Uncertainty in Multi-Commodity Routing Networks: When does it help?

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***Abstract*— We study the equilibrium quality in a multi-commodity selfish routing game with many types of users, where each user type experiences a different level of uncertainty. We consider a new model of uncertainty where each user-type over- or under-estimates their congestion costs by a multiplicative constant. We present a variety of theoretical results showing that when users under-estimate their costs, the network congestion decreases at equilibrium, whereas over-estimation of costs leads to increased equilibrium congestion. Motivated by applications in urban transportation networks, we perform simulations consisting of parking users and through traffic on synthetic and realistic network topologies. In light of the dynamic pricing policies aimed at tackling congestion, our results indicate that while users' perception of these prices can significantly impact the policy's efficacy, optimism in the face of uncertainty leads to favorable network conditions.**

I. Introduction

Multi-commodity routing networks that allocate resources to self-interested users lie at the heart of many systems such as communication, transportation, and power networks [1]. In all of these systems, users are *inherently heterogeneous* not only in their demands and objectives, but also in their belief about the state of the system and how they trade-off between time, money, and risk [2], [3]. Naturally, these private beliefs influence each user's decisions and as a consequence, the total welfare of the overall system. Therefore, understanding the effects of these heterogeneities is fundamental to characterizing network state and performance.

A motivating example of a routing network, which we use throughout this paper, is the *urban transportation network*. Travelers in road networks simultaneously trade-off between objectives such as total travel time, road taxes, parking costs, waiting delays, walking distance and environmental impact. At the same time, these users tend to possess varying levels of information and heterogeneous attitudes, and there is evidence to suggest that the routes adopted depend not on the true costs but on how they are perceived by users. For instance, users prefer safer routes over those with high variance [4], seek to minimize travel time over parking costs [5], and react adversely to per-mile road taxes [6].

Furthermore, the technological and economic incentives employed by network operators to tackle congestion may compound these effects by interacting with user beliefs in a 'perverse manner' [7]. For example, to limit the economic loss arising from urban congestion [8], cities across the world have introduced a number of solutions including road taxes [9], time-of-day-pricing, and road-side message signs [10]. However, the dynamic nature of these incentives (e.g., frequent price updates) and the limited availability of information dispersal mechanisms may add to users' uncertainties and asymmetries in beliefs. Therefore, to truly evaluate the efficacy of such solutions, it is crucial to understand how these changes impact the congestion level of the system.

The effect of uncertainties on network equilibrium was examined in a recent body of work [7], [11]. These *uncertainties are reflected through the beliefs of users*, where each user may perceive the network condition to be different than the 'true' conditions. Current results have largely focused on simple networks (e.g., parallel links) where a fixed percentage of the population is endowed with the same level of uncertainty. Given the complexity in most practical networks, it is natural to ask how uncertainty (i.e., user beliefs) affects equilibria when there are many many types of users, who are heterogeneous in their perceptions. Specifically, in this work we answer the following two questions: (i) *how do equilibria depend on the type and level of uncertainty in networks with a multitude of users*, and (ii) *when does uncertainty lead to an improvement or degradation in equilibrium quality?*

To address these questions, we turn to a *multi-commodity selfish routing* framework commonly employed by many disciplines (see, e.g., [12]–[14]). In our model, each user seeks to route some flow along links connecting two nodes in a network and faces congestion costs on each link. These congestion costs are perceived differently by each user in the network, representing the uncertainties in their beliefs. It is well-known that even in the presence of perfect information (every user knows the exact true cost), strategic behavior by the users can result in considerably worse congestion at equilibrium when compared to an optimum routing solution [15]. Against this backdrop, we analyze what happens when users have imprecise views about the congestion costs. A surprising outcome arises: in the presence of uncertainty, if users under-estimate the costs and select routes based on these perceived costs, *the equilibrium quality is better compared to the full information case*. Conversely, if the users are overly conservative and over-estimate the costs, the equilibrium quality becomes worse.

A. Contributions

We introduce the notion of *type-dependent uncertainty* in multi-commodity routing networks, where the uncertainty of users belonging to type θ is captured by a single parameter $r_\theta > 0$. Specifically, for each user of type θ, if her true cost on edge e is given by $C_e(x) = a_e x + b_e$, where x is the total

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population of users on this edge, then her *perceived cost* is $a_e x + r_\theta b$.

We are interested in studying the quality of the equilibrium routing as well as the socially optimal routing solution, using *social cost* as a metric. Specifically, given an allocation $\mathbf{x}$ where each user routes an infinitesimal amount of flow between pre-specified nodes in the network, the social cost of this solution is given by

$$\mathrm{C}(\mathbf{x}) = \sum_{e \in \mathcal{E}} x_e C_e(x_e) \tag{1}$$

where x_e is the total population mass, summed over all user types, allocated to edge e under the allocation rule $\mathbf{x}$ (see Section II for formal definitions).

We consider two types of uncertainties: *pessimism* where users over-estimate the costs ($r_\theta \geq 1$) and *optimism* where users under-estimate the costs ($r_\theta \leq 1$), for all types θ.

Under this model of type-based uncertainty in congestion costs, we have the following contributions:

(a) The social cost of the equilibrium solution where all users have the same level of uncertainty ($r_\theta = r$ for all θ) is *always smaller than or equal to* the cost of the equilibrium solution without uncertainty when $r \in [0.5, 1]$ and vice-versa when $r \geq 1$.

(b) The worst-case ratio of the social cost of the equilibrium to that of the socially optimal solution (i.e., the price of anarchy [15]) is $\frac{4r_{\min}^2}{4r_{\min}\gamma - 1}$, where $r_{\min} = \min_\theta r_\theta$ and $\gamma \leq 1$ is the ratio of the minimum to the maximum uncertainty over user types.

(c) In systems having users with and without uncertainty, the routing choices adopted by the uncertain users always results in an improvement in the costs experienced by users without uncertainty.

Under some additional model assumptions, the above results also extend to a more general model where users possess different uncertainties on each edge (these extensions are covered in a longer version of this work [16]).

To validate the theoretical results, we present a number of simulation results. We focus specifically on the application of parking in urban transportation networks (see, e.g., Fig. 2) and consider simple networks as well as realistic urban network topologies with two types of users: through traffic and parking users. We demonstrate the effects of an uncertain parking population on equilibrium quality. We show via simulations that *optimism improves equilibrium quality while pessimism degrades it* both when uncertainty is asymmetric across user types and when users face different levels of uncertainty on different network edges.

B. Comparison with Other Models of Uncertainties

Our work is closely related to the extensive body of work on risk-averse selfish routing [17], [18] and pricing tolls in congestion networks [2], [13]. The former line of research focuses on the well known *mean-standard deviation* model where each individual user selects a path that minimizes a linear combination of their expected travel time and standard deviation. While such an objective is desirable from a central planner's perspective, experimental studies suggest that individuals tend to employ simpler heuristics when faced with uncertainty [19]. Motivated by this, we adopt a multiplicative model of uncertainty similar to [20], [21].

In regards to the latter line of work, the literature on computing tolls for heterogeneous users is driven by the need to *implement the optimum routing* by adjusting the toll amount, which is often interpreted as the *time–money* tradeoff, on each edge as a function of the congestion. We, on the other hand, assume that the user beliefs are independent of the congestion in the network, and aim for a more nuanced understanding of the dependence of network congestion on the level of uncertainty. Moreover, we study the effect of both optimistic and pessimistic attitudes, whereas much of the existing work focuses strictly on pessimistic user behavior.

C. Organization

The rest of the paper is structured as follows. In Section II, we formally introduce our model followed by our main results in Section III. Section IV presents our simulation results on urban transportation networks with parking and routing users who face different levels of uncertainty. Finally, we conclude with some discussion and comments on future directions in Section V.

II. Model and Preliminaries

We consider a non-atomic, multi-commodity selfish routing game with multiple types of users. Specifically, we consider a network represented as $G = (V, \mathcal{E})$ where V is the set of nodes and $\mathcal{E}$ is the set of edges. For each edge $e \in \mathcal{E}$, we define a linear cost function

$$C_e(x_e) = a_e x_e + b_e, \tag{2}$$

where $x_e \geq 0$ is the total population (or flow) of users on that edge and $a_e, b_e \geq 0$. One can interpret $C_e(\cdot)$ as the true cost or expected congestion felt by the users on this edge. However, due to uncertainty, users may perceive the cost on each edge $e \in \mathcal{E}$ to be different from its true cost.

To capture that users may have different perceived uncertainties, we introduce the notion of *type*. Specifically, we consider a finite set of user types $\mathcal{T}$, where each type $\theta \in \mathcal{T}$ is uniquely defined by the following tuple $(s_\theta, t_\theta, \mu_\theta, r_\theta)$. We assume that $\mu_\theta > 0$ denotes the total population of users belonging to type θ such that each of these infinitesimal users seeks to route *some flow* from its source node $s_\theta \in V$ to the destination node $t_\theta \in V$. Moreover, the parameter $r_\theta > 0$ captures the beliefs or uncertainties associated with users of type θ and affects the edge cost in the following way: users of type θ perceive the cost of edge $e \in \mathcal{E}$ to be

$$\hat{C}_e^\theta(x) = a_e x + r_\theta b_e. \tag{3}$$

If we interpret b_e as a price or a tax, then $r_\theta < 1$ denotes the case where users of type θ under-estimate prices compared to their actual value and $r_\theta > 1$ captures situations where users over-estimate prices or view them adversely compared to their other costs.

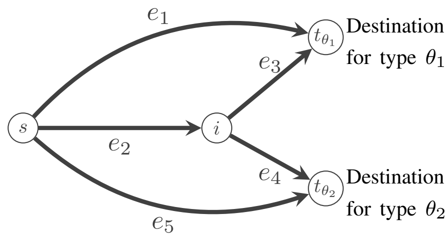


Fig. 1: An example of a two-commodity routing game, where all traffic originates at node s. Users of type θ_1 select a route between nodes s and t_{θ_1}, whereas users of type θ_2 route traffic between s and t_{θ_2}.

Let $\mathcal{P}_\theta$ denote the set of all s_θ–t_θ paths in G where s_θ is the source and t_θ is the destination. A path $p \in \mathcal{P}_\theta$ is a seqence of edges connecting s_θ to t_θ.

Formally, let $x_p^\theta \in \mathbb{R}$ be the total flow routed by users of type θ on path $p \in \mathcal{P}_\theta$. We define a *feasible flow* to be a flow $\mathbf{x} = (x_p^\theta)_{\theta\in\mathcal{T}, p\in\mathcal{P}_\theta} \in \mathbb{R}^{|\mathcal{T}|\cdot|\mathcal{P}_\theta|}$ such that for all $\theta \in \mathcal{T}$, $\sum_{p\in\mathcal{P}_\theta} x_p^\theta = \mu_\theta$, and $x_p^\theta \geq 0$ for all paths $p \in \mathcal{P}_\theta$. Path flows are related to edge flows. Indeed, let $x_e^\theta \in \mathbb{R}$ be the flow on edge e of users of type θ. The edge and path flow for users of type θ are related by

$$x_e^\theta = \textstyle\sum_{p\in\mathcal{P}_\theta, p\ni e} x_p^\theta \tag{4}$$

Define the total flow on edge e to be $x_e = \sum_{\theta\in\mathcal{T}} x_e^\theta$. Then, in this notation, we can write the path cost in terms of edge flow; indeed for any path p in the graph,

$$C_p(\mathbf{x}) = \textstyle\sum_{e\in p} C_e(x_e) = \sum_{e\in p}(a_e x_e + b_e). \tag{5}$$

Similarly, the perceived path costs are given by

$$\hat{C}_p^\theta(\mathbf{x}) = \textstyle\sum_{e\in p} \hat{C}_e^\theta(x_e) = \sum_{e\in p} a_e x_e + r_\theta b_e. \tag{6}$$

Users of type θ choose a feasible flow that minimizes their perceived cost under the allocation $\mathbf{x}^\theta$:

$$\textstyle\sum_{p\in\mathcal{P}_\theta} x_p^\theta \hat{C}_p^\theta(\mathbf{x}). \tag{7}$$

We define a game instance as the tuple

$$\mathcal{G} = \{(V,\mathcal{E}), \mathcal{T}, (s_\theta, t_\theta, \mu_\theta, r_\theta)_{\theta\in\mathcal{T}}, (C_e)_{e\in\mathcal{E}}\}. \tag{8}$$

In the full version of this work [16], we generalize this model to consider situations where users belonging to a given type experience different uncertainties on each of the edges.

A. Illustrative Example

A simple example of a multi-commodity routing model that captures the conflict between two types of users is depicted in Fig. 1. Such a situation could occur in transportation networks, for instance, when θ_1 represents travelers who choose between public transportation (path $\{e_2, e_3\}$) and simply walking to their final destination (edge e_1). On the other hand, θ_2 could denote drivers with personal vehicles (paths $\{e_2, e_4\}$ and $\{e_5\}$). It is conceivable that users of type θ_1 are averse to (or even favor) walking long distances and therefore $r_{\theta_2} \neq 1$. Clearly, the biases that are intrinsic to users of type θ_1 affect the number of these users on edge e_2 and hence, the congestion experienced by users of type θ_2. We explore these effects further via more nuanced transportation related examples in Section IV.

B. Nash Equilibrium Concept and its Efficiency

We assume that the users in the system are self-interested and route their flow with the goal of minimizing their individual cost. Therefore, the solution concept of interest in such a setting is that of a Nash equilibrium, where each user type routes their flow on minimum cost paths with respect to their perceived cost functions and the actions of the other users.

Definition 2.1 (Nash Equilibrium): Given a game instance $\mathcal{G}$, a feasible flow $\mathbf{x}$ is said to be a *Nash equilibrium* if for every $\theta \in \mathcal{T}$, for all $p \in \mathcal{P}_\theta$ with positive flow, $x_p^\theta > 0$,

$$\hat{C}_p^\theta(\mathbf{x}) \leq \hat{C}_{p'}^\theta(\mathbf{x}), \quad \forall\ p' \in \mathcal{P}_\theta. \tag{9}$$

For the rest of this work, we will assume that all the flows considered are feasible.

Remark 1 (User Beliefs): In order to employ the classical notion of Nash equilibrium, we assume that all uncertainty levels are known by all of the users. That is, a user of type θ knows the values $(r_{\theta'} b_e)_{e\in\mathcal{E}}$ for all $\theta' \in \mathcal{T}$. While a user knowing $(r_\theta b_e)_{e\in\mathcal{E}}$ within its own type θ may not be unreasonable, full knowledge of $(r_\theta b_e)_{e\in\mathcal{E}}$ for all $\theta \in \mathcal{T}$ is a strong assumption.

This being said, for the types of games we consider, a number of *myopic learning rules*[1] converge to Nash equilibria independent of the beliefs held by users regarding other user types. (see, e.g., [22] and references therein).

C. Social Cost and Price of Anarchy

We measure the quality of a solution using its social cost, which is defined to be the aggregate (true) cost incurred by all of the users in the system. Formally, the social cost of a flow $\mathbf{x}$ is given by

$$\mathrm{C}(\mathbf{x}) = \textstyle\sum_{e\in\mathcal{E}} C_e(x_e) x_e. \tag{10}$$

The social cost is only measured with respect to the true congestion costs and does not reflect users' beliefs.

To capture inefficiencies, we leverage the well-studied notion of the *price of anarchy* which is the ratio of the social cost of the *worst-case Nash equilibrium* to that of the socially optimal solution [14]. Formally, given an instance $\mathcal{G}$ of a multi-commodity routing game belonging to some class $\mathcal{C}$ (a class refers to a set of instances that usually share some property) suppose that $\mathbf{x}_\mathcal{G}^*$ is the flow that minimizes the social cost $\mathrm{C}(\mathbf{x})$ and that $\tilde{\mathbf{x}}_\mathcal{G}$ is the Nash equilibrium for the given instance, then the price of anarchy is:

Definition 2.2 (Price of Anarchy): Given a class of instances $\mathcal{C}$, the *price of anarchy* for this class is

$$\max_{\mathcal{G}\in\mathcal{C}} \frac{\mathrm{C}(\tilde{\mathbf{x}}_\mathcal{G})}{\mathrm{C}(\mathbf{x}_\mathcal{G}^*)}. \tag{11}$$

[1]By *myopic learning rules*, we mean rules for iterated play that require each player to have minimal-to-no knowledge of other players' cost functions and/or strategies.

Of course, the price of anarchy is always greater than or equal to one.

III. Main Results

The first step to analyzing the multi-commodity game is to characterize the Nash equilibria. If the games fall into the general class of potential games, then the equilibria have "nice properties" in terms of existence, uniqueness, and computability [22]. General multi-commodity, selfish routing games with heterogeneous users, however, do not belong to the class of potential games unless certain assumptions on the edge cost structure are met [12].

Due to the fact that we have linear latencies for each type and the type-dependent uncertainty appears on the b_e terms for each edge, all game instances of the form we consider admit a potential function and hence, there always exists a Nash equilibrium [12].

Proposition 3.1: A feasible flow $\mathbf{x}$ is a Nash equilibrium for a given instance $\mathcal{G}$ of a multi-commodity routing game if and only if it minimizes the following potential function:

$$\Phi_r(\mathbf{x}) = \sum_{e\in\mathcal{E}} \left(\frac{a_e x_e^2}{2} + b_e \sum_{\theta\in\mathcal{T}} r_\theta x_e^\theta\right) \qquad (12)$$

Moreover, for any two minimizers $\mathbf{x}, \mathbf{x}'$, $C_e(x_e) = C_e(x'_e)$ for every edge $e \in \mathcal{E}$.

The proof of the proposition follows from standard arguments pertaining to the minimizer of a convex function and from the definition of Nash equilibrium as in (9) and we refer the reader to [22] for more details. The second part of the proposition indicates that the equilibria are essentially unique as the cost on every edge is the same across solutions.

With the above proposition in hand, we now derive three main results on (i) the impact of uncertainty on social cost, (ii) the impact of uncertainty on players, and (iii) bounds on the price of anarchy.

A. Effect of Uncertainty on Equilibrium Quality

Our first main result identifies a special case of the general multi-commodity game for which uncertainty helps improve equilibrium quality—i.e. decreases the social cost—whenever users under-estimate costs for *every instance* and vice-versa when they over-estimate costs. For a given instance of the multi-commodity routing game, we say that the users are optimistic if $r_\theta \leq 1$ for all $\theta \in \mathcal{T}$. Similarly, the users are pessimistic if $r_\theta \geq 1$ for all user types.

Given an instance $\mathcal{G}$ of the multi-commodity routing game, we define $\mathcal{G}^1$ to be the corresponding game instance with no uncertainty—that is, $\mathcal{G}^1$ has the same graph, cost functions, and user types as $\mathcal{G}$, yet $r_\theta = 1$ for all $\theta \in \mathcal{T}$.

Theorem 3.2: Consider any given instance $\mathcal{G}$ of the multi-commodity routing game with Nash equilibrium $\tilde{\mathbf{x}}$ and corresponding game instance $\mathcal{G}^1$, having no uncertainty, whose Nash equilibrium is $\mathbf{x}^1$. Suppose all users experience the same uncertainty, where $r_\theta = r$ for all $\theta \in \mathcal{T}$. Then, the following hold:

1) $\mathrm{C}(\tilde{\mathbf{x}}) \leq \mathrm{C}(\mathbf{x}^1)$ if $0.5 \leq r \leq 1$.
2) $\mathrm{C}(\tilde{\mathbf{x}}) \geq \mathrm{C}(\mathbf{x}^1)$ if $r \geq 1$.

We provide the proof in Appendix A. The following corollary identifies a specific level of (optimistic) uncertainty at which the equilibrium solution is actually optimal.

Corollary 3.3: Given an instance $\mathcal{G}$ of the multi-commodity routing game, let $\tilde{\mathbf{x}}$ denote its Nash equilibrium and $\mathbf{x}^*$ denote the socially optimal flow. If $r_\theta = 0.5$ for all $\theta \in \mathcal{T}$, then $\mathrm{C}(\tilde{\mathbf{x}}) = \mathrm{C}(\mathbf{x}^*)$—i.e. the equilibrium is socially optimal.

We address the issue of heterogeneous uncertainties across user types in [16]. Specifically, under some additional model assumptions, we can prove that (mild) optimism helps lower equilibrium costs and pessimism increases equilibrium congestion for a more general model where users possess different uncertainties on each edge.

B. Impact of Uncertain Users on Those without Uncertainty

Now that we have a better understanding of how uncertainty affects the performance of the entire system as measured by the social cost, we now tackle a more nuanced question: *in systems where only some users are uncertain, how does their behavior impact the social cost of the users who do not have uncertainty?* This question is of considerable interest in a number of settings, e.g., in urban transportation networks, where it is believed that [23]–[26] inefficient behavior by the parking users (such as cruising or searching for parking spots) can often cascade into increased congestion for other drivers leading to a detrimental effect on the overall congestion cost.

To answer this question, we restrict our attention to a two-commodity routing game $\mathcal{G}$, where both types of users seek to route their flow from a common source node s to sink node t. Moreover, we assume that for the first type θ_1, $r_{\theta_1} = 1$. For the second type θ_2, r_{θ_2} is not necessarily 1. We refer to this as the *two-commodity game having users with and without uncertainty.*

We now define some additional notation. Suppose that $\mathbf{x}$ denotes a feasible flow for a given instance of the two-commodity game with uncertain and full information users, we use $\mathrm{C}^{\theta_1}(\mathbf{x}) = \sum_{e\in\mathcal{E}} C_e(x_e)x_e^{\theta_1}$ to denote the aggregate cost of users of type θ_1. Finally, we also restrict our focus to series-parallel networks, which denote a popular class of network topologies in the literature pertaining to network routing [27], [28]. Informally, a graph is said to be series-parallel if it does not contain an embedded *Wheatstone network* or equivalently if, in the undirected version of this graph two routes never pass through any edge in opposite directions. The reader is referred to [28] for a more rigorous definition.

We present a surprising result in Theorem 3.4: the behavior under uncertainty by one type of users always decreases the congestion costs of other types of users who do not face any uncertainty. This result holds for both optimistic and pessimistic behavior.

Theorem 3.4: Given an instance $\mathcal{G}$ of the two-commodity game having users with and without uncertainty such that the graph G is series-parallel, let $\mathcal{G}^1$ denote a modified version of this instance with no uncertainty (i.e. $r_{\theta_1} = r_{\theta_2} = 1$). Let

$\tilde{\mathbf{x}}$ and $\mathbf{x}^1$ denote the Nash equilibrium for the two instances, respectively. Then,

$$\mathbf{C}^{\theta_1}(\tilde{\mathbf{x}}) \leq \mathbf{C}^{\theta_1}(\mathbf{x}^1). \tag{13}$$

We provide the proof of the above theorem in Appendix B.

C. Price of Anarchy Under Uncertainty

In Theorem 3.2, we showed that the equilibrium cost under uncertainty decreases (resp. increases) when users are optimistic (resp. pessimistic) and all user types have the same level of uncertainty. This naturally raises the question of quantifying the improvement (or degradation in equilibrium) and whether uncertainty helps when the uncertainty parameter can differ between user types. In the following theorem, we address both of these questions by providing price of anarchy bounds as a function of the minimum uncertainty in the system and γ, which is the ratio between the minimum and maximum uncertainty among user types.

Let $r_{\min} = \min_{\theta \in \mathcal{T}} r_\theta$ and $\gamma = \frac{\min_\theta r_\theta}{\max_\theta r_\theta}$.

Theorem 3.5: If $1 < 4r_{\min}\gamma$, the price of anarchy for multi-commodity routing games is given by

$$\frac{4r_{\min}^2}{4r_{\min}\gamma - 1}. \tag{14}$$

We provide the proof of the above theorem in [16].

The above result broadly validates our message that uncertainty helps equilibria when users under-estimate their costs and hurts equilibrium when users over-estimate their costs. To understand why, let us consider the case of $\gamma = 1$—i.e. the uncertainty is the same across user types. We already know that in the absence of uncertainty, the price of anarchy of multi-commodity routing games with linear costs is given by $\frac{4}{3}$ [15]; this can also be seen by substituting $r_{\min} = 1$, $\gamma = 1$ in (11). We observe that the price of anarchy is strictly smaller than $\frac{4}{3}$ for $r_{\min} < 1$ and reaches the optimum value of one at $r_{\min} = 0.5$ thereby confirming Corollary 3.3.

Similarly, as $r_{\min}$ increases from one, the price of anarchy also increases nearly linearly. In fact, our price of anarchy result goes one step beyond Theorem 3.2 as it provides guarantees even when different user types have different uncertainty levels. For example, when $r_{\min} = 0.6$, and $\gamma = 0.9$—i.e. $\max_\theta r_\theta \approx 0.67$—the price of anarchy is 1.24, which is still better than the price of anarchy without uncertainty. Furthermore, the price of anarchy result reveals a surprising dichotomy: as long as $r_{\min} < 1$ and γ is not too large, for any given instance $\mathcal{G}$ of the multi-commodity routing game, either the equilibrium quality is already good or uncertainty helps lower congestion by a significant amount.

IV. Numerical Examples

In this section, we present our main simulation results on both stylistic as well as realistic urban network topologies comprising of two types of users (two commodities)—i.e. *through traffic, parking users* (types θ_1, θ_2, respectively). We consider a more general model of uncertainty for our simulations, where the parking users have different uncertainty levels on different parts of the network and the through traffic does not suffer from uncertainty at all. We vary the level of

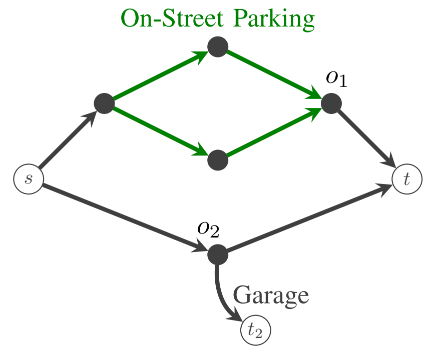


Fig. 2: A special case of our general multi-commodity network with two types of users- parking users and through traffic. All of the network traffic originates at the source node s. Users belonging to the through traffic simply select a (minimum-cost) path from s to t and incur only the latencies on each link. The parking users select between one of two parking structures: on-street parking (indicated in green) with additional circling costs and off-street (e.g., parking garage).

uncertainty faced by the parking users, and observe its effect on the social cost at equilibrium.

Despite the generality of the model considered here—different user types have different beliefs and their level of uncertainty depends on the edge under consideration—our simulations validate the theoretical results presented in the previous section.

A. Effect of Uncertainty on On-Street vs Garage Parking

Inspired by work in [23] which provides a framework for integrating parking into a classical routing game that abstracts route choices in urban networks, we begin with a somewhat stylized example of an urban network, depicted and described in Fig. 2. The users looking for a parking spot are faced with two options: (i) *on-street parking* which, as in reality, is cheaper but leads to larger wait times due to cruising in search of parking; (ii) an *off-street or a private garage option* that is much easier to access (in terms of wait times) at the expense of a higher price.

To understand the costs faced by the parking users (type θ_2), let $\mathcal{E}_{os}$ be the set of edges in the on-street parking structure (the green edges in Fig. 2). For parking users that select the on-street parking option, the cost on edges $e \in \mathcal{E}_{os}$ are of the form $C_e^{\theta_2}(x_e) = C_{e,\ell}^{\theta_2}(x_e) + C_{e,os}^{\theta_2}(x_e)$ where $C_{e,\ell}^{\theta_2}(x_e) = a_e x_e + b_e$ is the travel latency and $C_{e,os}^{\theta_2}(x_e) = a_{e,os}x + b_{e,os}$ is the parking cost.

Fig. 2 easily transforms into a two-commodity network by creating a *fake* edge $\tilde{e}$ from node o_1 to t_2, having the accumulated parking costs from edges $\mathcal{E}_{os}$—i.e. $C_{\tilde{e}}(x_{\tilde{e}}) = \sum_{e \in \mathcal{E}_{os}} C_{e,os}^{\theta_2}(x_e)$ Then the costs on edges in $\mathcal{E}_{os}$ are re-defined to only contain the travel latency cost, and this is the same for both types of users: for $e \in \mathcal{E}_{os}$, $C_e^{\theta_1}(x_e) = C_e^{\theta_2}(x_e) = a_e x_e + b_e$. For the off-street parking structure, the edge, say e', from o_2 to t_2 has cost $C_{e'}(x_{e'}) = a_{pg}x_{e'} + b_{pg}$.

The uncertainty is only faced by the parking users (and only on $\tilde{e}, e'$) who perceive the cost of the two parking

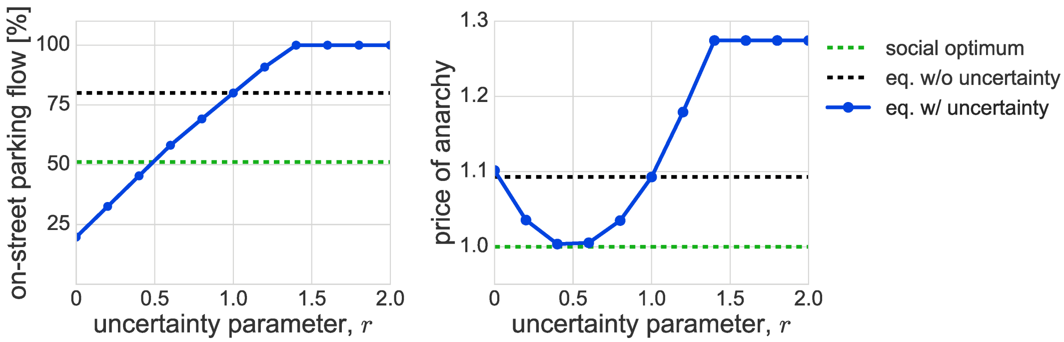


Fig. 3: The left plot shows the on-street parking population flow (mass) under the social optimum and the Nash equilibrium with and without uncertainty as r, the uncertainty parameter, varies. The right plots shows the equilibrium quality as measured by the price of anarchy for each of these solutions. As r, the level of uncertainty faced by the parking users increases, we observe from the left plot more users move towards on-street parking, which in turn affects the social cost as seen from the right plot. Particularly, when $0.5 \leq r \leq 1$, the users under-estimate both on-street and garage prices leading to a decrease in social cost as more users choose the garage option. On the other hand, for $r \geq 1$, users view the garage option adversely, which leads to more congestion. For a sufficiently large r, all of the parking users end up using the on-street option and the social cost saturates around $r = 1.4$ with further increase in uncertainty having no effect on equilibrium cost.

structures to be $\hat{C}_{\tilde{e}}^{\theta_2}(x_e) = a_{os}x_e + rb_{os}$ and $\hat{C}_{e'}^{\theta_2}(x_e) = a_{pg}x + rb_{pg}$, respectively, for some parameter $r > 0$. The through traffic (type θ_1) does not suffer from uncertainty on any of its edges and therefore, $r_{\theta_1} = 1$ on all edges. Moreover, $r_{\theta_2} = 1$ on all edges in the network except e, e'.

For the simulations, we assume that there is an equal mass of parking and through traffic users originating at the source node s. The parameters of the edge congestion functions are selected uniformly at random from a suitable range. For both on-street and off-street parking, the cost of the parking are set as follows: (i) The parameter a_{os} was chosen to be inversely proportional to the number of on-street parking slots typically available on roads, whereas we set $a_{pg} = 0$, since garages have a large number of parking slots; (ii) The parameters b_{os} and b_{pg} equal the price of on-street and garage parking (respectively) commonly used in cities multiplied by a constant that captures how users trade-off between time and money.

Fig. 3 shows how the parking users divide themselves among the on-street and garage option (left plot) and how this affects uncertainty as r varies (right plot). From the left plot, we observe that at the social optimum, approximately 77% of the parking population prefers on-street parking. With no uncertainty (i.e. when $r = 1$), at the Nash equilibrium more parking users (around 84%) gravitate towards the on-street option leading to higher congestion and inefficiency—that is, even without uncertainty, the system is inefficient as is expected. This is reminiscent of the classic Pigou example [14] in traffic networks.

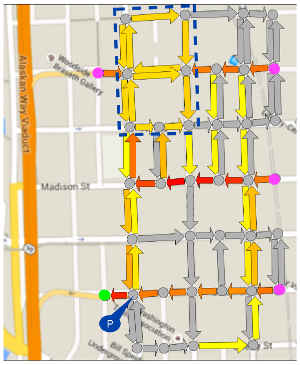
As r decreases—that is, as users become more optimistic in their beliefs about prices—more users start flocking to the off-street option as they perceive a multiplicative decrease in price. The result is an improvement in efficiency. On the other hand, for users who tend to over-estimate prices ($r > 1$), the appeal of off-street parking decreases and more users move to on-street options leading to increased congestion and poor equilibrium quality. The effect of the above behavior on equilibrium inefficiency is quantified in the right-hand plot in Fig. 3.

B. *Parking vs. Through Traffic in Downtown Seattle*

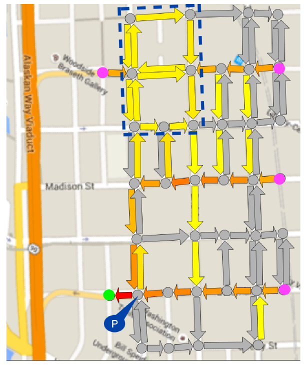
In a similar manner to the toy example, we take a real-world urban traffic network (depicted in Fig. 4) that captures a slice of a highly congested area in downtown Seattle. The network contains both on-street and off-street parking options and the parking population experiences both a travel latency cost and a parking cost, both of which we model as affine functions. Once again, this can be converted to a standard two-commodity instance by adding a common destination node t_{θ_2} and including *fake edges* from (i) every node from which users can enter the on-street parking area (boundary nodes of the blue colored dotted area in Fig. 4a) to t_{θ_2}; (ii) the node containing the parking garage (marked with a 'P' symbol in Fig. 4a) to t_{θ_2}. As with our previous example, we assume that the through traffic faces no uncertainty ($r_{\theta_1} = 1$ for all edges) and the parking users face an uncertainty parameter of $r > 0$ only on the fake edges, which model their perception of the parking costs.

For the simulations, we assume that the parking traffic originates at a few select nodes in the network (indicated in pink in Fig. 4a) and wishes to route their flow to either an on-street parking slot or a garage. On the other hand, the through traffic originates at every node in the network and has a single destination, which represents drivers seeking to leave the downtown area via state highway 99.

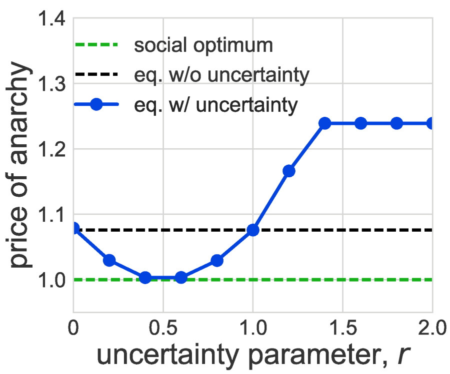
The cost functions for the two parking structures were based on an estimation of the number of available parking slots (which influence the wait times), and the hourly price for parking for both on-street and garage parking in Seattle. Furthermore, owing to the uniformity of the downtown roads, we assumed that all edges in the network have the same congestion cost function, which were sufficiently scaled in



(a) Flow intensity for $r = 1$



(b) Flow intensity for $r = 0.4$



(c) Inefficiency of Equilibrium vs r

Fig. 4: Parking and through traffic example. (a)–(b) Network superimposed on the corresponding area in downtown Seattle. The parking population originates at the pink nodes, whereas the through traffic routes from every grey node to the green node. The blue dotted box represents the on-street parking zone and the parking symbol ('P') is the parking garage. The color on each edge depicts the intensity of flow on that edge: the intensity increases as the color transitions from gray to red. (c) Equilibrium inefficiency as a function of the parking uncertainty level r.

order to ensure that the parking costs are comparable to the transit costs.

As seen in Fig. 4a, in the downtown Seattle network, the equilibrium without uncertainty is sub-optimal as more parking users select the cheaper on-street option. This leads to heavy congestion in the middle of the network (indicated by the red edges) as parking users distributed across the network approach the on-street parking area.

In the presence of uncertainty, we observe an interesting phenomenon. When the parking users are optimistic, the garage option becomes more preferable to users who are equidistant from both parking locations. The parking users distribute themselves more evenly across the two options, which in turn leads to lesser congestion in the middle of the network. It is well-known that sub-optimal behavior by the parking users can cascade and lead to increased congestion for through traffic [25]. Our simulation results indicate that the routes adopted by the parking users under uncertainty helps alleviate some of this congestion.

Fig. 4c shows the inefficiency of each of the equilibria as a function of the uncertainty of the parking users. Specifically, at $r \approx 0.5$, the equilibrium solution coincides with the socially optimal flow. On the other hand, as r increases above one, the social cost of the equilibrium solution increases sharply since more users select the on-street option. This, in turn, leads to heavier congestion in the rest of the network. Finally, we remark that even though optimism results only leads to a small improvement in the price of anarchy (see Fig. 4c), even minor improvements in daily congestion in downtown areas could result in economic gains [8].

V. Conclusions and Future Work

In this work, we considered a multi-commodity selfish routing game where different types of users face different levels of uncertainty quantified by a multiplicative parameter r_θ. Broadly classifying the user attitudes as optimistic and pessimistic, we showed several theoretical and experimental results highlighting the effect that when users under-estimate their network costs, equilibrium quality tends to improve and vice-versa when users over-estimate the costs.

Although, we focused on linear congestion cost functions in this work, we remark that *all* of our results extend naturally to polynomial cost functions of the form $C_e(x) = a_e x^d + b$, albeit with different price of anarchy bounds. Finally, on a somewhat abstract note, the central message of this paper that 'optimism in the face of uncertainty leads to near-optimal solutions' bears thematic similarities to the popular Upper-Confidence Bound (UCB) algorithm [29] used for Multi-Armed Bandit problems. It would be interesting to see if this connection could be exploited to show similar bounds on equilibrium in routing games with learning.

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APPENDIX

A. Proof of Theorem 3.2

Proof: Let $\Phi_r(\mathbf{x})$ denote the potential function for the instance $\mathcal{G}$ and $\Phi_1(\mathbf{x})$ denote the potential function for $\mathcal{G}^1$ where Φ_1 is given in (12) with $r_\theta = 1$ for all θ. By definition of the potential function, we know that $\Phi_r(\tilde{\mathbf{x}}) - \Phi_r(\mathbf{x}^1) \le 0$ and $\Phi_1(\mathbf{x}^1) - \Phi_1(\tilde{\mathbf{x}}) \le 0$. Adding these two inequalities and rearranging the terms, we get that $\Phi_r(\tilde{\mathbf{x}}) - \Phi_1(\tilde{\mathbf{x}}) - (\Phi_r(\mathbf{x}^1) - \Phi_1(\mathbf{x}^1)) \le 0$. Applying the definition of the potential function (Equation 14) and expanding the terms, the a_e terms cancel out giving us:

$$\sum_{e \in \mathcal{E}} b_e \sum_{\theta \in \mathcal{T}} (r_\theta - 1)\Delta x_e^\theta \le 0, \tag{15}$$

where $\Delta x_e^\theta = \tilde{x}_e^\theta - x_e^{1,\theta}$, and $\Delta x_e = \tilde{x}_e - x_e^1$. Recall that $x_e^{1,\theta}$ denotes the total flow on edge e by users of type θ in the solution $\mathbf{x}^1$.

By the conditions of the theorem, we know that $r_\theta = r$ for all $\theta \in \mathcal{T}$. Substituting this into (15) and using the fact that $\sum_\theta \Delta x_e^\theta = \Delta x_e$ for all edges, we get:

$$(r-1)\sum_{e \in \mathcal{E}} b_e \Delta x_e \le 0. \tag{16}$$

Hence,

1) When $r \in [0.5, 1)$, we have that $(r-1) < 0$, and therefore, $\sum_{e \in \mathcal{E}} b_e \Delta x_e \ge 0$.
2) When $r > 1$, we have that $(r-1) > 0$, and therefore, $\sum_{e \in \mathcal{E}} b_e \Delta x_e \le 0$.

We can finish the proof by consider each of these cases.

Case I: $r \in [0.5, 1)$: Applying Lemma 3.2 [16] to the instance $\mathcal{G}$ with $\mathbf{x} = \mathbf{x}^1$ and $r_\theta = r$ for all $\theta \in \mathcal{T}$, we obtain:

$$\mathrm{C}(\tilde{\mathbf{x}}) - \mathrm{C}(\mathbf{x}^1) \le (1-2r)\sum_{e \in \mathcal{E}} b_e \Delta x_e. \tag{17}$$

We claim that when $r \in [0.5, 1)$, the right-hand side of (17) is lesser than or equal to zero. This is not particularly hard to deduce owing to the fact that $(1-2r) < 0$ in the given range and that $\sum_{e \in \mathcal{E}} b_e \Delta x_e \ge 0$ as deduced from (16). Therefore, $\mathrm{C}(\tilde{\mathbf{x}}) - \mathrm{C}(\mathbf{x}^1) \le 0$, which proves the first of the claim that uncertainty with a limited amount of optimism helps lower equilibrium costs.

Case II: $r > 1$: Applying Lemma 3.2 of [16] to the instance $\mathcal{G}^1$ with $\mathbf{x} = \tilde{\mathbf{x}}$, and using the fact that $\Delta x_e = \tilde{x}_e - x_e^1$, we have:

$$\mathrm{C}(\mathbf{x}^1) - \mathrm{C}(\tilde{\mathbf{x}}) \le (2r-1)\sum_{e \in \mathcal{E}} b_e \Delta x_e. \tag{18}$$

Once again when $r \ge 1$, we know that $2r - 1 > 0$ and from (9), we can deduce that $\sum_{e \in \mathcal{E}} b_e \Delta x_e \le 0$ in the given range. This completes the proof that uncertainty coupled with pessimism hurts equilibrium quality. ∎

B. Proof of Theorem 3.4

Proof: Consider the flows $\mathbf{x}^1$ and $\tilde{\mathbf{x}}$. Applying a result for series-parallel graphs [28, Lemma 3], we get that there exists a path p with $x_p^1 > 0$ such that for all $e \in p$, $\tilde{x}_e \le x_e^1$.

We can now bound both $\mathrm{C}^{\theta_1}(\tilde{\mathbf{x}})$ and $\mathrm{C}^{\theta_1}(\mathbf{x}^1)$ in terms of the cost of the path p. Specifically note that in the solution $\mathbf{x}^1$, the path p has non-zero flow on it, we get that $\mathrm{C}^{\theta_1}(\mathbf{x}^1) = \ell_{\theta_1}\sum_{e \in p} C_e(x_e^1)$.

However, in the solution $\tilde{\mathbf{x}}$, we know that every user of type θ_1 is using a minimum cost path with respect to the true costs and therefore, the cost of any path used by users of type θ_1 is at least that of the path p. Formally,

$$\mathrm{C}^{\theta_1}(\tilde{\mathbf{x}}) \le \ell_{\theta_1}\sum_{e \in p} C_e(\tilde{x}_e) \le \ell_{\theta_1}\sum_{e \in p} C_e(x_e^1). \tag{19}$$

The final inequality follows from the monotonicity of the cost functions and the fact that $\tilde{x}_e \le x_e^1$ for all $e \in p$. Therefore, we conclude that $\mathrm{C}^{\theta_1}(\tilde{\mathbf{x}}) \le \mathrm{C}^{\theta_1}(\mathbf{x}^1)$. ∎