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Some Properties and Invariants of Multivariate Difference-Differential Dimension Polynomials

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Multivariate dimension polynomials associated with finitely generated differential and difference field extensions arise as natural generalizations of the univariate differential and difference dimension polynomials defined respectively in [1] and [2]. It turns out, however, that they carry more information about the corresponding extensions than their univariate counterparts (see [3, Theorem 4.2.17] and [4]). In this presentation we extend the known results on multivariate dimension polynomials to the case of difference-differential field extensions with arbitrary partitions of sets of basic operators. We also describe some properties of multivariate dimension polynomials and their invariants. The following is the outline of the talk.

Let K be a difference-differential field, $\mathrm{Char}\, K = 0$, and let $\Delta = \{\delta_1, \dots, \delta_m\}$ and $\sigma = \{\alpha_1, \dots, \alpha_n\}$ be basic sets of derivations and automorphisms of K, respectively. Below we often use the prefix Δ-σ- instead of "difference-differential". Suppose that the sets Δ and σ are represented as unions of disjoint subsets: $\Delta = \cup_{i=1}^{p}\Delta_i$ and $\sigma = \cup_{j=1}^{q}\sigma_j$ where $\mathrm{Card}\,\Delta_i = m_i$ $(1 \le i \le p)$ and $\mathrm{Card}\,\sigma_i = n_i$ $(1 \le i \le q)$. Let Λ denote the free commutative semigroup of all power products of the form $\lambda = \delta_1^{k_1} \dots \delta_m^{k_m}\alpha_1^{l_1} \dots \alpha_n^{l_n}$ where $k_\mu \in \mathbb{N}$, $l_\nu \in \mathbb{Z}$ and for every such λ, let

$$\mathrm{ord}_{\Delta_i}\lambda = \sum_{\mu \in \Delta_i} k_\mu \ \text{ and } \ \mathrm{ord}_{\sigma_j}\lambda = \sum_{\nu \in \sigma_j} |l_\nu|$$

$(1 \le i \le p,\ 1 \le j \le q)$. Furthermore, for any $(r_1, \dots, r_{p+q}) \in \mathbb{N}^{p+q}$, let $\Lambda(r_1, \dots, r_{p+q}) = \{\lambda \in \Lambda \mid \mathrm{ord}_{\Delta_i}\lambda \le r_i \text{ for } i = 1, \dots, p \text{ and } \mathrm{ord}_{\sigma_j}\lambda \le r_{p+j} \text{ for } j = 1, \dots, q\}$. The following theorem generalizes the main result of [4].

Theorem 1. *Let $L = K\langle\eta_1, \dots, \eta_s\rangle$ be a Δ-σ-field extension generated by a set $\eta = \{\eta_1, \dots, \eta_s\}$. Then there exists a polynomial $\Phi_\eta \in \mathbb{Q}[t_1, \dots, t_{p+q}]$ (called the Δ-σ-dimension polynomial of the extension L/K) such that*

(i) $\Phi_\eta(r_1, \dots, r_{p+q}) = \mathrm{tr.deg}_K K(\bigcup_{j=1}^{s}\Lambda(r_1, \dots, r_{p+q})\eta_j)$

for all sufficiently large $(r_1, \dots, r_{p+q}) \in \mathbb{N}^{p+q}$ (it means that there exist $s_1, \dots, s_{p+q} \in \mathbb{N}$ such that the equality holds for all $(r_1, \dots, r_{p+q}) \in \mathbb{N}^{p+q}$ with $r_1 \ge s_1, \dots, r_{p+q} \ge s_{p+q}$);

(ii) $\deg_{t_i}\Phi_\eta \le m_i$ *$(1 \le i \le p)$,* $\deg_{t_{p+j}}\Phi_\eta \le n_j$ *$(1 \le j \le q)$ and $\Phi_\eta(t_1, \dots, t_{p+q})$ can be represented as*

$$\Phi_\eta = \sum_{i_1=0}^{m_1} \dots \sum_{i_p=0}^{m_p} \sum_{i_{p+1}=0}^{n_1} \dots \sum_{i_{p+q}=0}^{n_q} a_{i_1 \dots i_{p+q}} \binom{t_1 + i_1}{i_1} \dots \binom{t_{p+q} + i_{p+q}}{i_{p+q}}$$

where $a_{i_1 \dots i_{p+q}} \in \mathbb{Z}$ and $2^n \mid a_{m_1 \dots m_p n_1 \dots n_q}$.

We sketch the proof of this theorem and present a method of computation of the polynomial Φ_η based on a generalization of the Ritt-Kolchin method of characteristic sets. Furthermore, we determine invariants of a Δ-σ-dimension polynomial, i. e., numerical characteristics of the Δ-σ-field extension that are carried by such a polynomial and that do not depend on the set of Δ-σ-generators this Δ-σ-dimension polynomial is associated with. We also give conditions under which the Δ-σ-dimension polynomial is of the simplest possible form.

Keywords: Difference-differential field extension, Dimension polynomial, Characteristic set

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