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Multi-dimensional Lévy Processes for Multiple Dependent Degradation Processes in Lifetime Analysis

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Abstract

The analysis of multiple dependent degradation processes is a challenging research work in the reliability field, especially for complex degradation with random jumps. To integrally handle the jump uncertainties in degradation and the dependence among degradation processes, we construct multi-dimensional Lévy processes to describe multiple dependent degradation processes in engineering systems. The evolution of each degradation process can be modeled by a one-dimensional Lévy subordinator with a marginal Lévy measure, and the dependence among all dimensions can be described by Lévy copulas and the associated multiple-dimensional Lévy measure. This Lévy measure is obtained from all its one-dimensional marginal Lévy measures and the Lévy copula. We develop the Fokker-Planck equations to describe the probability density in stochastic systems. The Laplace transforms of both reliability function and lifetime moments are derived. Numerical examples are used to demonstrate our models in lifetime analysis.

Keywords

Multi-dimensional Lévy processes, Lévy copulas, degradation modeling, reliability function

1. Introduction

In the financial and actuarial science fields, many applications involve stochastic processes with random jumps in both one-dimensional and multi-dimensional cases. Similarly, these phenomena happen in the engineering fields, especially the degradation-based reliability engineering field. It is common to observe degradations with random jumps in one-dimensional or multiple-dimensional cases. Typically, the multi-dimensional degradation processes are dependent with each other due to the internal and external conditions of the systems.

Lévy processes have been widely applied to describe financial and insurance time series with jumps. More specifically, non-decreasing financial time series can be modeled as Levy subordinators. This process can be well used in describing degradation processes of engineering systems. The first study of Lévy processes in degradation analysis was introduced by Cinlar [13]. Shu et al. derived the reliability function of degradation processes described by Lévy subordinators and the results of life time distributions of degradation processes, respectively, based on Fokker Planck Equation (FPE) [2]. However, their results consider only one-dimensional degradation processes. For modeling multiple-dimensional Lévy degradation processes, the challenge is how to consider the internal dependence structures.

Lévy copula was introduced here since it separates the margins and the dependence structures of multiple-dimensional Lévy measures and described by Sklar's theorem [1]. Sklar's theorem explains that the tail integral of any multiple-dimensional Lévy process can be expressed by a function of all its marginal tail integrals; and the rule of function is the Lévy copula. Moreover, pair copulas need to be introduced between each pair of two dimensions in three of higher dimensional Lévy processes [4]. Multiple dimensional Lévy processes with Lévy copulas have been widely applied in actuarial and insurance analysis. In reliability studies, multiple degradation processes have not been commonly represented by multivariate Lévy processes with Lévy copulas.

This paper is organized as follows. Some concepts and preliminaries of multiple dimensional Lévy processes and Lévy copulas are introduced in Section 2. In Section 3, we build high-dimensional copulas for multiple dependent degradation processes described by Lévy subordinators. Section 4 presents the high-dimensional Lévy measures, the reliability function and lifetime moment based on the high-dimensional copulas in Section 3. Some simulations are given in Section 5, and conclusions are given in Section 6.

2. Preliminaries

In this section, we introduce some fundamental concepts and related properties of Lévy processes and Lévy copulas, based on R_+^m space.

Definition 1 [6]. *A $m(m \geq 1)$-dimensional Lévy process $L_t^m (t > 0)$ is a stochastic process with $L_0^m = 0$ and satisfies the following properties:*

1. *Independent increments: for a time sequence $t_0,.....,t_n$, the increment random variables $L_{t_0}, L_{t_1} - L_{t_0},....., L_{t_n} - L_{t_{n-1}}$ are independent.*
2. *Stationary increment: the law of $L_{t+h} - L_t$ does not depend on t.*
3. *Stochastic continuity: $\forall \varepsilon > 0$, $\lim_{h \to 0} P(|L_{t+h} - L_t| \geq \varepsilon) = 0$.*

From the first and second properties, we can generate a random walk $S_n(\Delta) = \sum_{k=0}^{n-1} L_{(k+1)\Delta} - L_{k\Delta}$.

If $n\Delta = t$, then $L_t = S_n(\Delta)$ can be represented as a sum of n i.i.d. random variables. Therefore, the distribution of L_t can be divided into n i.i.d. parts with the same distribution. More specifically, Lévy subordinators are a type of Lévy processes having values on $[0, \infty)$ with a non-decreasing path.

3. Lévy Copulas and Lévy Measures

[7] indicated that the definition of regular copulas can be extended to Lévy copulas for multi-dimensional Lévy measures, with different domains and ranges.

Definition 2 [6]. *A function C: $\overline{R}^m \to \overline{R}$ is called Lévy copula if:*

1. *$C(u_1,....,u_m) \neq \infty$ for $(u_1,....,u_m) \neq (\infty,\infty,...,\infty)$,*
2. *$C(u_1,....,u_m) = 0$ if $u_i = 0$ for at least one $i \in \{1,....,m\}$ (grounded)*
3. *C is m-increasing,*
4. *$C^{(i)}(u) = u$ for any $i \in \{1,2,....,m\}, u \in R$ (uniform marginal)*

If we change the domain and range from $\overline{R}^m \to \overline{R}$ to $\overline{R}_+^m \to \overline{R}_+$, it becomes a *positive Lévy copula*.

[8] found out that the relationship between Lévy copulas and the associated Lévy measures can be considered as image mapping. First, define a bijection under a multi-dimensional space:

$$Q_m : [0,\infty]^m \to [0,\infty]^m, \qquad (x_1, x_2,....,x_m) \mapsto (x_1^{-1}, x_2^{-1},....,x_m^{-1}) . \tag{1}$$

Using the fact that a Lévy copula C is uniform marginal, we can find the measure:

$$\chi_C([0,\infty]^{k-1} \times [0,x_k] \times [0,\infty]^{m-k}) = C(\infty,....,x_k,....,\infty) = x_k, k = 1,2,...,m . \tag{2}$$

The Clayton Lévy copula family is widely considered in multi-dimensional Lévy subordinators. We assume the positive Lévy copula of a Lévy subordinator $L_t = \{L_t^1,....,L_t^m\}$ is a m-dimensional Clayton Lévy copula C defined on $\overline{R}_+^m \to \overline{R}_+$:

$$C(u_1,u_2,....,u_m)=(\sum_i^m u_i^{-\theta})^{\frac{-1}{\theta}}, \tag{3}$$

where $\theta \in (0,\infty)$ is the dependence parameter. The higher value of θ represents the stronger dependence.

From the Sklar's theorem, we know Lévy copulas can separate the margins and the dependence structures of multi-dimensional Lévy measures. It gives us the Sklar's theorem for the case of positive Clayton Lévy copulas such that:

$$F(x_1,....,x_m)=C(F_1(x_1),.....,F_m(x_m))=[F_1(x_1)^{-\theta}+.....+F_m(x_m)^{-\theta}]^{\frac{-1}{\theta}}. \tag{4}$$

For any Boral set $B=[x_1,\infty]*...*[x_m,\infty]$ in $[0,\infty]^m$,

$$v_C([x_1,\infty]\times...\times[x_m,\infty])=\chi_C\times Q_m^{-1}([x_1,\infty]...[x_m,\infty])=\chi_C([0,x_1^{-1}]....[0,x_m^{-1}])=C(x_1^{-1},.....,x_m^{-1})=F(x_1,....,x_m). \tag{5}$$

This is a *Lévy measure* for the *m*-dimensional Lévy subordinator $L_t=\{L_t^1,....,L_t^m\}$.

Moreover, for any *m*-dimensional Lévy measure v_C, its *m* margins, $v_1,....,v_m$ are all one-dimensional Lévy measures. We can get one dimensional Lévy measure that

$$v_i=\chi_i\times Q_1^{-1}, i=1,2,...,m. \tag{6}$$

4. Reliability Function and Lifetime Moments

For a system/device with multiple degradation processes modeled by a multi-dimensional Lévy subordinator $X(t)=[X_1(t),....,X_m(t)]$, it fails when any one of the degradation processes exceeds a failure threshold. The lifetime of the system can be defined as $T_x=\inf\{t: X_1(t)>x_1,....,X_m(t)>x_m\}$. The multi-dimensional Laplace transform of both reliability function and lifetime moments $R_X(x,t)$ and $M(T_X^n,x)$ ($x=(x_1,......,x_m)$) can be derived by Fokker-Planck Equation (FPE) and represented by Lévy measures [2].

Lemma 1 *For any multiple dimensional degradation processes with random jumps described by Lévy subordinators, the Fokker-Planck equation is*

$$\begin{aligned}\frac{\partial p(x,t)}{\partial t}&=-b_1\frac{\partial p(x,t)}{\partial x_1}-......-b_m\frac{\partial p(x,t)}{\partial x_m}+\int_{R_+^m}(p(x_1-y_1,....,x_m-y_m,t)-p(x,t))v(dy)\\&+\int_{R_+^m}(1_{|y|<1}y_1\frac{\partial p(x,t)}{\partial x_1}+....+1_{|y|<1}y_m\frac{\partial p(x,t)}{\partial x_m})v(dy),\end{aligned} \tag{7}$$

Where $x=(x_1,....,x_m)$, *and* $v(dy)=v(dy_1,...,dy_m)$ *is a m-dimensional Lévy measure.*

Shu et al. derived the relationship between the reliability function $R^{LL}(u,w)$ and the probability density function $p^{LL}(u,w)$ in terms of Laplace transform, and the relationship between the lifetime moments and the derivative of reliability function in one-dimensional case. These can be naturally extended to multi-dimensional situations.

Theorem 1. *For any multiple dependent degradation processes with random jumps described by a multi-dimensional Lévy subordinator, the multi-dimensional Laplace transform of reliability function* $R_X(x,t)$ *is*

$$R^{LL}(u,w)=u^{-1}\times\{w+\langle b^*,u\rangle-\int_{R_+^m}(e^{-\langle u,y\rangle}-1)v(dy)\}^{-1}, \tag{8}$$

where $b^* = b - \int_{|y|<1} y\nu(dy) = (b_1^*, \ldots\ldots, b_m^*)$ *is a m-dimensional constant vector,* $\nu(dy) = \nu(dy_1 dy_2 \ldots\ldots dy_m)$ *is a m-dimensional Lévy measure, and m is the number of dimensions.*

The Laplace transform of reliability function is a monotonous function with time transform *w* in a sense that it decreases monotonously as *w* increases when the threshold is fixed.

5. Simulation and Numerical Examples

To illustrate the multiple degradation processes, we implement the simulation and sampling of the subordinators' series representation using the pair copula construction algorithm [5, 6]. We begin with two-dimensional case $\{L_1(t), L_2(t)\}$ where $L_1(t)$ and $L_2(t)$ are dependent based on their two-dimensional Clayton Lévy copula $C(u_1, u_2, \theta) = (u_1^{-\theta} + u_2^{-\theta})^{-1/\theta}$ where θ is a parameter representing the dependence of absolute values of jumps in two marginal processes. We also assume both marginal processes are α-stable such that $F_1(x_1) = x_1^{-\alpha} = y_1$ and $F_2(x_2) = x_2^{-\alpha} = y_2$ with the notation that $\nu(dx) = \dfrac{\kappa}{\Gamma(1-\kappa)} \dfrac{1}{x^{\kappa+1}} dx$, $0 < \kappa < 1$ .The simulation results are shown in Figure 1 and 2. We choose $\theta = 2$ and $\theta = 5$ to represent the weak and strong dependence relationship, respectively. These figures show that a larger dependence parameter θ indicates stronger dependence between two degradation processes than a smaller dependence parameter does.

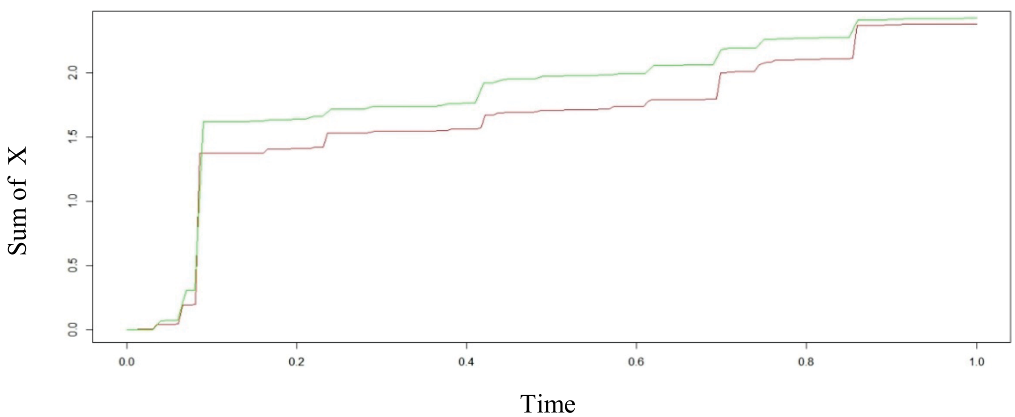

Figure 1: Two-dimensional Lévy subordinator ($\theta = 5$)

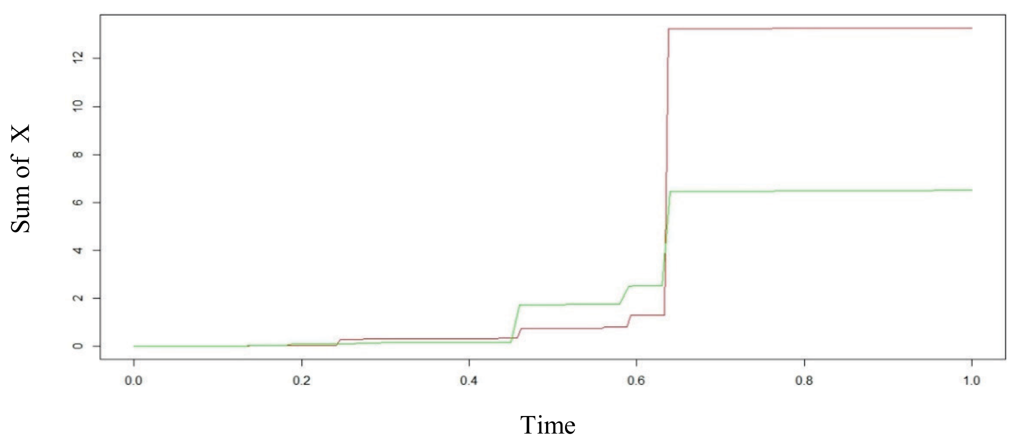

Figure 2: Two-dimensional Lévy subordinator ($\theta = 2$)

From (4) and (5), the two-dimensional Lévy measure can be described as the second derivative of the Lévy copula

$$v(x_1,x_2)=\frac{\partial C(y_1,y_2)}{\partial y_1\partial y_2}\Big|_{y_1=F_1(x_1),y_2=F_2(x_2)}v_1v_2=(1+\theta)\left[\int_{x_1}^{\infty}v_1dz_1\right]^{-1-\theta}\left[\int_{x2}^{\infty}v_2dz_2\right]^{-1-\theta}\left[\int_{x_1}^{\infty}v_1dz_1^{-\theta}+\int_{x2}^{\infty}v_2dz_2^{-\theta}\right]^{-\frac{1}{\theta}-2}v_1v_2\,. \quad (9)$$

From Theorem 1, we can derive the Laplace transform of reliability function:

$$R^{LL}(u,w)=(u_1u_2)^{-1}*\left\{w+\left(b_1^*u_1+b_2^*u_2\right)-\int_{R_+^2}\left(e^{-(u_1y_1+u_2y_2)}-1\right)(1+\theta)\frac{\kappa_1\kappa_2}{y_1y_2}\left(\Gamma(1-\kappa_1)y_1^{\kappa1}\right)^{1+\theta}\left(\Gamma(1-\kappa_2)y_2^{\kappa2}\right)^{1+\theta}\left[\left(\Gamma(1-\kappa_1)y_1^{\kappa1}\right)^{\theta}+\left(\Gamma(1-\kappa_2)y_2^{\kappa2}\right)^{\theta}\right]^{-2-\frac{1}{\theta}}dy_1dy_2\right\}^{-1}. \quad (10)$$

We choose a simple case where $\Gamma(\frac{1}{2})=\sqrt{\pi}$, which means $\kappa_1=\kappa_2=0.5$. When $\theta=2$, the integral part of the Laplace transform of reliability function can be derived as:

$$\begin{aligned}&\int_{R_+^2}\left(e^{-(u_1y_1+u_2y_2)}-1\right)(1+\theta)\frac{\kappa_1\kappa_2}{y_1y_2}\left(\Gamma(1-\kappa_1)y_1^{\kappa1}\right)^{1+2}\left(\Gamma(1-\kappa_2)y_2^{\kappa2}\right)^{1+2}\left[\left(\Gamma(1-\kappa_1)y_1^{\kappa1}\right)^{2}+\left(\Gamma(1-\kappa_2)y_2^{\kappa2}\right)^{2}\right]^{-2-\frac{1}{2}}dy_1dy_2\\&=\frac{3}{4\sqrt{\pi}}\int_{R_+^2}\left(e^{-(u_1y_1+u_2y_2)}-1\right)\left(y_1+y_2\right)^{\frac{-5}{2}}dy_1dy_2=-\frac{u_1^{3/2}-u_2^{3/2}}{u_1-u_2}\end{aligned} \quad (11)$$

When θ takes other values, the formula in (11) can be derived in the similar manner.

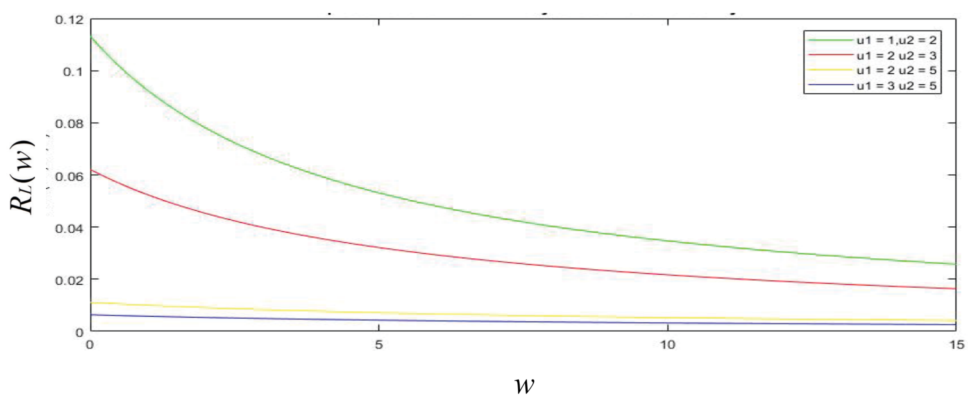

Figure 3: Laplace transform of reliability function for 2-D Lévy subordinator ($\theta=2$)

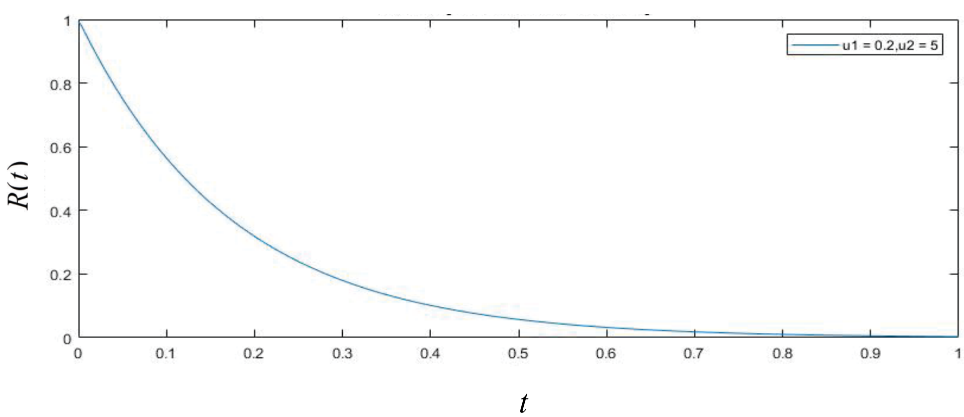

Figure 4: Reliability function for 2-D Levy subordinator ($\theta=2$)

Figure 3 illustrates the Laplace transform of the reliability function for a 2-D Lévy subordinator, which decreases monotonously as the time transform w increases, as we discussed in Theorem 1. Figure 4 shows the reliability function for a 2-D Lévy subordinator that decreases as time t increases when we fix the failure thresholds for the two degradation processes.

6. Conclusions

In this article, we presented a new model describing the multiple dependent degradation processes by extending the one-dimensional case in [2]. The multiple degradation processes can be represented as multi-dimensional Lévy subordinators, while each marginal dimension is a one-dimensional Lévy subordinator. We derived the reliability function in Laplace transform based on Fokker Planck equation. In this article, we represent the dependence among different dimensions using Clayton Lévy copula. In the case of three or higher dimensions, however, some dependence between each pair of dimensions needs to be considered. For future research, pair Lévy copulas representing the dependence relationship between each pair of dimensions can be used to construct multiple dependent degradation processes.

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