Applications of Computer Algebra – ACA2018
Santiago de Compostela, June 18–22, 2018

Dimension Polynomials and the Einstein's Strength of Some Systems of Quasi-linear Algebraic Difference Equations

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We present a difference algebraic technique for the evaluation of the Einstein's strength of quasi-linear partial difference equations and some systems of such equations. Our approach is based on the properties of difference dimension polynomials that express the Einstein's strength and on the characteristic set method for computing such polynomials. The obtained results are applied to the comparative analysis of difference schemes for some chemical reaction-diffusion equations.

Keywords: Difference dimension polynomial, Autoreduced set, Einstein's strength

Mathematics Subject Classification 2010: 12H10, 35K57

1 Preliminaries

Let K be an inversive difference field with a basic set of automorphisms $\sigma = \{\alpha_1, \dots, \alpha_m\}$ and Γ the free commutative group generated by σ. If $\gamma = \alpha_1^{k_1} \dots \alpha_m^{k_m} \in \Gamma$, then the number $\operatorname{ord}\gamma = \sum_{i=1}^{m} |k_i|$ is called the *order* of γ; if $r \in \mathbb{N}$, we set $\Gamma(r) = \{\gamma \in \Gamma \mid \operatorname{ord}\gamma \leq r\}$. In what follows we denote the set $\{\alpha_1, \dots, \alpha_m, \alpha_1^{-1}, \dots, \alpha_m^{-1}\}$ by σ^* and use the prefix σ^*-instead of "inversive difference". A reflexive difference ideal will be refer to as a σ^*-ideal.

Let $R = K\{y_1, \dots, y_n\}^*$ be the ring of σ^*-polynomials in n σ^*-indeterminates over K. (As a ring, $R = K[\{\gamma y_i \mid \gamma \in \Gamma,\ 1 \leq i \leq n\}]$) An n-tuple $\xi = (\xi_1, \dots, \xi_n)$ with coordinates in some σ^*-overfield K' of K is said to be a solution of the set of σ^*-polynomials $F = \{f_j \mid j \in J\} \subseteq R$ or a solution of the system of algebraic difference equations

$$f_j(y_1, \dots, y_n) = 0 \ \ (j \in J) \tag{1}$$

if F is contained in the kernel of the natural difference K-homomorphism ("substitution") $R \to K'$ $(y_i \mapsto \xi_i)$. The system (1) is called *prime* if the σ^*-ideal P generated by the set F in R (it is denoted by $[F]^*$) is prime. In this case the quotient field L of R/P has a natural structure of a finitely generated σ^*-field extension of K: $L = K\langle \eta_1, \dots, \eta_n \rangle^*$ where η_i is the canonical image of y_i in L. (As a field, $L = K(\{\gamma(\eta_i) \mid \gamma \in \Gamma,\ 1 \leq i \leq n\})$.) As it is proven in [3, Section 6.4], there exists a polynomial $\phi_{\eta|K}(t) \in \mathbb{Q}[t]$ such that

$$\phi_{\eta|K}(r) = \operatorname{tr.deg}_K K(\{\gamma\eta_j | \gamma \in \Gamma(r), 1 \leq j \leq n\}) \text{ for all sufficiently large } r \in \mathbb{Z}.$$

This polynomial is called the σ^*-*dimension polynomial* of the σ^*-field extension L/K associated with the system of σ^*-generators $\eta = \{\eta_1, \dots, \eta_n\}$. It is also said to be the σ^*-dimension polynomial of system (1). We refer to [3, Chapter 6] and [4, Chapters 4 and 7] for properties, invariants, and methods of computation of σ^*-dimension polynomials.

Let us consider a system of equations in finite differences with respect to unknown functions of m real (or complex) variables $x_1, \dots, x_m$ that induces a prime system of algebraic difference equations. (The m basic automorphisms are defined by the shifts of the arguments: for any function $g(x_1, \dots, x_m)$, $\alpha_i : g(x_1, \dots, x_m) \mapsto g(x_1, \dots, x_{i-1}, x_i + h_i, x_{i+1}, \dots, x_m)$ where $h_1, \dots, h_m$ are some real (or complex) numbers.) It is shown in [4, Section 7.7] that the σ^*-dimension polynomial of such a system expresses its strength in the sense of A. Einstein. This important characteristic of the system is a difference counterpart the concept of strength of a system of PDEs introduced in [1], see [4, Section 7.7] for details.

2 Autoreduced sets of quasi-linear σ^*-polynomials. Computation of the Einstein's Strength

With the above notation, let $\Gamma Y = \{\gamma y_i | \gamma \in \Gamma, 1 \le i \le n\} \subseteq R$; the elements of this set are called *terms*. The order $\operatorname{ord} u$ of a term $u = \gamma y_j$ is defined as the order of γ.

In what follows we consider the set $\mathbb{Z}^m$ as the union of 2^m *orthants* $\mathbb{Z}_j^{(m)}$ $(1 \le j \le 2^m)$, that is, Cartesian products of m factors each of which is either $\mathbb{N} = \{k \in \mathbb{Z}, k \ge 0\}$ or $\bar{\mathbb{Z}}_- = \{k \in \mathbb{Z}, k \le 0\}$. We set $\Gamma_j = \{\alpha_1^{k_1} \dots \alpha_m^{k_m} \in \Gamma \,|\, (k_1, \dots, k_m) \in \mathbb{Z}_j^{(m)}\}$ and $(\Gamma Y)_j = \{\gamma y_i \,|\, \gamma \in \Gamma_j, 1 \le i \le n\}$, so that $\Gamma Y = \bigcup_{j=1}^{2^m} (\Gamma Y)_j$. A term $v \in \Gamma Y$ is called a *transform* of a term $u \in \Gamma Y$ if u and v belong to the same set $(\Gamma Y)_j$ and $v = \gamma u$ for some $\gamma \in \Gamma_j$. We also fix an *orderly ranking* on ΓY, that is, a well-ordering $\le$ of ΓY such that
(i) If $u \in (\Gamma Y)_j$ and $\gamma \in \Gamma_j$, then $u \le \gamma u$; (ii) If $u, v \in (\Gamma Y)_j$, $u \le v$ and $\gamma \in \Gamma_j$, then $\gamma u \le \gamma v$; (iii) If $u, v \in \Gamma Y$ and $\operatorname{ord} u < \operatorname{ord} v$, then $u < v$.

If $A \in R$, then the greatest (with respect to $\le$) term in A is called the *leader* of A; it is denoted by u_A. If $d = \deg_{u_A} A$ and A is written as a polynomial in u_A, then the coefficient of u_A^d is called the *initial* of A and is denoted by I_A. If $d = 1$ then the σ^*-polynomial A is called *quasi-linear*.

Let $A, B \in R$. The σ^*-polynomial A is said to be *reduced* with respect to B if A does not contain any power of a transform γu_B whose exponent is greater than or equal to $\deg_{u_B} B$. If $\mathcal{A} \subseteq R \setminus K$, then a σ^*-polynomial $A \in R$, is said to be reduced with respect to $\mathcal{A}$ if A is reduced with respect to every element of $\mathcal{A}$. A set $\mathcal{A} \subseteq R$ is said to be *autoreduced* if either $\mathcal{A} = \emptyset$ or $\mathcal{A} \bigcap K = \emptyset$ and the elements of $\mathcal{A}$ are reduced with respect to each other. As it is shown in [3, Section 3.4], distinct elements of an autoreduced set $\mathcal{A}$ have distinct leaders and every autoreduced set is finite. Furthermore, if $A \in R$, then there exists a σ^*-polynomial $B \in R$ such that B is reduced with respect to $\mathcal{A}$ and $IB \equiv A (mod[\mathcal{A}]^*)$ where I is a product of transforms of initials of elements of $\mathcal{A}$. (We say that A *reduces to* B *modulo* $\mathcal{A}$.)

Let $A, B \in R$. We say that A has higher rank than B and write $\operatorname{rk} A > \operatorname{rk} B$ if either $A \notin K$, $B \in K$, or $u_B < u_A$, or $u_A = u_B$ and $\deg_{u_A} B < \deg_{u_A} A$. If $u_A = u_B$ and $\deg_{u_A} A = \deg_{u_A} B$, we say that A and B have the same rank and write $\operatorname{rk} A = \operatorname{rk} B$. Assuming that elements of an autoreduced set in R are arranged in the order of increasing rank, we compare such sets as follows: if $\mathcal{A} = \{A_1, \dots, A_p\}$ and $\mathcal{B} = \{B_1, \dots, B_q\}$, then $\mathcal{A}$ is said to have lower rank than $\mathcal{B}$ if either there exists $k \in \mathbb{N}$, $1 \le k \le \min\{p, q\}$, such that $\operatorname{rk} A_i = \operatorname{rk} B_i$ for $i < k$ and $\operatorname{rk} A_k < \operatorname{rk} B_k$, or $p > q$ and $\operatorname{rk} A_i = \operatorname{rk} B_i$ for $i = 1, \dots, q$.

By [3, Proposition 3.4.30], every nonempty family of autoreduced sets contains an autoreduced set of lowest rank. If P is an ideal of R, then an autoreduced subset of P of lowest rank is called a *characteristic set* of P. Basic properties of characteristic sets are described in [4, Section 2.4]. In particular, it is shown that if P is generated by the σ^*-polynomials in the left-hand sides of a prime system of difference equations (1) and $\mathcal{A}$ is a characteristic set of P, then the σ^*-dimension polynomial of the system is determined by the leaders of elements of $\mathcal{A}$. Therefore, the strength of a prime system of difference equations is determined by a characteristic set of the associated σ^*-ideal in the ring of σ^*-polynomials.

An autoreduced subset $\mathcal{A}$ of R consisting of quasi-linear σ^*-polynomials is called *coherent* if it satisfies the following two conditions: (i) If $A \in \mathcal{A}$ and $\gamma \in \Gamma$, then γA reduces to zero modulo $\mathcal{A}$; (ii) If $A, B \in \mathcal{A}$ and $v = \gamma_1 u_A = \gamma_2 u_B$ is a common transform of u_A and u_B, then the σ^*-polynomial $(\gamma_2 I_B)(\gamma_1 A) - (\gamma_1 I_A)(\gamma_2 B)$ reduces to zero modulo $\mathcal{A}$.

The following two statements are the main results that allow one to evaluate the Einstein's strength of difference equations that arise from difference schemes for some chemical reaction-diffusion equations arising in many problems of transfusion, see [2].

Theorem 1. *If a characteristic set $\mathcal{A}$ of some σ^*-ideal in R consists of quasi-linear σ^*-polynomials, then $\mathcal{A}$ is a coherent autoreduced set. Conversely, if $\mathcal{A}$ is a coherent autoreduced set consisting of quasi-linear σ^*-polynomials, then it is a characteristic set of $[\mathcal{A}]^*$.*

Theorem 2. *Let $\preccurlyeq$ be a preorder on R such that $A_1 \preccurlyeq A_2$ iff u_{A_2} is a transform of u_{A_1}. Let A be a quasi-linear σ^*-polynomial and $\Gamma A = \{\gamma A \mid \gamma \in \Gamma\}$. Then the σ^*-ideal $[\mathcal{A}]^*$ is prime and all minimal (with respect to $\preccurlyeq$) elements of ΓA form a characteristic set of $[\mathcal{A}]^*$.*

Using the last two theorems and the expression of the σ^*-dimension polynomial given in [3, Theorem 6.4.8], we obtain σ^*-dimension polynomials that express the Einstein's strength of difference schemes for some quasi-linear reaction-diffusion PDEs (e. g., the Murray's equation and its particular cases), the system of PDEs of chemical reaction kinetics with the diffusion phenomena and the mass balance PDEs of chromatography. The results of the corresponding computations allow one to do comparative analysis of alternative difference schemes from the point of view of their strength.

This work was supported by the NSF grant CCF-1714425.

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