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Approximation of diffusion processes on solvable Lie groups by random walks. Local and quasi-local limit theorems

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Abstract. This note states several results on the exponential functionals of the Brownian motion and their approximations by Markov chains. Starting from M. Yor, such functionals were studied in mathematical finance. At the same time, they play a significant role in different settings: the analysis of diffusions on the class of solvable Lie groups, in particular on the group of (2×2) upper triangular matrices, with positive diagonal elements. The discrete random walks cannot properly describe the local structure of diffusion. However, instead of the usual local limit theorem (which is not applicable) its weaker form, namely quasi-local form is given.

Keywords: Exponential functionals, Brownian motion, Asian option, solvable Lie groups, random walks, quasi-local theorems.

1. Introduction

In the well known paper [1], see also selected works by Marc Yor on the exponential functionals [2], the author studied the moments and the distribution density for particular functional of the Brownian motion $B(s)$, $s \geq 0$:

$$A_t^\nu = \int_0^t \exp(2B(s) + \nu s) ds,$$

which corresponds up to normalization in t^{-1} to the process associated with the Asian option in the Black and Scholes model. The general case $\nu \neq 0$ can be reduced to $\nu = 0$ using the Girsanov transformation and the central object is now $A_t = \int_0^t \exp(2B(s))ds$.

This and more general exponential functionals (which have a transparent financial meaning) also appear in a completely different setting: the Brownian motion on the solvable matrix groups. In this note we present several results in the area but to simplify the notations we consider one particular case, namely the group G_2 of 2×2 upper triangular matrices with positive diagonal elements.

We will use the parametrization

$$g = \begin{bmatrix} \exp(x) & z \\ 0 & \exp(y) \end{bmatrix}, \ x, y, z \in \mathbb{R}, \ g \in G_2.$$

2. Main section

The simplest random walk on G_2 with short steps can be constructed as the product of random matrices containing a parameter $\varepsilon = \varepsilon_n = \frac{1}{\sqrt{n}}$:

$$A_{\varepsilon,k} = \begin{bmatrix} \exp(\varepsilon X_k) & \varepsilon Z_k \\ 0 & \exp(\varepsilon Y_k) \end{bmatrix},$$

where the random variables (X_k, Y_k, Z_k) are independent for fixed k and the triples for different k are also independent. In addition

$$\mathsf{E}X_k = \mathsf{E}Y_k = \mathsf{E}Z_k = 0, \ VarX_k = VarY_k = VarZ_k = 1.$$

Let

$$g_\varepsilon(r) = \prod_{k=1}^{r} A_{\varepsilon,k} = \begin{bmatrix} \exp(x(r)) & z(r) \\ 0 & \exp(y(r)) \end{bmatrix}, \ g_\varepsilon(r+1) = g_\varepsilon(r) A_{\varepsilon,r+1}. \quad (1)$$

From the last relation

$$\begin{aligned} x(r+1) &= x(r) + \varepsilon X_{r+1} \Rightarrow \exp(x(r)) = \exp(\varepsilon(X_1 + \cdots + X_r)), \\ y(r+1) &= y(r) + \varepsilon Y_{r+1} \Rightarrow \exp(y(r)) = \exp(\varepsilon(Y_1 + \cdots + Y_r)), \\ z(r+1) &= \varepsilon Z_{r+1} \exp(\varepsilon(X_1 + \cdots + X_r)) + z(r) \exp(\varepsilon Y_{r+1}). \end{aligned}$$

After iteration in the last formula

$$\begin{aligned} z(r) = \varepsilon[Z_r \exp(\varepsilon(X_1 + \cdots + X_{r-1})) + Z_{r-1} \exp(\varepsilon(X_1 + \cdots + X_{r-2} + Y_r)) \\ + Z_{r-2} \exp(\varepsilon(X_1 + \cdots + X_{r-3} + Y_{r-1} + Y_r)) + \cdots \\ + Z_1 \exp(\varepsilon(Y_2 + \cdots + Y_r))]. \end{aligned}$$

The successive sums of the random variables X_k, Y_k, Z_k are attracted (due to the functional CLT by Donsker-Prokhorov) to the Brownian motion i.e for $n \to \infty$,

$$\{\varepsilon(X_1 + \cdots + X_r) = \frac{(X_1 + \cdots + X_r)}{\sqrt{n}}, \ r = [tn], t \le 1\} \stackrel{\text{(law)}}{\Longrightarrow} b_1(t), \ t \le 1.$$

Similarly,

$$\{\varepsilon(Y_1+\cdots+Y_r),\ r=[tn] \overset{\text{(law)}}{\Longrightarrow} b_2(t),\ t\le 1\},$$
$$\{\varepsilon(Z_1+\cdots+Z_r),\ r=[tn] \overset{\text{(law)}}{\Longrightarrow} b_3(t),\ t\le 1\},$$

and the processes $b_i(t)$, $i\in\{1,2,3\}$ are three independent standard one dimensional Brownian motions.

Theorem 1 *(Brownian motion on G_2) For the process introduced in* (1), *the following convergence holds:*

$$g_{\frac{1}{\sqrt{n}}}([tn]) \underset{n\to\infty}{\overset{\text{(law)}}{\Rightarrow}} g(t),\ t\le 1,$$

and

$$g(t)=\begin{bmatrix}\exp(b_1(t)) & \int_0^t \exp(b_1(s)+b_2(t)-b_2(s))db_3(s)\\ 0 & \exp(b_2(t))\end{bmatrix}\quad (2).$$

The generator of the process:

$$\Theta_t=\big(\exp(b_1(t)),\exp(b_2(t)),\exp(b_2(t))\int_0^t \exp(b_1(s)-b_2(s))db_3(s)\big),$$

writes:

$$\begin{aligned}(Lf)(x,y,z) &= \frac{1}{2}\left[x^2\frac{\partial^2 f}{\partial x^2}+y^2\frac{\partial^2 f}{\partial y^2}+(z^2+x^2)\frac{\partial^2 f}{\partial z^2}+2yz\frac{\partial^2 f}{\partial y\partial z}\right.\\ &\left. +x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}\right](x,y,z).\end{aligned}$$

Usually in the study of the approximations of the diffusions by the Markov chains we want to prove not only the integral theorems (on the convergence of the distribution functions) but also some kind of the local theorems. In our case everything is nice for the diagonal elements of the matrices. For instance, if X_i are symmetric Bernoulli random variable then for $k=O(\sqrt{n})$,

$$P\{X_1+\cdots+X_n=k\}=P\{\frac{1}{\sqrt{n}}(X_1+\cdots+X_n)=\frac{1}{\sqrt{n}}k\}\underset{n\to\infty}{\sim}\frac{e^{-\frac{k^2}{2n}}}{\sqrt{2\pi n}}.$$

The same is true for arbitrary integer valued random variables X_i such that $\mathsf{E}X_i=0$, $VarX_i=1$.

What happens in the Bernoulli case for the non diagonal entry? Consider the random variable

$$\frac{z(n)}{\varepsilon} = Z_n e^{\varepsilon(X_1+\cdots X_{n-1})} + Z_{n-1} e^{\varepsilon(X_1+\cdots X_{n-2}+Y_n)} + \cdots + Z_1 e^{\varepsilon(Y_2+\cdots Y_n)}.$$

For fixed X_i, Y_j it has a form

$$S = \sum_{i=1}^{n} a_i Z_i,$$

where a_i are typically (say, for rational ε) jointly transcendental. It gives either $P\{S = 0\} = 0$ or (like in our case of particular a_i, $i = 1, \cdots, n$) $P\{S_n = 0\} = o(e^{-\sqrt{n}})$ (see e.g. [6]), i.e the local CLT is not working. But there is a bit weaker quasi-local theorem. Let us formulate the following result.

Lemma 1 *Let $X_{1n}, \cdots, X_{nn}$ be independent random variables (in each group) and $\mathsf{E}X_{in} = 0$, $Var X_{in} = \sigma_{in}^2$, $B_n^2 = \sum_{i=1}^n \sigma_{in}^2$, $\mu_{in} = \mathsf{E}|X_{in}|^3$ and $M_n = \sum_{i=1}^n \mu_{in}$. Assume that $B_n \to \infty$ and Lyapunov function $L(n) := \frac{M_n}{B_n^3} = O(\frac{1}{B_n})$. Then for any sequence $d_n \to \infty$ and $\frac{d_n}{B_n} \to 0$ for a fixed integer b,*

$$P\left\{\frac{X_{1n} + \cdots + X_{nn}}{B_n} \in \left(b, \frac{d_n}{B_n} + b\right)\right\} \sim \frac{d_n}{B_n}\phi(b).$$

Here $\phi(b) = \frac{e^{-b^2/2}}{\sqrt{2\pi}}$.

Using this lemma one can prove the quasi-local theorem for the element $z(n)$. If the random variables X_k, Y_k, Z_k, $k \le n$ have the densities, then the following results hold.

Theorem 2 *If X_i, Y_j, Z_k have characteristic functions in $L^1(R)$, so that in particular they have densities, then the joint distribution density for the elements of $g_{\frac{1}{\sqrt{n}}}([tn])$ exists and converges uniformly to the joint distribution density for elements of the $g(t)$.*

The recent progress in the study of limit theorems for non-lattice and non absolutely continuous distributions (see e.g. [3], [4]), opens the possibility to prove stronger quasi-local theorems for singular distributions of Z_i with Diophantine property.

Finally we want to formulate two results on the *generalized* Yor's type exponential functional on G_2:

$$A(t) = \frac{1}{t}\int_0^t \exp[2(b_1(s) + b_2(t) - b_2(s))]ds.$$

Theorem 3 *For $t \to \infty$*

$$\frac{\ln A(t)}{2\sqrt{t}} \overset{\text{(law)}}{\Longrightarrow} M_1 := \max_{s \le 1}[b_1(s) + b_2(1) - b_2(s)].$$

For the distribution of M_1 there exists an explicit formula. This formula shows that the generalized Asian option is very large for $t \to \infty$ with high probability.

However, the following result holds.

Theorem 4 *Consider*

$$\alpha(t) = \mathsf{E}[A(t) |\, b_1(s) \le 1,\, b_2(s) \le 1,\, s \le t]$$

(i.e prices of both stocks did not grow). Then $\alpha(t) \to 0$, $t \to \infty$, a.s.

3. Conclusions

The proofs of all results presented above as well as the theorems announced by the authors in [5] will be given in the forthcoming paper by the authors. In particular, it will include the quasi-local theorems for the random walks on the dense subgroups of R^1 and such groups as H_3 (Heisenberg group) and $Aff(R^1)$. We also expect to prove new limit theorems for the general exponential functionals given by the multiple Itô integrals.

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