



Global Regularity for the Fractional Euler Alignment System

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Abstract

We study a pressureless Euler system with a non-linear density-dependent alignment term, originating in the Cucker–Smale swarming models. The alignment term is dissipative in the sense that it tends to equilibrate the velocities. Its density dependence is natural: the alignment rate increases in the areas of high density due to species discomfort. The diffusive term has the order of a fractional Laplacian $(-\partial_{xx})^{\alpha/2}$, $\alpha \in (0, 1)$. The corresponding Burgers equation with a linear dissipation of this type develops shocks in a finite time. We show that the alignment nonlinearity enhances the dissipation, and the solutions are globally regular for all $\alpha \in (0, 1)$. To the best of our knowledge, this is the first example of such regularization due to the non-local nonlinear modulation of dissipation.

1. Introduction

The Cucker–Smale Model Modeling of the self-organized collective behavior, or swarming, has attracted a large amount of attention over the last few years. Even an attempt at a brief review of this field is well beyond the scope of this introduction, and we refer to the recent reviews [15, 18, 42]. A remarkable phenomenon commonly observed in biological systems is flocking, or velocity alignment by near-by individuals. One of the early flocking models, discrete in time and two-dimensional, is commonly referred to as the Vicsek model: the angle $\theta_i(t)$ of the velocity of i -th particle satisfies

$$\theta_i(t+1) = \frac{1}{|\mathcal{N}_i(t)|} \sum_{j \in \mathcal{N}_i(t)} \theta_j(t) + \eta \Delta \theta. \quad (1.1)$$

Here, $\mathcal{N}_i(t) = \{j : |x_i(t) - x_j(t)| \leq r\}$, with some $r > 0$ fixed, $\Delta \theta$ is a uniformly distributed random variable in $[-1, 1]$, and $\eta > 0$ is a parameter measuring the

strength of the noise. This model preserves the modulus of the particle velocity and only affects its direction. First via numerical simulations and then by mathematical tools, it has been shown that this model has a rich behavior, ranging from flocking when η is small, to a completely chaotic motion for large η , with a phase transition at a certain critical value η_c .

A natural generalization of the Vicsek model was introduced by CUCKER AND SMALE [21]:

$$\dot{x}_i = v_i, \quad \dot{v}_i = \frac{1}{N} \sum_{j=1}^N \phi(|x_i - x_j|)(v_j - v_i). \quad (1.2)$$

Here, $\{x_i, v_i\}_{i=1}^N$ represent, respectively, the locations and the velocities of the agents. Individuals align their velocity to their neighbors, with the interaction strength characterized by a non-negative influence function $\phi(x) \geq 0$. The relative influence is typically taken as a decreasing function of the distance between individuals. An important flexible element of the Cucker–Smale model is that it both does not impose a constraint on the velocity magnitude and it allows to analysis of the behavior based on the decay properties of the kernel $\phi(r)$. One of the main results of the Cucker–Smale paper was that, roughly, provided that $\phi(r)$ decays slower than r^{-1} as $r \rightarrow +\infty$, then all velocities $v_i(t)$ converge to a common limit $\bar{v}(t)$, and the relative particle positions $x_i(t) - x_j(t) \rightarrow \bar{x}_{ij}$ also have a common limit—the particles form a swarm moving with a uniform velocity. This is what we would call a global flocking: all particles move with nearly identical velocities.

One potential shortcoming of the Cucker–Smale model is that an “isolated clump” of particles may be more affected by a “far away” large mass than by its own neighbors. Essentially, the dynamics inside a small clump would be suppressed by the presence of a large group of particles “far away”. This can be balanced by a different kind of averaging, rather than simple division by N in (1.2), as was done by MOTSCH AND TADMOR [35]:

$$\dot{x}_i = v_i, \quad \dot{v}_i = \frac{\lambda}{\Phi_i} \sum_{j=1}^N \phi(|x_i - x_j|)(v_j - v_i), \quad \Phi_i = \sum_{k=1}^N \phi(|x_i - x_k|), \quad (1.3)$$

with some $\lambda > 0$. This modification reinforces the local alignment over the long distance interactions.

A Kinetic Cucker–Smale Model Kinetic models are also commonly used to describe the collective behavior when the number of particles is large, in terms of the particle density $f(t, x, v)$, with $x \in \mathbb{R}^d$, $v \in \mathbb{R}^d$. A kinetic limit of the Cucker–Smale model was obtained by HA AND TADMOR [27], as a nonlinear and non-local kinetic equation:

$$f_t + v \cdot \nabla_x f + \nabla_v \cdot (L[f]f) = 0, \quad (1.4)$$

with

$$L[f](t, x, v) = \int_{\mathbb{R}^{2d}} \phi(x - y)(v' - v)f(t, y, v')dv'dy. \quad (1.5)$$

Together, (1.4)–(1.5) give a nonlinear kinetic version of the Cucker–Smale system.

It was shown in [16] that its solutions exhibit global flocking, in the sense that the size of the support in x ,

$$S(t) = \sup\{|x - y| : (x, v), (y, v') \in \text{supp}(f(t, \cdot, \cdot))\},$$

remains uniformly bounded in time, and the support in v shrinks:

$$V(t) = \sup\{|v - v'| : (x, v), (y, v') \in \text{supp}(f(t, \cdot, \cdot))\} \rightarrow 0 \text{ as } t \rightarrow +\infty, \quad (1.6)$$

under the assumption that $\phi(r)$ decays slower than r^{-1} as $r \rightarrow +\infty$. A similar result was obtained in [41] for the kinetic Motsch–Tadmor system.

A kinetic model that combines the features of the Cucker–Smale and Motsch–Tadmor models was proposed in a paper by KAPER ET AL. [28]:

$$f_t + v \cdot \nabla_x f + \nabla_v \cdot (L[f]f) + \lambda \nabla_v \cdot ((u(t, x) - v)f) = \Delta_v f, \quad (1.7)$$

with $L[f]$ as in (1.5), $\lambda > 0$, and the local average velocity $u(t, x)$ defined as

$$u(t, x) = \frac{1}{\rho(t, x)} \int_{\mathbb{R}^d} v f(t, x, v) dv, \quad \rho(t, x) = \int_{\mathbb{R}^d} f(t, x, v) dv. \quad (1.8)$$

The Laplacian in the right side of (1.7) takes into account the possible Brownian noise in the velocity.

One should also mention a large body of literature on the kinetic versions of the Vicsek model and its modifications, and their hydrodynamic limits; see [12, 22–25] and references therein.

An Euler Alignment Model The kinetic Cucker–Smale model can be further “macroscopized” as a hydrodynamic model for the local density $\rho(t, x)$ and local average velocity $u(t, x)$ defined in (1.8). The standard formal derivation of the hydrodynamic limit for nonlinear kinetic equations often relies on a (often hard to justify) moment closure procedure. An alternative is to consider the “monokinetic” solutions of (1.4)–(1.5) of the form

$$f(t, x, v) = \rho(t, x) \delta(v - u(t, x)). \quad (1.9)$$

In a sense, this is a “local alignment” (as opposed to global flocking) ansatz—the particles move locally with just a single velocity but the velocity does vary in space. Inserting this expression into (1.4)–(1.5) gives the Euler alignment system, which we write in one dimension as

$$\partial_t \rho + \partial_x(\rho u) = 0, \quad (1.10)$$

$$\partial_t(\rho u) + \partial_x(\rho u^2) = \int_{\mathbb{R}} \phi(x - y)(u(t, y) - u(t, x)) \rho(t, y) \rho(t, x) dy. \quad (1.11)$$

The presence of the density ρ under the integral in the right side of (1.11) has a very reasonable biological interpretation: the alignment effect between the individual

agents becomes stronger where the density is high (assuming that the interaction kernel ϕ is localized). As far as a rigorous derivation of the hydrodynamic limit is concerned, the aforementioned paper [28] derives the hydrodynamic limit starting from the “combined” Cucker–Smale–Motsch–Tadmor kinetic system (1.7):

$$\partial_t \rho + \partial_x(\rho u) = 0, \quad (1.12)$$

$$\partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x \rho = \int_{\mathbb{R}} \phi(x - y)(u(t, y) - u(t, x))\rho(t, y)\rho(t, x)dy. \quad (1.13)$$

This system has an extra term $\partial_x \rho$ in the left side of (1.11) that can be thought of as pressure, with the constitutive law $p(\rho) = \rho$. The pressure appears as a result of the balance between the local interaction term in the left side of (1.7) and the Laplacian in the right side. In particular, the starting point of the derivation is not the single local velocity ansatz (1.9) but its smooth Maxwellian version (setting $\lambda = 1$ in (1.7) for convenience)

$$f(t, x, v) = \rho(t, x) \exp\left(-\frac{(v - u(t, x))^2}{2}\right), \quad (1.14)$$

taken together with the assumption that the interaction is weak, $\phi \rightarrow \varepsilon\phi$, and a large time–space rescaling $(t, x) \rightarrow t/\varepsilon, x/\varepsilon$.

Another version of the Euler equations as a model for swarming has been proposed in [34], and formally justified in [19]:

$$\begin{aligned} \partial_t \rho + \partial_x(\rho u) &= 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x \rho &= \alpha \rho u - \beta \rho |u|^2 u - \int_{\mathbb{R}} \nabla V(x - y)\rho(t, y)\rho(t, x)dy. \end{aligned} \quad (1.15)$$

The key difference between models like (1.15) and the ones we consider here is the absence of the regularizing term $u(t, y) - u(t, x)$ in the right side, so one does not expect the regularizing effect of the interactions that we will observe here.

The Euler Alignment System for Lipschitz Interaction Kernels When particles do not interact, that is, $\phi(x) \equiv 0$, the system (1.10)–(1.11) is simply the pressureless Euler equations. In particular, in that case, (1.11) is the Burgers equation:

$$\partial_t u + u \partial_x u = 0. \quad (1.16)$$

Its solutions develop a shock singularity in a finite time if the initial condition $u_0(x)$ has a point where $\partial_x u_0(x) < 0$. In particular, if $u_0(x)$ is periodic and not identically equal to a constant, then $u(x, t)$ becomes discontinuous in a finite time. The function $z(x, t) = -\partial_x u(x, t)$ satisfies the continuity equation

$$\partial_t z + \partial_x(zu) = 0, \quad (1.17)$$

and becomes infinite at the shock location.

The singularity in the Burgers equation does not mean that there is a singularity in the solution of kinetic equation; it only means that the ansatz (1.9) breaks down, and we cannot associate a single velocity to a given position. This is a version of “a shock implies no local alignment”. To illustrate this point, consider the solution of free transport equation

$$\partial_t f + v \partial_x f = 0, \quad (1.18)$$

with the initial condition $f_0(x) = \delta(v + x)$. The solution of the kinetic equation is

$$f(t, x, v) = f_0(x - vt, v) = \delta(v + x - vt), \quad (1.19)$$

hence the ansatz (1.9) fails at $t = 1$. This is the time when the corresponding Euler equation

$$\partial_t u + u \partial_x u = 0, \quad (1.20)$$

with the initial condition $u(0, x) = -x$, develops a shock: $u(t, x) = -x/(1 - t)$.

The integral term in the right side of (1.11) has a dissipative nature when $\phi \neq 0$; it tries to regularize the velocity discontinuity. When the function $\phi(x)$ is Lipschitz, this system has been investigated in [14] and [40] that show two results. First, a version of global flocking: if ϕ decays slower than $|x|^{-1}$ at infinity, and the solution remains smooth for all $t \geq 0$ and the initial density ρ_0 is compactly supported, then the support S_t of $\rho(t, \cdot)$ remains uniformly bounded in time, and

$$\sup_{x, y \in S_t} |u(t, x) - u(t, y)| \rightarrow 0 \text{ as } t \rightarrow +\infty. \quad (1.21)$$

An improvement in global regularity compared to the Burgers equation (1.16) was also obtained in [14] and [40]. As we have mentioned, solutions of the latter become discontinuous in a finite time provided there is a point $x \in \mathbb{R}$ where the initial condition $u_0(x)$ has a negative derivative: $\partial_x u_0(x) < 0$. On the other hand, solutions of the Euler alignment equations remain regular for initial data such that

$$\partial_x u_0(x) \geq -(\phi \star \rho_0)(x) \text{ for all } x \in \mathbb{R}, \quad (1.22)$$

while the solution blows up in a finite time if there exists $x_0 \in \mathbb{R}$ such that

$$\partial_x u_0(x) < -(\phi \star \rho_0)(x). \quad (1.23)$$

Thus, the presence of the dissipative term in (1.11) leads to global regularity for some initial data that blows up for the Burgers equation: the right side of (1.22) may be negative. However, a Lipschitz interaction kernel $\phi(x)$ arrests the shock singularity for the Euler alignment equations only for some initial conditions.

Singular Alignment Kernels Our interest is in singular interaction kernels of the form $\phi(x) = |x|^{-\beta}$, with $\beta > 0$. One reason to consider such kernels is to strengthen the effect of the local interactions compared to the effect of “far-away” particles, in the spirit of the Motsch–Tadmor correction. The well-posedness of the finite number of particles Cucker–Smale system with such interactions is a delicate issue—the difficulty is in either ruling out the possibility of particle collisions, or understanding the behavior of the system at and after a collision. This problem was addressed in [37, 38] for $\beta \in (0, 1)$ —it was shown that in this range, particles may get stuck together but a weak solution of the ODE system can still be defined. When $\beta \geq 1$, a set of initial conditions that has no particle collisions was described in [1]. The absence of collisions was proved very recently for general initial configurations in [13]. As far as flocking is concerned, unconditional flocking was proved in [26] for $\beta \in (0, 1)$, while for $\beta \geq 1$ there are initial configurations that do not lead to global flocking—the long distance interaction is too weak. The well-posedness of the kinetic Cucker–Smale system for $\beta \in (0, 1/2)$ was established in [36].

We consider here the alignment kernels $\phi(x)$ with $\beta > 1$:

$$\phi_\alpha(x) = \frac{c_\alpha}{|x|^{1+\alpha}}, \quad (1.24)$$

with $\alpha > 0$. In particular, the decay of $\phi(x)$ at large $|x|$ is faster than the $1/|x|$ decay required for the Cucker–Smale and other proofs of flocking. It is compensated by a very strong alignment for $|x| \rightarrow 0$. The constant c_α is chosen so that

$$\Lambda^\alpha f = c_\alpha \int_{\mathbb{R}} \frac{f(x) - f(y)}{|x - y|^{1+\alpha}} dy, \quad \Lambda = (-\partial_{xx})^{1/2}.$$

Then the strong form of the Euler alignment system is

$$\partial_t \rho + \partial_x(\rho u) = 0 \quad (1.25)$$

$$\partial_t u + u \partial_x u = c_\alpha \int_{\mathbb{R}} \frac{u(t, y) - u(t, x)}{|y - x|^{1+\alpha}} \rho(t, y) dy. \quad (1.26)$$

Let us first compare the Euler alignment system (1.25)–(1.26) to the Burgers equation with a fractional dissipation

$$\partial_t u + u \partial_x u = -\Lambda^\alpha u, \quad (1.27)$$

obtained by formally setting $\rho(t, x) \equiv 1$ in (1.26) and dropping (1.25) altogether. This neglects the nonlinear mechanism of the dissipation. Global regularity of the solutions of the fractional Burgers equation has been studied in [32]. One can distinguish three regimes: first, when $\alpha > 1$, the dissipative term in the right side has a higher order derivative than the nonlinear term in the left side. This is the sub-critical regime: the dissipation dominates the nonlinearity, and global existence of the strong solutions can be shown in a reasonably straightforward manner using the energy methods. On the other hand, when $0 < \alpha < 1$, the dissipation is too weak to compete with the nonlinear term, which has a higher derivative, and solutions with smooth initial conditions may develop a shock, as in the inviscid case. The critical case is $\alpha = 1$ when the dissipation and the nonlinearity contain

derivatives of the same order. One may expect that then the nonlinearity may win over the dissipation for some large data. This, however, is not the case: solutions with smooth initial conditions remain regular globally in time. The proof of the global regularity when $\alpha = 1$ is much less straightforward than for $\alpha > 1$ and does not rely solely on the energy methods.

One may hope that the nonlinearity in the dissipative term in the right side of (1.26) is actually beneficial, compared to the fractional Burgers equation (1.27). Indeed, on the qualitative level, as the shock would form, the density ρ would be expected to increase near the point of the shock. This, in turn, would increase the dissipation in (1.26), moving the problem from “like a super-critical Burgers” to “like a sub-critical Burgers”. This intuition, however, may be slightly misleading—for instance, as we will see, strengthening the dissipation by increasing α does not appear to make the problem any easier, or change its critical character. The competition between the Burgers nonlinearity in the left side of (1.26) and the nonlinear dissipation in the right side is rather delicate.

The aforementioned results of [14, 40] may lead to a conjecture that a dissipation term involving the convolution kernel $\phi \notin L^1$, as in (1.26), should lead to global regularity. However, this is far from obvious. The global regularity argument of [14, 40] uses two ingredients: first, if, initially,

$$\partial_x u_0 + \phi \star \rho_0 \geq 0 \quad (1.28)$$

for all $x \in \mathbb{R}$, then

$$\partial_x u + \phi \star \rho \geq 0 \quad (1.29)$$

for all $x \in \mathbb{R}$ and $t \geq 0$. Second, an L^∞ -bound on ρ is established. When ϕ is an L^1 -function, one deduces a lower bound $\partial_x u > -C_0$, which is crucial for global regularity. One may combine an argument of [14] with the Constantin–Vicoll nonlinear maximum principle to establish the L^∞ -bound on ρ in our case, as well. However, in our case, the analogous inequality to (1.29) is

$$\partial_x u - \Lambda^\alpha \rho \geq 0. \quad (1.30)$$

This fails to give the required lower bound on $\partial_x u$ based on just the L^∞ control of ρ , and the global regularity does not follow easily from the uniform bound on the density. Instead, we have to deploy a much subtler argument involving both upper and lower bounds on the density and a non-trivial modification of the modulus of continuity technique of [33].

The Main Result We consider here the Euler alignment system (1.25)–(1.26) on the torus \mathbb{T} , for $\alpha \in (0, 1)$. In particular, this range of α corresponds to the supercritical case for the fractional Burgers equation (1.27). We prove that the nonlinear, density modulated dissipation qualitatively changes the behavior of the solutions; instead of blowing up in a finite time, solutions are globally regular.

Theorem 1.1. *For $\alpha \in (0, 1)$, the Euler alignment system (1.25)–(1.26) with periodic smooth initial data (ρ_0, u_0) such that $\rho_0(x) > 0$ for all $x \in \mathbb{T}$, has a unique global smooth solution.*

The regularizing effect of a non-linear diffusion has been observed before, for instance, in the chemotaxis problems with a nonlinear diffusion—see [5–9]. The main novelties here are that the nonlinearity is non-local, and that, as we will see, increasing α does not, contrary to a naive intuition, and unlike what happens in the fractional Burgers equation, strengthen the regularization effect.

To explain the ideas behind the result and its proof, it is convenient to reformulate the Euler alignment system (1.25)–(1.26) as the following system for ρ and $G = \partial_x u - \Lambda^\alpha \rho$:

$$\partial_t \rho + \partial_x(\rho u) = 0, \quad (1.31)$$

$$\partial_t G + \partial_x(Gu) = 0, \quad (1.32)$$

with the velocity u related to ρ and G via

$$\partial_x u = \Lambda^\alpha \rho + G. \quad (1.33)$$

We show in Section 2 that (1.25)–(1.26) and (1.31)–(1.33) are, indeed, equivalent for regular solutions. Note that (1.33) only defines u up to its mean, which is determined from the conservation of the momentum:

$$\int_{\mathbb{T}} \rho(x, t) u(x, t) dx = \int_{\mathbb{T}} \rho_0(x) u_0(x) dx. \quad (1.34)$$

Somewhat paradoxically, (1.33) seems to indicate that increasing the dissipation α makes the velocity more singular in terms of the density rather than more regular.

The solutions of (1.31)–(1.32) with the initial conditions $\rho_0(x)$, $u_0(x)$ such that

$$G_0(x) = \partial_x u_0(x) - \Lambda^\alpha \rho_0(x) \equiv 0, \quad (1.35)$$

preserve the constraint $G = 0$ for all $t > 0$, and (1.31)–(1.32) then reduces to a single equation

$$\partial_t \rho + \partial_x(\rho u) = 0, \quad \partial_x u = \Lambda^\alpha \rho, \quad (1.36)$$

that is simpler to analyze. Note that (1.36) defines $u(x, t)$ only up to its spatial average—we assume that it has mean-zero for all $t > 0$. Equation (1.36) is interesting in its own right, as a model for the 1D nonlinear porous medium flow with fractional potential pressure. Global existence of the weak solutions was studied by CAFFARELLI AND VAZQUEZ [11], for $\alpha \in (0, 2)$ and $\rho_0 \geq 0$. It has been further investigated in [10], where it is shown that the weak solutions with L^1 initial data instantly become Hölder continuous and stay Hölder for all time.

When $\alpha = 1$, so that the velocity is the Hilbert transform of the density, the model (1.36) was introduced as a 1D vortex sheet model in [4], and has been extensively studied in [17] as a 1D model of the 2D quasi-geostrophic equation. In particular, the global existence of classical solution if $\rho_0 > 0$ is proved in [17] using the algebraic properties of the Hilbert transform. A counterexample is also provided that says if the initial data ρ_0 consists a vacuum, solution could lose C^1 regularity in finite time. It indicates that one can not extend the result in [10] to gain higher regularity, and the assumption $\rho_0 > 0$ is crucial to obtain global wellposedness.

Our results in this paper can be directly applied to (1.36), and show the global regularity of the solutions for all $\alpha \in (0, 1)$. The strategy of the regularity proof here is very different from that in [10, 17]. A quintessential feature of (1.36) is that increasing α does not help the dissipation in its competition with the Burgers nonlinearity. Indeed, the toy model (1.36) can be written as

$$\partial_t \rho + \left(\partial_x^{-1} \Lambda^\alpha \rho \right) \partial_x \rho = -\rho \Lambda^\alpha \rho. \quad (1.37)$$

Thus, the scalings of the dissipation in the right side and of the nonlinear transport term in the left side are exactly the same, both in ρ and in x , no matter what $\alpha \in (0, 1)$ is. While the proof of global regularity for (1.37) is inspired by the nonlocal maximum principle arguments of [32, 33], the nonlinear nature of dissipative term necessitates significant changes and new estimates. The upgrade of the proof from global regularity of the model equation to the full system is also highly non-trivial and requires new ideas.

We note that our results can be applied to the case $\alpha \in [1, 2)$, where the global behavior is the same as for the fractional Burgers equation. This case was recently studied by [39] using different techniques. One can also extend our results to the influence kernels of the form

$$\phi(x) = \frac{\chi(|x|)}{|x|}, \quad (1.38)$$

with a non-negative smooth compactly supported function $\chi(r)$. This is the analog of the kernels in (1.24) for $\alpha = 0$. We expect that as soon as the influence kernel is not integrable, solutions remain regular. The proofs of these extensions require some nontrivial adjustments and further technicalities compared to the arguments in this paper, and will be presented elsewhere.

Our results also lead to global flocking behavior for (1.25)–(1.26). The periodical influence function

$$\phi_p(x) = \sum_{m \in \mathbb{R} \setminus \mathbb{T}} \phi(x + m)$$

has a positive lower bound for all $x \in \mathbb{T}$. Since the solution is smooth, one can use the argument in [40] to obtain asymptotic flocking behavior in the sense that

$$\sup_{x, y \in \mathbb{T}} |u(t, x) - u(t, y)| \rightarrow 0 \text{ as } t \rightarrow +\infty. \quad (1.39)$$

This paper is organized as follows. In Section 2 we prove an a priori L^∞ -bound on ρ , that is the key estimate for the regularity of the solutions, as well as lower bound on ρ . The local well-posedness of the solutions is proved in Section 3. Section 4 contains the proof of our main result, Theorem 1.1. Appendix A contains the proof of an auxiliary technical estimate. Throughout the paper we denote by C , C' , etc. various universal constants, and by C_0 , C'_0 etc. constants that depend only on the initial conditions.

2. Bounds on the Density

In this section, we prove the upper and lower bounds on the density $\rho(t, x)$. The upper bound is uniform in time, and is crucial for the global regularity. The lower bound will deteriorate in time but will be sufficient for our purposes.

2.1. The Reformulation of the Euler Alignment System

We first explain how the Euler alignment system (1.25)–(1.26) is reformulated as (1.31)–(1.32), as we will mostly use the latter. We only need to obtain (1.32) for G defined in (1.33). The idea comes from [14]. We apply the operator Λ^α to (1.25), and use the identity

$$u(y)\rho(y) - u(x)\rho(x) = [u(y) - u(x)]\rho(y) + u(x)[\rho(y) - \rho(x)]$$

to obtain

$$\partial_t \Lambda^\alpha \rho = -\partial_x \Lambda^\alpha (\rho u) = c_\alpha \partial_x \int_{\mathbb{R}} \frac{u(y) - u(x)}{|y - x|^{1+\alpha}} \rho(y) dy - \partial_x (u(x) \Lambda^\alpha \rho) \quad (2.1)$$

On the other hand, applying ∂_x to (1.26), we get

$$\partial_t (\partial_x u) + \partial_x (u \partial_x u) = c_\alpha \partial_x \int_{\mathbb{R}} \frac{u(y) - u(x)}{|y - x|^{1+\alpha}} \rho(y) dy. \quad (2.2)$$

Subtracting (2.1) from (2.2) gives an equation for the function $G = \partial_x u - \Lambda^\alpha \rho$:

$$\partial_t G + \partial_x (Gu) = 0,$$

which is (1.32).

Let us comment on how to recover u from (1.33). Let us denote by

$$\kappa = \frac{1}{|\mathbb{T}|} \int_{\mathbb{T}} \rho(x, t) dx \quad (2.3)$$

the average of ρ in \mathbb{T} , which is preserved in time by (1.31), at least as long as ρ remains smooth. Note that $G(x, t)$ has mean zero automatically:

$$\int_{\mathbb{T}} G(x, t) dx = \int_{\mathbb{T}} G_0(x) dx = 0. \quad (2.4)$$

We also define

$$\theta(x, t) = \rho(x, t) - \kappa, \quad (2.5)$$

so that

$$\int_{\mathbb{T}} \theta(x, t) dx = 0.$$

Thus, the primitive functions of $\theta(x, t)$ and $G(x, t)$ are periodic. We denote by (φ, ψ) the mean-zero primitive functions of (θ, G) , respectively:

$$\theta(x, t) = \partial_x \varphi(x, t), \quad \int_{\mathbb{T}} \varphi(x, t) dx = 0, \quad (2.6)$$

and

$$G(x, t) = \partial_x \psi(x, t), \quad \int_{\mathbb{T}} \psi(x, t) dx = 0. \quad (2.7)$$

Then, u can be written as

$$u(x, t) = \Lambda^\alpha \varphi(x, t) + \psi(x, t) + I_0(t). \quad (2.8)$$

To determine $I_0(t)$, we use the conservation of the momentum. Note that the conservation law form of (1.26) is

$$\partial_t(\rho u) + \partial_x(\rho u^2) = c_\alpha \int_{\mathbb{R}} \frac{u(t, y) - u(t, x)}{|y - x|^{1+\alpha}} \rho(t, y) dy. \quad (2.9)$$

Integrating (2.9) gives

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{T}} \rho u dx &= c_\alpha \int_{\mathbb{T}} \int_{\mathbb{R}} \frac{u(y, t) - u(x, t)}{|y - x|^{1+\alpha}} \rho(y, t) \rho(x, t) dy dx \\ &= \sum_{m \in \mathbb{R} \setminus \mathbb{T}} c_\alpha \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{u(y, t) - u(x, t)}{|y + m - x|^{1+\alpha}} \rho(y, t) \rho(x, t) dy dx = 0, \end{aligned} \quad (2.10)$$

thus

$$\int_{\mathbb{T}} \rho(x, t) u(x, t) dx = \int_{\mathbb{T}} \rho_0(x) u_0(x) dx.$$

Together with (2.8), u is now uniquely defined, with $I_0(t)$ given by

$$I_0(t) = \frac{1}{\kappa |\mathbb{T}|} \left[\int_{\mathbb{T}} \rho_0(x) u_0(x) dx - \int_{\mathbb{T}} \rho(x, t) (\Lambda^\alpha \varphi(x, t) + \psi(x, t)) dx \right]. \quad (2.11)$$

Note that we have

$$\int_{\mathbb{T}} \rho(x, t) \Lambda^\alpha \varphi(x, t) dx = \kappa \int_{\mathbb{T}} \Lambda^\alpha \varphi(x, t) dx + \int_{\mathbb{T}} (\partial_x \varphi(x, t)) \Lambda^\alpha \varphi(x, t) dx = 0, \quad (2.12)$$

thus

$$I_0(t) = \frac{1}{\kappa |\mathbb{T}|} \left[\int_{\mathbb{T}} \rho_0(x) u_0(x) dx - \int_{\mathbb{T}} \rho(x, t) \psi(x, t) dx \right]. \quad (2.13)$$

In particular, $I_0(t)$ is time-independent in the special case $G \equiv 0$, that leads to (1.36), and then we have

$$I_0(t) \equiv I_0(0). \quad (2.14)$$

2.2. The Upper Bound on the Density

We now prove an a priori L^∞ bound on ρ .

Theorem 2.1. *Let $\rho(x, t), u(x, t)$ be a strong solution to (1.25)–(1.26) for $0 \leq t \leq T$, with smooth periodic initial conditions $\rho_0(x), u_0(x)$ such that $\rho_0(x) > 0$ on \mathbb{T} . Then, there exists a constant $C_0 > 0$ that depends on ρ_0 and u_0 but not on T , so that $\|\rho(\cdot, t)\|_{L^\infty} \leq C_0$ for all $t \geq 0$.*

This bound already indicates that the Euler alignment system behaves not as the fractional Burgers equation. Indeed, if we couple fractional Burgers equation with (1.25), the density may blow up for $\alpha \in (0, 1)$ for suitable smooth initial conditions.

The Proof of Theorem 2.1 As the functions ρ and G obey the same continuity equation, their ratio $F = G/\rho$ satisfies

$$\partial_t F + u \partial_x F = 0. \quad (2.15)$$

It follows that F is uniformly bounded:

$$\|F(\cdot, t)\|_{L^\infty} \leq \|F_0\|_{L^\infty} = \left\| \frac{\partial_x u_0 - \Lambda^\alpha \rho_0}{\rho_0} \right\|_{L^\infty} < +\infty,$$

as ρ_0 and u_0 are smooth, and ρ_0 is strictly positive.

In order to prove the upper bound on ρ , for a fixed $t \geq 0$, let \bar{x} be such that

$$\rho(\bar{x}, t) = \max_{x \in \mathbb{R}} \rho(x, t). \quad (2.16)$$

It follows from (1.31) that

$$\partial_t \rho(\bar{x}, t) = -u(\bar{x}, t) \partial_x \rho(\bar{x}, t) - \rho(\bar{x}, t) \partial_x u(\bar{x}, t) = -\rho(\bar{x}, t) \partial_x u(\bar{x}, t). \quad (2.17)$$

Thus, to obtain an a priori upper bound on ρ , it suffices to show that there exists C_0 that depends on the initial conditions ρ_0 and u_0 so that if $\rho(\bar{x}, t) > C_0$, then

$$\partial_x u(\bar{x}, t) > 0. \quad (2.18)$$

To obtain (2.18), note that

$$\partial_x u = \Lambda^\alpha \rho + F\rho \geq \Lambda^\alpha \rho - \|F_0\|_{L^\infty} \rho. \quad (2.19)$$

In order to bound $\Lambda^\alpha \rho$ in the right side of (2.19) from below, we use the nonlinear maximum principle for the fractional Laplacian, see [20, Theorem 2.3]:

$$\text{either } \Lambda^\alpha \rho(\bar{x}) = \Lambda^\alpha \theta(\bar{x}) \geq \frac{\theta^{1+\alpha}(\bar{x})}{c \|\varphi\|_{L^\infty}^\alpha} \quad \text{or} \quad \theta(\bar{x}) \leq c \|\varphi\|_{L^\infty}. \quad (2.20)$$

Here, the constant $c > 0$ only depends on α . Recall that we denote by $\theta(x, t)$ the mean-zero shift of $\rho(x, t)$, as in (2.3) and (2.5), and by $\varphi(x, t)$ the mean-zero primitive of $\theta(x, t)$, as in (2.6). Note that $\|\varphi(\cdot, t)\|_{L^\infty}$ is uniformly bounded:

$$\|\varphi(\cdot, t)\|_{L^\infty} \leq C \|\theta(\cdot, t)\|_{L^1} \leq C \|\rho(\cdot, t)\|_{L^1} = C \|\rho_0\|_{L^1}. \quad (2.21)$$

Therefore, if

$$\rho(\bar{x}, t) \geq 2\kappa + C\|\rho_0\|_{L^1}, \quad (2.22)$$

with a sufficiently large C , which depends only on ρ_0 and u_0 , then

$$\theta(\bar{x}, t) = \rho(\bar{x}, t) - \kappa \geq 2c\|\varphi(\cdot, t)\|_{L^\infty},$$

and the second possibility in (2.20) can not hold. Thus, as soon as (2.22) holds, we have

$$\Lambda^\alpha \rho(\bar{x}, t) \geq C \frac{(\rho(\bar{x}, t) - \kappa)^{1+\alpha}}{\|\rho_0\|_{L^1}^\alpha} \geq C_0 \rho(\bar{x}, t)^{1+\alpha}, \quad (2.23)$$

with a constant C_0 that depends on the initial condition ρ_0 . Going back to (2.19), this implies

$$\partial_x u(\bar{x}, t) \geq C_0 \rho(\bar{x}, t)^{1+\alpha} - \|F_0\|_{L^\infty} \rho(\bar{x}, t) > 0.$$

Thus, (2.18) indeed holds if $\rho(\bar{x}, t) > C'_0$, where C'_0 is a constant that depends only on ρ_0 and u_0 , and the proof of Theorem 2.1 is complete. \square

One immediate consequence of Theorem 2.1 is that $I_0(t)$ in (2.13) is uniformly bounded for all time. Indeed, it suffices to bound

$$\left| \int_{\mathbb{T}} \rho(x, t) \psi(x, t) dx \right| \leq \|\rho(\cdot, t)\|_{L^2} \|\psi(\cdot, t)\|_{L^2},$$

while

$$\|\psi(\cdot, t)\|_{L^2} \leq C\|G(\cdot, t)\|_{L^2} \leq C\|G(\cdot, t)\|_{L^\infty} \leq C\|\rho(\cdot, t)\|_{L^\infty} \|F_0\|_{L^\infty} \leq C, \quad (2.24)$$

where C is a universal constant independent of t . Summarizing, we have

$$|I_0(t)| \leq C_0, \quad (2.25)$$

with a constant C_0 that depends only on ρ_0 and u_0 .

Thus, we have the following a priori bound on $\|u\|_{L^2}$.

Corollary 2.2. *Let $\rho(x, t)$, $u(x, t)$ be a strong solution to (1.25)–(1.26) for $0 \leq t \leq T$, with smooth periodic initial conditions $\rho_0(x)$, $u_0(x)$ such that $\rho_0(x) > 0$ on \mathbb{T} . There exists a constant C_0 that depends only on ρ_0 and u_0 but not on T so that $\|u(\cdot, t)\|_{L^2} \leq C_0$ for all $0 \leq t \leq T$.*

Proof. This follows immediately from the bound

$$\|u(\cdot, t)\|_{L^2} \leq \|\Lambda^\alpha \varphi(\cdot, t)\|_{L^2} + \|\psi(\cdot, t)\|_{L^2} + |I_0(t)|,$$

together with the bound

$$\|\Lambda^\alpha \varphi(\cdot, t)\|_{L^2} \leq C\|\theta(\cdot, t)\|_{L^2} \leq C\|\rho(\cdot, t)\|_{L^\infty}, \quad (2.26)$$

and (2.24)–(2.25). \square

The uniform upper bound on the density also implies a uniformly Lipschitz bound on F .

Lemma 2.3. *The function $F = G/\rho$ is Lipschitz, and the Lipschitz bound is uniform in time.*

Proof. Recall that F satisfies (2.15), thus $p = \partial_x F$ satisfies the same continuity equation as ρ :

$$\partial_t p + \partial_x (up) = 0, \quad (2.27)$$

and $w = p/\rho$ is a solution of

$$\partial_t w + u \partial_x w = 0.$$

It follows that $\|w(\cdot, t)\|_{L^\infty} = \|w_0\|_{L^\infty}$, and therefore,

$$\|\partial_x F(\cdot, t)\|_{L^\infty} \leq \|w_0\|_{L^\infty} \|\rho(\cdot, t)\|_{L^\infty}.$$

Theorem 2.1 implies now that F is Lipschitz, with a time-independent Lipschitz bound. \square

2.3. A Lower Bound on the Density

A uniform lower bound on ρ plays an important role as it keeps the dissipation active. The following lemma ensures no creation of vacuum in finite time.

Lemma 2.4. *Let $\rho(x, t), u(x, t)$ be a strong solution to (1.25)–(1.26) for $0 \leq t \leq T$, with smooth periodic initial conditions $\rho_0(x), u_0(x)$ such that $\rho_0(x) > 0$ on \mathbb{T} . There exists a positive constant $C_0 > 0$ that depends on ρ_0 and u_0 but not on T , so that*

$$\rho(x, t) \geq \frac{1}{C_0(1+t)}, \quad \text{for all } x \in \mathbb{T} \text{ and } 0 \leq t \leq T. \quad (2.28)$$

Proof. Fix some $t > 0$ and let \underline{x} be such that

$$\rho(\underline{x}, t) = \min_x \rho(x, t).$$

Then we have

$$\Lambda^\alpha \rho(\underline{x}, t) \leq 0,$$

and thus

$$\rho_m(t) = \rho(\underline{x}, t) = \min_{x \in \mathbb{T}} \rho(x, t), \quad (2.29)$$

satisfies

$$\begin{aligned} \frac{d\rho_m(t)}{dt} &= \partial_t \rho(\underline{x}, t) = [-\partial_x u(\underline{x}, t)] \rho(\underline{x}, t) \geq -(\Lambda^\alpha \rho(\underline{x}, t) + \|F_0\|_{L^\infty} \rho_m(t)) \rho_m(t) \\ &\geq -\|F_0\|_{L^\infty} \rho_m(t)^2. \end{aligned}$$

If the minimum is achieved at more than one point, we just need to take a minimum over all of them in the above estimate, which leads to the same bound. Notice that $\rho_m(t)$ is Lipschitz in time, so the estimate is valid for a.e. t , and $d\rho_m/dt$ determines $\rho_m(t)$. Integrating this differential inequality, we get

$$\rho_m(t) \geq \frac{1}{[\rho_m(0)]^{-1} + t\|F_0\|_{L^\infty}}, \quad (2.30)$$

finishing the proof. \square

In particular, in the special case $G \equiv 0$, that is, for (1.36) we have the following.

Corollary 2.5. *Let $\rho(x, t)$ be the solution of (1.36). Then, we have*

$$\rho(x, t) \geq \min_{x \in \mathbb{T}} \rho_0(x), \text{ for all } t > 0 \text{ and } x \in \mathbb{T}. \quad (2.31)$$

3. The Local Wellposedness

The a priori bounds on ρ established in the previous section rule out some kinds of finite time blow up, but do not imply that there is no finite time shock formation. This remains to be shown. To proceed further, we first establish a local well-posedness theory for solutions of the Euler alignment system with smooth initial conditions.

Theorem 3.1. *Let $\alpha \in (0, 1)$. Assume that the initial conditions ρ_0 and u_0 satisfy*

$$\rho_0 \in H^s(\mathbb{T}), \quad \min_{x \in \mathbb{T}} \rho_0(x) > 0, \quad \partial_x u_0 - \Lambda^\alpha \rho_0 \in H^{s-\frac{\alpha}{2}}(\mathbb{T}), \quad (3.1)$$

with a sufficiently large even integer $s > 0$. Then, there exists $T_0 > 0$ such that the system (1.25)–(1.26) has a unique strong solution $\rho(x, t), u(x, t)$ on $[0, T_0]$, with

$$\rho \in C([0, T_0], H^s(\mathbb{T})) \times L^2([0, T_0], H^{s+\frac{\alpha}{2}}(\mathbb{T})), \quad u \in C([0, T_0], H^{s+1-\alpha}(\mathbb{T})). \quad (3.2)$$

Moreover, a necessary and sufficient condition for the solution to exist on a time interval $[0, T]$ is

$$\int_0^T \|\partial_x \rho(\cdot, t)\|_{L^\infty}^2 dt < \infty. \quad (3.3)$$

Condition (3.3) is a Beale–Kato–Majda type criterion. It indicates that the solution is globally regular if $\partial_x \rho$ is uniformly bounded in the L^∞ norm. We will show that such bound actually does hold in Section 4, using the modulus of continuity method.

3.1. The Commutator Estimates

We will need some commutator estimates for the local well-posedness theory. We will use the following notation:

$$\begin{aligned} [\mathcal{L}, f, g] &= \mathcal{L}(fg) - f\mathcal{L}g - g\mathcal{L}f, \\ [\mathcal{L}, f]g &= \mathcal{L}(fg) - f\mathcal{L}g. \end{aligned}$$

Lemma 3.2. *The following commutator estimates hold:*

(i) *for any $n \geq 1$, we have*

$$\|[\partial_x^n, f, g]\|_{L^2} \leq C(\|\partial_x f\|_{L^\infty}\|g\|_{H^{n-1}} + \|\partial_x g\|_{L^\infty}\|f\|_{H^{n-1}}); \quad (3.4)$$

(ii) *for any $\gamma \in (0, 1)$ and $\varepsilon > 0$, we have*

$$\|[\Lambda^\gamma, f, g]\|_{L^2} \leq C\|f\|_{L^2}\|g\|_{C^{\gamma+\varepsilon}}; \quad (3.5)$$

(iii) *for any $\gamma > 0$, we have*

$$\|[\Lambda^\gamma, f]g\|_{L^2} \leq C(\|\partial_x f\|_{L^\infty}\|g\|_{H^{\gamma-1}} + \|f\|_{H^\gamma}\|g\|_{L^\infty}). \quad (3.6)$$

Let us comment briefly on the proof of these estimates. Estimate (3.4) can be obtained by the standard Gagliardo–Nirenberg interpolation inequality. As $\Lambda^2 = -\partial_{xx}^2$, this estimate holds if we replace the operator ∂_x^n by Λ^s with an even integer s .

A version of (3.5) is discussed in [30, Theorem A.8]. We sketch the proof in Appendix A. Finally, estimate (3.6) is due to KATO AND PONCE [29]. The proof is similar to that of (3.5).

3.2. The Proof of the Local Well-Posedness

It will be convenient to use the variables (θ, G) , so that equations (1.31)–(1.32) take the form

$$\partial_t \theta + \partial_x(\theta u) = -\kappa \partial_x u, \quad \partial_t G + \partial_x(Gu) = 0, \quad (3.7)$$

$$\partial_x u = \Lambda^\alpha \theta + G. \quad (3.8)$$

Here κ is the constant in time mean of ρ , as in (2.3).

Let us fix $T > 0$ and take a sufficiently large even integer $s > 0$. We will aim to obtain a differential inequality on

$$Y(t) := 1 + \|\theta(\cdot, t)\|_{H^s}^2 + \|G(\cdot, t)\|_{H^{s-\frac{\alpha}{2}}}^2, \quad (3.9)$$

that will have bounded solutions on a time interval $[0, T_0]$, with a sufficiently small T_0 depending on the initial conditions. To this end, we apply the operator Λ^s to the equation for θ in (3.7), multiply the result by $\Lambda^s \theta$ and integrate in x :

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\theta(\cdot, t)\|_{H^s}^2 &= - \int (\Lambda^s \theta \cdot \Lambda^s \partial_x(\theta u)) dx - \kappa \|\theta(\cdot, t)\|_{\dot{H}^{s+\frac{\alpha}{2}}}^2 \\ &\quad - \kappa \int (\Lambda^s \theta \cdot \Lambda^s G) dx. \end{aligned} \quad (3.10)$$

The second term in the right side produces the dissipation. We shall use it to control the other two terms.

We split the first term in the right side of (3.10) into three pieces:

$$\begin{aligned} \int \Lambda^s \theta \cdot \Lambda^s \partial_x (\theta u) dx &= \int (\Lambda^s \theta \cdot \Lambda^s \partial_x u) \theta dx + \int (\Lambda^s \theta \cdot u) (\Lambda^s \partial_x \theta) dx \\ &\quad + \int \Lambda^s \theta \cdot [\Lambda^s \partial_x, u, \theta] dx \\ &= I + II + III. \end{aligned} \quad (3.11)$$

Let us start with I :

$$\begin{aligned} I &= \int \left(\Lambda^{s-\frac{\alpha}{2}} \partial_x u \right) \cdot \Lambda^{\frac{\alpha}{2}} (\theta \cdot \Lambda^s \theta) dx \\ &= \int \left(\Lambda^{s-\frac{\alpha}{2}} \partial_x u \right) \cdot \left(\Lambda^{s+\frac{\alpha}{2}} \theta \right) \cdot \theta dx + \int \left(\Lambda^{s-\frac{\alpha}{2}} \partial_x u \right) \cdot (\Lambda^s \theta) \cdot \left(\Lambda^{\frac{\alpha}{2}} \theta \right) dx \\ &\quad + \int \left(\Lambda^{s-\frac{\alpha}{2}} \partial_x u \right) \cdot \left[\Lambda^{\frac{\alpha}{2}}, \Lambda^s \theta, \theta \right] dx \\ &= I_1 + I_2 + I_3. \end{aligned} \quad (3.12)$$

For I_1 , we have, using (3.8):

$$I_1 = \int |\Lambda^{s+\frac{\alpha}{2}} \theta|^2 \cdot \theta dx + \int \left(\Lambda^{s-\frac{\alpha}{2}} G \right) \cdot \left(\Lambda^{s+\frac{\alpha}{2}} \theta \right) \cdot \theta dx = I_{11} + I_{12}.$$

The term I_{11} is controlled by the dissipation in the right side of (3.10): set

$$\rho_m(t) = \inf_{0 \leq \tau \leq t, x \in \mathbb{T}} \rho(x, \tau).$$

Note that $\rho_m(t) > 0$ by Lemma 2.4. Then we have, using Lemma 2.4:

$$-I_{11} - \kappa \|\theta\|_{\dot{H}^{s+\frac{\alpha}{2}}}^2 \leq (\|\theta\|_{L^\infty} - \kappa) \|\theta\|_{\dot{H}^{s+\frac{\alpha}{2}}}^2 \leq -\rho_m(t) \|\theta\|_{\dot{H}^{s+\frac{\alpha}{2}}}^2. \quad (3.13)$$

To bound I_{12} we use the Hölder inequality,

$$|I_{12}| \leq \|G\|_{\dot{H}^{s-\frac{\alpha}{2}}} \|\theta\|_{\dot{H}^{s+\frac{\alpha}{2}}} \|\theta\|_{L^\infty} \leq \frac{\rho_m}{6} \|\theta\|_{\dot{H}^{s+\frac{\alpha}{2}}}^2 + \frac{3}{2\rho_m} \|\theta\|_{L^\infty}^2 \|G\|_{\dot{H}^{s-\frac{\alpha}{2}}}^2. \quad (3.14)$$

In order to control the term I_2 in (3.12), we, once again, use (3.8), and the Hölder inequality:

$$\begin{aligned} |I_2| &\leq \left(\|\theta\|_{\dot{H}^{s+\frac{\alpha}{2}}} + \|G\|_{\dot{H}^{s-\frac{\alpha}{2}}} \right) \|\theta\|_{\dot{H}^s} \|\Lambda^{\frac{\alpha}{2}} \theta\|_{L^\infty} \\ &\leq \frac{\rho_m}{6} \|\theta\|_{\dot{H}^{s+\frac{\alpha}{2}}}^2 + \left(\frac{3}{2\rho_m} + \frac{1}{2} \right) \|\Lambda^{\frac{\alpha}{2}} \theta\|_{L^\infty}^2 \|\theta\|_{\dot{H}^s}^2 + \frac{1}{2} \|G\|_{\dot{H}^{s-\frac{\alpha}{2}}}^2. \end{aligned} \quad (3.15)$$

The contribution of I_3 in (3.12) is bounded using the commutator estimate (3.5):

$$\begin{aligned}
 |I_3| &\leq \left(\|\theta\|_{\dot{H}^{s+\frac{\alpha}{2}}} + \|G\|_{\dot{H}^{s-\frac{\alpha}{2}}} \right) \left\| \left[\Lambda^{\frac{\alpha}{2}}, \Lambda^s \theta, \theta \right] \right\|_{L^2} \\
 &\leq C \left(\|\theta\|_{\dot{H}^{s+\frac{\alpha}{2}}} + \|G\|_{\dot{H}^{s-\frac{\alpha}{2}}} \right) \|\theta\|_{\dot{H}^s} \|\theta\|_{C^{\frac{\alpha}{2}+\varepsilon}} \\
 &\leq \frac{\rho_m}{6} \|\theta\|_{\dot{H}^{s+\frac{\alpha}{2}}}^2 + \left(\frac{3}{2\rho_m} + \frac{1}{2} \right) C^2 \|\theta\|_{C^{\frac{\alpha}{2}+\varepsilon}}^2 \|\theta\|_{\dot{H}^s}^2 + \frac{1}{2} \|G\|_{\dot{H}^{s-\frac{\alpha}{2}}}^2. \quad (3.16)
 \end{aligned}$$

Next, we estimate the term II in (3.11), integrating by parts

$$|II| = \frac{1}{2} \left| \int (\Lambda^s \theta)^2 \cdot \partial_x u \, dx \right| \leq C (\|\Lambda^\alpha \theta\|_{L^\infty} + \|G\|_{L^\infty}) \|\theta\|_{\dot{H}^s}^2. \quad (3.17)$$

For the term III in (3.11), we apply the commutator estimate (3.4) and get

$$\begin{aligned}
 |III| &\leq \|\theta\|_{\dot{H}^s} \left\| [\Lambda^s \partial_x u, \theta] \right\|_{L^2} \\
 &\leq C \|\theta\|_{H^s} (\|\partial_x u\|_{L^\infty} \|\theta\|_{H^s} + \|\partial_x \theta\|_{L^\infty} \|u\|_{H^s}). \quad (3.18)
 \end{aligned}$$

To estimate $\|u\|_{H^s}$ in the right side, we apply Corollary 2.2 to get

$$\|u\|_{H^s} = \|u\|_{L^2} + \|\partial_x u\|_{H^{s-1}} \leq C (1 + \|\theta\|_{H^{s-1+\alpha}} + \|G\|_{H^{s-1}}). \quad (3.19)$$

We also have, using the uniform bound on the density,

$$\begin{aligned}
 |\Lambda^\alpha \theta| &\leq c_\alpha \int_{\mathbb{R}} \frac{|\theta(x) - \theta(y)| \, dy}{|x - y|^{1+\alpha}} \leq C (\|\theta\|_{L^\infty} + \|\partial_x \theta\|_{L^\infty}) \\
 &\leq C_0 (1 + \|\partial_x \theta\|_{L^\infty}), \quad (3.20)
 \end{aligned}$$

with a constant C_0 that depends on ρ_0 and u_0 . Therefore, $\partial_x u$ satisfies

$$\|\partial_x u\|_{L^\infty} \leq \|\Lambda^\alpha \theta\|_{L^\infty} + \|G\|_{L^\infty} \leq C (1 + \|\partial_x \theta\|_{L^\infty} + \|G\|_{L^\infty}). \quad (3.21)$$

Together, (3.18)–(3.21) give

$$|III| \leq C (1 + \|\partial_x \theta\|_{L^\infty} + \|G\|_{L^\infty}) \left(1 + \|\theta\|_{\dot{H}^s}^2 + \|G\|_{\dot{H}^{s-1}}^2 \right). \quad (3.22)$$

The third term in the right side of (3.10) can be estimated as

$$\kappa \left| \int (\Lambda^s \theta) \cdot (\Lambda^s G) \, dx \right| \leq \kappa \|\theta\|_{\dot{H}^{s+\frac{\alpha}{2}}} \|G\|_{\dot{H}^{s-\frac{\alpha}{2}}} \leq \frac{\rho_m}{6} \|\theta\|_{\dot{H}^{s+\frac{\alpha}{2}}}^2 + \frac{3\kappa^2}{2\rho_m} \|G\|_{\dot{H}^{s-\frac{\alpha}{2}}}^2.$$

Putting the above estimates together, we end up with the following inequality:

$$\begin{aligned}
 \frac{1}{2} \frac{d}{dt} \|\theta\|_{\dot{H}^s}^2 &\leq C \left(1 + \frac{1}{\rho_m} \right) \left(1 + \|\partial_x \theta\|_{L^\infty}^2 + \|G\|_{L^\infty}^2 \right) \left(\|\theta\|_{\dot{H}^s}^2 + \|G\|_{\dot{H}^{s-\frac{\alpha}{2}}}^2 + 1 \right) \\
 &\quad - \frac{\rho_m}{3} \|\theta\|_{\dot{H}^{s+\frac{\alpha}{2}}}^2. \quad (3.23)
 \end{aligned}$$

In order to close the estimate, and obtain a bound on $Y(t)$ defined in (3.9), we write

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|G\|_{\dot{H}^{s-\frac{\alpha}{2}}}^2 &= - \int \left(\Lambda^{s-\frac{\alpha}{2}} G \right) \cdot \left(\Lambda^{s-\frac{\alpha}{2}} \partial_x (Gu) \right) dx \\ &= - \int \left(\Lambda^{s-\frac{\alpha}{2}} G \right) \cdot \left(u \Lambda^{s-\frac{\alpha}{2}} \partial_x G \right) dx \\ &\quad - \int \left(\Lambda^{s-\frac{\alpha}{2}} G \right) \cdot \left[\Lambda^{s-\frac{\alpha}{2}} \partial_x, u \right] G dx = IV + V. \end{aligned} \quad (3.24)$$

The term IV can be treated as II via integration by parts, together with (3.20):

$$|IV| = \frac{1}{2} \left| \int \left(\Lambda^{s-\frac{\alpha}{2}} G \right)^2 \cdot \partial_x u \, dx \right| \leq C (1 + \|\partial_x \theta\|_{L^\infty} + \|G\|_{L^\infty}) \|G\|_{\dot{H}^{s-\frac{\alpha}{2}}}^2. \quad (3.25)$$

To bound V , we apply the commutator estimate (3.6), as well as (3.21):

$$\begin{aligned} |V| &\leq \|G\|_{\dot{H}^{s-\frac{\alpha}{2}}} \left\| \left[\Lambda^{s-\frac{\alpha}{2}} \partial_x, u \right] G \right\|_{L^2} \\ &\leq C \|G\|_{\dot{H}^{s-\frac{\alpha}{2}}} \left(\|\partial_x u\|_{L^\infty} \|G\|_{\dot{H}^{s-\frac{\alpha}{2}}} + \|G\|_{L^\infty} \|\partial_x u\|_{\dot{H}^{s-\frac{\alpha}{2}}} \right) \\ &\leq C (1 + \|\partial_x \theta\|_{L^\infty} + \|G\|_{L^\infty}) \|G\|_{\dot{H}^{s-\frac{\alpha}{2}}}^2 + C \|G\|_{L^\infty} \|G\|_{\dot{H}^{s-\frac{\alpha}{2}}} \|\theta\|_{\dot{H}^{s+\frac{\alpha}{2}}} \\ &\leq \frac{\rho_m}{6} \|\theta\|_{\dot{H}^{s+\frac{\alpha}{2}}}^2 + C \left(1 + \frac{1}{\rho_m} \|G\|_{L^\infty}^2 + \|\partial_x \theta\|_{L^\infty} + \|G\|_{L^\infty} \right) \|G\|_{\dot{H}^{s-\frac{\alpha}{2}}}^2. \end{aligned} \quad (3.26)$$

Now, estimates (3.23)–(3.26), together with the uniform bound on $\|G\|_{L^\infty}$, yield an inequality

$$\frac{d}{dt} Y(t) \leq C \left(1 + \frac{1}{\rho_m(t)} \right) (1 + \|\partial_x \theta(\cdot, t)\|_{L^\infty}^2) Y(t) - \frac{\rho_m(t)}{6} \|\theta(\cdot, t)\|_{\dot{H}^{s+\frac{\alpha}{2}}}^2. \quad (3.27)$$

For $s > 3/2$, H^s is embedded in $W^{1,\infty}$. This, together with Lemma 2.4, implies

$$\frac{d}{dt} Y(t) \leq C(1+t)(1+Y(t))Y(t), \quad (3.28)$$

and the local in time well-posedness for solutions with H^s initial data follows. Moreover, it follows from (3.27) that

$$Y(T) \leq Y(0) \exp \left[C \int_0^T (1+t) \left(1 + \|\partial_x \theta(\cdot, t)\|_{L^\infty}^2 \right) dt \right]. \quad (3.29)$$

For all finite $T > 0$, if the Beale–Kato–Majda criterion (3.3) is satisfied, the right side of (3.29) is finite, whence

$$\theta \in C([0, T], H^s(\mathbb{T})), \quad G(\cdot, t) \in C\left([0, T], H^{s-\frac{\alpha}{2}}(\mathbb{T})\right),$$

and thus $\rho \in C([0, T], H^s(\mathbb{T}))$. Furthermore, integrating (3.27) in $[0, T]$, we see that if (3.3) holds, then

$$\frac{\rho_m(T)}{6} \|\theta\|_{L^2([0, T], H^{s+\frac{\alpha}{2}})}^2 < +\infty,$$

thus $\rho \in L^2([0, T], H^{s+\frac{\alpha}{2}})$. To recover the conditions on u in (3.2), we apply Corollary 2.2 and get

$$\|u(\cdot, t)\|_{H^{s+1-\alpha}}^2 = \|u(\cdot, t)\|_{L^2}^2 + \|\partial_x u(\cdot, t)\|_{H^{s-\alpha}}^2 \leq C + CY(t) < \infty.$$

This ends the proof of Theorem 3.1.

4. The Global Regularity

In this section, we derive a uniform L^∞ -bound on $\partial_x \rho$, using a variant of the modulus of continuity method. Together with the Beale–Kato–Majda type criterion (3.3), this will imply the global well-posedness of the Euler alignment system (1.25)–(1.26), and prove Theorem 1.1. We will first consider the special case $G \equiv 0$, that is, the system (1.36). The nonlinear diffusive term makes the problem subtler than in the SQG or Burgers equation case. Finally, we prove the result to the general Euler alignment system, using a combination of an appropriate scaling argument, estimate on the minimum of ρ , and additional regularity estimates. In this case, the bound on $\partial_x \rho$ will depend on time and may grow, but remains finite for every $t > 0$.

For convenience, we work on \mathbb{R} , and extend ρ and u periodically in space.

4.1. The Modulus of Continuity

We say that a function f obeys modulus of continuity ω if

$$f(x) - f(y) < \omega(|x - y|), \quad \text{for all } x, y \in \mathbb{R}.$$

We will work with the following modulus of continuity for the density ρ :

$$\omega(\xi) = \begin{cases} \xi - \xi^{1+\alpha/2}, & 0 \leq \xi < \delta \\ \gamma \log(\xi/\delta) + \delta - \delta^{1+\alpha/2}, & \xi \geq \delta, \end{cases} \quad (4.1)$$

so that ω is continuous at $\xi = \delta$. The parameters δ and γ are sufficiently small positive numbers to be specified later. The modulus ω is continuous, piecewise differentiable, increasing and concave, and satisfies

$$\omega''(0) = -\infty. \quad (4.2)$$

The following proposition describes the only possible modulus breakthrough scenario for evolution equations.

Proposition 4.1. ([33]). *Suppose ρ_0 obeys a modulus of continuity ω that satisfies (4.2). If the solution $\rho(x, t)$ violates ω at some positive time, then there must exist $t_1 > 0$ and $x_1 \neq y_1$ such that*

$$\rho(x_1, t_1) - \rho(y_1, t_1) = \omega(|x_1 - y_1|), \text{ and } \rho(\cdot, t) \text{ obeys } \omega \text{ for every } 0 \leq t < t_1. \quad (4.3)$$

Thus, to prove that ρ obeys a modulus of continuity ω for all times $t > 0$, it is sufficient to prove that if (4.3) holds, then

$$\partial_t(\rho(x_1, t_1) - \rho(y_1, t_1)) < 0. \quad (4.4)$$

As a remark on the notation, we will again use C as a notation for various universal constants that do not depend on T, δ and γ .

4.2. The Global Regularity for the Special System with $G \equiv 0$

Let us first consider the special case $G \equiv 0$, or, equivalently, the system (1.36):

$$\partial_t \rho + \partial_x(\rho u) = 0, \quad \partial_x u = \Lambda^\alpha \rho. \quad (4.5)$$

As the mean of u is preserved by the evolution—see (2.14), we may assume without loss of generality that

$$\int_{\mathbb{T}} u(x, t) dx = 0, \quad (4.6)$$

for otherwise we would simply consider (4.5) in a frame moving the speed equal to the mean of u_0 . Thus, we have

$$u(x, t) = \Lambda^\alpha \varphi(x, t). \quad (4.7)$$

Here, $\varphi(x, t)$ is the mean-zero primitive of $\theta(x, t)$, as in (2.6). We will prove the following result:

Theorem 4.2. *The system (4.5) with a smooth periodic initial condition ρ_0 such that $\rho_0(x) > 0$ for all $x \in \mathbb{T}$ has a unique global smooth solution.*

The key step in the proof is

Lemma 4.3. *Suppose that $m = \min_{x \in \mathbb{T}} \rho_0(x) > 0$. Then there exist δ_m and γ_m , independent of the period of the initial data, such that if $\rho_0(x)$ obeys the modulus of continuity ω given by (4.1), then $\rho(x, t)$ obeys ω for all $t > 0$.*

Theorem 4.2 is a consequence of Lemma 4.3. Indeed, suppose that Lemma 4.3 is true. Notice that the equation (4.5) has a scaling invariance: if $\rho(x, t)$ is a solution, then so is

$$\rho_\lambda(x, t) = \rho(\lambda x, \lambda^\alpha t), \quad (4.8)$$

for any $\lambda > 0$. From the properties of the modulus of continuity ω given by (4.1) (in particular its growth at infinity) it follows that we can find $\lambda > 0$ sufficiently small such that $\rho_\lambda^0(x) = \rho_0(\lambda x)$ obeys ω with $\delta = \delta_m$, $\gamma = \gamma_m$ provided by Lemma 4.3. Note that the rescaling (4.8) does not change the minimum of ρ . As δ_m and γ_m do not depend on the period, Lemma 4.3 shows that $\rho_\lambda(x, t)$ obeys ω for all $t > 0$. In particular, it follows that

$$|\partial_x \rho_\lambda(t, x)| \leq 1, \quad \text{for all } t > 0 \text{ and } x \in \mathbb{T}. \quad (4.9)$$

As we have mentioned, (4.9) together with the Beale–Kato–Majda type criterion (3.3), implies that $\rho_\lambda(t, x)$ is a global in time solution of (4.5), and thus so is $\rho(t, x)$.

Therefore, we only need to prove Lemma 4.3. Our strategy is as follows. Let us assume that a modulus of continuity ω , with some δ and γ is broken at a time t_1 , in the sense that (4.3) holds for some $x_1, y_1 \in \mathbb{T}$. We denote

$$\xi = |x_1 - y_1| > 0, \quad (4.10)$$

and, for simplicity, drop the time variable t_1 in the notation. We compute

$$\begin{aligned} \partial_t(\rho(x_1) - \rho(y_1)) &= -\partial_x(\rho(x_1)u(x_1)) + \partial_x(\rho(y_1)u(y_1)) \\ &= -(u(x_1)\partial_x \rho(x_1) - u(y_1)\partial_x \rho(y_1)) \\ &\quad -(\rho(x_1) - \rho(y_1))\partial_x u(x_1) - \rho(y_1)(\partial_x u(x_1) - \partial_x u(y_1)) \\ &= I + II + III. \end{aligned} \quad (4.11)$$

We will obtain the following estimates for the three terms in the right side of (4.11): to bound the first term we note that if $\Omega(\xi)$ is a modulus of continuity for u , then it follows from [33] that

$$|I| = |u(x_1)\partial_x \rho(x_1) - u(y_1)\partial_x \rho(y_1)| \leq \omega'(\xi)\Omega(\xi). \quad (4.12)$$

The modulus $\Omega(\xi)$ for u is given by the following:

Lemma 4.4. *Let ρ obey the modulus of continuity ω as in (4.1). There exists a universal constant $C > 0$ so that then $u(x)$ obeys a modulus of continuity*

$$\Omega(\xi) \leq \begin{cases} C\xi, & 0 < \xi < \delta, \\ C\xi^{1-\alpha}\omega(\xi), & \xi \geq \delta. \end{cases} \quad (4.13)$$

We will prove Lemma 4.4 later in this section.

As $\omega'(\xi) \leq 1$ for $0 \leq \xi < \delta$, and $\omega'(\xi) = \gamma/\xi$ for $\xi > \delta$, we conclude that

$$|I| \leq \omega'(\xi)\Omega(\xi) \leq \begin{cases} C\xi, & 0 < \xi < \delta, \\ C\gamma \frac{\omega(\xi)}{\xi^\alpha}, & \xi \geq \delta, \end{cases} \quad (4.14)$$

again, with the constant $C > 0$ that does not depend on ρ_0 .

To bound the last two terms in the right side of (4.11) purely in terms of $\xi = |x_1 - y_1|$ we will use the following lemma:

Lemma 4.5. *Let ρ obey the modulus of continuity ω as in (4.1), and let x_1, y_1 be the breakthrough points as in (4.3). There exists a constant $C > 0$ that may only depend on α such that*

$$\Lambda^\alpha \rho(x_1) \geq -A(\xi), \quad A(\xi) := \begin{cases} C & \text{if } 0 \leq \xi \leq \delta, \\ C\gamma\xi^{-\alpha} & \text{if } \xi > \delta, \end{cases} \quad (4.15)$$

and

$$\Lambda^\alpha \rho(x_1) - \Lambda^\alpha \rho(y_1) \geq D_1(\xi), \quad D_1(\xi) := \begin{cases} C\xi^{1-\alpha/2}, & 0 < \xi \leq \delta, \\ C\omega(\xi)\xi^{-\alpha}, & \xi \geq \delta. \end{cases} \quad (4.16)$$

The first estimate in the above lemma gives a bound for the second term in (4.11):

$$II = -(\rho(x_1) - \rho(y_1))\Lambda^\alpha \rho(x_1) \leq \omega(\xi)A(\xi), \quad (4.17)$$

while (4.16) leads to:

$$III = -\rho(y)(\Lambda^\alpha \rho(x_1) - \Lambda^\alpha \rho(y_1)) \leq -mD_1(\xi). \quad (4.18)$$

Here, m is the minimum of ρ_0 and is preserved in time; see Corollary 2.5. Putting (4.12), (4.17) and (4.18) together, we obtain

$$\partial_t(\rho(x_1, t_1) - \rho(y_1, t_1)) \leq \omega'(\xi)\Omega(\xi) + \omega(\xi)A(\xi) - mD_1(\xi). \quad (4.19)$$

For $0 \leq \xi < \delta$, using (4.13), (4.15) and (4.16), as well as the inequalities

$$\omega(\xi) \leq \xi, \quad \omega'(\xi) \leq 1, \quad 0 \leq \xi < \delta, \quad (4.20)$$

we see that

$$\omega'(\xi)\Omega(\xi) + \omega(\xi)A(\xi) - \frac{1}{2}mD_1(\xi) \leq C\xi - Cm\xi^{1-\alpha/2} < 0, \quad (4.21)$$

provided that

$$\delta < Cm^{2/\alpha}. \quad (4.22)$$

On the other hand, for $\xi \geq \delta$, the above bounds tell us

$$\omega'(\xi)\Omega(\xi) + \omega(\xi)A(\xi) - \frac{1}{2}mD_1(\xi) \leq \frac{C\gamma\omega(\xi)}{\xi^\alpha} - \frac{Cm\omega(\xi)}{\xi^\alpha} < 0, \quad (4.23)$$

if

$$\gamma < Cm. \quad (4.24)$$

Therefore, for δ and γ sufficiently small, we have

$$\partial_t(\rho(x_1, t_1) - \rho(y_1, t_1)) < 0, \quad (4.25)$$

which is a contradiction to the assumption that t_1 is the first breakthrough time. Thus, ω can never be broken, and the proof of Lemma 4.3 is complete, except for the proof of Lemmas 4.4 and 4.5. \square

4.2.1. The Dissipation Bound in Lemma 4.5 We first prove the dissipation bound (4.16) in Lemma 4.5. It was shown in [31] that

$$\Lambda^\alpha \rho(x_1) - \Lambda^\alpha \rho(y_1) \geq D(\xi) \quad (4.26)$$

with

$$D(\xi) = c_\alpha \left[\int_0^{\xi/2} \frac{2\omega(\xi) - \omega(\xi + 2\eta) - \omega(\xi - 2\eta)}{\eta^{1+\alpha}} d\eta + \int_{\xi/2}^\infty \frac{2\omega(\xi) - \omega(\xi + 2\eta) + \omega(2\eta - \xi)}{\eta^{1+\alpha}} d\eta \right]. \quad (4.27)$$

Both terms in the right side are positive due to the concavity of ω .

To obtain a lower bound for $D(\xi)$, we consider two cases. For $\xi \leq \delta$, we only keep the first term. Note that

$$\omega(\xi + 2\eta) \leq \omega(\xi) + 2\omega'(\xi)\eta$$

due to the concavity of ω , and

$$\omega(\xi - 2\eta) \leq \omega(\xi) - 2\omega'(\xi)\eta + 2\omega''(\xi)\eta^2,$$

due to the second order Taylor formula and the monotone growth of

$$\omega''(\xi) = -\frac{\alpha(2+\alpha)}{4}\xi^{-1+\alpha/2}.$$

This gives

$$D(\xi) \geq C \int_0^{\xi/2} \frac{(-\omega''(\xi))\eta^2}{\eta^{1+\alpha}} d\eta = C\xi^{1-\alpha/2}, \quad \text{for } 0 \leq \xi \leq \delta, \quad (4.28)$$

which is the first bound in (4.16).

For $\xi > \delta$, we only keep the second term in (4.27). Due to the concavity of ω , we have

$$\omega(2\eta + \xi) - \omega(2\eta - \xi) \leq \omega(2\xi) = \omega(\xi) + \gamma \log 2 \leq \frac{3}{2}\omega(\xi), \quad (4.29)$$

if

$$\gamma \leq \frac{\omega(\delta)}{2 \log 2} = \frac{\delta - \delta^{1+\alpha/2}}{2 \log 2}. \quad (4.30)$$

In that case, we have, using (4.29), that

$$D(\xi) \geq c_\alpha \int_{\xi/2}^\infty \frac{2\omega(\xi) - \omega(2\xi)}{\eta^{1+\alpha}} d\eta \geq C\omega(\xi) \cdot \frac{1}{\alpha} \left(\frac{\xi}{2} \right)^{-\alpha} = C \frac{\omega(\xi)}{\xi^\alpha}, \quad \text{for } \xi > \delta, \quad (4.31)$$

and the proof of (4.16) is complete.

4.2.2. A Lower Bound on $\Lambda^\alpha \rho$ in Lemma 4.5 The next step is obtain the lower bound (4.15) for $\Lambda^\alpha \rho(x_1, t_1)$. As ω is a modulus of ρ , we have for any $z \in \mathbb{R}$

$$\rho(z) \leq \rho(y) + \omega(|y - z|), \quad (4.32)$$

while

$$\rho(x_1) = \rho(y_1) + \omega(|x_1 - y_1|). \quad (4.33)$$

This implies a lower bound

$$\begin{aligned} \Lambda^\alpha \rho(x_1) &= c_\alpha \int_{\mathbb{R}} \frac{\rho(x_1) - \rho(y_1) + \rho(y_1) - \rho(z)}{|x_1 - z|^{1+\alpha}} dz \\ &\geq c_\alpha \int_{\mathbb{R}} \frac{\omega(\xi) - \omega(|y_1 - z|)}{|x_1 - z|^{1+\alpha}} dz \\ &= c_\alpha \int_{\mathbb{R}} \frac{\omega(\xi) - \omega(|\xi - \eta|)}{|\eta|^{1+\alpha}} d\eta =: -A(\xi). \end{aligned} \quad (4.34)$$

Our goal is to bound $A(\xi)$ from above. Let us decompose the integral in the second line of (4.34) as

$$\begin{aligned} -A(\xi) &= c_\alpha \int_{\mathbb{R}} \frac{\omega(\xi) - \omega(|\xi - \eta|)}{|\eta|^{1+\alpha}} d\eta = \int_{-\infty}^{-\xi} + \int_{-\xi}^{\xi} + \int_{\xi}^{2\xi} + \int_{2\xi}^{\infty} \\ &= A_1 + A_2 + A_3 + A_4. \end{aligned}$$

We claim that A_2 and A_3 are positive, so that their contribution to $A(\xi)$ is negative. Indeed, we can estimate A_2 using the concavity of ω :

$$A_2 = \int_0^{\xi} \frac{2\omega(\xi) - \omega(\xi - \eta) - \omega(\xi + \eta)}{\eta^{1+\alpha}} d\eta \geq 0. \quad (4.35)$$

In addition, $A_3 \geq 0$ simply due to the monotonicity of ω , which implies

$$\omega(\xi) \geq \omega(|\eta - \xi|), \text{ for } \eta \in [\xi, 2\xi].$$

It remains to bound A_1 and A_4 from below. We first consider $0 \leq \xi \leq \delta$. In this region, we can estimate A_4 as follows:

$$\begin{aligned} A_4 &\geq - \int_{2\xi}^{\infty} \frac{\omega(\eta - \xi)}{\eta^{1+\alpha}} d\eta \geq - \int_{2\xi}^{\xi+\delta} \frac{\eta}{\eta^{1+\alpha}} d\eta \\ &\quad - \int_{\xi+\delta}^{\infty} \frac{\gamma \log((\eta - \xi)/\delta) + \delta - \delta^{1+\alpha/2}}{\eta^{1+\alpha}} d\eta \\ &\geq - \int_0^{2\delta} \frac{d\eta}{\eta^\alpha} - (\delta - \delta^{1+\alpha/2}) \int_{\delta}^{\infty} \frac{d\eta}{\eta^{1+\alpha}} \\ &\quad - \gamma \int_{\delta}^{\infty} \frac{\log(\eta/\delta)}{\eta^{1+\alpha}} d\eta \geq -C\delta^{1-\alpha} - C\gamma\delta^{-\alpha}. \end{aligned} \quad (4.36)$$

Thus, if we choose $\delta < 1$ and $\gamma < \delta$, as in (4.30), we obtain

$$A_4 \geq -C, \quad \text{for } 0 \leq \xi \leq \delta. \quad (4.37)$$

The term A_1 can be estimated similarly for $0 \leq \xi \leq \delta$. Indeed, for $\xi < \delta/2$, we have

$$\begin{aligned} A_1 &\geq - \int_{\xi}^{\infty} \frac{\omega(\eta + \xi)}{\eta^{1+\alpha}} d\eta \geq - \int_{\xi}^{\delta-\xi} \frac{\eta}{\eta^{1+\alpha}} d\eta - \int_{\delta-\xi}^{\infty} \frac{\gamma \log((\eta + \xi)/\delta) + \delta}{\eta^{1+\alpha}} d\eta \\ &\geq -C\delta^{1-\alpha} - C\gamma \int_{\delta/2}^{\infty} \frac{\log(\eta/\delta)}{\eta^{1+\alpha}} d\eta \geq -C\delta^{1-\alpha} - C\gamma\delta^{-\alpha} \geq -C, \end{aligned} \quad (4.38)$$

provided that γ satisfies (4.30). On the other hand, for $\delta/2 \leq \xi \leq \delta$, we have

$$\begin{aligned} A_1 &\geq - \int_{\xi}^{\infty} \frac{\gamma \log((\eta + \xi)/\delta) + \delta}{\eta^{1+\alpha}} d\eta \geq - \int_{\xi}^{\infty} \frac{\gamma \log(2\eta/\delta) + \delta}{\eta^{1+\alpha}} d\eta \\ &\geq - \int_{\delta/2}^{\infty} \frac{\gamma \log(2\eta/\delta) + \delta}{\eta^{1+\alpha}} d\eta \\ &\geq -C\delta^{1-\alpha} - C\gamma\delta^{-\alpha} \geq -C. \end{aligned} \quad (4.39)$$

Summing up the above computation, we conclude that

$$\Lambda^{\alpha} \rho(x_1) \geq -A(\xi) \geq -C \text{ if } 0 \leq \xi \leq \delta. \quad (4.40)$$

On the other hand, if $\xi > \delta$, we have the following estimates on A_1 and A_4 :

$$A_1 = \gamma \int_{-\infty}^{-\xi} \frac{\log \xi - \log(\xi - \eta)}{|\eta|^{1+\alpha}} d\eta = -\frac{\gamma}{\xi^{\alpha}} \int_{-\infty}^{-1} \frac{\log(1 - \zeta)}{|\zeta|^{1+\alpha}} d\zeta \geq -\frac{C\gamma}{\xi^{\alpha}}, \quad (4.41)$$

and

$$A_4 = \gamma \int_{2\xi}^{\infty} \frac{\log \xi - \log(\eta - \xi)}{|\eta|^{1+\alpha}} d\eta = -\frac{\gamma}{\xi^{\alpha}} \int_2^{\infty} \frac{\log(\zeta - 1)}{\zeta^{1+\alpha}} d\zeta \geq -\frac{C\gamma}{\xi^{\alpha}}, \quad (4.42)$$

Thus, we have the bound

$$\Lambda^{\alpha} \rho(x_1) \geq -A(\xi) \geq -C\gamma\xi^{-\alpha} \text{ for } \xi > \delta, \quad (4.43)$$

finishing the proof of (4.15), as well as of Lemma 4.5. \square

4.2.3. The Proof of Lemma 4.4 Next, we find a modulus of continuity Ω for u , if ρ obeys ω given by (4.1). We start with (4.7):

$$u(x) = c_{\alpha} \lim_{\varepsilon \downarrow 0} \int_{|y| > \varepsilon} \frac{\varphi(x) - \varphi(x + y)}{|y|^{1+\alpha}} dy. \quad (4.44)$$

The first term in the right side can be evaluated explicitly:

$$\int_{|y| > \varepsilon} \frac{\varphi(x)}{|y|^{1+\alpha}} dy = \frac{2}{\alpha} \frac{\varphi(x)}{\varepsilon^{\alpha}}. \quad (4.45)$$

The second term in the right side of (4.44) can be re-written using integration by parts as

$$\int_{|y|>\varepsilon} \frac{\varphi(x+y)}{|y|^{1+\alpha}} dy = \frac{1}{\alpha} \frac{\varphi(x+\varepsilon) + \varphi(x-\varepsilon)}{\varepsilon^\alpha} + \frac{1}{\alpha} \int_{|y|>\varepsilon} \frac{\theta(x+y)}{\operatorname{sgn}(y)|y|^\alpha} dy. \quad (4.46)$$

As $\theta \in L^\infty$, so that φ is uniformly Lipschitz, we can combine (4.45) and (4.46), pass to the limit $\varepsilon \downarrow 0$, and obtain

$$u(x) = -\frac{c_\alpha}{\alpha} \int_{\mathbb{R}} \frac{\theta(x+y)}{\operatorname{sgn}(y)|y|^\alpha} dy. \quad (4.47)$$

Let us note that, since $\theta(x)$ is a periodic mean-zero function, the integral in the right side of (4.47) converges as $|y| \rightarrow +\infty$, and

$$u(x) = \frac{c_\alpha}{\alpha} \int_{\mathbb{R}} \frac{\theta(x) - \theta(x+y)}{\operatorname{sgn}(y)|y|^\alpha} dy = \frac{c_\alpha}{\alpha} \int_{\mathbb{R}} \frac{\rho(x) - \rho(x+y)}{\operatorname{sgn}(y)|y|^\alpha} dy. \quad (4.48)$$

Using an argument similar to that in the appendix of [33], one can show that, as long as $\rho(x)$ obeys a modulus of continuity ω , the function $u(x)$ given by (4.48) obeys the modulus of continuity

$$\Omega(\xi) = C \left(\int_0^\xi \frac{\omega(\eta)}{\eta^\alpha} d\eta + \xi \int_\xi^\infty \frac{\omega(\eta)}{\eta^{1+\alpha}} d\eta \right), \quad (4.49)$$

with a universal constant $C > 0$.

Thus, for $0 \leq \xi \leq \delta$, we get

$$\begin{aligned} \Omega(\xi) &\leq C \left(\int_0^\xi \eta^{1-\alpha} d\eta + \xi \int_\xi^\delta \eta^{1-\alpha} d\eta + \xi \int_\delta^\infty \frac{\gamma \log(\eta/\delta) + \delta}{\eta^{1+\alpha}} d\eta \right) \\ &\leq C \left(\xi^{2-\alpha} + \xi \delta^{2-\alpha} + \xi \gamma \delta^{-\alpha} + \xi \delta^{1-\alpha} \right) \leq C \xi, \end{aligned} \quad (4.50)$$

as long as we take $\gamma < \delta$. This is the first inequality in (4.13).

For $\xi > \delta$, we use (4.49) to write

$$\begin{aligned} \Omega(\xi) &\leq C \left(\int_0^\delta \eta^{1-\alpha} d\eta + \int_\delta^\xi \frac{\gamma \log(\eta/\delta) + \delta - \delta^{1+\alpha/2}}{\eta^\alpha} d\eta \right. \\ &\quad \left. + \xi \int_\xi^\infty \frac{\gamma \log(\eta/\delta) + \delta - \delta^{1+\alpha/2}}{\eta^{1+\alpha}} d\eta \right) \\ &\leq C \left(\delta^{2-\alpha} + \xi^{1-\alpha} (\delta - \delta^{1+\alpha/2}) \right) + C \gamma \delta^{1-\alpha} \int_1^{\xi/\delta} \frac{\log \eta}{\eta^\alpha} d\eta \\ &\quad + C \gamma \xi \delta^{-\alpha} \int_{\xi/\delta}^\infty \frac{\log \eta}{\eta^{1+\alpha}} d\eta \\ &\leq C \left(\delta^{2-\alpha} + \xi^{1-\alpha} (\delta - \delta^{1+\alpha/2}) \right) + C \gamma \xi^{1-\alpha} (1 + \log(\xi/\delta)) \\ &\leq C \left(\delta^{2-\alpha} + \xi^{1-\alpha} \omega(\xi) \right) \leq C \xi^{1-\alpha} \omega(\xi), \end{aligned} \quad (4.51)$$

finishing the proof of Lemma 4.4.

4.3. The Global Regularity for the Full System

We now consider the full system (1.31)–(1.33):

$$\partial_t \rho + \partial_x(\rho u) = 0, \quad (4.52)$$

$$\partial_t G + \partial_x(Gu) = 0, \quad (4.53)$$

$$\partial_x u = \Lambda^\alpha \rho + G, \quad (4.54)$$

without the extra assumption $G \equiv 0$. Let us recall representation (2.8):

$$u(x) = \Lambda^\alpha \varphi(x) + (\psi(x) + I_0) =: u^{(1)}(x) + u^{(2)}(x). \quad (4.55)$$

Here, $\phi(x)$ and $\psi(x)$ are the mean-zero primitives of θ and G , respectively, as in (2.6)–(2.7), and I_0 is given by (2.13).

Note that if $\rho(x, t)$ and $G(x, t)$ are solutions of (4.52)–(4.54), with the corresponding velocity $u(x, t)$, then

$$\rho_\lambda(x, t) = \rho(\lambda x, \lambda^\alpha t), \quad G_\lambda(x, t) = \lambda^\alpha G(\lambda x, \lambda^\alpha t), \quad (4.56)$$

are also solutions, with the corresponding velocity

$$u_\lambda(x, t) = \lambda^{-(1-\alpha)} u(\lambda x, \lambda^\alpha t), \quad (4.57)$$

and

$$F_\lambda(x, t) = \lambda^\alpha F(\lambda x, \lambda^\alpha t), \quad F(x, t) = \frac{G(x, t)}{\rho(x, t)}. \quad (4.58)$$

Note that if $\rho_\lambda(x, t)$ obeys a modulus of continuity ω , then $\rho(x, t)$ obeys the modulus of continuity

$$\omega_\lambda(\xi) = \omega(\lambda^{-1}\xi). \quad (4.59)$$

The proof of the global regularity for the solutions of (4.52)–(4.54) is based on the following lemma.

Lemma 4.6. *Let ω and ω_λ be as in (4.1) and (4.59), respectively. Given a smooth periodic initial condition (ρ_0, u_0) for (4.52)–(4.54) and $T > 0$, there exist $\delta > 0$, $\gamma > 0$ and $\lambda > 0$ so that $\rho(x, t)$ obeys the modulus of continuity $\omega_\lambda(\xi)$ for all $0 \leq t \leq T$. The parameters δ , γ and λ may depend on α , ρ_0 , u_0 , and T .*

This will imply a uniform bound on $\|\partial_x \rho\|_{L^\infty}$ on $0 \leq t \leq T$. As T is arbitrary, this is sufficient for the global regularity of the solutions, according to (3.3). Note that $\rho(x, t)$ obeys ω_λ until a time T if and only if $\rho_\lambda(x, t)$ obeys the modulus of continuity ω until the time $T_\lambda = \lambda^{-\alpha}T$, and this is what we will show. That is, given ρ_0 and u_0 , and $T > 0$, we will find $\lambda > 0$, $\delta > 0$ and $\gamma > 0$ sufficiently small, so that (i) $\rho_\lambda(0, x) = \rho_0(\lambda x)$ obeys ω , and (ii) $\rho_\lambda(x, t)$ obeys ω at least until the time $\lambda^{-\alpha}T$. The a priori bounds on $\rho(x, t)$ and $F(x, t)$ will play a crucial role in the proof.

As in the case $G \equiv 0$ considered above, we assume that a modulus of continuity ω of the form (4.1), with some δ and γ , is broken by ρ_λ at a time t_1 , at some

$x_1, y_1 \in \mathbb{R}$, in the sense of (4.3). If $\mathbb{T} = [0, L]$, then ρ_λ is $\lambda^{-1}L$ -periodic, and we can restrict our attention to $x_1, y_1 \in \mathbb{T}_\lambda := \lambda^{-1}\mathbb{T}$. We also set

$$\xi = |x_1 - y_1| > 0, \quad (4.60)$$

and drop the time variable t_1 in the notation. We decompose as in (4.11):

$$\partial_t(\rho_\lambda(x_1) - \rho_\lambda(y_1)) = -\partial_x(\rho_\lambda(x_1)u_\lambda(x_1)) + \partial_x(\rho_\lambda(y_1)u_\lambda(y_1)) = R_1 + R_2, \quad (4.61)$$

with the terms R_1 and R_2 coming from the contributions of $u_\lambda^{(1)}$ and $u_\lambda^{(2)}$ in (4.55). We treat R_1 as before:

$$\begin{aligned} R_1 &= -(u_\lambda^{(1)}(x_1)\partial_x\rho_\lambda(x_1) - u_\lambda^{(1)}(y_1)\partial_x\rho_\lambda(y_1)) \\ &\quad -(\rho_\lambda(x_1) - \rho_\lambda(y_1))\partial_x u_\lambda^{(1)}(x_1) - \rho_\lambda(y_1)(\partial_x u_\lambda^{(1)}(x_1) - \partial_x u_\lambda^{(1)}(y_1)) \\ &= I + II + III. \end{aligned} \quad (4.62)$$

Note that I and II can be estimated exactly as before: first, as in (4.14), we have

$$|I| \leq \begin{cases} C\xi, & 0 < \xi < \delta, \\ C\gamma \frac{\omega(\xi)}{\xi^\alpha}, & \xi \geq \delta, \end{cases} \quad (4.63)$$

with a constant $C > 0$ that does not depend on ρ_0 or u_0 . The term II can be bounded as in (4.17):

$$II \leq \omega(\xi)A(\xi), \quad (4.64)$$

with $A(\xi)$ defined in (4.15). The term III is bounded slightly differently from (4.18)

$$III \leq -\rho_m^{(\lambda)}(T)D_1(\xi). \quad (4.65)$$

Here, $\rho_m^{(\lambda)}(T)$ is the minimum of $\rho_\lambda(x, t)$ over $0 \leq t \leq \lambda^{-\alpha}T$, and $D_1(\xi)$ is defined in (4.16). The lower bound (2.30) in Lemma 2.4 implies that

$$\begin{aligned} \rho_m^{(\lambda)}(T) &\geq \frac{1}{\left[\rho_m^{(\lambda)}(0)\right]^{-1} + \lambda^{-\alpha}T\|F_0^\lambda\|_{L^\infty}} = \frac{1}{[\rho_m(0)]^{-1} + T\|F_0\|_{L^\infty}} \\ &\geq \frac{\rho_m(0)}{1 + T\|\partial_x u_0\|_{L^\infty} + T\|\Delta^\alpha \rho_0\|_{L^\infty}} := \bar{\rho}_m(T), \end{aligned} \quad (4.66)$$

as follows from (4.58). That is, even though now, unlike in the special case $G \equiv 0$, the function $\rho(x, t)$ does not necessarily obey the minimum principle, and $\rho_m(t)$ may decrease in time, the value of $\rho_m^{(\lambda)}(t)$ does not depend on $\lambda > 0$. Thus, we may first choose the parameters δ and γ in the definition (4.1) of the modulus of continuity ω so that (4.22) and (4.24) hold with m replaced by $\bar{\rho}_m(T)$, and, in addition, they satisfy (4.30). Next, we choose λ sufficiently small, so that $\rho_\lambda^0(x) = \rho_0(\lambda x)$ obeys the modulus of continuity ω with the above choice of δ and γ .

It remains to take into account the contribution of $u_\lambda^{(2)}$ to the right side of (4.61). The goal is to control the corresponding terms in (4.11) by the dissipation, namely, to show that

$$\begin{aligned} R_2 &= |u_\lambda^{(2)}(x_1)\partial_x \rho_\lambda(x_1) - u_\lambda^{(2)}(y_1)\partial_x \rho_\lambda(y_1)| \\ &\quad + |\rho_\lambda(x_1)\partial_x u_\lambda^{(2)}(x_1) - \rho_\lambda(y_1)\partial_x u_\lambda^{(2)}(y_1)| \\ &= R_{21} + R_{22} < \frac{1}{2}\bar{\rho}_m(T)D_1(\xi). \end{aligned} \quad (4.67)$$

Note that the flow $u_\lambda^{(2)}(x)$ is Lipschitz, as

$$|\partial_x u_\lambda^{(2)}(t, x)| = |G_\lambda(t, x)| \leq |\rho_\lambda(t, \cdot)|_{L^\infty} \|F_\lambda(t, \cdot)\|_{L^\infty} \leq C_0 \lambda^\alpha, \quad (4.68)$$

with a constant C_0 that depends on the initial conditions ρ_0 and u_0 but not on $\lambda > 0$. Therefore, $u_\lambda^{(2)}$ obeys the modulus of continuity

$$\Omega_2(\xi) = C_0 \lambda^\alpha \xi, \quad (4.69)$$

and the first term in (4.67) can be bounded by

$$R_{21} := |u_\lambda^{(2)}(x_1)\partial_x \rho_\lambda(x_1) - u_\lambda^{(2)}(y_1)\partial_x \rho_\lambda(y_1)| \leq C_0 \lambda^\alpha \xi \omega'(\xi). \quad (4.70)$$

Let us recall from (4.1) and (4.16) that

$$\omega'(\xi) \leq 1, \quad D_1(\xi) = C_1 \xi^{1-\alpha/2}, \quad \text{for } 0 \leq \xi \leq \delta, \quad (4.71)$$

hence, we have

$$R_{21} \leq C_0 \lambda^\alpha \xi \omega'(\xi) \leq C_0 \lambda^\alpha \xi < \frac{C_1 \bar{\rho}_m(T)}{4} \xi^{1-\frac{\alpha}{2}} < \frac{1}{4} \bar{\rho}_m(T) D_1(\xi) \quad \text{for } 0 \leq \xi \leq \delta, \quad (4.72)$$

provided that δ and λ are sufficiently small. On the other hand, we see from (4.1) and (4.16) again that

$$\omega'(\xi) = \frac{\gamma}{\xi}, \quad D_1(\xi) = \frac{C_1 \omega(\xi)}{\xi^\alpha}, \quad \text{for } \delta \leq \xi \leq L\lambda^{-1}. \quad (4.73)$$

It is also straightforward to check that $D_1(\xi)$ is decreasing for $\xi > \delta$, provided that

$$\gamma < c\delta, \quad (4.74)$$

with a sufficiently small constant $c > 0$ that depends only on α . We also have

$$\frac{\omega(\lambda^{-1}L)}{L^\alpha} \rightarrow +\infty, \quad \text{as } \lambda \rightarrow 0, \text{ with } L > 0 \text{ fixed.} \quad (4.75)$$

Hence, taking λ sufficiently small, depending on L as well, we have the inequality

$$\begin{aligned} R_{21} &\leq C_0 \lambda^\alpha \xi \omega'(\xi) = C_0 \lambda^\alpha \gamma < \frac{C_1 \bar{\rho}_m(T)}{4} \frac{\omega(\lambda^{-1}L)}{(\lambda^{-1}L)^\alpha} \\ &\leq \frac{C_1 \bar{\rho}_m(T)}{4} \frac{\omega(\xi)}{\xi^\alpha} \leq \frac{1}{4} \bar{\rho}_m(T) D_1(\xi), \end{aligned} \quad (4.76)$$

for $\delta \leq \xi \leq L\lambda^{-1}$. Together, (4.72) and (4.76) show that

$$R_{21} \leq \frac{1}{4} \bar{\rho}_m(T) D_1(\xi). \quad (4.77)$$

For the second term in (4.67), we write

$$\begin{aligned} R_{22} &= |\rho_\lambda(x_1) \partial_x u_\lambda^{(2)}(x_1) - \rho_\lambda(y_1) \partial_x u_\lambda^{(2)}(y_1)| \\ &= |\rho_\lambda(x_1)^2 F_\lambda(x_1) - \rho_\lambda(y_1)^2 F_\lambda(y_1)| \\ &\leq 2\lambda^\alpha \|\rho\|_{L^\infty}^2 \|F\|_{L^\infty} \leq C_0 \lambda^\alpha, \end{aligned} \quad (4.78)$$

with a constant C_0 that depends only on the initial condition ρ_0 and u_0 . Then, for λ sufficiently small, we have, once again using the fact that $\omega(\xi)/\xi^\alpha$ is decreasing for $\xi > \delta$ and (4.75):

$$\begin{aligned} R_{22} &\leq C_0 \lambda^\alpha \leq \frac{C_1 \bar{\rho}_m(T)}{4} \frac{\omega(\lambda^{-1}L)}{(\lambda^{-1}L)^\alpha} \leq \frac{C_1 \bar{\rho}_m(T)}{4} \frac{\omega(\xi)}{\xi^\alpha} \\ &= \frac{1}{4} \bar{\rho}_m(T) D_1(\xi), \text{ for } \delta \leq \xi \leq \lambda^{-1}L. \end{aligned} \quad (4.79)$$

To bound R_{22} in the region $0 \leq \xi \leq \delta$, we write

$$\begin{aligned} R_{22} &= |\rho_\lambda(x_1)^2 F_\lambda(x_1) - \rho_\lambda(y_1)^2 F_\lambda(y_1)| \\ &\leq |\rho_\lambda(x_1)^2 F_\lambda(x_1) - \rho_\lambda(y_1)^2 F_\lambda(x_1)| + |\rho_\lambda(y_1)^2 F_\lambda(x_1) - \rho_\lambda(y_1)^2 F_\lambda(y_1)| \\ &\leq 2\|\rho_\lambda\|_{L^\infty} \|F_\lambda\|_{L^\infty} \omega(\xi) + \|\rho_\lambda\|_{L^\infty}^2 \|\partial_x F_\lambda\|_{L^\infty} \xi \end{aligned} \quad (4.80)$$

Lemma 2.3 guarantees that F is Lipschitz, and the Lipschitz bound is uniform in time, thus (4.58) implies

$$\|\partial_x F_\lambda\|_{L^\infty} \leq C_0 \lambda^{1+\alpha},$$

with a constant C_0 that depends only on the initial conditions. In addition, it follows from (4.58) that

$$\|F_\lambda\|_{L^\infty} \leq C_0 \lambda^\alpha.$$

Inserting the last two bounds in (4.80), together with the expression for $D_1(\xi)$ in (4.71), gives

$$R_{22} \leq C_0 \lambda^\alpha (\omega(\xi) + \xi) \leq \frac{C_1 \bar{\rho}_m(T)}{4} \xi^{1-\alpha/2} = \frac{\bar{\rho}_m(T)}{4} D_1(\xi). \quad (4.81)$$

Here the constant C_0 depends only on the initial conditions ρ_0 and u_0 , and the second inequality holds provided that δ and λ are sufficiently small. This proves (4.67), and finishes the proof of Lemma 4.6.

Let us recap the order in which we choose the parameters. The value of α is fixed throughout the argument. Given the initial data, we also fix its period, L . We can also assume that λ does not exceed one. Next we choose δ sufficiently small so that (4.22) (with m replaced by $\bar{\rho}_m(T)$), (4.72), and (4.81) hold. Then we choose γ so that (4.24) (with m replaced by $\bar{\rho}_m(T)$), (4.30) and (4.74) hold. Finally, we choose λ so that $\rho_\lambda(0, x)$ obeys ω with the above choice of δ , γ and so that (4.76) and (4.79) hold. The proof of Theorem 1.1 is now complete. \square

A The Proof of a Commutator Estimate

In this section, we prove the commutator estimate (3.5),

$$\|[\Lambda^\gamma, f, g]\|_{L^2} \lesssim \|f\|_{L^2} \|g\|_{C^{\gamma+\varepsilon}}, \quad \gamma \in (0, 1).$$

The proof is for $x \in \mathbb{R}^n$, though it can be easily adapted to periodic case. Let (χ, η) be smooth functions such that χ is supported in a ball $\{\xi : |\xi| \leq 4/3\}$, η is supported in an annulus $\{\xi : 3/4 \leq |\xi| \leq 8/3\}$, and

$$\chi(\xi) + \sum_{q=0}^{\infty} \eta(2^{-q}\xi) \equiv 1, \quad \forall \xi \in \mathbb{R}^n.$$

It is standard to take

$$\eta(\xi) = \chi(\xi/2) - \chi(\xi),$$

which we will assume. Denote the Littlewood–Paley decomposition of f as $\sum_{q=-1}^{\infty} \Delta_q f$, where $\Delta_q f = \eta(2^{-q}D)f$ for $q \geq 0$, and $\Delta_{-1}f = \chi(D)f$. The Besov norm is defined as [3]

$$\|f\|_{B_{p,r}^s} = \left(\sum_q 2^{rs} \|\Delta_q f\|_{L^p}^r \right)^{1/r}.$$

Let the partial sum $S_q f = \sum_{p \leq q-1} \Delta_p f = \chi(2^{-q}D)f$. The Bony decomposition states

$$fg = T_f g + T_g f + R(f, g),$$

where

$$T_f g = \sum_q S_{q-1} f \cdot \Delta_q g, \quad R(f, g) = \sum_q \tilde{\Delta}_q f \cdot \Delta_q g, \quad \tilde{\Delta}_q f = \sum_{p=q-1}^{q+1} \Delta_p f.$$

Proof of the commutator estimate. First, we observe

$$\|f \Lambda^\gamma g\|_{L^2} \leq \|f\|_{L^2} \|\Lambda^\gamma g\|_{L^\infty} \lesssim \|f\|_{L^2} \|g\|_{C^{\gamma+\varepsilon}}.$$

Therefore, it suffices to prove

$$\|\Lambda^\gamma(fg) - g \Lambda^\gamma f\|_{L^2} \lesssim \|f\|_{L^2} \|g\|_{C^{\gamma+\varepsilon}}.$$

We apply the Bony decomposition to both terms, to get

$$\begin{aligned} \Lambda^\gamma(fg) &= \Lambda^\gamma(T_f g) + \Lambda^\gamma(T_g f) + \Lambda^\gamma(R(f, g)) = \text{I}_1 + \text{I}_2 + \text{I}_3, \\ g \Lambda^\gamma f &= T_{(\Lambda^\gamma f)} g + T_g(\Lambda^\gamma f) + R(\Lambda^\gamma f, g) = \text{II}_1 + \text{II}_2 + \text{II}_3. \end{aligned}$$

The terms I_1, Π_1, I_3, Π_3 can be estimated with standard paraproduct calculus, sketched as follows:

$$\begin{aligned}
\|I_1\|_{L^2}^2 &= \sum_q \|\Delta_q \Lambda^\gamma (T_f g)\|_{L^2}^2 \lesssim \sum_q 2^{2q\gamma} \|\Delta_q (T_f g)\|_{L^2}^2 \\
&\lesssim \sum_q 2^{2q\gamma} \|S_{q-1} f \cdot \Delta_q g\|_{L^2}^2 \leq \|f\|_{L^2}^2 \|g\|_{B_{\infty,2}^\gamma}^2, \\
\|\Pi_1\|_{L^2}^2 &= \sum_q \|\Delta_q T(\Lambda^\gamma f) g\|_{L^2}^2 \lesssim \sum_q \|S_{q-1} \Lambda^\gamma f \cdot \Delta_q g\|_{L^2}^2 \\
&\lesssim \sum_q \|S_{q-1} \Lambda^\gamma f\|_{L^2}^2 \|\Delta_q g\|_{L^\infty}^2 \\
&\lesssim \sum_q 2^{2q\gamma} \|S_{q-1} f\|_{L^2}^2 \|\Delta_q g\|_{L^\infty}^2 \leq \|f\|_{L^2}^2 \|g\|_{B_{\infty,2}^\gamma}^2, \\
\|I_3\|_{L^2}^2 &\leq \sum_q \|\Lambda^\gamma (\tilde{\Delta}_q f \cdot \Delta_q g)\|_{L^2}^2 \\
&\lesssim \sum_q 2^{2q\gamma} \|\tilde{\Delta}_q f \cdot \Delta_q g\|_{L^2}^2 \leq \sum_q 2^{2q\gamma} \|\tilde{\Delta}_q f\|_{L^2}^2 \|\Delta_q g\|_{L^\infty}^2 \\
&\leq \|f\|_{L^2}^2 \|g\|_{B_{\infty,2}^\gamma}^2, \\
\|\Pi_3\|_{L^2}^2 &\leq \sum_q \|\tilde{\Delta}_q (\Lambda^\gamma f) \cdot \Delta_q g\|_{L^2}^2 \leq \sum_q \|\tilde{\Delta}_q (\Lambda^\gamma f)\|_{L^2}^2 \|\Delta_q g\|_{L^\infty}^2 \\
&\lesssim \sum_q 2^{2q\gamma} \|\tilde{\Delta}_q f\|_{L^2}^2 \|\Delta_q g\|_{L^\infty}^2 \\
&\leq \|f\|_{L^2}^2 \|g\|_{B_{\infty,2}^\gamma}^2,
\end{aligned}$$

as $C^{\gamma+\varepsilon}$ is embedded in $B_{\infty,2}^\gamma$. These terms are nicely controlled.

The commutator structure is mainly used to estimate $I_2 - \Pi_2$. Let us denote the difference as III . Given any $q \in \mathbb{N}$,

$$\Delta_q \text{III} = \sum_p \Delta_q (\Lambda^\gamma (S_{p-1} g \cdot \Delta_p f) - S_{p-1} g \cdot \Lambda^\gamma (\Delta_p f)) =: \sum_p \text{III}_p.$$

Note that $\text{III}_p \equiv 0$ for $|p - q| \geq 5$. Therefore, it is a finite sum. We discuss III_q and the other terms can be treated similarly.

Following from [30], we estimate III_q in the Fourier side as

$$\begin{aligned}
\text{III}_q(x) &= \iint (|\xi + \zeta|^\gamma \\
&\quad - |\xi|^\gamma) \eta(2^{-q}(\xi + \zeta)) \chi(2^{-(q-2)}\zeta) \eta(2^{-q}\xi) \hat{f}(\xi) \hat{g}(\zeta) e^{i(\xi+\zeta)x} d\xi d\zeta.
\end{aligned}$$

Define a multiplier $m(\xi, \zeta)$ as

$$m(\xi, \zeta) = \frac{|\xi + \zeta|^\gamma - |\xi|^\gamma}{|\zeta|^\gamma} \eta(\xi + \zeta) \chi(4\zeta) \eta(\xi).$$

It is easy to check that m is uniformly bounded, compactly supported and C^∞ . Let $m_q(\xi, \zeta) = m(2^{-q}\xi, 2^{-q}\zeta)$, then

$$\begin{aligned} \text{III}_q(x) &= \iint m_q(\xi, \zeta) \hat{f}(\xi) |\zeta|^\gamma \hat{g}(\zeta) e^{i(\xi+\zeta)x} d\xi d\zeta \\ &= \iint h_q(y, z) \cdot \Delta_q f(x-y) \cdot \Lambda^\gamma S_{q-1} g(x-z) dy dz, \end{aligned}$$

where

$$h_q(y, z) = C \iint m_q(\xi, \zeta) e^{i(\xi y + \zeta z)} d\xi d\zeta.$$

Compute

$$\iint |h_q(y, z)| dy dz = 2^{2q} \iint |h_1(2^q y, 2^q z)| dy dz = \iint |h_1(y, z)| dy dz \leq C,$$

where the last integral is bounded due to smoothness of m , and the constant C does not depend on q . Then, applying Young's inequality, we get

$$\|\text{III}_q\|_{L^2} \lesssim \|h_q(\cdot, \cdot)\|_{L^1} \|\Delta_q f\|_{L^2} \|\Lambda^\gamma S_{q-1} g\|_{L^\infty} \lesssim \|\Delta_q f\|_{L^2} \sum_{p < q-1} 2^{p\gamma} \|\Delta_p g\|_{L^\infty}.$$

We collect all modes and conclude

$$\begin{aligned} \|\text{III}\|_{L^2}^2 &= \sum_q \|\Delta_q \text{III}\|_{L^2}^2 \lesssim \sum_q \|\Delta_q f\|_{L^2}^2 \left(\sum_{p < q-1} 2^{p\gamma} \|\Delta_p g\|_{L^\infty} \right)^2 \\ &\lesssim \sum_q \|\Delta_q f\|_{L^2}^2 \sum_{p < q-1} 2^{2p(\gamma + \frac{\alpha}{2})} \|\Delta_p g\|_{L^\infty}^2 \\ &= \sum_p 2^{2p(\gamma + \frac{\alpha}{2})} \|\Delta_p g\|_{L^\infty}^2 \sum_{q > p+1} \|\Delta_q f\|_{L^2}^2 \leq \|f\|_{L^2}^2 \|g\|_{B_{\infty,2}^{\gamma + \frac{\alpha}{2}}}^2 \\ &\lesssim \|f\|_{L^2}^2 \|g\|_{C^{\gamma + \varepsilon}}^2. \end{aligned}$$

□

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