\int_0^T \langle \varphi'\varphi, \psi^{-1}\psi'\rangle\, dt.
\end{aligned}$$

Applying this computation to $\tau\left(U_\psi^R U_\varphi^R\right)$ completes the proof. □

§5. Energy representation

Let (H, W, Γ) be an abstract Wiener space, that is, H is a real separable Hilbert space densely continuously embedded into a real separable Banach space W, and Γ is the Gaussian measure defined by the characteristic functional

$$\int_W e^{i\varphi(x)} d\Gamma(x) = \exp\left(-\frac{|\varphi|_{H^*}^2}{2}\right)$$

for any $\varphi \in W^* \subset H^*$. We will identify W^* with a dense subspace of H such that for any $h \in W^*$ the linear functional $\langle \cdot, h\rangle$ extends continuously from H to W. We will usually write $\langle \varphi, w\rangle := \varphi(w)$ for $\varphi \in W^*$, $w \in W$. More details can be found in [5].

It is known that Γ is a Borel measure, that is, it is defined on the Borel σ-algebra $\mathcal{B}(W)$ generated by the open subsets of W. The Gaussian measure Γ is quasi-invariant under the translations from H and invariant under orthogonal transformations of H. We want to be more precise here.

Notation 5.1. *We call an orthogonal transformation of H which is a topological homeomorphism of W^* a* ***rotation*** *of W^*. The space of all such rotations is denoted by $\mathrm{O}(W^*)$. For any $R \in \mathrm{O}(W^*)$ its adjoint, R^*, is defined by*

$$\langle \varphi, R^* w\rangle := \langle R^{-1}\varphi, w\rangle, \ \ w \in W, \varphi \in W^*.$$

Theorem 5.2. *For any $R \in \mathrm{O}(W^*)$ the map R^* is a $\mathcal{B}(W)$-measurable map from W to W and*

$$\Gamma \circ (R^*)^{-1} = \Gamma.$$

Proof. The measurability of R^* follows from the fact that R is continuous on H. For any $\varphi \in W^*$

$$\begin{aligned} \int_W e^{i\varphi(x)} d\Gamma\left((R^*)^{-1} x\right) &= \int_W e^{i\langle\varphi, x\rangle} d\Gamma\left((R^*)^{-1} x\right) = \int_W e^{i\langle\varphi, R^* x\rangle} d\Gamma(x) \\ &= \exp\left(-\frac{|R^{-1}\varphi|^2_{H^*}}{2}\right) = \exp\left(-\frac{|\varphi|^2_{H^*}}{2}\right) \\ &= \int_W e^{i\varphi(x)} d\Gamma(x) \end{aligned}$$

since R is an isometry. □

Corollary 5.3. *Any $R \in \mathrm{O}(W^*)$ extends to a unitary map on $L^2(W, \Gamma)$.*

The Cameron–Martin theorem states that Γ is quasi-invariant under translations by elements in H, namely, $T_h : W \to W$, $T_h(w) = w + h$. The Radon–Nikodym derivative is given by

$$\frac{d(T_h)_* \Gamma}{d\Gamma}(w) = \frac{d\left(\Gamma \circ T_h^{-1}\right)}{d\Gamma}(w) = \frac{d(\Gamma \circ T_{-h})}{d\Gamma}(w) = e^{-\langle h, w\rangle - \frac{|h|^2}{2}},$$
$$w \in W, \quad h \in H.$$

Following [9] we consider the Gaussian regular representation of the Euclidean group of transformations $w \mapsto R^* w + h$, $x \in H, h \in H, R \in \mathrm{O}(W^*)$ on $L^2(W, \Gamma)$ defined as

$$
\begin{aligned}
\left(U_{R,h}f\right)(w) &:= \left(\frac{d\left(\Gamma\circ\left(T_hR^*\right)\right)}{d\Gamma}(w)\right)^{1/2} f\left(\left(T_hR^*\right)^{-1}(w)\right) \\
&= \left(\frac{d\left(\Gamma\circ T_h\right)}{d\Gamma}(w)\right)^{1/2} f\left(\left(R^*\right)^{-1}(w-h)\right) \\
&= e^{\langle h,w\rangle-\frac{|h|^2}{2}} f\left(\left(R^*\right)^{-1}(w-h)\right),\ w\in W
\end{aligned}
\tag{5.1}
$$

which is well-defined by Corollary 5.3. It is clear that this is a unitary representation.

Now we need to define the Fourier-Wiener transform $\mathcal{F}$ on $L^2(W,\Gamma)$. This can be done in several ways, and for now we refer to Definition 17 in [9] with the parameter $r=1/2$. In particular, one can check that $\mathcal{F}^4\equiv I$ on $L^2(W,\Gamma)$ by doing a computation on Hermite functions.

The following formula is very convenient for computations, but some care should be taken over its applicability. One of the ways of making this formula rigorous is to define it on Hermite functions using the Fock space, as it is done in [16].

$$
\left(\mathcal{F}f\right)(w)=\int_W f\left(iw+\sqrt{2}u\right)d\Gamma(u),\ f\in L^2(W,\Gamma).
$$

In particular, identities in Proposition 5.4 follow from this formula quite easily.

Proposition 5.4. 1. *Let* $\mathcal{E}:=\operatorname{Span}_{\mathbb{C}}\{\widehat{\varphi}(w):=e^{i\langle\varphi,w\rangle},\ \varphi\in W^*,\ w\in W\}$. *Then* $\mathcal{E}$ *is an algebra which is dense in* $L^2(W,\Gamma)$.

2. *For any* $\varphi\in W^*$ *we have*

$$
\begin{gathered}
\int_W \widehat{\varphi}(w)\,d\Gamma(w)=e^{-\frac{|\varphi|^2_{H^*}}{2}}, \\
\left(\mathcal{F}\widehat{\varphi}\right)(w)=e^{-|\varphi|^2_{H^*}}e^{-\langle\varphi,w\rangle},\ \textit{and}\ \left(\mathcal{F}e^{\langle\varphi,\cdot\rangle}\right)(w)=e^{|\varphi|^2_{H^*}}\widehat{\varphi}(w).
\end{gathered}
\tag{5.2}
$$

Proof. The first statement is proven in a number of references, one of which is [18], Theorem 4.1, so we omit the proof for now. Identities in (5.2) follow from similar finite-dimensional calculations using the methods in [9] or approximations by Hermite functions. □

Proposition 5.5 (Proposition 18 [9]). *If* $f \in L^2(W,\Gamma)$, $R \in \mathrm{O}(W^*)$, $h \in W^*$, *then*

$$\left(\mathcal{F}U_{R,h}\mathcal{F}^{-1}f\right)(w) = e^{-\frac{i\langle h,w\rangle}{2}} f(R^*w) \text{ for } w \in W.$$

Proof. By Proposition 5.4 it is enough to check the statement for $f(w) = \widehat{\varphi}(w)$. First, let us compute $\mathcal{F}^3\widehat{\varphi}(w)$ using (5.2)

$$\begin{aligned}\left(\mathcal{F}^3\widehat{\varphi}\right)(w) &= e^{-|\varphi|^2_{H^*}}\left(\mathcal{F}^2 e^{-\langle\varphi,\cdot\rangle}\right)(w)\\ &= e^{-|\varphi|^2_{H^*}} e^{|\varphi|^2_{H^*}}\left(\mathcal{F}e^{-i\langle\varphi,\cdot\rangle}\right)(w) = e^{-|\varphi|^2_{H^*}} e^{\langle\varphi,w\rangle}.\end{aligned}$$

Then

$$\begin{aligned}\left(\mathcal{F}U_{R,h}\mathcal{F}^{-1}\widehat{\varphi}\right)(w) &= \left(\mathcal{F}U_{I,h}U_{R,0}\mathcal{F}^3\widehat{\varphi}\right)(w)\\ &= e^{-|\varphi|^2_{H^*}}\left(\mathcal{F}U_{I,h}U_{R,0}e^{\langle\varphi,\cdot\rangle}\right)(w)\\ &= e^{-|\varphi|^2_{H^*}} e^{-\frac{|h|^2}{4}}\left(\mathcal{F}e^{\frac{\langle h,\cdot\rangle}{2}} e^{\langle R\varphi,\cdot+h\rangle}\right)(w)\\ &= e^{-|\varphi|^2_{H^*}} e^{-\frac{|h|^2}{4}} e^{\langle R\varphi,h\rangle}\left(\mathcal{F}e^{i\frac{\langle -i(h+2R\varphi),\cdot\rangle}{2}}\right)(w)\\ &= e^{-\frac{|h+2R\varphi|^2_{H^*}}{4}} e^{\frac{|h+2R\varphi|^2_{H^*}}{4}} e^{i\langle\frac{h}{2}+R\varphi,w\rangle}\\ &= e^{i\langle\frac{h}{2},w\rangle}\widehat{\varphi}(R^*w),\end{aligned}$$

where we used the fact that $|R\varphi|_{H^*} = |\varphi|_{H^*}$. □

Corollary 5.6. *By taking* $f \equiv 1$ *in Proposition* 5.5, *we see that for any* $h \in H$

$$\mathcal{F}e^{\langle h,w\rangle-\frac{|h|^2}{2}} = e^{-\frac{i\langle h,w\rangle}{2}}.$$

We now work on the measure space $\left(W(\mathfrak{g}), \mathcal{B}_{W(\mathfrak{g})}, \nu\right)$ and let $w_s : W(\mathfrak{g}) \to \mathfrak{g}$ be the projection map, $w_s(\omega) = \omega_s$ for all $0 \leqslant s \leqslant T$ and $\omega \in W(\mathfrak{g})$. [Note, we may also view w as the identity map from $W(\mathfrak{g})$ to $W(\mathfrak{g})$.] The energy representation is a unitary representation of $H(G)$ on the space $L^2(W(\mathfrak{g}), \nu)$. First we introduce an operator on $W(\mathfrak{g})$ used to define the energy representation. Note that since the inner product on $\mathfrak{g}$ is Ad-invariant, the operator O_φ defined by

$$O_{\varphi}(w) := \int_0^{\cdot} \mathrm{Ad}_{\varphi}\, \delta w_s, w \in W(\mathfrak{g}), \varphi \in H(G) \tag{5.3}$$

is well-defined on $W(\mathfrak{g})$ by Lévy's criterion as we indicated in (2.10). Moreover, since the Itô and Stratonovich integrals of deterministic integrands are equal, we see that

$$O_{\varphi}(w) = \int_0^{\cdot} \mathrm{Ad}_{\varphi}\, \delta w_s = \int_0^{\cdot} \mathrm{Ad}_{\varphi}\, dw_s.$$

Definition 5.7. For any $\varphi \in H(G)$

$$(E_{\varphi} f)(w) := e^{i \int_0^T \langle \varphi^{-1} \varphi'(s), dw_s \rangle} f\left(O_{\varphi^{-1}} w\right).$$

for any $f \in L^2(W(\mathfrak{g}), \nu)$. Then E_{φ} is called the *energy representation* of $H(G)$.

Again using the fact that the Itô and Stratonovich integrals are equal for deterministic integrands, we see that

$$(E_{\varphi} f)(w) = e^{i \int_0^T \langle \varphi^{-1} \varphi'(s), dw_s \rangle} f\left(O_{\varphi^{-1}} w\right).$$

It is easy to see that $E_{\varphi}^{*} = E_{\varphi^{-1}}$, so it is a unitary representation of $H(G)$ on $L^2(W(\mathfrak{g}), \nu)$. For our future results using Itô integrals will be more convenient, so this is what we will be using from now on mostly.

Theorem 5.8. *Both U^R and U^L are unitarily equivalent to the energy representation E.*

Proof. As we noted in Theorem 4.6, U^R and U^L are unitarily equivalent. Using (2.13) we see that under the inverse Itô map B^L the left multiplication is mapped to the following operator

$$\left(\left(B^L\right)^{*} R_{\varphi}^{*}\right) f(w) = f\Big(O_{\varphi^{-1}} w + \int_0^{\cdot} \varphi^{-1} d\varphi\Big), \tag{5.4}$$

where $f \in L^2\big(W(\mathfrak{g}), \nu\big)$, $w \in W(\mathfrak{g})$, and R_{φ}^{*} is the adjoint operator.

Then the representation U_{φ}^R corresponds to the following representation on $L^2(W(\mathfrak{g}), \nu)$

$$\begin{aligned}\left(u_\varphi^R f\right)(w) &:= \Big(\left(B^L\right)^* U_\varphi^R f\Big)(w) \qquad (5.5)\\ &= e^{\frac{1}{2}\int_0^T \langle \varphi^{-1}\varphi'(s), dw_s\rangle - \frac{1}{4}\|\varphi\|^2_{H,\cdot}} f\Big(O_{\varphi^{-1}} w + \int_0^\cdot \varphi^{-1} d\varphi\Big).\end{aligned}$$

Here we used $O_{\varphi^{-1}}$ to denote the operator introduced by (5.3). Note that $\left(u_\varphi^R f\right)(w) = U_{R,h}$, where $U_{R,h}$ is defined by (5.1) with $R^*(w) = O_{\varphi^{-1}} w$ and $h = -\varphi^{-1} d\varphi$. The adjoint representation of G on $\mathfrak{g}$ is unitary, and therefore $O_{\varphi^{-1}}$ is a continuous unitary transformation on $H(\mathfrak{g})$. Thus we can apply Proposition 5.5 to see that u_φ^R is unitarily equivalent to E_φ. The intertwining operator here is the Fourier–Wiener transform $\mathcal{F}$, and the intertwining map between U^L and E is then $\mathcal{F} \circ \left(B^L\right)^*$. □

Corollary 5.9. *Theorem* 3.1 *implies that* 1 *is a cyclic vector for the energy representation.*

References

1. S. Albeverio, R. Høegh-Krohn, D. Testard, A. Vershik, *Factorial representations of path groups.* — J. Funct. Anal. **51**, No. 1 (1983), 115–131.
2. S. Albeverio, R. Høegh-Krohn, D. Testard, A. Vershik, *Note: "Factorial representations of path groups"*, J. Funct. Anal. **52** (1983), no. 1, 148.
3. S. S. Albeverio, R. Høegh-Krohn, *The energy representation of Sobolev-Lie groups.* — Compositio Math. **36**, No. 1 (1978), 37–51.
4. S. Albeverio, R. Høegh-Krohn, J. A. Marion, D. H. Testard, B. S. Torrésani, *Noncommutative distributions.* — Monographs and Textbooks in Pure and Applied Mathematics, vol. 175, Marcel Dekker Inc., New York, 1993.
5. V. I. Bogachev, *Gaussian measures*, Mathematical Surveys and Monographs, vol. 62, American Mathematical Society, Providence, RI, 1998. MR MR1642391 (2000a:60004)
6. Bruce K. Driver, *A Cameron–Martin type quasi-invariance theorem for Brownian motion on a compact Riemannian manifold.* — J. Funct. Anal. **110**, No. 2 (1992), 272–376.
7. Bruce K. Driver, *A Cameron–Martin type quasi-invariance theorem for pinned Brownian motion on a compact Riemannian manifold.* — Trans. Amer. Math. Soc. **342**, No. 1 (1994), 375–395.
8. Bruce K. Driver, *Towards calculus and geometry on path spaces.* — Stochastic analysis (Ithaca, NY, 1993), Proc. Sympos. Pure Math., vol. 57, Amer. Math. Soc., Providence, RI, 1995, pp. 405–422.
9. Bruce K. Driver, Brian C. Hall, *The energy representation has no non-zero fixed vectors*, Stochastic processes, physics and geometry: new interplays, II (Leipzig,

1999), CMS Conf. Proc., vol. 29, Amer. Math. Soc., Providence, RI, 2000, pp. 143–155.

10. Gerald B. Folland, *A course in abstract harmonic analysis*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995.
11. I. M. Gel′fand, M. I. Graev, A. M. Veršik,*Representations of the group of smooth mappings of a manifold X into a compact Lie group.* — Compositio Math. **35**, No. 3 (1977), 299–334.
12. I. M. Gel′fand, M. I. Graev, A. M. Veršik, *Representations of the group of functions taking values in a compact Lie group.* — Compositio Math. **42**, No. 2 (1980/81), 217–243.
13. M. Gordina, *Taylor map on groups associated with a* II_1*-factor.* — Infin. Dimens. Anal. Quantum Probab. Relat. Top. **5**, No. 1 (2002), 93–111.
14. M. Gordina, *Stochastic differential equations on noncommutative* L^2*.* — Finite and infinite dimensional analysis in honor of Leonard Gross (New Orleans, LA, 2001), Contemp. Math., vol. 317, Amer. Math. Soc., Providence, RI, 2003, pp. 87–98.
15. L. Gross, *Uniqueness of ground states for Schrödinger operators over loop groups.* — J. Funct. Anal. **112**, No. 2 (1993), 373–441.
16. A. Guichardet, *Symmetric Hilbert spaces and related topics*, Lecture Notes in Mathematics, Vol. 261, Springer-Verlag, Berlin, 1972, Infinitely divisible positive definite functions. Continuous products and tensor products. Gaussian and Poissonian stochastic processes.
17. Brian C. Hall, Ambar N. Sengupta, *The Segal-Bargmann transform for path-groups.* — J. Funct. Anal. **152**, No. 1 (1998), 220–254.
18. T. Hida, *Brownian motion*, Applications of Mathematics, vol. 11, Springer-Verlag, New York, 1980, Translated from the Japanese by the author and T. P. Speed.
19. Richard V. Kadison, John R. Ringrose, *Fundamentals of the theory of operator algebras. Vol. II*, Pure and Applied Mathematics, vol. 100, Academic Press Inc., Orlando, FL, 1986, Advanced theory.
20. M.-P. Malliavin, P. Malliavin, *Integration on loop groups. I. Quasi invariant measures.* — J. Funct. Anal. **93**, No. 1 (1990), 207–237.
21. J. Marion, *Wiener functionals on spaces of Lie algebra valued* 1*-currents, and unitary representations of current groups.* — Publ. Mat. **33**, No. 1 (1989), 99–121.
22. Derek W. Robinson, *Elliptic operators and Lie groups*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1991, Oxford Science Publications.
23. I. E. Segal, *The two-sided regular representation of a unimodular locally compact group.* — Ann. of Math. (2) **51** (1950), 293–298.
24. Ichirō Shigekawa, *Transformations of the Brownian motion on a Riemannian symmetric space.* — Z. Wahrsch. Verw. Gebiete **65**, No. 4 (1984), 493–522.
25. Daniel W. Stroock, S. R. Srinivasa Varadhan, *Multidimensional diffusion processes*, Classics in Mathematics, Springer-Verlag, Berlin, 2006, Reprint of the 1997 edition.

26. V. S. Sunder, *An invitation to von Neumann algebras*, Universitext, Springer-Verlag, New York, 1987.
27. D. Testard, *Representations of the group of equivariant loops in* $\mathrm{SU}(N)$, Stochastic processes—mathematics and physics, II (Bielefeld, 1985), Lecture Notes in Math., vol. 1250, Springer, Berlin, 1987, pp. 326–341.
28. Nolan R. Wallach, *On the irreducibility and inequivalence of unitary representations of gauge groups.* — Compositio Math. **64**, No. 1 (1987), 3–29.

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