Multipulse Subspace Detectors

Louis L. Scharf, and Pooria Pakrooh
Department of Mathematics, Colorado state University, Fort Collins, CO, USA

***Abstract*—In this paper we frame a fairly comprehensive set of spacetime detection problems, where a subspace signal modulates the mean-value vector of a multivariate normal measurement and nonstationary additive noise determines the covariance matrix. The measured spacetime data matrix consists of multiple measurements in time. As time advances, the signal component moves around in a subspace and the noise covariance matrix changes in scale.**

I. Introduction

Consider an L-element array of sensors, each of which records an M-sample time series. We denote the mth measurement, $\mathbf{x}[m] = \alpha\mathbf{H}\mathbf{s}[m] + \mathbf{n}[m]$, with $\mathbf{x} \in \mathcal{C}^L$, $\alpha \in \mathcal{C}$, $\mathbf{H} \in \mathcal{C}^{L\times p}$, $\mathbf{s} \in \mathcal{C}^p$, $\mathbf{n} \in \mathcal{C}^L$. When M copies of such a measurement are organized into an $L \times M$ matrix, the result is $\mathbf{X} = \alpha\mathbf{H}\mathbf{S}^H + \mathbf{N}$, where $\mathbf{X} = [\mathbf{x}[1], \mathbf{x}[2], \ldots, \mathbf{x}[M]]$, $\mathbf{S}^H = [\mathbf{s}[1], \mathbf{s}[2], \ldots, \mathbf{s}[M]]$, $\mathbf{N} = [\mathbf{n}[1], \mathbf{n}[2], \ldots, \mathbf{n}[M]]$. At time index m, the ith column of $\mathbf{H}$ determines the channel response $\mathbf{h}_i s_i[m]$ to the signal transmitted from source i.

We shall test $H_0 : \mathbf{x}[m] = \mathbf{n}[m],\ m = 1, 2, \ldots, M$ versus $H_1 : \mathbf{x}[m] = \alpha\mathbf{H}\mathbf{s}[m] + \mathbf{n}[m],\ m = 1, 2, \ldots, M$, for various assumptions on $\mathbf{\Sigma}_m$ in the noise model $\mathbf{n}[m] : \mathcal{CN}_L[\mathbf{0}, \mathbf{\Sigma}_m]$, and for various assumptions about what is known in the signal model $\alpha\mathbf{H}\mathbf{s}[m]$. The distribution of $\mathbf{X}$ is the distribution of independent $\mathcal{CN}_L[\alpha\mathbf{H}\mathbf{s}[m], \mathbf{\Sigma}_m]$ random vectors, assumed to be proper. We term this a problem of *multipulse subspace detection*, as the signal component of the measurement is trapped in a subspace $\langle\mathbf{H}\rangle$, and there is a sequence of temporal measurements, as in SAR, STAP, and pulsed Doppler radar. The term *multi-snapshot subspace detector* would have been just as descriptive for passive problems, where the sequence of measurements corresponds to snapshots in time of a spacetime field. Throughout our developments we shall maximize likelihood with respect to unknown parameters, to obtain generalized likelihood ratios (GLRs). In the statistics literature these are termed ordinary likelihoods.

II. Temporally stationary noise of known covariance

The measurement model is $\mathcal{CN}_L[\alpha\mathbf{H}\mathbf{s}[m], \mathbf{\Sigma}[m]]$, with $\mathbf{\Sigma}[m] = \mathbf{\Sigma}$ for all $m = 1, 2, \ldots, M$. The covariance $\mathbf{\Sigma}$ is known. Without loss of generality we may assume $\mathbf{\Sigma} = \mathbf{I}_{L\times L}$.

There are four cases to be considered for the subspace signal model $\alpha\mathbf{H}\mathbf{s}[m]$.

Multipulse matched filter. In this case, $\mathbf{H}$ is known, $\mathbf{s}[m]$ is known, and α is unknown. Write $\mathbf{H}\mathbf{S}^H = \mathbf{G}$. It is a simple matter to show that the likelihood ratio is the *multipulse incoherent matched filter*

$$\log\Lambda = \frac{|\ \mathrm{tr}[\mathbf{G}^H\mathbf{X}]\ |^2}{\mathrm{tr}[\mathbf{G}^H\mathbf{G}]}. \tag{1}$$

This statistic is invariant to translation of measurements $\mathbf{X}$ in $\langle\mathbf{H}\rangle^\perp$. It's null distribution is χ^2_2. This generalizes a related result in [1].

Multipulse matched subspace detector. In this case, $\mathbf{H}$ is known, but $\alpha\mathbf{s}[m]$ is unknown. When likelihood is maximized with respect to the unknown $\alpha\mathbf{s}[m]$, the resulting GLR is

$$\log\Lambda = \mathrm{tr}[\mathbf{X}^H\mathbf{P}_{\mathbf{H}}\mathbf{X}] \tag{2}$$

This statistic is invariant to rotation of measurements $\mathbf{X}$ in $\langle\mathbf{H}\rangle$, $\langle\mathbf{H}\rangle^\perp$, and to translation in $\langle\mathbf{H}\rangle^\perp$. This is a cylinder. The null distribution is χ^2_{2Mp}. This generalizes a related result in [1].

Dual of Matched subspace detector. In this case, $\alpha\mathbf{H}$ is unknown, but $\mathbf{s}[m]$ is known. When likelihood is maximized with respect to the subspace basis $\alpha\mathbf{H}$, the resulting GLR is

$$\log\Lambda = \mathrm{tr}[\mathbf{X}\mathbf{P}_{\mathbf{S}}\mathbf{X}^H] \tag{3}$$

This statistic is invariant to rotation in $\langle\mathbf{S}\rangle$, $\langle\mathbf{S}\rangle^\perp$, and translation in $\langle\mathbf{S}\rangle^\perp$. Its null distribution is χ^2_{2Lp}.

Matched direction detector. In this case $\alpha\mathbf{H}$, and $\mathbf{s}[m]$ are unknown. When likelihood is maximized with respect to these unknowns, the resulting GLR is

$$\log\Lambda = \sum_{r=1}^{p} ev_r[\mathbf{X}\mathbf{X}^H] \tag{4}$$

where ev_r is the rth largest eigenvalue. This statistic could as well have been written $\mathrm{tr}[\mathbf{U}_p^H\mathbf{X}\mathbf{X}^H\mathbf{U}_p]$, where $\mathbf{U}_p$ contains the p dominant eignevectors for $\mathbf{X}\mathbf{X}^H$. So the measurements are matched to the p dominant eigenvectors, or orthogonal directions, of the sample covariance matrix. This statistic is invariant to unitary transformation of the measurement matrix $\mathbf{X} \to \mathbf{U}\mathbf{X}\mathbf{V}$, for $\mathbf{U} \in \mathcal{C}^{L\times L}$, $\mathbf{V} \in \mathcal{C}^{M\times M}$ unitary. Under the null and for $p = 1$, $\log\Lambda$ is the largest eigenvalue of a central complex Wishart distribution, and its pdf is given in [2].

This work is supported in part by NSF under grant CCF-1712788, and by the Air Force Office of Scientific Research under award number FA9550-14-1-0185. Consequently the U.S. Government is authorized to reproduce and distribute reprints for Governmental purposes notwithstanding any copyright notation thereon.

III. Temporally stationary noise of known covariance structure, but unknown and stationary scale

The measurement model is $\mathcal{CN}_L[\alpha\mathbf{Hs}[m], \mathbf{\Sigma}[m]]$, with $\mathbf{\Sigma}[m] = \sigma^2\mathbf{\Sigma}$ for all $m = 1, 2, \ldots, M$. The nominal covariance $\mathbf{\Sigma}$ is known, but its stationary scale σ^2 is unknown. Without loss of generality we may assume $\mathbf{\Sigma} = \mathbf{I}_{L\times L}$. There are four cases to be considered for the subspace signal model $\alpha\mathbf{Hs}[m]$.

CFAR incoherent matched filter. In this case, $\mathbf{H}$ is known, and $\mathbf{s}[m]$ is known. Write $\mathbf{HS} = \mathbf{G}$. The generalized likelihood ratio, obtained by maximizing likelihood with respect to σ^2 and α, under each of $\mathcal{H}_0$ and $\mathcal{H}_1$ is

$$1 - \frac{1}{\Lambda^{1/LM}} = \frac{|\operatorname{tr}[\mathbf{G}^H\mathbf{X}]|^2}{\operatorname{tr}[\mathbf{G}^H\mathbf{G}]\operatorname{tr}[\mathbf{X}\mathbf{X}^H]}. \tag{5}$$

This statistic is invariant to translation of measurements $\mathbf{X}$ in $\langle\mathbf{H}\rangle^\perp$. This generalizes a related result in [1].

Multipulse CFAR matched subspace detector. In this case, $\mathbf{H}$ is known, and $\alpha\mathbf{s}[m]$ is unknown. Likelihood is maximized with respect to the unknown scale σ^2 and the sequence of unknown signal vectors $\alpha\mathbf{s}[m], m = 1, \ldots, M$. The result is

$$1 - \frac{1}{\Lambda^{1/LM}} = \frac{\operatorname{tr}[\mathbf{X}^H\mathbf{P_H}\mathbf{X}]}{\operatorname{tr}[\mathbf{X}^H\mathbf{X}]} \tag{6}$$

This is a multipulse CFAR matched subspace detector, generalizing the result of [1]. It is invariant to rotation in $\langle\mathbf{H}\rangle$, $\langle\mathbf{H}\rangle^\perp$, scaling of $\mathbf{X}$, and unitary transformation $\mathbf{X} \to \mathbf{XU}$, for $\mathbf{U} \in \mathcal{C}^{\mathbf{M}\times\mathbf{M}}$ unitary. Under the null it is distributed as $\beta(Mp, M(L-p))$. This generalizes a related result in [1].

Dual of multipulse CFAR matched subspace detector. In this case, $\alpha\mathbf{H}$ is unknown, but $\mathbf{s}[m]$ is known. The GLR is

$$1 - \frac{1}{\Lambda^{1/LM}} = \frac{\operatorname{tr}[\mathbf{X}\mathbf{P_S}\mathbf{X}^H]}{\operatorname{tr}[\mathbf{X}\mathbf{X}^H]} \tag{7}$$

This statistic is invariant to rotation in $\langle\mathbf{S}\rangle$, $\langle\mathbf{S}\rangle^\perp$, and to scaling of $\mathbf{X}$. Its null distribution is $\beta(Lp, L(M-p))$.

Multipulse CFAR matched direction detector. In this case, $\alpha\mathbf{H}$ is unknown, and $\mathbf{s}[m]$ is unknown. The GLR is

$$1 - \frac{1}{\Lambda^{1/LM}} = \frac{\sum_{r=1}^{p} ev_r(\mathbf{X}\mathbf{X}^H)}{\operatorname{tr}[\mathbf{X}\mathbf{X}^H]} \tag{8}$$

This statistic is invariant to unitary transformation $\mathbf{X} \to \mathbf{UXV}$, for $\mathbf{U} \in \mathcal{C}^{\mathbf{L}\times\mathbf{L}}$, $\mathbf{V} \in \mathcal{C}^{\mathbf{M}\times\mathbf{M}}$ unitary, and to scaling of $\mathbf{X}$, making it CFAR. This statistic generalizes the matched direction detector of [3]. For $p = 1$ and $L = 2$, the null and alternate distributions of Λ are studied in [3].

IV. Noise of known covariance, with unknown and time-varying scale

The measurement model is $\mathcal{CN}_L[\alpha\mathbf{Hs}[m], \mathbf{\Sigma}[m]]$, with $\mathbf{\Sigma}[m] = \sigma^2[m]\mathbf{\Sigma}$ for $m = 1, 2, \ldots, M$. The nominal covariance $\mathbf{\Sigma}$ is known, but its time-varying scale $\sigma^2[m]$ is unknown. Without loss of generality we may assume $\mathbf{\Sigma} = \mathbf{I}_{L\times L}$. In this model there is just one assumption about the signal model $\alpha\mathbf{Hs}[m]$ that leads to a tractable GLR, namely $\mathbf{H}$ is known, but $\alpha\mathbf{s}[m]$ is unknown.

Multipulse Coherence Estimator (MPCE). In this case, $\mathbf{H}$ is known, but $\alpha\mathbf{s}[m]$ is unknown. Likelihood may be maximized over unknown $\sigma^2[m]$ and $\alpha\mathbf{s}[m]$ to produce the GLR

$$1 - \frac{1}{\Lambda^{1/L}} = 1 - \prod_{m=1}^{M}(1 - \rho_m^2); \rho_m^2 = \frac{\mathbf{x}^H[m]\mathbf{P_H}\mathbf{x}[m]}{\mathbf{x}^H[m]\mathbf{x}[m]} \tag{9}$$

This multipulse coherence statistic in invariant to rotation in $\langle\mathbf{H}\rangle$, $\langle\mathbf{H}\rangle^\perp$, and time-varying scalings of the $\mathbf{x}[m]$'s. Its null distribution is the distribution of the random variable $1 - \prod u_m$, where the random variables u_m are independent random variables distributed as $u_m \sim \beta(L-p, p)$. This MPCE has been derived by Abramovich and Besson [4] by assigning a Gamma prior to an i.i.d. sequence of $\sigma^2[m]$. They then marginalized for a Bayesian likelihood, and approximated the result to obtain (9). In [5] this result is generalized to obtain a multipulse adaptive coherence estimator (MPACE) that uses secondary data to estimate an unknown covariance matrix.

V. Stationary noise of unknown covariance

The measurement model is $\mathcal{CN}_L[\alpha\mathbf{Hs}[m], \mathbf{\Sigma}[m]]$, with $\mathbf{\Sigma}[m] = \mathbf{\Sigma}$ for all $m = 1, 2, \ldots, M$. The covariance $\mathbf{\Sigma}$ is unknown, but stationary in time. In this model there is just one assumption about the signal model $\alpha\mathbf{Hs}[m]$ that leads to a tractable GLR, namely $\alpha\mathbf{H}$ is unknown, but $\mathbf{s}[m]$ known. This is the generalization of the RX problem [6] considered by Bliss and Parker [7].

Multi-rank RX. In this case, $\alpha\mathbf{H}$ is unknown, and $\mathbf{s}[m]$ is known. When likelihood is maximized over the unknown basis $\alpha\mathbf{H}$, the following GLR results:

$$\frac{1}{\Lambda^{1/M}} = \frac{\det(\mathbf{X}(\mathbf{I}_M - \mathbf{P_S})\mathbf{X}^H)}{\det(\mathbf{X}\mathbf{X}^H)}. \tag{10}$$

This statistic is invariant to transformation $\mathbf{X} \to \mathbf{TX}$, for $\mathbf{T} \in \mathcal{C}^{L\times L}$ nonsingular. In a related paper [8], it is shown that this statistics may be written as a variation on the Wilks Λ. Its null distribution is shown to be distributed as the random variable $\prod_{\ell=1}^{L} u_\ell$, where the independent u_ℓ's are $u_\ell \sim \beta(M-p-\ell-1, p)$. This result specializes to the null distribution found by Reed and Yu for the case $p = 1$. This exact null distribution for finite values of L and M may be compared with the large random matrix approximation of Hiltunen and Loubaton [9] for asymptotically large L and M.

VI. Conclusions

The detectors of this paper may be considered to be first-order detectors, where the signal component to be detected is carried in the mean value vectors of a sequence of temporal measurements in a multi-sensor array. The second-order versions of these problems would assign a multivariate normal prior distribution to unknowns $\alpha \mathbf{s}[m]$, leading to variations on standard problems in factor analysis [10], [11].

Acknowledgments

The authors thank Sonja Hiltunen for describing her large matrix theory results to LLS at the 2017 IEEE ICASSP in New Orleans. They thank Douglas Cochran for his interest and helpful suggestions.

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