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A MULTISCALE METHOD FOR COMPUTING EFFECTIVE PARAMETERS OF COMPOSITE ELECTROMAGNETIC MATERIALS WITH MEMORY EFFECTS

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Abstract. We consider the problem of computing (macroscopic) effective properties of composite materials that are mixtures of complex dispersive dielectrics described by polarization and magnetization laws. We assume that the micro-structure of the composite material is described by spatially periodic and deterministic parameters. Mathematically, the problem is to *homogenize* Maxwell's equations along with constitutive laws that describe the material response of the micro-structure comprising the mixture, to obtain an equivalent effective model for the composite material with constant effective parameters. The novel contribution of this paper is the homogenization of a hybrid model consisting of the Maxwell partial differential equations along with ordinary (auxiliary) differential equations modeling the evolution of the polarization and magnetization, as a model for the complex dielectric material. This is in contrast to our previous work (2006) in which we employed a convolution in time of a susceptibility kernel with the electric field to model the delayed polarization effects in the dispersive material. In this paper, we describe the auxiliary differential equation approach to modeling material responses in the composite material and use the periodic unfolding method to construct a homogenized model.

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1. Introduction. Computational simulations of the propagation and scattering of transient electromagnetic (EM) waves in complex materials are important for constructing prediction tools that are reliable [29]. Examples are applications involving radar, environmental and medical imaging, such as the noninvasive detection of degradation in materials, the detection of cancerous tumors [17], the investigation of the effect of precursors on the human body, and the design of engineered composites such as ceramic matrix composites (CMCs) and metamaterials [35]. Thus, the development and analysis of efficient forward numerical methods which are accurate, consistent, and stable has been and continues to be an important area of research in computational electromagnetics.

For some time there has been interest in the integration of silicon nitride carbon based ceramic matrix composites (CMCs) for their use in high temperature turbine engines [2, 25]. As these materials are being investigated for their use in these applications, there is also a need to nondestructively monitor the material's degradation. In recent efforts [5–7], the authors were specifically interested in a silicon nitride carbon based CMC. This SiC/SiCN CMC has a silicon carbon fiber and a silicon nitride carbon matrix. Exposure to high temperature environments induces oxidation in the CMC, producing SiO_2 and SiN. As discussed in [5–7, 14, 15], multiple Lorentz polarization laws (based on the Lorentz model for polarization (95) which results from the displacement of electrons from equilibrium under the effect of an interrogating electromagnetic field) are appropriate to use in describing these components.

More recently, there has been increased activity in the design and development of new materials called *metamaterials*, with tailored EM properties. These materials are a broad class of micro- or nano-structures made up of tailored building blocks that are smaller than the wavelength of the interrogating electromagnetic field, thus enabling dense packing into an effective material [34]. Metamaterials are fabricated through engineering design as building blocks for devices with unique EM responses, from the microwave to the optical frequency range [35].

For a complex dispersive dielectric that exhibits a heterogeneous micro-structure described by spatially periodic parameters, we consider an electromagnetic interrogation technique for identifying the response of this composite material by subjecting it to electromagnetic fields generated by currents of varying frequencies. When the period of the structure is small compared to the wavelength of the interrogating field, the coefficients in Maxwell's equations oscillate rapidly, and are difficult to treat numerically in simulations. We use the mathematical theory of homogenization to produce an equivalent model for efficient simulation of the response of the heterogeneous material to the interrogating electromagnetic field. Homogenization is a mathematical method in which a model for a composite material involving spatially dependent parameters (depending on the microscopic structure) is replaced with a model for an equivalent material with macroscopic, homogeneous properties. The limiting homogeneous model with effective constant coefficients is easier to numerically discretize. The approach to homogenization that we take here is based on the periodic unfolding method presented in [11].

For the computation of effective parameters of composite materials, traditional mixture formulas based on physical arguments [30] are available in the literature. Some of the most popular mixing formulas are the Maxwell Garnett formula, the Böttcher mixture rule or Bruggeman formula, and the coherent potential formula. The mathematical theory of homogenization has also been applied to Maxwell's equations in composite materials. In [1, 16, 20, 21, 24, 32, 33, 36] using a variety of techniques, including two scale convergence, homogenized models for Maxwell's equations in composite materials having anisotropy and memory effects are constructed. In [23], a singular limit approximation of the constitutive laws for chiral media in the time domain are studied, while in [26], an asymptotic homogenization approach for 3D periodic lattices of complex media inclusions with bianisotropic properties is constructed. Additional constructions can be found in [27, 28]. In [9, 37, 38] multiscale numerical methods based on finite elements and finite differences are constructed for the time-dependent Maxwell's equations with memory effects in composite materials (linear dispersive dielectrics). Numerical homogenization using the heterogeneous multiscale method, and based on two scale convergence for Maxwell's equations is presented in [10].

In this paper, we use the periodic unfolding method, introduced in [12] in the abstract framework of stationary elliptic equations, to homogenize the time-dependent Maxwell's equations in complex materials that are described by constitutive laws involving the time evolution of the electric polarization and magnetization. We continue our efforts from [3, 8] in which we considered composite materials described by convolutions in time of a susceptibility kernel with the electric and/or magnetic field. The periodicity of the composite material results in spatially periodic parameters in the convolution description of the polarization and magnetization, as well as in the electric permittivity, magnetic permeability, and conductivity of the material. In this paper, we use a different but equivalent approach called the *auxiliary differential equation* (ADE) approach, in which the time evolution of the polarization is modeled by a system of first order ordinary differential equations (ODEs) forced by the electric field. Using an analogous approach for the magnetization, we also include systems of ODEs for the time evolution of the magnetization dependent on the magnetic field. Appending these ODEs to the Maxwell partial differential equations (PDEs) gives a hybrid PDE-ODE model for the composite material with spatially varying (periodic) parameters and fields. The homogenization of this hybrid PDE-ODE model provides an alternative homogenized model to the convolution approach in [3, 8] for computing the effective response of the complex composites considered here.

The outline of the rest of the paper is as follows. In Section 2, we describe Maxwell's equations, and present constitutive laws in ODE form for the polarization and magnetization in Section 3. In Section 4, we present a priori estimates and in Section 5, using the periodic unfolding method and the theory developed in [8], we develop the homogenized limit model for Maxwell's equations considered in this paper. In Section 6, we consider some specific models for linear dispersive and metamaterials, and in Section 7, we use the general theory developed in Section 5, to develop the limit homogenized model for a composite material described by the Debye model for orientational polarization with spatially periodic parameters.

2. Maxwell's equations in a complex dielectric. We start with Maxwell's equations for a linear and isotropic dielectric that includes terms for the electric polarization and magnetization. Consider a time $T > 0$ and let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary $\partial\Omega$. Maxwell's curl equations are given as:

$$\frac{\partial \mathbb{D}}{\partial t} = \nabla\times\mathbb{H} - \mathbb{J}_E \text{ in } (0,T)\times\Omega, \tag{1}$$

$$\frac{\partial \mathbb{B}}{\partial t} = -\nabla\times\mathbb{E} - \mathbb{J}_H \text{ in } (0,T)\times\Omega, \tag{2}$$

along with zero Gauss divergence laws

$$\nabla\cdot\mathbb{D} = 0,\ \nabla\cdot\mathbb{B} = 0 \text{ in } (0,T)\times\Omega, \tag{3}$$

in a region Ω with no free charges. The vector valued functions $\mathbb{E}$ and $\mathbb{H}$ represent the strengths of the electric and magnetic fields, respectively, while $\mathbb{D}$ and $\mathbb{B}$ are the electric and magnetic flux densities, respectively. The external electric and magnetic source current densities are given by $\mathbb{J}_E$, and $\mathbb{J}_H$, respectively.

We assume perfect conducting boundary conditions on the boundary $\partial\Omega$ given by

$$\mathbf{n}\times\mathbb{E} = 0 \text{ on } [0,T]\times\partial\Omega, \tag{4}$$

where $\mathbf{n}$ is the unit normal vector to $\partial\Omega$. We have the initial conditions

$$\mathbb{E}(0,\mathbf{x}) = \mathbb{E}_0,\ \mathbb{H}(0,\mathbf{x}) = \mathbb{H}_0 \text{ in } \Omega. \tag{5}$$

The fields $\mathbb{E}_0, \mathbb{H}_0$ are the initial electric and magnetic fields. We assume that these initial fields satisfy the Gauss divergence laws.

System (1) is completed by constitutive laws that embody the behavior of the material in response to the electromagnetic fields. These are given in the form

$$\mathbb{D}(t,\mathbf{x}) = \epsilon_0\epsilon_r(\mathbf{x})\mathbb{E}(t,\mathbf{x}) + \mathbb{P}_R(t,\mathbf{x}) \text{ in } (0,T)\times\Omega, \tag{6}$$

$$\mathbb{B}(t,\mathbf{x}) = \mu_0\mu_r(\mathbf{x})\mathbb{H}(t,\mathbf{x})\ + \mathbb{M}_R(t,\mathbf{x}),\ \text{ in } (0,T)\times\Omega. \tag{7}$$

To describe the behavior of the media's macroscopic retarded (or delayed) electric polarization $\mathbb{P}_R$, and magnetization $\mathbb{M}_R$, we employ a general integral equation model in which the polarization, and magnetization, explicitly depend on the past history of the electric, and magnetic fields, respectively. We assume that there are no free electric charges unaccounted for in the electric polarization $\mathbb{P}$. The model for polarization is sufficiently general to include microscopic polarization mechanisms such as dipole or orientational polarization [19] as well as ionic and electronic polarization [18] and other frequency dependent polarization mechanisms leading to linear models [4]. In addition, with nonzero magnetization we can also include the case of the Drude and Lorentz metamaterial models [22]. The resulting constitutive laws for the polarization and magnetization can be given in terms of an electric susceptibility kernel $\nu^{\mathbf{E}}$, and magnetic susceptibility kernel $\nu^{\mathbf{H}}$ in the form

$$\mathbb{P}_R(t, \mathbf{x}) = \int_0^t \nu^E(t-s, \mathbf{x})\mathbb{E}(s, \mathbf{x}) \text{ ds}, \tag{8}$$

$$\mathbb{M}_R(t, \mathbf{x}) = \int_0^t \nu^H(t-s, \mathbf{x})\mathbb{H}(s, \mathbf{x}) \text{ ds}. \tag{9}$$

3. The auxiliary differential equation (ADE) technique: ODE models for polarization and magnetization. We consider composite materials (see Figure 1) which have a periodic micro-structure, α in which the inclusions inside the host matrix, and the host matrix are both described as materials with responses governed by (8) and (9). Thus, the polarization, and magnetization vectors $\mathbb{P}_R$, and $\mathbb{M}_R$ are modeled by different susceptibility kernels, one inside the inclusion and another within the host material.

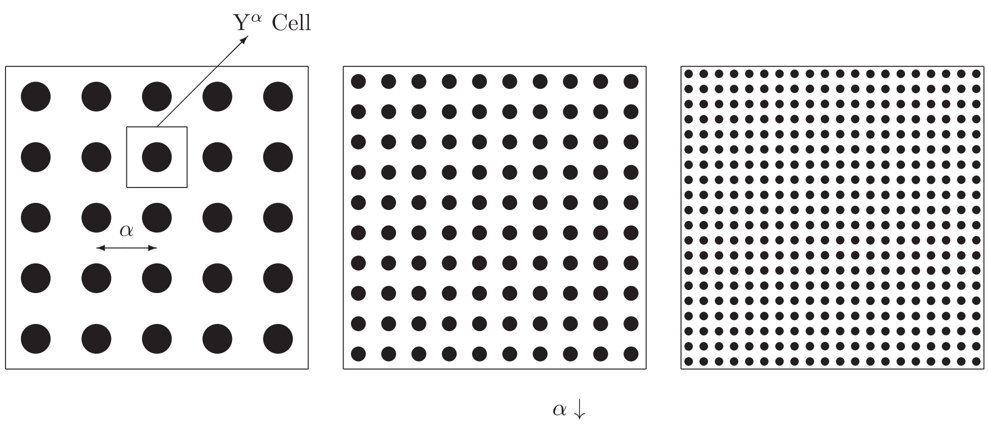

Fig. 1. A two dimensional periodic composite material presenting a circular micro-structure with periodicity α. The figure shows α decreasing from left to right. The picture also depicts the cell Y^α.

We will consider the case where the polarization, and magnetization vectors, $\mathbb{P}_R$ and $\mathbb{M}_R$ modeled by convolutions of susceptibility kernels with the electric field $\mathbb{E}$, and magnetic field $\mathbb{H}$, respectively, can be equivalently described by laws in which the dynamic evolution of $\mathbb{P}_R$ and $\mathbb{M}_R$ is given in the time domain, by nth order ordinary differential equations (ODEs) in the form

$$\sum_{j=0}^{N_E} a_j^E(\mathbf{x})\frac{\partial^j \mathbb{P}_R}{\partial t^j}(t, \mathbf{x}) + \beta_0^E(\mathbf{x})\mathbb{E}(t, \mathbf{x}) = 0, \tag{10}$$

$$\sum_{j=0}^{N_H} a_j^H(\mathbf{x})\frac{\partial^j \mathbb{M}_R}{\partial t^j}(t, \mathbf{x}) + \beta_0^H(\mathbf{x})\mathbb{H}(t, \mathbf{x}) = 0, \tag{11}$$

on $(0,T) \times \Omega$ with $a^E_{N_E}(\mathbf{x})$ and $a^H_{N_H}(\mathbf{x})$ strictly positive functions of $\mathbf{x}$ on Ω. Thus, in the most general case both the host matrix and the inclusions are modeled by different constitutive laws given in the form of equations (6), (10), and (11). We refer the reader to the books [4, 22] for a discussion on the class of polarization and magnetization laws given by susceptibility kernals that can be equivalently defined using systems of ODEs.

In this paper, we will assume that the host matrix and inclusions are modeled by the same system of ODEs, with the material coefficients $a^E_i, i = 0, \dots, N_E$, and $a^H_i, i = 0, \dots, N_H$, given as functions of the spatial variable $\mathbf{x}$ to accommodate treatment of composite materials and structures. The ODE models for $\mathbb{P}_R$ and $\mathbb{M}_R$ can be written as a system of first order ordinary differential equations. If $N_E = N_H = 1$, then this will already be the case. Thus, if $N_E \geq 2$ and $N_H \geq 2$, we can rewrite the models as systems of ODEs. To do this, we define $\mathbb{P}^{(0)} = \mathbb{P}_R$ and $\mathbb{M}^{(0)} = \mathbb{M}_R$, and define the time derivatives

$$\mathbb{P}^{(\ell+1)} = \frac{\partial^\ell \mathbb{P}_R}{\partial t^\ell}, \ \mathbb{M}^{(j+1)} = \frac{\partial^j \mathbb{M}_R}{\partial t^j}, \tag{12}$$

for $\ell = 0, 1, \dots, N_E - 2$ and $j = 0, 1, \dots, N_H - 2$. Next, for $(t, \mathbf{x}) \in [0,T] \times \Omega$, we define the vector functions

$$\mathbb{P}(t,\mathbf{x}) = \left(\mathbb{P}^{(0)}(t,\mathbf{x}), \mathbb{P}^{(1)}(t,\mathbf{x}), \dots, \mathbb{P}^{(N_E-1)}(t,\mathbf{x})\right)^T, \tag{13}$$

$$\mathbb{M}(t,\mathbf{x}) = \left(\mathbb{M}^{(0)}(t,\mathbf{x}), \mathbb{M}^{(1)}(t,\mathbf{x}), \dots, \mathbb{M}^{(N_H-1)}(t,\mathbf{x})\right)^T. \tag{14}$$

Using (13) and (14) we can rewrite (10) and (11) as

$$(S^E_1(\mathbf{x}) \otimes \mathbf{I}_3)\frac{\partial \mathbb{P}}{\partial t} + (S^E_2(\mathbf{x}) \otimes \mathbf{I}_3)\mathbb{P} + (S^E_3(\mathbf{x}) \otimes \mathbf{I}_3)\mathbb{E} = \mathbf{0}, \tag{15}$$

$$(S^H_1(\mathbf{x}) \otimes \mathbf{I}_3)\frac{\partial \mathbb{M}}{\partial t} + (S^H_2(\mathbf{x}) \otimes \mathbf{I}_3)\mathbb{M} + (S^H_3(\mathbf{x}) \otimes \mathbf{I}_3)\mathbb{H} = \mathbf{0}. \tag{16}$$

The form of the matrices in (15) and (16) depend on the values of N_E and N_H.

Case ($N_E = N_H = 1$). If the evolution of $\mathbb{P}$ and $\mathbb{M}$ are governed by first order ODEs, then, for $V \in \{E, H\}$, i.e., $V = E$ or $V = H$, the matrices in (15) and (16) are of order 1×1, i.e., scalar functions given as

$$S^V_1(\mathbf{x}) = a^V_1(\mathbf{x}), \tag{17a}$$

$$S^V_2(\mathbf{x}) = a^V_0(\mathbf{x}), \tag{17b}$$

$$S^V_3(\mathbf{x}) = \beta^V_0(\mathbf{x}). \tag{17c}$$

Case ($N_E \geq 2, N_H \geq 2$). Let $n, k \in \mathbb{N}$. We define $\mathbf{I}_n$ to be the $n \times n$ identity matrix, and $\mathbf{0}_{n\times k}$ to be the $n \times k$ matrix of zeros. We denote $\mathbf{0}_n = \mathbf{0}_{n\times n}$.

In this case, $S^V_1(\mathbf{x})$ is an $N_V \times N_V$ diagonal matrix, with

$$S^V_1(\mathbf{x}) = \begin{bmatrix} \mathbf{I}_{N_V-1} & 0 \\ \mathbf{0}_{N_V-1} & a^V_{N_V}(\mathbf{x}) \end{bmatrix}. \tag{18}$$

The matrix $S_2^V(\mathbf{x})$ is the $N_V \times N_V$ matrix with -1 on the super-diagonal, $(a_0(\mathbf{x}), a_1(\mathbf{x}), \ldots, a_{N_V-1}(\mathbf{x}), 0)$ in the N_Vth row and zeros elsewhere. The matrix $S_3^V(\mathbf{x})$ is an $N_V \times 1$ matrix, with zeros in all rows except the N_Vth element which is $\beta_0^V(\mathbf{x})$.

3.1. *Maxwell's equations with polarization and magnetization laws.* Define

$$n = 3(2 + N_E + N_H).$$

We define the vector function $\mathbf{u} : (0, T) \times \Omega \to \mathbb{R}^n$ given as

$$\mathbf{u}(t, \mathbf{x}) = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4)^T = (\mathbb{E}(t, \mathbf{x}), \mathbb{P}(t, \mathbf{x}), \mathbb{H}(t, \mathbf{x}), \mathbb{M}(t, \mathbf{x}))^T. \tag{19}$$

We can rewrite Maxwell's equations along with the constitutive laws (10) and (11) in the form

$$\begin{cases} \text{(i)} \quad \mathbf{A}(\mathbf{x})\dfrac{\partial \mathbf{u}}{\partial t}(t, \mathbf{x}) + \mathbf{B}(\mathbf{x})\mathbf{u}(t, \mathbf{x}) = \mathbf{F}\mathbf{u}(t, \mathbf{x}) - \mathbf{J}_s(t, \mathbf{x}) \;\text{ in } (0, T) \times \Omega, \\ \text{(ii)} \quad \mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0 \text{ in } \Omega, \\ \text{(iii)} \quad \mathbf{u}_1(t, \mathbf{x}) \times \mathbf{n}(\mathbf{x}) = \mathbf{0} \text{ on } (0, T) \times \partial\Omega. \end{cases} \tag{20}$$

In the above, $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ diagonal matrices with the following forms:

$$\mathbf{A}(\mathbf{x}) = \begin{bmatrix} \epsilon_0\epsilon_r(\mathbf{x}) & \mathbf{0}_{1\times N_E} & 0 & \mathbf{0}_{1\times N_H} \\ \mathbf{0}_{N_E\times 1} & S_1^{\mathbf{E}}(\mathbf{x}) & \mathbf{0}_{N_E\times 1} & \mathbf{0}_{N_E\times N_H} \\ 0 & \mathbf{0}_{1\times N_E} & \mu_0\mu_r(\mathbf{x}) & \mathbf{0}_{1\times N_H} \\ \mathbf{0}_{N_H\times 1} & \mathbf{0}_{N_H\times N_E} & \mathbf{0}_{N_H\times 1} & S_1^{\mathbf{H}}(\mathbf{x}) \end{bmatrix} \otimes \mathbf{I}_3, \tag{21}$$

while the matrix $\mathbf{B}$ is the $n \times n$ matrix given by

$$\mathbf{B}(\mathbf{x}) = \begin{bmatrix} S_5^E(\mathbf{x}) & S_4^E(\mathbf{x}) & 0 & \mathbf{0}_{1\times N_H} \\ S_3^E(\mathbf{x}) & S_2^E(\mathbf{x}) & \mathbf{0}_{N_E\times 1} & \mathbf{0}_{N_E\times N_H} \\ 0 & \mathbf{0}_{1\times N_E} & S_5^H(\mathbf{x}) & S_4^H(\mathbf{x}) \\ \mathbf{0}_{N_H\times 1} & \mathbf{0}_{N_H\times N_E} & S_3^H(\mathbf{x}) & S_2^H(\mathbf{x}) \end{bmatrix} \otimes \mathbf{I}_3. \tag{22}$$

CASE ($N_E = N_H = 1$). In this case $n = 12$. The matrices $S_j^V(\mathbf{x})$ for $j = 1, 2, 3$ and $V \in \{E, H\}$ are given in Section 3. The additional matrices $S_4^V(\mathbf{x})$ and $S_5^V(\mathbf{x})$ are scalars defined as;

$$S_5^E(\mathbf{x}) = -\frac{\beta_0^E(\mathbf{x})}{a_1^E(\mathbf{x})}, \; S_4^E(\mathbf{x}) = -\frac{a_0^E(\mathbf{x})}{a_1^E(\mathbf{x})}, \tag{23}$$

$$S_5^H(\mathbf{x}) = -\frac{\beta_0^H(\mathbf{x})}{a_1^H(\mathbf{x})}, \; S_4^H(\mathbf{x}) = -\frac{a_0^H(\mathbf{x})}{a_1^H(\mathbf{x})}. \tag{24}$$

CASE ($N_E \geq 2, N_H \geq 2$). In this case, matrices S_4^E and S_4^H are of order $1 \times N_E$ and $1 \times N_H$, respectively, each with a 1 in the second column and zeros elsewhere. Also, $S_5^E(\mathbf{x}) = 0$ and $S_5^H(\mathbf{x}) = 0$ are both scalar quantities.

Next, we define the formal (extended) Maxwell operator $\mathcal{M}$ by

$$\mathbf{F}\mathbf{u}(t, \mathbf{x}) = \left((\nabla\times\mathbb{H})(t, \mathbf{x}), \mathbf{0}_{3N_E\times 1}, -(\nabla\times\mathbb{E})(t, \mathbf{x}), \mathbf{0}_{3N_H\times 1}\right)^T, \tag{25}$$

and the vector $\mathbf{J}_s$ as

$$\mathbf{J}_s(t,\mathbf{x}) = (J_{s,1}, J_{s,2}, J_{s,3}, J_{s,4})^T = (\mathbb{J}_E(t,\mathbf{x}), \mathbf{0}_{3N_E\times 1}, \mathbb{J}_H(t,\mathbf{x}), \mathbf{0}_{3N_H\times 1})^T. \tag{26}$$

4. A priori estimates. We consider the space

$$\mathbf{V}(\Omega) = \mathbf{H}_0(\mathrm{curl},\Omega) \times (\mathbf{H}(\mathrm{curl},\Omega))^{N_E} \times \mathbf{H}(\mathrm{curl},\Omega) \times (\mathbf{H}(\mathrm{curl},\Omega))^{N_H}, \tag{27}$$

where we define

$$\mathbf{H}(\mathrm{curl},\Omega) = \{\mathbf{v} \in L^2(\Omega;\mathbb{R}^3); \mathrm{curl}\ \mathbf{v} \in L^2(\Omega;\mathbb{R}^3)\},$$

equipped with the norm $||v||^2 = |v|^2 + |\mathrm{curl}\ v|^2$, and

$$\mathbf{H}_0(\mathrm{curl},\Omega) = \{\mathbf{v} \in \mathbf{H}(\mathrm{curl},\Omega);\ \mathbf{n}\times\mathbf{v} = 0 \text{ in } H^{-\frac{1}{2}}(\partial\Omega;\mathbb{R}^3)\}.$$

Let $n = 3(2 + N_E + N_H)$.

LEMMA 1. Consider the Maxwell system (20) along with (21) and (22) in which the matrices $\mathbf{A}, \mathbf{B} \in L^\infty(\Omega;\mathbb{R}^{n^2})$, with the matrix $\mathbf{A}$ symmetric and uniformly coercive. Assuming that the initial condition $\mathbf{u}_0 \in V(\Omega)$ and the source term $\mathbf{J}_s \in W^{1,1}(0,T;L^2(\Omega;\mathbb{R}^n))$, system (20) has a unique solution $\mathbf{u} = (\mathbf{E},\mathbf{P},\mathbf{H},\mathbf{M})^T$ with the property

$$\mathbf{E} \in \mathcal{C}^1([0,T];L^2(\Omega,\mathbb{R}^3)) \cap \mathcal{C}^0([0,T];\mathbf{H}_0(\mathrm{curl},\Omega)), \tag{28}$$

$$\mathbf{P} \in \mathcal{C}^1([0,T];L^2(\Omega,\mathbb{R}^{3N_E})) \cap \mathcal{C}^0([0,T];(\mathbf{H}_0(\mathrm{curl},\Omega))^{N_E}), \tag{29}$$

$$\mathbf{H} \in \mathcal{C}^1([0,T];L^2(\Omega,\mathbb{R}^3)) \cap \mathcal{C}^0([0,T];\mathbf{H}(\mathrm{curl},\Omega)), \tag{30}$$

$$\mathbf{M} \in \mathcal{C}^1([0,T];L^2(\Omega,\mathbb{R}^{3N_H}) \cap \mathcal{C}^0([0,T];(\mathbf{H}(\mathrm{curl},\Omega))^{N_H}). \tag{31}$$

We also have the a priori estimate

$$||\mathbf{u}||_{L^\infty(0,T;V(\Omega))} + ||\frac{d\mathbf{u}}{dt}||_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^n))} \leq C\mathrm{e}^{C_1 t}(||\mathbf{J}_s||_{W^{1,1}(0,T;L^2(\Omega;\mathbb{R}^n))} + ||\mathbf{u}_0||_{V(\Omega)}), \tag{32}$$

where the constants C, C_1 are strictly positive and depend only on the data $\mathbf{A}, \mathbf{B}$.

Proof. The proof of this result can be constructed by using the Faedo-Galerkin method. There are three steps involved in the proof: 1) Showing the existence of an approximate solution $\mathbf{u}_m$ of $\mathbf{u}$, 2) Estimates on $\mathbf{u}_m$, and 3) existence of the solution $\mathbf{u}$ by proving convergence of $\mathbf{u}_m$ to $\mathbf{u}$ and of $\frac{d\mathbf{u}_m}{dt}$ in $L^\infty(0,T;L^2(\Omega;\mathbb{R}^n))$, which can be done based on results in [8]. The a priori estimates are based on a *conservation law* that is satisfied in these linear media. This law is given as

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \mathbf{A}(\mathbf{y})\mathbf{u}(t,\mathbf{x})\cdot\mathbf{u}(t,\mathbf{x})\mathrm{dx} + \int_0^t\int_\Omega \mathbf{B}(\mathbf{y})\mathbf{u}(s,\mathbf{x})\cdot\mathbf{u}(s,\mathbf{x})\ \mathrm{dx\ ds} \\
&\quad + \int_0^t\int_\Omega \mathbf{J}_s(s,\mathbf{x})\cdot\mathbf{u}(s,\mathbf{x})\ \mathrm{dx\ ds} \\
&\quad = \frac{1}{2}\int_\Omega \mathbf{A}(\mathbf{y})\mathbf{u}_0(\mathbf{x})\cdot\mathbf{u}_0(\mathbf{x})\ \mathrm{dx}.
\end{aligned}$$

□

5. Homogenization. We assume that the material that occupies the domain Ω contains periodic micro-structures characterized by matrices $\mathbf{A}$, and $\mathbf{B}$ with periodically oscillating spatial coefficients. We assume that the periodic structure of the material is characterized by an *elementary micro-structure* with size $\alpha > 0$ as seen in Figure 1. (The small parameter, generally denoted by ϵ in the literature, is denoted here by α to avoid any confusion with permittivity, ϵ.) The parameter dependent constitutive matrices $(\mathbf{A}^\alpha, \mathbf{B}^\alpha)$ and the data $(\mathbf{u}_0^\alpha, \mathbf{J}_s^\alpha)$ that depend on α, are assumed to have the regularity needed by Lemma 1. Given $\alpha > 0$, we obtain a family of electromagnetic fields $\mathbf{u}^\alpha$ (indexed by α) which are solutions to the evolution problems

$$\mathbf{A}^\alpha(\mathbf{x})\frac{\partial}{\partial t}\mathbf{u}^\alpha(t,\mathbf{x}) + \mathbf{B}^\alpha(\mathbf{x})\mathbf{u}^\alpha(t,\mathbf{x}) = \mathbf{F}\mathbf{u}^\alpha(t,\mathbf{x}) - \mathbf{J}_s^\alpha(t,\mathbf{x}) \text{ in } (0,T)\times\Omega, \tag{33a}$$

$$\mathbf{u}^\alpha(0,\mathbf{x}) = \mathbf{u}_0^\alpha(\mathbf{x}) \text{ in } \Omega, \tag{33b}$$

$$\mathbf{n}\times\mathbf{u}_1^\alpha(t,\mathbf{x}) = 0 \text{ on } (0,T)\times\partial\Omega. \tag{33c}$$

Our aim is to obtain the asymptotic behavior of the solution $\mathbf{u}^\alpha$ when the periodicity α goes to zero. This requires understanding the asymptotic behavior of the initial data and of the source which depend on α. We make the following assumptions of strong convergence on the data. Assume:

$$\begin{cases} \mathbf{u}_0^\alpha & \longrightarrow \mathbf{u}_0^L \quad \text{in } V(\Omega), \\ \mathbf{J}_s^\alpha & \longrightarrow \mathcal{J}_s^L \quad \text{in } W^{1,1}(0,T;L^2(\Omega;\mathbb{R}^n)). \end{cases} \tag{34}$$

Thus, we aim to obtain a homogenized version of problem (20) in which an ordinary differential equation (ODE) or systems of ODEs describe the hysteretic part of the polarization and magnetization terms instead of homogenized electric and magnetic susceptibility kernels as in [8]. As discussed in Section 3, these two approaches are equivalent in the continuous setting. However, in the discrete setting, the numerical discretizations and computational simulations of the limit model are different in the two approaches. The ODE approach involves the construction of volume discretizations like the finite difference and finite element approaches, while the approach with susceptibility kernels requires the discretizations of convolutions.

5.1. *Periodic geometry.* We assume here that the micro-structure is of cubic form. Denote by $Y = (0,1)^3$ the reference cell. For a.e. $\mathbf{z} \in \mathbb{R}^3$ let $[\mathbf{z}]$ be the unique element belonging to $\mathbb{Z}^3$ such that $\mathbf{z} - [\mathbf{z}] \in Y$, so that we may write $\mathbf{z} = [\mathbf{z}] + \{\mathbf{z}\}$ for a.e. $\mathbf{z} \in \mathbb{R}^3$. Consequently, for all $\alpha > 0$, we get the unique decomposition

$$\mathbf{x} = \alpha([\frac{\mathbf{x}}{\alpha}] + \{\frac{\mathbf{x}}{\alpha}\}) \quad \text{for a.e. } \mathbf{x} \in \mathbb{R}^3. \tag{35}$$

To be consistent with this geometry, the constitutive parameter matrices $(\mathbf{A}^\alpha, \mathbf{B}^\alpha)$ are assumed to be periodic with period α; more precisely, we assume that according to the previous decomposition there exists two matrices $(\mathbf{A}, \mathbf{B})$ such that

$$\mathbf{A}^\alpha(\mathbf{x}) = \mathbf{A}(\{\frac{\mathbf{x}}{\alpha}\}), \quad \mathbf{B}^\alpha(\mathbf{x}) = \mathbf{B}(\{\frac{\mathbf{x}}{\alpha}\}) \qquad \text{for a.e. } \mathbf{x} \in \mathbb{R}^3. \tag{36}$$

5.2. *Periodic unfolding operator.* We study the limit, when α goes to 0, of the family $\mathbf{u}^\alpha$, by using *the periodic unfolding method* [11]. Set

$$\Xi_\alpha = \big\{\xi \in \mathbb{Z}^3 \mid \alpha(\xi + Y) \subset \Omega\big\}, \qquad \widehat{\Omega}_\alpha = \text{interior}\Big(\bigcup_{\xi\in\Xi_\alpha} \alpha(\xi + \overline{Y})\Big).$$

The periodic unfolding operator $\mathcal{T}_\alpha : v \in L^2(\Omega;\mathbb{R}^m) \longrightarrow L^2(\Omega \times Y;\mathbb{R}^m)$ is defined by

$$\mathcal{T}_\alpha(v)(\mathbf{x},\mathbf{y}) = \begin{cases} v(\alpha[\dfrac{\mathbf{x}}{\alpha}] + \alpha\mathbf{y}) & \text{for a.e. } (\mathbf{x},\mathbf{y}) \in \widehat{\Omega}_\alpha \times Y, \\ 0 & \text{otherwise.} \end{cases}$$

Hence the periodicity of the constitutive parameters yields

$$\mathcal{T}_\alpha(\mathbf{A}^\alpha)(\mathbf{x},\mathbf{y}) = \mathbf{A}(\mathbf{y}), \qquad \mathcal{T}_\alpha(\mathbf{B}^\alpha)(\mathbf{x},\mathbf{y}) = \mathbf{B}(\mathbf{y}) \qquad \text{a.e. in } \widehat{\Omega}_\alpha \times Y. \tag{37}$$

For our purpose all functions defined in $L^2(\Omega)$ are extended by 0 outside Ω and we denote by $H^1_{\text{per}}(Y)$ the space of periodic functions with vanishing mean value. We refer the reader to [8] for discussion of properties of the operator $\mathcal{T}_\alpha$, which are essential for obtaining the limiting model.

Notation. Recall $n = 3(2+N_E+N_H)$. Let $N = n/3$. We extend to $\mathbb{R}^N$ some notation defined in $\mathbb{R}^3$. Let $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)^T$, where $\mathbf{v}_1$ is synonymous with an electric field, $\mathbb{E}$, $\mathbf{v}_2$ with polarization $\mathbb{P}$, $\mathbf{v}_3$ with $\mathbb{H}$ and $\mathbf{v}_4$ with $\mathbb{M}$. To be precise, for $\ell = 1, 3$, $\mathbf{v}_\ell \in \mathbb{R}^3$, $\mathbf{v}_2 = (\mathbf{v}_{2,\ell})_{\ell=1}^{N_E} \in \mathbb{R}^{3N_E}, \mathbf{v}_{2,\ell} \in \mathbb{R}^3$, $\mathbf{v}_4 = (\mathbf{v}_{4,\ell})_{\ell=1}^{N_H} \in \mathbb{R}^{3N_H}$, with $\mathbf{v}_{4,\ell} \in \mathbb{R}^3$. Next, define $\mathbf{w} = (w_1, \mathbf{w}_2, w_3, \mathbf{w}_4)^T$ with $w_1, w_3 \in \mathbb{R}, \mathbf{w}_2 = (w_{2,\ell})_{\ell=1}^{N_E} \in \mathbb{R}^{N_E}$, with $w_{2,\ell} \in \mathbb{R}$ for $\ell = 1, \ldots, N_E$, and $\mathbf{w}_4 = (w_{4,\ell})_{\ell=1}^{N_H} \in \mathbb{R}^{N_H}$, with $w_{4,\ell} \in \mathbb{R}$ for $\ell = 1, \ldots, N_H$. Then we define

$$\text{curl } \mathbf{v} := (\text{curl } \mathbf{v}_\ell)_{\ell=1}^4, \tag{38}$$

$$\text{div } \mathbf{v} := (\text{div } \mathbf{v}_\ell)_{\ell=1}^4, \tag{39}$$

$$\mathbf{n} \times \mathbf{v} := (\mathbf{n} \times \mathbf{v}_\ell)_{\ell=1}^4, \tag{40}$$

$$\nabla\mathbf{w} := (\nabla w_1, \nabla\mathbf{w}_2, \nabla w_3, \nabla\mathbf{w}_4,)^T, \tag{41}$$

where for $\mathbf{v}_2$ ($\mathbb{P}$) and $\mathbf{v}_4$ ($\mathbb{H}$) the curl, divergence, cross product and gradient are defined component wise. For example, curl $\mathbf{v}_2 = (\text{curl } \mathbf{v}_{2,\ell})_{\ell=1}^{N_E}$. The other quantities are defined in a similar fashion.

For $\alpha > 0$, let $\mathbf{u}^\alpha$ be the solution to (33); as established in Lemma 1. Then $\mathbf{u}^\alpha$ satisfies the uniform bound

$$||\mathbf{u}^\alpha||_{L^\infty(0,T;V(\Omega))} + ||\frac{d\mathbf{u}^\alpha}{dt}||_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^n))} \le C\mathrm{e}^{C_1 t}(||J_s^\alpha||_{W^{1,1}(0,T;L^2(\Omega;\mathbb{R}^n))} + ||\mathbf{u}_0^\alpha||_{V(\Omega)}) \tag{42}$$

from which we will prove the convergence of the family $\mathbf{u}^\alpha$ and identify its limit.

Theorem 5.1. Let $\mathbf{A}^\alpha \in L^\infty(\Omega;\mathbb{R}^{n^2})$ and $\mathbf{B}^\alpha \in L^\infty(\Omega;\mathbb{R}^{n^2})$ be two matrix sequences given by (21) and (22), respectively, with $\mathbf{A}^\alpha$ symmetric and uniformly coercive. Assume that the initial condition $\mathbf{u}_0^\alpha \in \mathbf{V}(\Omega)$ and the source $\mathbf{J}_s^\alpha \in W^{1,1}(0,T;L^\infty(\Omega;\mathbb{R}^n))$. Let

$\mathbf{u}^\alpha$ be the solution to the Maxwell problem (33). Then, there exist three fields

$$\mathbf{u}^L \in \; W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^n)) \cap L^\infty(0,T;\mathbf{V}(\Omega)), \tag{43}$$

$$\bar{\mathbf{u}}^L \in \; W^{1,\infty}(0,T;L^2(\Omega;H^1_{\text{per}}(Y;\mathbb{R}^N))), \tag{44}$$

$$\bar{\bar{\mathbf{u}}}^L \in \; L^\infty(0,T;L^2(\Omega;H^1_{\text{per}}(Y;\mathbb{R}^n))), \operatorname{div}_y(\bar{\bar{\mathbf{u}}}^L) = 0, \tag{45}$$

which are limits of the following sequences:

$$\mathbf{u}^\alpha \rightharpoonup \mathbf{u}^L \text{ weakly} * \quad \text{in} \quad L^\infty(0,T;\mathbf{V}(\Omega)), \tag{46}$$

$$\mathcal{T}_\alpha(\mathbf{u}^\alpha) \longrightarrow \mathbf{u}^L + \nabla_y \bar{\mathbf{u}}^L \text{ strongly} \;\text{ in } H^1(0,T;L^2(\Omega\times Y;\mathbb{R}^n)), \tag{47}$$

$$\mathcal{T}_\alpha(\operatorname{curl}_x \mathbf{u}^\alpha_j) \longrightarrow \operatorname{curl}_x \mathbf{u}^L_j + \operatorname{curl}_y \bar{\bar{\mathbf{u}}}^L_j, j=1,3 \;\text{ strongly in } L^2((0,T)\times\Omega\times Y;\mathbb{R}^n), \tag{48}$$

which solve the evolution problem:

$$\begin{aligned} &\mathbf{A}(\mathbf{y})\frac{\partial}{\partial t}(\mathbf{u}^L(t,\mathbf{x}) + \nabla_{\mathbf{y}}\bar{\mathbf{u}}^L(t,\mathbf{x},\mathbf{y})) + \mathbf{B}(\mathbf{y})(\mathbf{u}^L(t,\mathbf{x}) + \nabla_{\mathbf{y}}\bar{\mathbf{u}}^L(t,\mathbf{x},\mathbf{y})) \\ &\qquad = \mathbf{F}_{\mathbf{x}}\mathbf{u}^L(t,\mathbf{x}) + \mathbf{F}_{\mathbf{y}}\bar{\bar{\mathbf{u}}}^L(t,\mathbf{x},\mathbf{y}) - \mathcal{J}^L_s(t,\mathbf{x}) \quad \text{in } (0,T)\times\Omega\times Y, \end{aligned} \tag{49a}$$

$$\mathbf{u}^L(0) = \mathbf{u}^L_0 \quad \text{in } \Omega\times Y, \;\; \bar{\mathbf{u}}^L(0) = 0, \tag{49b}$$

$$\mathbf{n}\times\mathbf{u}^L_1 = 0 \quad \text{on } (0,T)\times\partial\Omega. \tag{49c}$$

Proof. The proof involves three steps: 1) establishing weak convergence of the family $\mathbf{u}^\alpha$, $\mathcal{T}_\alpha(\mathbf{u}^\alpha)$, $\mathcal{T}_\alpha(\operatorname{curl}_x \mathbf{u}^\alpha_j)$, 2) establish the limit of the evolution problem, and 3) establish strong convergence. These steps can be developed using ideas and tools presented in [8]. Problem (49) has a unique solution $\mathbf{u}^L, \bar{\mathbf{u}}^L, \bar{\bar{\mathbf{u}}}^L$. The field $\mathbf{u}^L + \nabla_y\bar{\mathbf{u}}^L$ and its derivative with respect to time satisfy a conservation law analogous to (2):

$$\begin{aligned} &\frac{1}{2}\int_{\Omega\times Y} \mathbf{A}(\mathbf{y})(\mathbf{u}^L(t,\mathbf{x}) + \nabla_{\mathbf{y}}\bar{\mathbf{u}}^L(t,\mathbf{x},\mathbf{y}))\cdot(\mathbf{u}^L(t,\mathbf{x}) + \nabla_{\mathbf{y}}\bar{\mathbf{u}}^L(t,\mathbf{x},\mathbf{y})) \\ &\quad + \int_0^t\int_{\Omega\times Y} \mathbf{B}(\mathbf{y})(\mathbf{u}^L(s,\mathbf{x}) + \nabla_{\mathbf{y}}\bar{\mathbf{u}}^L(s,\mathbf{x},\mathbf{y}))\cdot(\mathbf{u}^L(s,\mathbf{x}) + \nabla_{\mathbf{y}}\bar{\mathbf{u}}^L(s,\mathbf{x},\mathbf{y}))\;\mathrm{ds} \\ &\quad + \int_0^t\int_\Omega \mathbf{J}_s\cdot\mathbf{u}^L(s)\;\mathrm{ds} = \frac{1}{2}\int_\Omega \mathbf{A}(0,\mathbf{y})\mathbf{u}^L_0(\mathbf{x})\cdot\mathbf{u}^L_0(\mathbf{x}). \end{aligned} \tag{50}$$

□

We can rewrite Theorem 5.1 in vector form as follows. Assuming the conditions of Theorem 5.1 hold, then, there exist three sets of fields

$$\mathbf{E}^L \in \; W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^3)) \cap L^\infty(0,T;\mathbf{H}_0(\text{curl},\Omega)), \tag{51}$$

$$\mathbf{P}^L \in \; W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^{3N_E})) \cap L^\infty(0,T;(\mathbf{H}_0(\text{curl}))^{N_E}), \tag{52}$$

$$\mathbf{H}^L \in \; W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^3)) \cap L^\infty(0,T;\mathbf{H}(\text{curl},\Omega)), \tag{53}$$

$$\mathbf{M}^L \in \; W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^{3N_H})) \cap L^\infty(0,T;(\mathbf{H}(\text{curl},\Omega))^{N_H}), \tag{54}$$

the set

$$\bar{\mathbf{E}}^L, \bar{\mathbf{H}}^L \in \ W^{1,\infty}(0,T;L^2(\Omega;H^1_{\mathrm{per}}(Y;\mathbb{R}^3))), \tag{55}$$

$$\bar{\mathbf{P}}^L \in \ W^{1,\infty}(0,T;L^2(\Omega;H^1_{\mathrm{per}}(Y;\mathbb{R}^{3N_E}))), \tag{56}$$

$$\bar{\mathbf{M}}^L \in \ W^{1,\infty}(0,T;L^2(\Omega;H^1_{\mathrm{per}}(Y;\mathbb{R}^{3N_H}))), \tag{57}$$

and finally

$$\overline{\overline{\mathbf{E}}}^L, \overline{\overline{\mathbf{H}}}^L \in \ L^\infty(0,T;L^2(\Omega;H^1_{\mathrm{per}}(Y;\mathbb{R}^3))), \mathrm{div}_y(\overline{\overline{\mathbf{E}}}^L) = 0, \mathrm{div}_y(\overline{\overline{\mathbf{H}}}^L) = 0, \tag{58}$$

$$\overline{\overline{\mathbf{P}}}^L \in \ L^\infty(0,T;L^2(\Omega;H^1_{\mathrm{per}}(Y;\mathbb{R}^{3N_E}))), \mathrm{div}_y(\overline{\overline{\mathbf{P}}}^L) = 0, \tag{59}$$

$$\overline{\overline{\mathbf{M}}}^L \in \ L^\infty(0,T;L^2(\Omega;H^1_{\mathrm{per}}(Y;\mathbb{R}^{3N_H}))), \mathrm{div}_y(\overline{\overline{\mathbf{M}}}^L) = 0, \tag{60}$$

which are limits of the following sequences:

$$\mathbf{E}^\alpha \rightharpoonup \mathbf{E}^L \text{ weakly} * \quad \text{in} \quad L^\infty(0,T;\mathbf{H}_0(\mathrm{curl},\Omega))), \tag{61}$$

$$\mathbf{P}^\alpha \rightharpoonup \mathbf{P}^L \text{ weakly} * \quad \text{in} \quad L^\infty(0,T;(\mathbf{H}_0(\mathrm{curl},\Omega))^{N_E}), \tag{62}$$

$$\mathbf{H}^\alpha \rightharpoonup \mathbf{H}^L \text{ weakly} * \quad \text{in} \quad L^\infty(0,T;\mathbf{H}(\mathrm{curl},\Omega)), \tag{63}$$

$$\mathbf{M}^\alpha \rightharpoonup \mathbf{M}^L \text{ weakly} * \quad \text{in} \quad L^\infty(0,T;(\mathbf{H}(\mathrm{curl},\Omega))^{N_H}), \tag{64}$$

and

$$\mathcal{T}_\alpha(\mathbf{E}^\alpha) \longrightarrow \mathbf{E}^L + \nabla_y \bar{\mathbf{E}}^L \text{ strongly in } H^1(0,T;L^2(\Omega\times Y;\mathbb{R}^3)), \tag{65}$$

$$\mathcal{T}_\alpha(\mathbf{P}^\alpha) \longrightarrow \mathbf{P}^L + \nabla_y \bar{\mathbf{P}}^L \text{ strongly in } H^1(0,T;L^2(\Omega\times Y;\mathbb{R}^{3N_E})), \tag{66}$$

$$\mathcal{T}_\alpha(\mathbf{H}^\alpha) \longrightarrow \mathbf{H}^L + \nabla_y \bar{\mathbf{H}}^L \text{ strongly in } H^1(0,T;L^2(\Omega\times Y;\mathbb{R}^3)), \tag{67}$$

$$\mathcal{T}_\alpha(\mathbf{M}^\alpha) \longrightarrow \mathbf{M}^L + \nabla_y \bar{\mathbf{M}}^L \text{ strongly in } H^1(0,T;L^2(\Omega\times Y;\mathbb{R}^{3N_H})), \tag{68}$$

with

$$\mathcal{T}_\alpha(\mathrm{curl}_x\mathbf{E}^\alpha) \longrightarrow \mathrm{curl}_x\mathbf{E}^L + \mathrm{curl}_y\overline{\overline{\mathbf{E}}}^L \text{ strongly in } L^2((0,T)\times\Omega\times Y;\mathbb{R}^3), \tag{69}$$

$$\mathcal{T}_\alpha(\mathrm{curl}_x\mathbf{H}^\alpha) \longrightarrow \mathrm{curl}_x\mathbf{H}^L + \mathrm{curl}_y\overline{\overline{\mathbf{H}}}^L \text{ strongly in } L^2((0,T)\times\Omega\times Y;\mathbb{R}^3), \tag{70}$$

which solve the following evolution equations in $(0,T)\times\Omega\times Y$:

$$\begin{aligned}&\epsilon_0\epsilon_\infty(\mathbf{y})\frac{\partial}{\partial t}\left(\mathbf{E}^L(t,\mathbf{x})+\nabla_\mathbf{y}\bar{\mathbf{E}}^L(t,\mathbf{x},\mathbf{y})\right)+S_5^E(\mathbf{y})(\mathbf{E}^L(t,\mathbf{x})+\nabla_\mathbf{y}\bar{\mathbf{E}}^L(t,\mathbf{x},\mathbf{y}))\\&\quad+S_4^E(\mathbf{y})(\mathbf{P}^L(t,\mathbf{x})+\nabla_\mathbf{y}\bar{\mathbf{P}}^L(t,\mathbf{x},\mathbf{y}))=\mathrm{curl}_\mathbf{x}\mathbf{H}^L(t,\mathbf{x})+\mathrm{curl}_\mathbf{y}\bar{\bar{\mathbf{H}}}^L(t,\mathbf{x},\mathbf{y})-\mathcal{J}_E^L(t,\mathbf{x}),\end{aligned}\tag{71a}$$

$$\begin{aligned}&S_1^E(\mathbf{y})\frac{\partial}{\partial t}(\mathbf{P}^L(t,\mathbf{x})+\nabla_\mathbf{y}\bar{\mathbf{P}}^L(t,\mathbf{x},\mathbf{y}))+S_3^E(\mathbf{y})(\mathbf{E}^L(t,\mathbf{x})+\nabla_\mathbf{y}\bar{\mathbf{E}}^L(t,\mathbf{x},\mathbf{y}))\\&\quad+S_2^E(\mathbf{y})(\mathbf{P}^L(t,\mathbf{x})+\nabla_\mathbf{y}\bar{\mathbf{P}}^L(t,\mathbf{x},\mathbf{y}))=0,\end{aligned}\tag{71b}$$

$$\begin{aligned}&\mu_0\mu_r(\mathbf{y})\frac{\partial}{\partial t}(\mathbf{H}^L(t,\mathbf{x})+\nabla_\mathbf{y}\bar{\mathbf{H}}^L(t,\mathbf{x},\mathbf{y}))+S_5^H(\mathbf{y})(\mathbf{H}^L(t,\mathbf{x})+\nabla_\mathbf{y}\bar{\mathbf{H}}^L(t,\mathbf{x},\mathbf{y}))\\&\quad+S_4^H(\mathbf{y})(\mathbf{M}^L(t,\mathbf{x})+\nabla_\mathbf{y}\bar{\mathbf{M}}^L(t,\mathbf{x},\mathbf{y}))=-\mathrm{curl}_\mathbf{x}\mathbf{E}^L(t,\mathbf{x})+\mathrm{curl}_\mathbf{y}\bar{\bar{\mathbf{E}}}^L(t,\mathbf{x},\mathbf{y})-\mathcal{J}_H^L(t,\mathbf{x}),\end{aligned}\tag{71c}$$

$$\begin{aligned}&S_1^H(\mathbf{y})\frac{\partial}{\partial t}(\mathbf{M}^L(t,\mathbf{x})+\nabla_\mathbf{y}\bar{\mathbf{M}}^L(t,\mathbf{x},\mathbf{y}))+S_3^H(\mathbf{y})(\mathbf{H}^L(t,\mathbf{x})+\nabla_\mathbf{y}\bar{\mathbf{H}}^L(t,\mathbf{x},\mathbf{y}))\\&\quad+S_2^H(\mathbf{y})(\mathbf{M}^L(t,\mathbf{x})+\nabla_\mathbf{y}\bar{\mathbf{M}}^L(t,\mathbf{x},\mathbf{y}))=0,\end{aligned}\tag{71d}$$

along with the initial conditions

$$\mathbf{E}^L(0)=\mathbf{E}_0^L\quad\text{in }\Omega\times Y,\ \bar{\mathbf{E}}^L(0)=0,\tag{72}$$

$$\mathbf{P}^L(0)=\mathbf{P}_0^L\quad\text{in }\Omega\times Y,\ \bar{\mathbf{P}}^L(0)=0,\tag{73}$$

$$\mathbf{H}^L(0)=\mathbf{H}_0^L\quad\text{in }\Omega\times Y,\ \bar{\mathbf{H}}^L(0)=0,\tag{74}$$

$$\mathbf{M}^L(0)=\mathbf{M}_0^L\quad\text{in }\Omega\times Y,\ \bar{\mathbf{M}}^L(0)=0,\tag{75}$$

and the boundary condition

$$\mathbf{n}\times\mathbf{E}^L=0\quad\text{on }(0,T)\times\partial\Omega.\tag{76}$$

5.3. *Limit model: Computation of correctors and effective matrices.* In this section we show that the *limit solution* $\mathbf{u}^L$ given by (6) solves a *global Maxwell problem* posed in $(0,T)\times\Omega$, while the correctors $\bar{\mathbf{u}}^L$ and $\bar{\bar{\mathbf{u}}}^L$ solve *local diffusion problems* posed in $(0,T)\times Y$.

THEOREM 5.2. For $\alpha>0$, let $\mathbf{A}^\alpha\in L^\infty(\Omega;\mathbb{R}^{n^2})$, symmetric and uniformly coercive, and $\mathbf{B}^\alpha\in L^\infty(\Omega;\mathbb{R}^{n^2})$, be two families of matrices indexed by α be given as in (36). Assume that the initial condition $\mathbf{u}_0^\alpha$ and the source $\mathbf{J}_s^\alpha$ satisfy assumptions (34). Then, there exists a unique limit electromagnetic field

$$\mathbf{u}^L=(\mathcal{E},\mathcal{P},\mathcal{H},\mathcal{M})^T\in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^n))\cap L^\infty(0,T;V(\Omega))$$

solution to the homogenized problem

$$\mathcal{A}\frac{\partial \mathbf{u}^L}{\partial t}(t,\mathbf{x}) + \mathcal{B}\mathbf{u}^L(t,\mathbf{x}) + \frac{\partial}{\partial t}\left(\int_0^t \mathcal{C}(t-s)\mathbf{u}^L(s,\mathbf{x})\right)\,\mathrm{ds} \tag{77a}$$

$$= \mathcal{F}\mathbf{u}^L(t,\mathbf{x}) - \mathcal{J}_s^L(t,\mathbf{x}) - \mathcal{J}^0(t,\mathbf{x}) \text{ in } (0,T)\times\Omega,$$

$$\mathbf{u}^L(0) = \mathbf{u}_0^L \text{ in } \Omega, \tag{77b}$$

$$\mathbf{n}\times\mathbf{u}_1^L = 0 \text{ on } (0,T)\times\partial\Omega, \tag{77c}$$

where $\mathcal{A},\mathcal{B}$ are *effective matrices* independent of the space variable $\mathbf{x}$, $\mathcal{J}^0$ is an extra source which depends only upon the initial condition $\mathbf{u}_0$. The matrix

$$\mathcal{C} \in W^{1,1}(0,T;L^\infty(\Omega;\mathbb{R}^{n^2}))$$

is a new effective matrix incorporating additional polarization and magnetization (memory) effects that arises due to the homogenization.

Proof. We note that the effective constitutive laws involve convolution terms that were not in the original model which arise due to the homogenization. The proof relies on appropriate choices of test functions in the variational form (77). The complete proof is based on additional preliminary lemmas presented in [8]. We present details of the computation of the corrector terms here.

Computation of $\bar{\mathbf{u}}^L$: We consider the decompositions $\mathbf{u}^L(t,\mathbf{x}) = \mathbf{u}_k^L(t,\mathbf{x})e_k$, with the initial condition at $t=0$ given by $\mathbf{u}_0(\mathbf{x}) = \mathbf{u}_{0,k}^L(\mathbf{x})e_k$, where $e_k, k\in\{1,2,\dots,n\}$ is the canonical basis of $\mathbb{R}^n$ and introduce three families of elementary correctors $(\bar{\mathbf{w}}^{\mathcal{A}},\bar{\mathbf{w}}^0,\bar{\mathbf{w}})$ (with value in $\mathbb{R}^N$) which are solutions to different local *diffusion problems* posed in Y.

Since the matrices $\mathbf{A}$ and $\mathbf{B}$ are independent of t, the corrector $\bar{\mathbf{w}}^{\mathbf{A}} \in H^1_{\mathrm{per}}(Y;\mathbb{R}^N)$ is independent of t, $\bar{\mathbf{w}}^0 \in W^{2,1}(0,T;H^1_{\mathrm{per}}(Y;\mathbb{R}^N))$, and $\bar{\mathbf{w}} \in W^{1,1}(0,T;H^1_{\mathrm{per}}(Y;\mathbb{R}^N))$ depend on only one variable. They solve the following variational problems satisfied for all $\bar{\mathbf{v}} \in H^1_{\mathrm{per}}(Y;\mathbb{R}^N)$:

- Corrector $\bar{\mathbf{w}}_k^0 \in W^{2,1}(0,T;H^1_{\mathrm{per}}(Y;\mathbb{R}^N))$, associated to the initial condition $\mathbf{u}(0,.)$, solves

$$\int_Y \mathbf{A}(\mathbf{y})\nabla_{\mathbf{y}}\bar{\mathbf{w}}_k^0(t,\mathbf{y})\cdot\nabla_{\mathbf{y}}\bar{\mathbf{v}}(\mathbf{y})\,\mathrm{d}\mathbf{y} + \int_Y\int_0^t \mathbf{B}(\mathbf{y})\nabla_{\mathbf{y}}\bar{\mathbf{w}}_k^0(s,\mathbf{y})\cdot\nabla_{\mathbf{y}}\bar{\mathbf{v}}(\mathbf{y})\,\mathrm{d}s\,\mathrm{d}\mathbf{y}$$
$$= \int_Y \mathbf{A}(\mathbf{y})e_k\cdot\nabla_{\mathbf{y}}\bar{\mathbf{v}}(\mathbf{y})\,\mathrm{d}\mathbf{y}, \qquad \forall\bar{\mathbf{v}}\in H^1_{\mathrm{per}}(Y;\mathbb{R}^n). \tag{78}$$

- Corrector $\bar{\mathbf{w}}_k^{\mathbf{A}} \in H^1_{\mathrm{per}}(Y;\mathbb{R}^n)$, depends on operator $\mathbf{A}$ and is defined as

$$\bar{\mathbf{w}}_k^{\mathbf{A}}(\mathbf{y}) = -\bar{\mathbf{w}}_k^0(0,\mathbf{y}),\ \mathbf{y}\in Y. \tag{79}$$

- The kernel $\bar{\mathbf{w}}_k \in W^{2,1}(0,T;H^1_{\mathrm{per}}(Y;\mathbb{R}^N))$

$$\int_Y \mathbf{A}(\mathbf{y})\nabla_{\mathbf{y}}\bar{\mathbf{w}}_k(t,\mathbf{y})\cdot\nabla_{\mathbf{y}}\bar{\mathbf{v}}(\mathbf{y})\,d\mathbf{y} + \int_Y\int_0^t \mathbf{B}(\mathbf{y})\nabla_{\mathbf{y}}\bar{\mathbf{w}}_k(s,\mathbf{y})\cdot\nabla_{\mathbf{y}}\bar{\mathbf{v}}(\mathbf{y})\,\mathrm{d}s\,\mathrm{d}\mathbf{y}$$
$$= -\int_Y \mathbf{B}(\mathbf{y})\left(e_k + \nabla_{\mathbf{y}}\bar{\mathbf{w}}_k^{\mathbf{A}}(\mathbf{y})\right)\cdot\nabla_{\mathbf{y}}\bar{\mathbf{v}}(\mathbf{y}) \qquad \forall\bar{\mathbf{v}}\in H^1_{\mathrm{per}}(Y;\mathbb{R}^N) \quad \text{a.e. in } (0,T). \tag{80}$$

We note that the correctors $\bar{\mathbf{w}}_k^0$ and $\bar{\mathbf{w}}_k$ have higher regularity than stated. In particular, we note that $\bar{\mathbf{w}}_k^0, \bar{\mathbf{w}}_k \in C^\infty(0,T; H^1_{\text{per}}(Y;\mathbb{R}^N))$.

Next, there exists a corrector $\bar{\mathbf{u}}^L \in W^{2,1}(0,T; H^1_{\text{per}}(Y;\mathbb{R}^N))$ that can be written as

$$\bar{\mathbf{u}}^L(t,\mathbf{x},\mathbf{y}) = \bar{\mathbf{w}}_k^{\mathbf{A}}(\mathbf{y})\mathbf{u}_k^L(t,\mathbf{x}) + \bar{\mathbf{w}}_k^0(t,\mathbf{y})\mathbf{u}_{0,k}^L(\mathbf{x}) + \int_0^t \bar{\mathbf{w}}_k(s,\mathbf{y})\mathbf{u}_k^L(s,\mathbf{x})ds, \tag{81}$$

and $\bar{\mathbf{u}}^L(0) = 0$, which can be deduced by substituting (79) into (81).

Finally, the effective matrices can be computed from the following formulas:

$$\mathcal{A} = \int_Y \mathbb{A}(\mathbf{y})\ \mathrm{d}\mathbf{y},\ \mathcal{A} \in \mathbb{R}^{n^2}, \tag{82}$$

$$\mathcal{B} = \int_Y \mathbb{B}(\mathbf{y})\ \mathrm{d}\mathbf{y},\ \mathcal{B} \in \mathbb{R}^{n^2}, \tag{83}$$

$$\mathcal{C}(t) = \int_Y \mathbb{C}(t,\mathbf{y})\ \mathrm{d}\mathbf{y},\ \mathcal{C} \in W^{1,1}(0,T;\mathbb{R}^{n^2}), \tag{84}$$

$$\mathcal{J}^0 = \frac{d}{dt}\left(\int_Y \mathbb{L}^0(t,\mathbf{y})\mathrm{d}\mathbf{y}\right) u^0(x),\ \mathcal{J}^0 \in W^{1,1}(0,T;L^\infty(\Omega;\mathbb{R}^{n^2})), \tag{85}$$

with the columns of the matrices given as

$$\mathbb{A}_k(\mathbf{y}) = \mathbf{A}(\mathbf{y})(e_k + \nabla_{\mathbf{y}}\bar{\mathbf{w}}_k^{\mathbf{A}}(\mathbf{y})), \tag{86}$$

$$\mathbb{B}_k(\mathbf{y}) = \mathbf{B}\left(\mathbf{y})(e_k + \nabla_y\bar{\mathbf{w}}_k^{\mathbf{A}}(\mathbf{y})\right), \tag{87}$$

$$\mathbb{C}_k(t,\mathbf{y}) = \mathbf{A}(\mathbf{y})\nabla_y\bar{\mathbf{w}}_k(t,\mathbf{y}) + \int_0^t \mathbf{B}(\mathbf{y})\nabla_y\bar{\mathbf{w}}_k(s,\mathbf{y})\ \mathrm{ds}, \tag{88}$$

$$\mathbb{L}_k^0(t,\mathbf{y}) = \mathbf{A}(\mathbf{y})\nabla_y\bar{\mathbf{w}}_k^0(t,\mathbf{y}) + \int_0^t \mathbf{B}(\mathbf{y})\nabla_y\bar{\mathbf{w}}_k^0(s,\mathbf{y})\ \mathrm{ds}. \tag{89}$$

For the computation of the field $\bar{\bar{\mathbf{u}}}^L$ we refer the reader to [8]. □

Theorem 5.1 suggests the formal asymptotic expansion

$$\mathbf{u}^\alpha(\mathbf{x}) = \mathbf{u}^L(\mathbf{x}) + \nabla_{\mathbf{y}}\bar{\mathbf{u}}^L(\mathbf{x},\frac{\mathbf{x}}{\alpha}) + \alpha\bar{\bar{\mathbf{u}}}^L(\mathbf{x},\frac{\mathbf{x}}{\alpha})\cdots. \tag{90}$$

Hence the computation of the term of order 0 (with respect to α) has to take into account the first corrector $\bar{\mathbf{u}}^L$. Under assumptions of Theorem 5.2 and by following the same approach as in [11], we obtain the strong convergences of the electromagnetic field

$$\mathbf{u}^\alpha - (\mathbf{u}^L + \mathcal{U}_\alpha(\nabla_{\mathbf{y}}\bar{\mathbf{u}}^L)) \longrightarrow 0 \qquad \text{in} \quad H^1(0,T;L^2(\Omega;\mathbb{R}^n)), \tag{91}$$

$$\mathrm{curl}_{\mathbf{x}}\mathbf{u}^\alpha - (\mathrm{curl}_{\mathbf{x}}\mathbf{u}^L + \mathcal{U}_\alpha(\mathrm{curl}_y\bar{\bar{\mathbf{u}}}^L)) \longrightarrow 0 \qquad \text{in} \quad L^2((0,T)\times\Omega;\mathbb{R}^n), \tag{92}$$

where $\mathcal{U}_\alpha$ is the averaging operator

$$\mathcal{U}_\alpha(\mathbf{v})(\mathbf{x}) = \frac{1}{|Y|}\int_Y \mathbf{v}\left(\alpha[\frac{\mathbf{x}}{\alpha}] + \alpha z, \{\frac{\mathbf{x}}{\alpha}\}\right) dz \quad \forall v \in L^2(\Omega\times Y).$$

6. Special cases of composite materials. In this section, we give examples of different models for the micro-structure of the composite material. We will consider the cases of models given by first and second order evolution equations for the polarization and/or magnetization. Below, we describe some popular ODE models that are used to model complex dielectrics. The constitutive ODE laws in (10) and (11) are sufficiently general to include models based on differential equations and systems of differential equations whose solutions can be expressed through fundamental solutions (in general variation-of-parameters representation) [4].

6.1. *Debye model for orientational polarization.* Assume $\mathbb{M}(t, \mathbf{x}) = \mathbf{0}$. The choice of the kernel function

$$\nu^E(\mathbf{x}, t) = \frac{\epsilon_0(\epsilon_s(\mathbf{x}) - \epsilon_\infty(\mathbf{x}))}{\tau(\mathbf{x})} \mathrm{e}^{-t/\tau(\mathbf{x})}, \tag{93}$$

in the dielectric corresponds to the differential equation of the *Debye model* for *orientational* or *dipolar polarization* [4, 19] given by

$$\tau(\mathbf{x}) \frac{\partial \mathbb{P}_R}{\partial t}(t, \mathbf{x}) + \mathbb{P}_R(t, \mathbf{x}) - \epsilon_0(\epsilon_s(\mathbf{x}) - \epsilon_\infty(\mathbf{x}))\mathbb{E}(t, \mathbf{x}) = \mathbf{0}. \tag{94}$$

In this case $N_E = 1$. Thus $\mathbb{P} = \mathbb{P}_R = \mathbb{P}^{(0)}$. The matrices S_1^E, S_2^E, and S_3^E are as given in (17), with $a_1^E(\mathbf{x}) = \tau(\mathbf{x})$, $a_0^E(\mathbf{x}) = 1$, and $\beta_0^E(\mathbf{x}) = -\epsilon_0(\epsilon_s(\mathbf{x}) - \epsilon_\infty(\mathbf{x}))$.

Here, ϵ_s is the static relative permittivity. The presence of instantaneous polarization is accounted for in this case by the coefficient ϵ_∞ in the electric flux equation. That is, $\epsilon_r = \epsilon_\infty$ in the dielectric, and $\epsilon_r = 1$ in air. The remainder of the electric polarization is seen to be a decaying exponential, driven by the electric field, less the part included in the instantaneous polarization. This model was first proposed by Debye [13], to model the behavior of materials whose molecules possess permanent dipole moments. The magnitude of the polarization term $\mathbb{P}$ represents the degree of alignment of these individual moments. The choice of coefficients in (94) gives a physical interpretation to ϵ_s and ϵ_∞ as the relative permittivities of the medium in the limit of the static field and very high frequencies, respectively. In the static case, we have $\mathbb{P}_t = 0$, so that $\mathbb{P} = \epsilon_0(\epsilon_s - \epsilon_\infty)\mathbb{E}$ and $\mathbb{D} = \epsilon_s \epsilon_0 \mathbb{E}$. For very high frequencies, $\tau \mathbb{P}_t$ dominates $\mathbb{P}$ so that $\mathbb{P} \approx 0$ and $\mathbb{D} = \epsilon_\infty \epsilon_0 \mathbb{E}$.

6.2. *Lorentz model for electronic polarization.* Again assume that $\mathbb{M}(t, \mathbf{x}) = \mathbf{0}$. The *Lorentz model* for *electronic* polarization [18] which, in differential form, is represented with the second order equation:

$$\frac{\partial^2 \mathbb{P}_R}{\partial t^2} + \lambda(\mathbf{x}) \frac{\partial \mathbb{P}_R}{\partial t} + \omega_0^2(\mathbf{x}) \mathbb{P}_R = \epsilon_0 \omega_p^2(\mathbf{x}) \mathbb{E}. \tag{95}$$

Here $N_E = 2$. $\mathbb{P} = (\mathbb{P}^{(0)}, \mathbb{P}^{(1)})^T$, In this case, the matrices S_1^E, S_2^E, and S_3^E are given to be

$$S_1^E = \mathbf{I}_2;\ S_2^E(\mathbf{x}) = \begin{bmatrix} 0 & -1 \\ \omega_0^2(\mathbf{x}) & \lambda \end{bmatrix};\ S_3^E = \begin{bmatrix} 0 \\ \epsilon_0 \omega_p^2(\mathbf{x}) \end{bmatrix}. \tag{96}$$

In (95), ω_p is called the *plasma frequency* and is defined to be

$$\omega_p(\mathbf{x})^2 = \omega_0(\mathbf{x})^2(\epsilon_s(\mathbf{x}) - \epsilon_\infty(\mathbf{x})). \tag{97}$$

A simple variation of constants solution [4] yields the correct kernel function

$$\nu^E(t,\mathbf{x}) = \frac{\epsilon_0\omega_p^2(\mathbf{x})}{\nu_0(\mathbf{x})}\mathrm{e}^{-\lambda(\mathbf{x})t/2}\sin\left(\nu_0(\mathbf{x})t\right), \tag{98}$$

$$\nu_0(\mathbf{x}) = \sqrt{\omega_0^2(\mathbf{x}) - \frac{\lambda^2(\mathbf{x})}{4}}. \tag{99}$$

6.3. *The Lorentz metamaterial model.* By metamaterial, we mean a class of artificial materials that have simultaneous negative permittivity and permeability (negative refractive index) [22]. These are also known as left-handed materials (LHMs). The *Lorentz metamaterial model* in differential form is represented with the second order equations

$$\frac{\partial^2\mathbb{P}_R}{\partial t^2} + \lambda_E(\mathbf{x})\frac{\partial\mathbb{P}_R}{\partial t} + \omega_{0,E}^2(\mathbf{x})\mathbb{P}_R = \epsilon_0\omega_{p,E}^2(\mathbf{x})\mathbb{E}, \tag{100}$$

$$\frac{\partial^2\mathbb{M}_R}{\partial t^2} + \lambda_H(\mathbf{x})\frac{\partial\mathbb{M}_R}{\partial t} + \omega_{0,H}^2(\mathbf{x})\mathbb{M}_R = \epsilon_0\omega_{p,H}^2(\mathbf{x})\mathbb{H}. \tag{101}$$

Here $N_E = N_H = 2$, $\mathbb{P} = (\mathbb{P}^{(0)}, \mathbb{P}^{(1)})^T$, $\mathbb{M} = (\mathbb{M}^{(0)}, \mathbb{M}^{(1)})^T$ and, matrices S_1^V, S_2^V, and S_3^V are as in (96) with appropriate V labels on the parameters. The matrices S_1^E, S_2^E, and S_3^E are given to be

$$S_1^E = \mathbf{I}_2;\; S_2^E(\mathbf{x}) = \begin{bmatrix} 0 & -1 \\ \omega_{0,E}^2(\mathbf{x}) & \lambda \end{bmatrix}\; S_3^E = \begin{bmatrix} 0 \\ \epsilon_0\omega_{p,E}^2(\mathbf{x}) \end{bmatrix} \tag{102}$$

and matrices S_1^H, S_2^H, and S_3^H are given to be

$$S_1^H = \mathbf{I}_2;\; S_2^H(\mathbf{x}) = \begin{bmatrix} 0 & -1 \\ \omega_{0,H}^2(\mathbf{x}) & \lambda \end{bmatrix}\; S_3^E = \begin{bmatrix} 0 \\ \epsilon_0\omega_{p,H}^2(\mathbf{x}) \end{bmatrix}. \tag{103}$$

For example, when $\lambda_E = \lambda_H = 0$ and $\omega_{p,E} = \omega_{p,H} = \sqrt{2}\omega$, the refractive index $RI = -1$. If the frequencies $\omega_{0,E} = \omega_{0,H} = 0$, then the model is called a *Drude* metamaterial.

7. Homogenization for Debye mixtures. In this section, we consider the case of a composite material with inclusions described by the Debye model given in Section 6.1 in more detail. For a mixture of two Debye materials (both host and inclusions described as Debye media), we develop the homogenized limit model here. In this case, $N_E = 1$, $\mathbb{M} = 0$, and $n = 9$. The matrices $\mathbf{A}$ and $\mathbf{B}$ are

$$\mathbf{A}(\mathbf{y}) = \begin{bmatrix} \epsilon_0\epsilon_\infty(\mathbf{y}) & 0 & 0 \\ 0 & \tau(\mathbf{y}) & 0 \\ 0 & 0 & \mu_0\mu_r(\mathbf{y}) \end{bmatrix} \otimes \mathbf{I}_3, \tag{104}$$

while the matrix $\mathbf{B}$ is

$$\mathbf{B}(\mathbf{y}) = \begin{bmatrix} \dfrac{\epsilon_0(\epsilon_s(\mathbf{y}) - \epsilon_\infty(\mathbf{y}))}{\tau(\mathbf{y})} & -\dfrac{1}{\tau(\mathbf{y})} & 0 \\ -\epsilon_0(\epsilon_s(\mathbf{y}) - \epsilon_\infty(\mathbf{y})) & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \mathbf{I}_3. \tag{105}$$

Theorem 5.1 for this case gives us the following limits. There exist three sets of fields,

$$\mathbf{E}^L \in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^3)) \cap L^\infty(0,T;\mathbf{H}_0(\mathrm{curl},\Omega)), \tag{106}$$

$$\mathbf{P}^L \in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^3)) \cap L^\infty(0,T;\mathbf{H}_0(\mathrm{curl})), \tag{107}$$

$$\mathbf{H}^L \in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^3)) \cap L^\infty(0,T;\mathbf{H}(\mathrm{curl},\Omega)), \tag{108}$$

the set

$$\bar{\mathbf{E}}^L \in W^{1,\infty}(0,T;L^2(\Omega;H^1_{\mathrm{per}}(Y;\mathbb{R}^3))), \tag{109}$$

$$\bar{\mathbf{P}}^L \in W^{1,\infty}(0,T;L^2(\Omega;H^1_{\mathrm{per}}(Y;\mathbb{R}^3))), \tag{110}$$

$$\bar{\mathbf{H}}^L \in W^{1,\infty}(0,T;L^2(\Omega;H^1_{\mathrm{per}}(Y;\mathbb{R}^3))), \tag{111}$$

and finally

$$\overline{\overline{\mathbf{E}}}^L \in L^\infty(0,T;L^2(\Omega;H^1_{\mathrm{per}}(Y;\mathbb{R}^3))), \mathrm{div}_y(\overline{\overline{\mathbf{E}}}^L) = 0, \tag{112}$$

$$\overline{\overline{\mathbf{P}}}^L \in L^\infty(0,T;L^2(\Omega;H^1_{\mathrm{per}}(Y;\mathbb{R}^3))), \mathrm{div}_y(\overline{\overline{\mathbf{P}}}^L) = 0, \tag{113}$$

$$\overline{\overline{\mathbf{H}}}^L \in L^\infty(0,T;L^2(\Omega;H^1_{\mathrm{per}}(Y;\mathbb{R}^3))), \mathrm{div}_y(\overline{\overline{\mathbf{H}}}^L) = 0, \tag{114}$$

which are limits of the following sequences:

$$\mathbf{E}^\alpha \rightharpoonup \mathbf{E}^L \text{ weakly} * \quad \text{in} \quad L^\infty(0,T;\mathbf{H}_0(\mathrm{curl},\Omega))), \tag{115}$$

$$\mathbf{P}^\alpha \rightharpoonup \mathbf{P}^L \text{ weakly} * \quad \text{in} \quad L^\infty(0,T;\mathbf{H}_0(\mathrm{curl},\Omega)), \tag{116}$$

$$\mathbf{H}^\alpha \rightharpoonup \mathbf{H}^L \text{ weakly} * \quad \text{in} \quad L^\infty(0,T;\mathbf{H}(\mathrm{curl},\Omega)), \tag{117}$$

and

$$\mathcal{T}_\alpha(\mathbf{E}^\alpha) \longrightarrow \mathbf{E}^L + \nabla_y \bar{\mathbf{E}}^L \text{ strongly in } H^1(0,T;L^2(\Omega \times Y;\mathbb{R}^3)), \tag{118}$$

$$\mathcal{T}_\alpha(\mathbf{P}^\alpha) \longrightarrow \mathbf{P}^L + \nabla_y \bar{\mathbf{P}}^L \text{ strongly in } H^1(0,T;L^2(\Omega \times Y;\mathbb{R}^3)), \tag{119}$$

$$\mathcal{T}_\alpha(\mathbf{H}^\alpha) \longrightarrow \mathbf{H}^L + \nabla_y \bar{\mathbf{H}}^L \text{ strongly in } H^1(0,T;L^2(\Omega \times Y;\mathbb{R}^3)), \tag{120}$$

and

$$\mathcal{T}_\alpha(\mathrm{curl}_x \mathbf{E}^\alpha) \longrightarrow \mathrm{curl}_x \mathbf{E}^L + \mathrm{curl}_y \overline{\overline{\mathbf{E}}}^L \text{ strongly in } L^2((0,T) \times \Omega \times Y;\mathbb{R}^3), \tag{121}$$

$$\mathcal{T}_\alpha(\mathrm{curl}_x \mathbf{H}^\alpha) \longrightarrow \mathrm{curl}_x \mathbf{H}^L + \mathrm{curl}_y \overline{\overline{\mathbf{H}}}^L \text{ strongly in } L^2((0,T) \times \Omega \times Y;\mathbb{R}^3), \tag{122}$$

which solve the following evolution equations in $(0,T)\times\Omega\times Y$:

$$\begin{aligned}\epsilon_0\epsilon_\infty(\mathbf{y})\frac{\partial}{\partial t}(\mathbf{E}^L(t,\mathbf{x})+\nabla_{\mathbf{y}}\bar{\mathbf{E}}^L(t,\mathbf{x},\mathbf{y}))+\frac{\epsilon_0}{\tau(\mathbf{y})}(\epsilon_s(\mathbf{y})-\epsilon_\infty(\mathbf{y})(\mathbf{E}^L(t,\mathbf{x})+\nabla_{\mathbf{y}}\bar{\mathbf{E}}^L(t,\mathbf{x},\mathbf{y}))\\ -\frac{1}{\tau(\mathbf{y})}(\mathbf{P}^L(t,\mathbf{x})+\nabla_{\mathbf{y}}\bar{\mathbf{P}}^L(t,\mathbf{x},\mathbf{y}))=\mathrm{curl}_{\mathbf{x}}\mathbf{H}^L(t,\mathbf{x})+\mathrm{curl}_{\mathbf{y}}\bar{\bar{\mathbf{H}}}^L(t,\mathbf{x},\mathbf{y})-\mathcal{J}_E^L(t,\mathbf{x}),\end{aligned} \tag{123a}$$

$$\begin{aligned}\tau(\mathbf{y})\frac{\partial}{\partial t}(\mathbf{P}^L(t,\mathbf{x})+\nabla_{\mathbf{y}}\bar{\mathbf{P}}^L(t,\mathbf{x},\mathbf{y}))+(\mathbf{P}^L(t,\mathbf{x})+\nabla_{\mathbf{y}}\bar{\mathbf{P}}^L(t,\mathbf{x},\mathbf{y}))\\ -\epsilon_0(\epsilon_s(\mathbf{y})-\epsilon_\infty(\mathbf{y}))(\mathbf{E}^L(t,\mathbf{x})+\nabla_{\mathbf{y}}\bar{\mathbf{E}}^L(t,\mathbf{x},\mathbf{y}))=0,\end{aligned} \tag{123b}$$

$$\mu_0\mu_r(\mathbf{y})\frac{\partial}{\partial t}(\mathbf{H}^L(t,\mathbf{x})+\nabla_{\mathbf{y}}\bar{\mathbf{H}}^L(t,\mathbf{x},\mathbf{y}))=-\mathrm{curl}_{\mathbf{x}}\mathbf{E}^L(t,\mathbf{x})+\mathrm{curl}_{\mathbf{y}}\bar{\bar{\mathbf{E}}}^L(t,\mathbf{x},\mathbf{y})-\mathcal{J}_H^L(t,\mathbf{x}), \tag{123c}$$

along with the initial conditions

$$\mathbf{E}^L(0)=\mathbf{E}_0^L \quad \text{in } \Omega\times Y,\ \bar{\mathbf{E}}^L(0)=0, \tag{124}$$

$$\mathbf{P}^L(0)=\mathbf{P}_0^L \quad \text{in } \Omega\times Y,\ \bar{\mathbf{P}}^L(0)=0, \tag{125}$$

$$\mathbf{H}^L(0)=\mathbf{H}_0^L \quad \text{in } \Omega\times Y,\ \bar{\mathbf{H}}^L(0)=0, \tag{126}$$

and the boundary condition

$$\mathbf{n}\times\mathbf{E}^L=0 \quad \text{on } (0,T)\times\partial\Omega. \tag{127}$$

The homogenized model is then computed from Theorem 5.2.

7.1. *Homogenization model in two dimensions.* Let $\mathbf{x}=(x_1,x_2,x_3)^T$. We now assume our problem to possess uniformity in the x_2-direction. Thus, we assume all derivatives with respect to x_2 (or y_2) to be zero. In this case Maxwell's equations decouple into two different modes, the TE and TM modes. Here, we are interested in the TE_y mode. The TE_y mode involves the components E_x, E_z for the electric field, the components P_x, P_z for the electric polarization field and the component H_y of the magnetic field. As mentioned earlier, there are no additional magnetic effects in the Debye model and $\mathbb{M}=\mathbf{0}$. In the rest of this section we will denote $\mathbf{x}=(x_1,x_3)^T$.

In a similar manner to the three dimensional case, we may construct matrices $\mathbf{A}^{\mathrm{TE}}$, and $\mathbf{B}^{\mathrm{TE}}$, that represent the constitutive relations in two dimensions. Thus the constitutive matrices are

$$\mathbf{A}^{\mathrm{TE}}=\begin{bmatrix}\mathbf{A}_{11}^{\mathrm{TE}}(\mathbf{y}) & 0\\ 0 & \mu_0\mu_r\end{bmatrix};\quad \mathbf{B}^{\mathrm{TE}}=\begin{bmatrix}\mathbf{B}_{11}^{\mathrm{TE}}(\mathbf{y}) & 0\\ 0 & 0\end{bmatrix}. \tag{128}$$

$$\mathbf{A}_{11}^{\mathrm{TE}}(\mathbf{y})=\begin{bmatrix}\epsilon_0\epsilon_\infty(\mathbf{y}) & 0\\ 0 & \tau(\mathbf{y})\end{bmatrix}\otimes\mathbf{I}_2;\quad \mathbf{B}_{11}^{\mathrm{TE}}(\mathbf{y})=\begin{bmatrix}\dfrac{\epsilon_0(\epsilon_s(\mathbf{y})-\epsilon_\infty(\mathbf{y}))}{\tau(\mathbf{y})} & -\dfrac{1}{\tau(\mathbf{y})}\\ -\epsilon_0(\epsilon_s(\mathbf{y})-\epsilon_\infty(\mathbf{y})) & 1\end{bmatrix}\otimes\mathbf{I}_2. \tag{129}$$

The homogenized solution for the TE mode is obtained from the formal asymptotic expansion as

$$E^{\alpha}_{x_1} = E_{x_1} + \partial_{y_1}\bar{u}_1(\mathbf{x},\mathbf{y}) + \ldots, \tag{130a}$$

$$E^{\alpha}_{x_3} = E_{x_3} + \partial_{y_3}\bar{u}_1(\mathbf{x},\mathbf{y}) + \ldots, \tag{130b}$$

$$P^{\alpha}_{x_1} = P_{x_1} + \partial_{y_1}\bar{u}_2(\mathbf{x},\mathbf{y}) + \ldots, \tag{130c}$$

$$P^{\alpha}_{x_3} = P_{x_3} + \partial_{y_3}\bar{u}_2(\mathbf{x},\mathbf{y}) + \ldots, \tag{130d}$$

$$H^{\alpha}_{x_2} = H_{x_2} + \partial_{y_2}\bar{u}_3(\mathbf{x},\mathbf{y}) + \ldots. \tag{130e}$$

Hence the homogenized electric field and electric polarization for the TE mode can be expanded as

$$\mathbf{E}^{\alpha} = \mathbf{E} + \nabla_{\mathbf{y}}\bar{u}_1(\mathbf{x},\mathbf{y}) + \ldots, \tag{131}$$

$$\mathbf{P}^{\alpha} = \mathbf{P} + \nabla_{\mathbf{y}}\bar{u}_2(\mathbf{x},\mathbf{y}) + \ldots, \tag{132}$$

where the gradient operator in this case is $\nabla_{\mathbf{y}} = (\partial_{y_1}, \partial_{y_3})^T$. Therefore we need to solve for $\bar{u}_1(\mathbf{x},\mathbf{y})$, and $\bar{u}_2(\mathbf{x},\mathbf{y})$, which in turn only depend on the first two components of $\bar{\mathbf{w}}^A_k$, $\bar{\mathbf{w}}^0_k$, and $\bar{\mathbf{w}}_k$, for $k = 1,2,3,4$. We now assume the same notation of the corrector to mean the first two components.

Let us again denote by Y the reference cell of the periodic structure that occupies $\Omega \subset \mathbb{R}^2$. The construction of the two dimensional homogenized problem involves solving for the *corrector* sub terms $\bar{\mathbf{w}}^A_k \in H^1_{\mathrm{per}}(Y;\mathbb{R}^2)$, $\bar{\mathbf{w}}_k \in W^{1,1}(0,T;H^1_{\mathrm{per}}(Y;\mathbb{R}^2))$, and $\bar{\mathbf{w}}^0_k \in W^{2,1}(0,T;H^1_{\mathrm{per}}(Y;\mathbb{R}^2))$, solutions to the corrector equations

$$\left\{\begin{aligned}
&\text{(i)} \int_Y \mathbf{A}^{\mathrm{TE}}_{11}(\mathbf{y})\nabla_{\mathbf{y}}\bar{\mathbf{w}}^0_k(t,\mathbf{y})\cdot\nabla_{\mathbf{y}}\bar{\mathbf{v}}(\mathbf{y})\mathrm{d}\mathbf{y}\\
&\qquad\quad + \int_Y\int_0^t \mathbf{B}^{\mathrm{TE}}_{11}(\mathbf{y})\nabla_{\mathbf{y}}\bar{\mathbf{w}}^0_k(s,\mathbf{y})\cdot\nabla_{\mathbf{y}}\bar{\mathbf{v}}(\mathbf{y})\mathrm{d}s\,\mathrm{d}\mathbf{y}\\
&\qquad = \int_Y \mathbf{A}^{\mathrm{TE}}_{11}(\mathbf{y})\mathbf{e}_k\cdot\nabla_{\mathbf{y}}\bar{\mathbf{v}}(\mathbf{y})\,\mathrm{d}\mathrm{y},\\
&\text{(ii)}\ \bar{\mathbf{w}}^{\mathbf{A}}_k(\mathbf{y}) = -\bar{\mathbf{w}}^0_k(0,\mathbf{y}),\ \mathbf{y}\in Y,\\
&\text{(iii)} \int_Y \mathbf{A}^{\mathrm{TE}}_{11}(\mathbf{y})\nabla_{\mathbf{y}}\bar{\mathbf{w}}_k(t,\mathbf{y})\cdot\nabla_{\mathbf{y}}\bar{\mathbf{v}}(\mathbf{y})\mathrm{d}\mathbf{y}\\
&\qquad\quad + \int_Y\int_0^t \mathbf{B}^{\mathrm{TE}}_{11}(\mathbf{y})\nabla_{\mathbf{y}}\bar{\mathbf{w}}_k(s,\mathbf{y})\cdot\nabla_{\mathbf{y}}\bar{\mathbf{v}}(\mathbf{y})\,\mathrm{d}s\,\mathrm{d}\mathbf{y}\\
&\qquad = -\int_Y \mathbf{B}^{\mathrm{TE}}_{11}(\mathbf{y})\left\{\mathbf{e}_k + \nabla_{\mathbf{y}}\bar{\mathbf{w}}^A_k\right\}\cdot\nabla_{\mathbf{y}}\bar{\mathbf{v}}(\mathbf{y})\,\mathrm{d}\mathbf{y}
\end{aligned}\right. \tag{133}$$

$\forall \bar{\mathbf{v}} \in H^1_{\text{per}}(Y;\mathbb{R}^2)$ and $\mathbf{e}_k$, $k = 1,2,3,4$ are the basis vectors in $\mathbb{R}^4$, i.e., $\mathbf{e}_1 = [1,0,0,0]^T, \mathbf{e}_2 = [0,1,0,0]^T$, $\mathbf{e}_3 = [0,0,1,0]^T, \mathbf{e}_4 = [0,0,0,1]^T$. Once we have solved for the corrector terms, we can then construct the homogenized matrices from

$$\begin{cases} \text{(i)} \quad (\mathcal{A}^{\text{TE}}_{11})_k = \displaystyle\int_Y \mathbf{A}^{\text{TE}}_{11}(\mathbf{y}) \left\{ \mathbf{e}_k + \nabla_{\mathbf{y}} \bar{\mathbf{w}}^A_k(\mathbf{y}) \right\} \mathrm{d}\mathbf{y}, \\ \text{(ii)} \quad (\mathcal{B}^{\text{TE}}_{11})_k = \displaystyle\int_Y \mathbf{B}^{\text{TE}}_{11}(\mathbf{y}) \left\{ \mathbf{e}_k + \nabla_{\mathbf{y}} \bar{\mathbf{w}}^A_k(\mathbf{y}) \right\} \mathrm{d}\mathbf{y}, \\ \text{(iii)} \quad (\mathcal{C}^{\text{TE}}_{11})_k = \displaystyle\int_Y \mathbf{A}^{\text{TE}}_{11}(\mathbf{y}) \nabla_{\mathbf{y}} \bar{\mathbf{w}}_k(t,\mathbf{y}) \ \mathrm{d}\mathbf{y} + \int_Y \int_0^t \mathbf{B}^{\text{TE}}_{11}(\mathbf{y}) \nabla_{\mathbf{y}} \bar{\mathbf{w}}_k(s,\mathbf{y}) \ \mathrm{d}s \ \mathrm{d}\mathbf{y}, \end{cases} \tag{134}$$

where $(\mathcal{A}^{\text{TE}}_{11})_k, (\mathcal{B}^{\text{TE}}_{11})_k$, and $(\mathcal{C}^{\text{TE}}_{11})_k$ are the kth columns of the matrices $\mathcal{A}^{\text{TE}}_{11}, \mathcal{B}^{\text{TE}}_{11}$, and $\mathcal{C}^{\text{TE}}_{11}$, respectively, and the homogenized matrices are given as

$$\begin{aligned} \mathcal{A}^{\text{TE}} &= \begin{bmatrix} \mathcal{A}^{\text{TE}}_{11} & 0 \\ 0 & \mu_0\mu_r \end{bmatrix}; \\ \mathcal{B}^{\text{TE}} &= \begin{bmatrix} \mathcal{B}^{\text{TE}}_{11} & 0 \\ 0 & 0 \end{bmatrix}; \\ \mathcal{C}^{\text{TE}} &= \begin{bmatrix} \mathcal{C}^{\text{TE}}_{11} & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{135}$$

The homogenized model is then computed from Theorem 5.2 with appropriate assumptions and simplifications for the two dimensional case.

7.2. *Numerical example: Varying relative permittivity.* We consider the simple example of a composite material which possesses circular micro-structures in two dimensions, involving a cell problem in the reference cell $Y = [0,1] \times [0,1]$, in which the value of the infinite frequency relative permittivity ϵ_∞ is given as

$$\epsilon_r(\mathbf{x}) = \begin{cases} \epsilon_i = 2.7 & \text{if } \mathbf{x} \in S, \\ \epsilon_e = 1.03 & \text{if } \mathbf{x} \in Y/\bar{S}. \end{cases} \tag{136}$$

The composite and the reference cell are depicted in Figure 2. In this test case, we will assume that $\epsilon_s = \epsilon_\infty$, and τ is constant over the entire dielectric material. In this case, the polarization $\mathbb{P} = 0$ and the homogenized effective permittivity is the same as that developed in Section 7 in our previous work [3]. We repeat some of the main results here, and refer the reader to [3] for details of the numerical simulations.

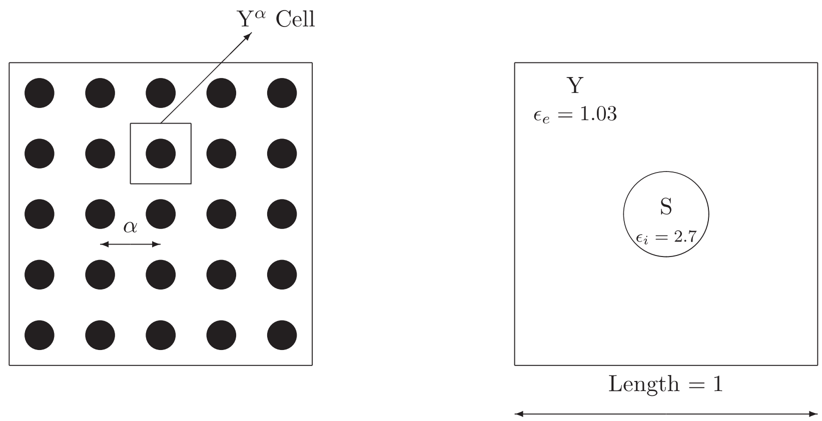

FIG. 2. (Left) Periodic composite material having inclusions with a circular micro-structure and with periodicity α. (Right) The reference cell $Y = [0,1] \times [0,1]$ with different relative permittivities inside and outside the circular inclusion.

Numerical simulation using continuous finite elements is performed on a 51×51 nodes mesh grid to solve the corrector system (133) and then to compute the homogenized matrices (134). Since there is no polarization in this problem, we only need to compute the first component of the corrector $\bar{\mathbf{w}}_k^A$, $k = 1, 2, 3, 4$ in order to compute the homogenized electric field in (131) and to compute the effective permittivity.

We define the inclusion volume fraction f as the ratio

$$f = \frac{\text{area of inclusion}}{\text{area of domain } Y}. \tag{137}$$

In Figure 3, we plot the relative effective permittivity versus the inclusion volume fraction for our periodic unfolding method and other theoretical mixture formulas [30,31], which are valid for the case of circular inclusions. The prediction of the effective relative permittivity of the composite mixture ϵ_{eff} by different mixture formulas is given as follows:

$$\epsilon_{\text{eff,MG}} = \epsilon_e + 2f\epsilon_e \frac{\epsilon_i - \epsilon_e}{\epsilon_i + \epsilon_e - f(\epsilon_i - \epsilon_e)}, \quad \text{(Maxwell Garnett)}, \tag{138}$$

$$(1-f)\frac{\epsilon_e - \epsilon_{\text{eff,B}}}{\epsilon_e + \epsilon_{\text{eff,B}}} + f\frac{\epsilon_i - \epsilon_{\text{eff,B}}}{\epsilon_i + \epsilon_{\text{eff,B}}} = 0, \quad \text{(Bruggeman)}, \tag{139}$$

$$\frac{\epsilon_{\text{eff,CP}} - \epsilon_e}{\epsilon_e + \epsilon_{\text{eff,CP}} + 2(\epsilon_{\text{eff,CP}} - \epsilon_e)} - \frac{\epsilon_i - \epsilon_e}{\epsilon_e + \epsilon_i + 2(\epsilon_{\text{eff,CP}} - \epsilon_e)} = 0, \quad \text{(Coherent potential)}, \tag{140}$$

$$\epsilon_{\text{eff,max}} = f\epsilon_i + (1-f)\epsilon_e, \quad \text{(Max Weiner bound)}, \tag{141}$$

$$\epsilon_{\text{eff,min}} = \frac{\epsilon_i \epsilon_e}{f\epsilon_e + (1-f)\epsilon_i}, \quad \text{(Min Weiner bound)}. \tag{142}$$

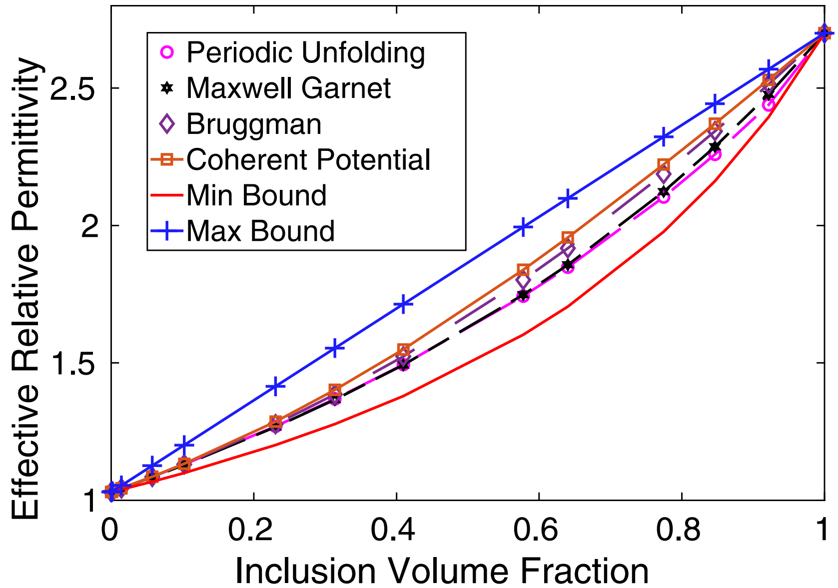

FIG. 3. Effective relative permittivities for periodic mixtures with circular inclusions against the corresponding inclusion volume fraction f. Effective values computed with the periodic unfolding method and various mixture formulas are plotted.

As seen from Figure 3, the mixture formulas given by Maxwell Garnett, Bruggeman, and the coherent potential along with the numerical simulations from our periodic unfolding method all produce effective permittivites that lie within the maximum and minimum Weiner bounds.

8. Conclusions. In this paper, we have presented an homogenization method based on the periodic unfolding technique for computing effective properties of a mixture of linear dispersive materials. Models for linear dispersive electromagnetic materials (materials with memory effects) can be built in a variety of ways. In [8], the constitutive laws for linear dispersive media included convolutions in time of electric and magnetic fields with appropriate kernels modeling the memory effects in the materials. In this paper, we explicitly model the evolution of the polarization and magnetization in time using ODEs forced by the electric field for the polarization, and by the magnetic field for the magnetization. The problem is to homogenize a hybrid system of PDEs-ODEs collectively given by Maxwell's PDEs combined with ODEs for the dynamic evolution of the dispersive medium's polarization and magnetization.

The method that we have presented in this paper is an alternate homogenization technique to the approach presented in [3, 8], in which the constitutive laws included convolutions in time of electric and magnetic fields. The numerical computation of effective parameters using our previous approach in [3] requires discretizing several convolutions in time, and in that paper we employed a recursive convolution approach for computing discrete homogenized susceptibility kernels.

The advantage of the method presented in this paper is that the model for each material in the mixture includes evolution equations that explicitly track the electric polarization and magnetization in time, rather than the implicit inclusion of polarization and magnetization via convolutions. Thus, the homogenized model has explicit equations for the effective polarization and magnetization. However, even though the constitutive laws for each material in the mixture are based on systems of ODEs, the homogenized system can include convolutions in time that arise through the homogenization process in addition to the systems of ODEs.

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