A New State-Space Representation for Coupled PDEs and Scalable Lyapunov Stability Analysis in the SOS Framework

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Abstract— We present a framework for stability analysis of systems of coupled linear Partial-Differential Equations (PDEs). The class of PDE systems considered in this paper includes parabolic, elliptic and hyperbolic systems with Dirichelet, Neuman and mixed boundary conditions. The results in this paper apply to systems with a single spatial variable and assume existence and continuity of solutions except in such cases when existence and continuity can be inferred from existence of a Lyapunov function. Our approach is based on a new concept of state for PDE systems which allows us to express the derivative of the Lyapunov function as a Linear Operator Inequality directly on L_2 and allows for any type of suitably well-posed boundary conditions. This approach obviates the need for integration by parts, spacing functions or similar mathematical encumbrances. The resulting algorithms are implemented in Matlab, tested on several motivating examples, and the codes have been posted online. Numerical testing indicates the approach has little or no conservatism for a large class of systems and can analyze systems of up to 20 coupled PDEs.

I. INTRODUCTION

Partial Differential Equations (PDEs) are used to model systems where the state depends continuously on both time and secondary independent variables. Common examples of such secondary dependence include space, as in, e.g. rigid bodies (Bernoulli-Euler beams) and fluid flow (Navier-Stokes), or maturation, as in, e.g. cell populations and predator-prey dynamics.

The most common method for stability analysis of PDEs is to project the state onto a finite-dimensional vector space using, e.g. [1], [2], [3] and to use the existing extensive literature on control of ODEs to test stability and design controllers for the resulting finite-dimensional system. However, such discretization approaches are often prone to instability and numerical ill-conditioning. Attempts to develop a rigorous state-space theory for PDEs without discretization includes the significant literature on Semigroup theory [4], [5], [6]. Perhaps the most well-known method for stabilization of PDEs without discretization is the backstepping approach to controller synthesis [7] (See the 2-state example in [8]). Unfortunately, however, backstepping cannot currently be used for direct construction of Lyapunov functions for the purpose of stability analysis. Additional work on the use of computational methods and LMIs for computing Lyapunov functions for PDEs can be found in the work of [9], [10], [11]. Other examples of LMI methods for stability analysis of PDEs include [12].

Recently, Sum-of-Squares (SOS) optimization methods have been applied to the problem of finding Lyapunov functions which prove stability of vector-valued PDEs. Examples of this work from our lab can be found in [13], [14], [15], [16] and work from our colleagues can be found in [13], [17], [18], [19]. While these previous works have proven remarkably effective, they suffered from high computational complexity and the lack of a unifying framework - deficiencies which limit the practical impact and scalability of these results. The goal of this paper is to provide such a unifying framework and significantly reduce computational complexity by re-evaluating the state-space framework on which these earlier works were based.

Specifically, in this paper, we consider the problem of stability analysis of multiple coupled linear PDEs in a single spatial variable. We write these systems in the universal form

$$\dot{x}(s,t) = A_0(s)x(s,t) + A_1(s)u_s(s,t) + A_2(s)x_{ss}(s,t)$$

where **x** is a **vector**-valued function $x : [a,b] \times \mathbb{R}^+ \to \mathbb{R}^n$ and with boundary constraints of the form

$$B \begin{bmatrix} x(a,t) & x(b,t) & x_s(a,t) & x_s(b,t) \end{bmatrix}^T = 0$$

where B is of row rank $2n$. These types of systems arise when there are multiple interacting spatially-distributed states and include wave equations, beam equations, et c.

The main technical result of this paper is to show that if **x** satisfies the boundary conditions and is suitably differentiable, then we have the following identities

$$x(s) = \int_a^b B_a(s,\eta)x_{ss}(\eta)d\eta + \int_a^s (s-\eta)x_{ss}(\eta)d\eta$$
$$x_s(s) = \int_a^b B_b(\eta)x_{ss}(\eta)d\eta + \int_a^s x_{ss}(\eta)d\eta,$$

where the matrix-valued functions B_a and B_b are uniquely determined by the matrix B and where $\mathbf{x}_{ss} \in L_2[a,b]$ need not satisfy any constraints in order to define a solution. This identity implies that for any $\mathbf{x}_{ss}$, the initial value problem is well-defined - implying that this is a boundary-condition independent representation of the state of the system.

We then use these identities to show that any Lyapunov function of the form

$$V(x) = \int_a^b x(s)^T \bigg(M(s)x(s)ds + \int_a^s N_1(s,\theta)x(\theta)d\theta + \int_s^b N_2(s,\theta)x(\theta)d\theta \bigg) ds$$

may be equivalently represented as

$$V(x) = \int_a^b x_{ss}(s)^T \left(\int_a^s R_1(s, \theta) x_{ss}(\theta) d\theta \right.$$
$$\left. + \int_s^b R_2(s, \theta) x_{ss}(\theta) d\theta \right) ds$$

for some R_1, R_2 and furthermore, the derivative of this functional, $\dot{V}$, may likewise be represented in the same form. We note that the structure of these quadratic Lyapunov functions are implied by the closed-loop stability conditions established via the backstepping transformation, as shown in [14]. Furthermore, these results imply that the problem of computing stability of linear PDEs is equivalent to the problem of determining positivity of Lyapunov functions of this form for arbitrary functions $x_{ss} \in L_2$.

In the remainder of this paper, we will establish the results listed above, provide a computational framework for enforcing positivity of Lyapunov functions of this form, and show that the results are non-conservative and scalable through the use of numerical examples. Note that the identities listed can also be extended to third and fourth-order spatial derivatives, if required.

II. NOTATION

In this paper, we define $L_2^n[X]$ to be space of $\mathbb{R}^n$-valued Lesbegue integrable functions defined on X and equipped with the standard inner product. We use $W^{k,p}[X]$ to denote the Sobolev subspace of $L_p[X]$ defined as $\{u \in L_p[X] : \frac{\partial^q}{\partial x^q} u \in L_p \text{ for all } q \leq k\}$.

III. PRELIMINARIES

In this paper we consider stability of solutions $\mathbf{x} : [a, b] \times \mathbb{R}^+ \to \mathbb{R}^n$ of PDEs of the form

$$x_t(s, t) = A_0(s)x(s, t) + A_1(s)x_s(s, t) + A_2(s)x_{ss}(s, t) \quad (1)$$

with boundary constraints of the form

$$B \begin{bmatrix} x(a, t)^T & x(b, t)^T & x_s(a, t)^T & x_s(b, t)^T \end{bmatrix}^T = 0. \quad (2)$$

These boundary conditions can be used to represent Dirichelet, Neumann, Robin, et c., with the only restriction that the row rank of B need be $2n$. In the semigroup framework, this translates to $\dot{\mathbf{x}} = \mathcal{A}\mathbf{x}$ with generator

$$\mathcal{A} := A_0(s) + A_1(s)\partial_s + A_2(s)\partial_{ss}$$

and domain

$$D_A :=$$
$$\{\mathbf{x} \in W^{2,2} : B \begin{bmatrix} \mathbf{x}(a)^T & \mathbf{x}(b)^T & \mathbf{x}_s(a)^T & \mathbf{x}_s(b)^T \end{bmatrix}^T = 0\}$$

IV. LYAPUNOV STABILITY

It seems that existence of a Lyapunov function does not guarantee existence and continuity of solutions for PDEs except in certain very limited special cases. Therefore, we must assume these properties hold and we give mathematical rigour to this assumption by assuming the existence of a "Semi-continuous semigroup", $S(t) : X \to X$ with domain

$D_A \subset X$ so that $S(\tau)\mathbf{x}(\cdot, t) = \mathbf{x}(\cdot, t + \tau)$ for any solution to Eqns. (1) and (2). See [4]. The following is from [5].

Theorem 1: Suppose that $\mathcal{A}$ generates a strongly-continuous semigroup on X with domain D_A and there exists $\alpha, \beta, \gamma > 0$ and $\mathcal{P} : X \to X$ such that $\alpha\|\mathbf{x}\|_X \leq \langle \mathbf{x}, \mathcal{P}\mathbf{x} \rangle_X \leq \beta\|\mathbf{x}\|_X$ and

$$\langle \mathbf{x}, \mathcal{P}\mathcal{A}\mathbf{x} \rangle_X + \langle \mathcal{A}\mathbf{x}, \mathcal{P}\mathbf{x} \rangle_X \leq -\gamma\|\mathbf{x}\|_X$$

for all $\mathbf{x} \in D_A$. Then the system defined by Eqns (1) and (2) is exponentially stable in $\|\cdot\|_X$.

In this paper, we show how these conditions may be enforced when $\mathcal{A}$ and D_A are as defined in Section III. In this case $\mathcal{A}$ is a differential operator. We will show in Sections V and VI that the stability conditions in Theorem 1 can be reformulated with on $D_A = L_2$ and in Sections VII and VIII we will show that these conditions can be enforced using LMIs based on an SOS-style approach.

V. FUNDAMENTAL IDENTITIES

In this section, we show that if

$$B \begin{bmatrix} x(a)^T & x(b)^T & x_s(a)^T & x_s(b)^T \end{bmatrix}^T = 0$$

where B is of row rank $2n$, then the following identities hold

$$x(s) = \int_a^b B_a(s, \eta)x_{ss}(\eta)d\eta + \int_a^s (s - \eta)x_{ss}(\eta)d\eta$$

$$x_s(s) = \int_a^b B_b(\eta)x_{ss}(\eta)d\eta + \int_a^s x_{ss}(\eta)d\eta,$$

where B_a and B_b are uniquely determined by the matrix B.

First, we establish the auxiliary identities:

Lemma 2: Suppose that x is twice continuously differentiable. Then

$$x_s(s) = x_s(a) + \int_a^s x_{ss}(\eta)d\eta$$

$$x(s) = x(a) + x_s(a)(s - a) + \int_a^s (s - \eta)x_{ss}(\eta)d\eta$$

Proof: The first identity is the fundamental theorem of calculus. The second identity is a repeated application of the fundamental theorem of calculus, combined with a change of variables. That is,

$$x(s) = x(a) + \int_a^s x_s(\eta)d\eta$$

$$= x(a) + \int_a^s x_s(a)ds + \int_a^s \int_a^\eta x_{ss}(\zeta)d\zeta d\eta$$

Examining the 3rd term, where $I(s)$ is the indicator function,

$$\int_a^s \int_a^\eta x_{ss}(\zeta)d\zeta d\eta = \int_a^b \int_a^b I(s - \eta)I(\eta - \zeta)x_{ss}(\zeta)d\zeta d\eta$$

$$= \int_a^b \left(\int_a^b I(s - \eta)I(\eta - \zeta)d\eta \right) x_{ss}(\zeta)d\zeta$$

$$= \int_a^b I(s - \zeta) \left(\int_s^\zeta d\eta \right) x_{ss}(\zeta)d\zeta = \int_a^s (s - \zeta)x_{ss}(\zeta)d\zeta$$

which is the desired result ∎

As an obvious corollary, we have

$$x_s(b) = x_s(a) + \int_a^b x_{ss}(\eta)d\eta$$

$$x(b) = x(a) + x_s(a)(b-a) + \int_a^b (b-\eta)x_{ss}(\eta)d\eta$$

The implication is that any boundary value can be expressed using two other boundary identities. We can now generalize this to the main result.

Theorem 3: Suppose $\mathbf{x} \in W^{3,2}[a,b]$ and

$$B\begin{bmatrix} x(a)^T & x(b)^T & x_s(a)^T & x_s(b)^T \end{bmatrix}^T = 0$$

where B has row rank $2n$, then

$$x(s) = \int_a^b B_a(s,\eta)x_{ss}(\eta)d\eta + \int_a^s (s-\eta)x_{ss}(\eta)d\eta$$

$$x_s(s) = \int_a^b B_b(\eta)x_{ss}(\eta)d\eta + \int_a^s x_{ss}(\eta)d\eta,$$

where

$$B_a(s,\eta) = B_4(s)(b-\eta) + B_5(s),$$
$$B_b(\eta) = B_6(b-\eta) + B_7$$
$$\begin{bmatrix} B_6 & B_7 \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}B_3,$$
$$\begin{bmatrix} B_4(s) & B_5(s) \end{bmatrix} = \begin{bmatrix} I & (s-a)I \end{bmatrix}B_3$$
$$B_3 = B_2^{-1}B\begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix}, \qquad B_2 = B\begin{bmatrix} I & 0 \\ I & (b-a)I \\ 0 & I \\ 0 & I \end{bmatrix}$$

Proof: Using Lemma 2, we can express all boundary terms using $x(a)$, $x_s(a)$, and $x_{ss}(s)$.

$$\begin{bmatrix} x(a) \\ x(b) \\ x_s(a) \\ x_s(b) \end{bmatrix} = \begin{bmatrix} I & 0 \\ I & (b-a)I \\ 0 & I \\ 0 & I \end{bmatrix}\begin{bmatrix} x(a) \\ x_s(a) \end{bmatrix} + \begin{bmatrix} 0 \\ \int_a^b(b-\eta)x_{ss}(\eta)d\eta \\ 0 \\ \int_a^b x_{ss}(\eta)d\eta \end{bmatrix}$$

Hence

$$B\begin{bmatrix} x(a) \\ x(b) \\ x_s(a) \\ x_s(b) \end{bmatrix} = B\begin{bmatrix} I & 0 \\ I & (b-a)I \\ 0 & I \\ 0 & I \end{bmatrix}\begin{bmatrix} x(a) \\ x_s(a) \end{bmatrix}$$

$$\underbrace{\phantom{B\begin{bmatrix} I & 0 \\ I & (b-a)I \\ 0 & I \\ 0 & I \end{bmatrix}}}_{B_2}$$

$$+ B\begin{bmatrix} 0 \\ \int_a^b(b-\eta)x_{ss}(\eta)d\eta \\ 0 \\ \int_a^b x_{ss}(\eta)d\eta \end{bmatrix}$$

$$= B_2\begin{bmatrix} x(a) \\ x_s(a) \end{bmatrix} + B\begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} \int_a^b(b-\eta)x_{ss}(\eta)d\eta \\ \int_a^b x_{ss}(\eta)d\eta \end{bmatrix} = 0$$

Since B has $2n$ row rank, B_2 is invertible and hence we have

$$\begin{bmatrix} x(a) \\ x_s(a) \end{bmatrix} = -B_2^{-1}\underbrace{B\begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix}}_{B_3}\begin{bmatrix} \int_a^b(b-\eta)x_{ss}(\eta)d\eta \\ \int_a^b x_{ss}(\eta)d\eta \end{bmatrix}$$

Now, using Lemma 2,

$$x(s) = x(a) + x_s(a)(s-a) + \int_a^s (s-\eta)x_{ss}(\eta)d\eta$$

$$= \begin{bmatrix} I & (s-a)I \end{bmatrix}\begin{bmatrix} x(a) \\ x_s(a) \end{bmatrix} + \int_a^s (s-\eta)x_{ss}(\eta)d\eta$$

$$= \underbrace{\begin{bmatrix} I & (s-a)I \end{bmatrix}B_3}_{\begin{bmatrix} B_4(s) & B_5(s) \end{bmatrix}}\begin{bmatrix} \int_a^b(b-\eta)x_{ss}(\eta)d\eta \\ \int_a^b x_{ss}(\eta)d\eta \end{bmatrix} + \int_a^s (s-\eta)x_{ss}(\eta)d\eta$$

$$= \int_a^b B_4(s)(b-\eta)x_{ss}(\eta)d\eta$$
$$+ B_5(s)\int_a^b x_{ss}(\eta)d\eta + \int_a^s (s-\eta)x_{ss}(\eta)d\eta$$

$$= \int_a^b \left(B_4(s)(b-\eta) + B_5(s)\right)x_{ss}(\eta)d\eta + \int_a^s (s-\eta)x_{ss}(\eta)d\eta$$

$$= \int_a^b B_a(s,\eta)x_{ss}(\eta)d\eta + \int_a^s (s-\eta)x_{ss}(\eta)d\eta.$$

Likewise, we have

$$x_s(s) = x_s(a) + \int_a^s x_{ss}(\eta)d\eta$$

$$= \begin{bmatrix} 0 & I \end{bmatrix}\begin{bmatrix} x(a) \\ x_s(a) \end{bmatrix} + \int_a^s x_{ss}(\eta)d\eta$$

$$= \underbrace{\begin{bmatrix} 0 & I \end{bmatrix}B_3}_{\begin{bmatrix} B_6 & B_7 \end{bmatrix}}\begin{bmatrix} \int_a^b(b-\eta)x_{ss}(\eta)d\eta \\ \int_a^b x_{ss}(\eta)d\eta \end{bmatrix} + \int_a^s x_{ss}(\eta)d\eta$$

$$= \int_a^b \left(B_6(b-\eta) + B_7\right)x_{ss}(\eta)d\eta + \int_a^s x_{ss}(\eta)d\eta$$

$$= \int_a^b B_b(\eta)x_{ss}(\eta)d\eta + \int_a^s x_{ss}(\eta)d\eta.$$

$\blacksquare$

VI. REFORMULATION OF THE LYAPUNOV FUNCTION

If we denote the class of operators $\mathcal{P}_{\{M,N_1,N_2\}} : L_2^n \to L_2^n$ by

$$\left(\mathcal{P}_{\{M,N_1,N_2\}}\mathbf{x}\right)(s)$$

$$= M(s)x(s)ds + \int_a^s N_1(s,\theta)x(\theta)d\theta + \int_s^b N_2(s,\theta)x(\theta)d\theta,$$

then we may compactly represent our Lyapunov candidate form as

$$V(\mathbf{x}) = \left\langle \mathbf{x}, \mathcal{P}_{\{M,N_1,N_2\}}\mathbf{x} \right\rangle_{L_2}$$

The derivative of the Lyapunov candidate may then be likewise compactly represented (w/ slight abuse of notation)

as

$$\frac{1}{2}\dot{V}(\mathbf{x}) = \langle \mathbf{x}, \mathcal{P}_{\{MA_0, N_1 A_0, N_2 A_0\}} \mathbf{x} \rangle$$
$$+ \langle \mathbf{x}, \mathcal{P}_{\{MA_1, N_1 A_1, N_2 A_1\}} \mathbf{x}_s \rangle + \langle \mathbf{x}, \mathcal{P}_{\{MA_2, N_1 A_2, N_2 A_2\}} \mathbf{x}_{ss} \rangle$$

The challenge, then, is to show that each of these terms may, in turn, be represented in the form

$$\langle \mathbf{x}_{ss}, \mathcal{P}_{\{0, R_1, R_2\}} \mathbf{x}_{ss} \rangle_{L_2}$$

through repeated use of the identities

$$x(s) = \int_a^b B_a(s, \eta) x_{ss}(\eta) d\eta + \int_a^s (s - \eta) x_{ss}(\eta) d\eta$$
$$x_s(s) = \int_a^b B_b(\eta) x_{ss}(\eta) d\eta + \int_a^s x_{ss}(\eta) d\eta.$$

For convenience, we leave off the A_i terms and address each inner product separately. Let use also define the following functions which are common to all three results.

$$Y_1(s, \eta) = B_a(\eta, s)^T M(\eta) + \int_\eta^b B_a(\theta, s)^T N_1(\theta, \eta) d\theta$$
$$+ \int_a^\eta B_8(\theta, s)^T N_2(\theta, \eta) d\theta$$

$$Y_2(\zeta) = M(\zeta) + \int_a^\zeta N_1(\zeta, \theta) d\theta + \int_\zeta^b N_2(\zeta, \theta) d\theta$$

$$Y_3(\zeta, \eta) = M(\zeta) B_a(\zeta, \eta) + \int_a^\zeta N_1(\zeta, \theta) B_a(\theta, \eta) d\theta$$
$$+ \int_\zeta^b N_2(\zeta, \theta) B_a(\theta, \eta) d\theta$$

Note that these functions are defined in terms of M, N_1, and N_2 and hence will vary if these terms are defined differently for Lemmas 4, 5, and 6.

Lemma 4: Suppose $\mathbf{x}$ satisfies the conditions of Thm. 3. Then
$$\langle \mathbf{x}, \mathcal{P}_{\{M, N_1, N_2\}} \mathbf{x}_{ss} \rangle = \langle \mathbf{x}_{ss}, \mathcal{P}_{\{0, R_1, R_2\}} \mathbf{x}_{ss} \rangle$$
where
$$R_1(s, \theta) = E_1(s, \theta) + E_3(s, \theta),$$
$$R_2(s, \theta) = E_2(s, \theta) + E_3(s, \theta)$$

$$E_1(s, \theta) = \int_s^b (\eta - s) N_1(\eta, \theta) d\eta$$

$$E_2(s, \theta) = (\theta - s) M(\theta) + \int_\theta^b (\eta - s) N_1(\eta, \theta) d\eta$$
$$+ \int_s^\theta (\eta - s) N_2(\eta, \theta) d\eta$$

$$E_3(s, \theta) = Y_1(s, \theta)$$

Proof: The proofs of these Lemmas cannot be included in conference format due to length constraints. Therefore, for the proof of these lemmas, we refer to an Arxiv Appendix, available online at [20] ∎

Notation: For convenience, we say
$$(R_1, R_2) = \mathcal{L}_1(M, N_1, N_2)$$
if R_1, R_2, M, N_1, and N_2 satisfy the conditions of Lemma 4.

Lemma 5: Suppose $\mathbf{x}$ satisfies the conditions of Thm. 3. Then
$$\langle \mathbf{x}, \mathcal{P}_{\{M, N_1, N_2\}} \mathbf{x}_s \rangle = \langle \mathbf{x}_{ss}, \mathcal{P}_{\{0, Q_1, Q_2\}} \mathbf{x}_{ss} \rangle$$
where
$$Q_1(s, \theta) = F_1(s, \theta) + F_3(s, \theta)$$
$$Q_2(s, \theta) = F_2(s, \theta) + F_3(s, \theta)$$

$$F_1(s, \theta) = \int_s^b ((\eta - s) F_4(\theta, \eta) + F_5(s, \eta)) d\eta$$

$$F_2(s, \theta) = \int_\theta^b ((\eta - s) F_4(\theta, \eta) + F_5(s, \eta)) d\eta$$

$$F_3(s, \eta) = \int_a^b B_a(\zeta, s)^T Y_2(\zeta) B_b(\eta) d\zeta + \int_\eta^b Y_1(s, \zeta) d\zeta$$
$$+ \int_s^b (\zeta - s) Y_2(\zeta) d\zeta B_b(\eta)$$

$$F_4(\theta, \eta) = M(\eta) + \int_\theta^\eta N_1(\eta, \zeta) d\zeta$$

$$F_5(s, \eta) = \int_s^\eta (\zeta - s) N_2(\zeta, \eta) d\zeta.$$

Notation: For convenience, we say
$$(Q_1, Q_2) = \mathcal{L}_2(M, N_1, N_2)$$
if Q_1, Q_2, M, N_1, and N_2 satisfy the conditions of Lemma 5.

Lemma 6: Suppose $\mathbf{x}$ satisfies the conditions of Thm. 3. Then
$$\langle \mathbf{x}, \mathcal{P}_{\{M, N_1, N_2\}} \mathbf{x} \rangle = \langle \mathbf{x}_{ss}, \mathcal{P}_{\{0, T_1, T_2\}} \mathbf{x}_{ss} \rangle$$
where
$$T_1(s, \theta) = G_1(s, \theta) + G_3(s, \theta)$$
$$T_2(s, \theta) = G_2(s, \theta) + G_3(s, \theta)$$

$$G_1(s, \theta) = \int_s^b ((\eta - s) G_4(\theta, \eta) + G_5(s, \theta, \eta)) d\eta$$

$$G_2(s, \theta) = \int_\theta^b ((\eta - s) G_4(\theta, \eta) + G_5(s, \theta, \eta)) d\eta$$

$$G_3(s, \theta) = \int_a^b B_a(\eta, s)^T Y_3(\eta, \theta) d\eta$$
$$+ \int_\theta^b (\eta - \theta) Y_1(s, \eta) d\eta + \int_s^b (\eta - s) Y_3(\eta, \theta) d\eta$$

$$G_4(\theta, \eta) = (\eta - \theta) M(\eta) + \int_\theta^\eta (\zeta - \theta) N_1(\eta, \zeta) d\zeta$$

$$G_5(s, \theta, \eta) = \int_s^\eta (\zeta - s)(\eta - \theta) N_2(\zeta, \eta) d\zeta.$$

Notation: For convenience, we say
$$(T_1, T_2) = \mathcal{L}_3(M, N_1, N_2)$$
if T_1, T_2, M, N_1, and N_2 satisfy the conditions of Lemma 6.

Note that the operators obtained here are not necessarily symmetric. However, we may construct an equivalent symmetric representation as $\mathcal{P} + \mathcal{P}^*$ using

$$\mathcal{P}^*_{\{0, Q_2(\theta, s)^T, Q_1(\theta, s)^T\}} = \mathcal{P}_{\{0, Q_1(s, \theta), Q_2(s, \theta)\}}.$$

That is, in the symmetric representation, $Q_1(s, \theta) = Q_2(\theta, s)^T$.

VII. Positivity of Operators

Now that we have shown how to represent our Lyapunov stability conditions as positivity of operators of the form $\mathcal{P}_{\{0,N_1,N_2\}}$, we must show how to use LMIs to enforce positivity of these operators when N_1 and N_2 are polynomials. This is a slight generalization of the result in [21].

Theorem 7: For any square-integrable functions $Z(s)$ and $Z(s,\theta)$, if $g(s) \geq 0$ for all $s \in [a,b]$ and

$$M(s) = g(s)Z(s)^T P_{11} Z(s)$$
$$N_1(s,\theta) = g(s)Z(s)^T P_{12} Z(s,\theta) + g(\theta)Z(\theta,s)^T P_{31} Z(\theta)$$
$$+ \int_0^\theta g(\nu)Z(\nu,s)^T P_{33} Z(\nu,\theta)d\nu + \int_\theta^s g(\nu)Z(\nu,s)^T P_{32} Z(\nu,\theta)d\nu$$
$$+ \int_s^L g(\nu)Z(\nu,s)^T P_{22} Z(\nu,\theta)d\nu$$
$$N_2(s,\theta) = g(s)Z(s)^T P_{13} Z(s,\theta) + g(\theta)Z(\theta,s)^T P_{21} Z(\theta)$$
$$+ \int_0^s g(\nu)Z(\nu,s)^T P_{33} Z(\nu,\theta)d\nu + \int_s^\theta g(\nu)Z(\nu,s)^T P_{23} Z(\nu,\theta)d\nu$$
$$+ \int_\theta^L g(\nu)Z(\nu,s)^T P_{22} Z(\nu,\theta)d\nu,$$

where

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \geq 0,$$

then $\langle \mathbf{x}, \mathcal{P}_{\{M,N_1,N_2\}}\mathbf{x}\rangle_{L_2} \geq 0$ for all $\mathbf{x} \in L_2[a,b]$.

Proof: Define the operator

$$(\mathcal{Z}\mathbf{x})(s) = \begin{bmatrix} \sqrt{g(s)}Z(s)\mathbf{x}(s) \\ \int_0^s \sqrt{g(s)}Z(s,\theta)\mathbf{x}(\theta)d\theta \\ \int_s^b \sqrt{g(s)}Z(s,\theta)\mathbf{x}(\theta)d\theta \end{bmatrix}.$$

Then

$$\langle \mathbf{x}, \mathcal{P}_{\{M,N_1,N_2\}}\mathbf{x}\rangle = \langle \mathcal{Z}\mathbf{x}, P\mathcal{Z}\mathbf{x}\rangle = \langle P^{\frac{1}{2}}\mathcal{Z}\mathbf{x}, P^{\frac{1}{2}}\mathcal{Z}\mathbf{x}\rangle \geq 0.$$

∎

For convenience, we define the cone of such operators as

$$\Phi := \{(M, N_1, N_2) : M, N_1 \text{ and } N_2 \text{ satisfy}$$
$$\text{the conditions of Thm. 7.}\} \quad (3)$$

where the dimension of the matrices M, N_1 and N_2 should be clear from context.

VIII. SOS Conditions for Stability

The stability conditions can now be written concisely using the definitions of Φ, $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$ as follows.

Theorem 8: Suppose there exist $\epsilon > 0$,

$$(M - \epsilon I, N_1, N_2) \in \Phi$$

and

$$\left(0, -H_1(s,\theta) - H_2(\theta,s)^T, -H_2(s,\theta) - H_1(\theta,s)^T\right) \in \Phi$$

where

$$(H_1, H_2) = \mathcal{L}_1(V_1, W_{11}, W_{12})$$
$$+ \mathcal{L}_2(V_2, W_{21}, W_{22}) + \mathcal{L}_3(V_3, W_{31}, W_{32})$$

$$V_1(s) = M(s)A_0(s) + \epsilon I, \qquad W_{11}(s,\theta) = N_1(s,\theta)A_0(\theta),$$
$$W_{12}(s,\theta) = N_2(s,\theta)A_0(\theta)$$
$$V_2(s) = M(s)A_1(s), \qquad W_{21}(s,\theta) = N_1(s,\theta)A_1(\theta),$$
$$W_{22}(s,\theta) = N_2(s,\theta)A_1(\theta))$$
$$V_3(s) = M(s)A_2(s), \qquad W_{31}(s,\theta) = N_1(s,\theta)A_2(\theta),$$
$$W_{32}(s,\theta) = N_2(s,\theta)A_2(\theta))$$

Then any solution of Eqns. (1) and (2) is exponentially stable.

Proof: Let

$$V(\mathbf{x}) = \langle \mathbf{x}, \mathcal{P}_{\{M,N_1,N_2\}}\mathbf{x}\rangle \geq \epsilon \|\mathbf{x}\|_{L_2}^2.$$

Then

$$\dot{V}(\mathbf{x}) + 2\epsilon\|\mathbf{x}\|_{L_2}^2 = 2\langle \mathbf{x}, P_{\{V_1,W_{11},W_{12}\}}\mathbf{x}\rangle$$
$$+ 2\langle \mathbf{x}, P_{\{V_2,W_{21},W_{22}\}}\mathbf{x}_s\rangle + 2\langle \mathbf{x}, P_{\{V_3,W_{31},W_{32}\}}\mathbf{x}_{ss}\rangle$$
$$= 2\langle \mathbf{x}_{ss}, \mathcal{P}_{\{0,H_1,H_2\}}\mathbf{x}_{ss}\rangle \leq 0$$

Therefore, we have exponential stability from Thm. 1. ∎

IX. Numerical Implementation and Analysis

In this section, we examine the accuracy and computational complexity of the proposed stability algorithm by applying the results to several well-studied problems. The algorithms are implemented using a Matlab toolbox which is an adaptation of SOSTOOLS [22] and which can be found online at http://control.asu.edu. In all cases, the conditions of Theorems 8 and 7 are applied by choosing Z to be a vector of monomial bases of degree d and less and either $g(s) = 1$ or $g(s) = (s-a)(b-s)$.

Example 1: We begin with several variations of the diffusion equation. The first is adapted from [18].

$$\dot{x}(t,s) = \lambda x(t,s) + x_{ss}(t,s)$$

where $x(0) = x(1) = 0$ and which is known to be stable if and only if $\lambda < \pi^2 = 9.8696$. For $d = 1$, the algorithm is able to prove stability for $\lambda = 9.8696$ with a computation time of .54s.

Example 2: The second example from [19] is the same, but changes the boundary conditions to $x(0) = 0$ and $x_s(1) = 0$ and is unstable for $\lambda > 2.467$. For $d = 1$, the algorithm is able to prove stability for $\lambda = 2.467$ with identical computation time.

Example 3: The third example from [14] is not homogeneous

$$\dot{x}(t,s) = (-.5s^3 + 1.3s^2 - 1.5s + .7 + \lambda)x(t,s)$$
$$+ (3s^2 - 2s)x_s(t,s) + (s^3 - s^2 + 2)x_{ss}(t,s)$$

where $x(0) = 0$ and $x_s(1) = 0$ and was estimated numerically to be unstable for $\lambda > 4.65$. For $d = 1$, the algorithm is able to prove stability for $\lambda = 4.65$ with similar computation time.

Example 4: In this example from [18], we have

$$\dot{x}(t,s) = \begin{bmatrix} 1 & 1.5 \\ 5 & .2 \end{bmatrix} x(t,s) + R^{-1}x_{ss}(t,s)$$

with $x(0) = 0$ and $x_s(1) = 0$. In this case, using $d = 1$, we can prove stability for $R = 2.93$ (improvement over $R = 2.45$ in [18]) with a computation time of $1.21s$.

Example 5: In this example from [19], we have

$$\dot{x}(t, s) = \begin{bmatrix} 0 & 0 & 0 \\ s & 0 & 0 \\ s^2 & -s^3 & 0 \end{bmatrix} x(t, s) + R^{-1} x_{ss}(t, s)$$

with $x(0) = 0$ and $x_s(1) = 0$. In this case, using $d = 1$, we prove stability for $R = 21$ (and greater) with a computation time of $4.06s$.

Example 6: Next, we consider a damped wave equation $x_{tt} = x_{ss} - kx_t$ with $x(0) = x(1) = 0$

$$\dot{x}(t, s) = \begin{bmatrix} 0 & 1 \\ 0 & -k \end{bmatrix} x(t, s) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x_{ss}(t, s)$$

This is shown to be stable for $k = .1$ with a computation time of $1.54s$.

Example 7: Finally, we explore computational complexity using a simple n-dimensional diffusion equation

$$\dot{x}(t, s) = x(t, s) + x_{ss}(t, s)$$

where $x(t, s) \in \mathbb{R}^n$. We then evaluate the computation time for different size problems, from $n = 1$ to $n = 20$.

n	1	5	10	20
CPU sec	.54	37.4	745	31620

X. Conclusion

In this paper, we have shown that stability of a large class of PDE systems can be represented compactly in LMI form using a variation of Sum-of-Squares optimization. To achieve this result, we proposed that the state of a PDE of the form of Equation (1) is actually $\mathbf{x}_{ss}$ and that all Lyapunov stability conditions may be represented on this state. A SOS-style algorithm to test these Lyapunov conditions is proposed and numerical examples indicate no conservatism in the stability conditions to at least 5 significant figures even for low polynomial degree. It is clear that these results can also be directly extended to: PDEs with uncertainty; H_∞-gain analysis of PDEs; H_∞-optimal observer synthesis for PDEs; H_∞-optimal control of PDEs. Nonlinear Stability analysis can likewise be considered. In addition, the identities proposed in Section V by be extended to multiple spatial dimensions. Some unanswered questions include how to repose several common stability problems in the proposed generalized framework. For example, the wave equation with $u_t(L) = -u_x(L)$ in its native form is not suitably well-posed as the B matrix does not have row rank $2n$.

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Appendix: Proofs from "A New State-Space Representation for Coupled PDEs and Scalable Lyapunov Stability Analysis in the SOS Framework"

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Abstract

Proofs of the Lemmas in "A New State-Space Representation for Coupled PDEs and Scalable Lyapunov Stability Analysis in the SOS Framework". These proofs are not compressed in order to allow for easier review and verification. The proofs are relatively straightforward and almost all manipulation in all proofs is based on three basic ways to use indicator functions to change the order of integration.

I. Proof of Lemma 4

First, we restate the Lemma where we define

$$Y_1(s,\eta) = B_a(\eta,s)^T M(\eta) + \int_\eta^b B_a(\theta,s)^T N_1(\theta,\eta)d\theta + \int_a^\eta B_a(\theta,s)^T N_2(\theta,\eta)d\theta$$

$$Y_2(\zeta) = M(\zeta) + \int_a^\zeta N_1(\zeta,\theta)d\theta + \int_\zeta^b N_2(\zeta,\theta)d\theta$$

$$Y_3(\zeta,\eta) = M(\zeta)B_a(\zeta,\eta) + \int_a^\zeta N_1(\zeta,\theta)B_a(\theta,\eta)d\theta + \int_\zeta^b N_2(\zeta,\theta)B_a(\theta,\eta)d\theta.$$

Lemma 1: Suppose $\mathbf{x}$ satisfies the conditions of Thm. 3. Then
$$\langle \mathbf{x}, \mathcal{P}_{\{M,N_1,N_2\}}\mathbf{x}_{ss}\rangle = \langle \mathbf{x}_{ss}, \mathcal{P}_{\{0,R_1,R_2\}}\mathbf{x}_{ss}\rangle$$
where

$$R_1(s,\theta) = E_1(s,\theta) + E_3(s,\theta),$$

$$R_2(s,\theta) = E_2(s,\theta) + E_3(s,\theta)$$

$$E_1(s,\theta) = \int_s^b (\eta - s)N_1(\eta,\theta)d\eta$$

$$E_2(s,\theta) = (\theta - s)M(\theta) + \int_\theta^b (\eta - s)N_1(\eta,\theta)d\eta + \int_s^\theta (\eta - s)N_2(\eta,\theta)d\eta$$

$$E_3(s,\theta) = Y_1(s,\theta)$$

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Proof:

First, we decompose $\mathcal{P}_{\{M,N_1,N_2\}}$ as

$$\left(\mathcal{P}_{\{M,N_1,N_2\}}\psi\right)(s) = M(s)\psi(s) + \int_a^s N_1(s,\theta)\phi(\theta)d\theta + \int_s^b N_2(s,\theta)\phi(\theta)d\theta,$$

$$= M(s)\psi(s) + \int_a^b I(s-\theta)N_1(s,\theta)\phi(\theta)d\theta + \int_a^b I(\theta-s)N_2(s,\theta)\phi(\theta)d\theta,$$

$$= (\mathcal{P}_1\psi)(s) + (\mathcal{P}_2\psi)(s) + (\mathcal{P}_3\psi)(s),$$

Similarly, we decompose x as

$$x(s) = \int_a^b B_a(s,\eta)x_{ss}(\eta)d\eta + \int_a^s (s-\eta)x_{ss}(\eta)d\eta = \int_a^b \left(B_a(s,\eta) + I(s-\eta)(s-\eta)\right)x_{ss}(\eta)d\eta$$

$$= x_1(s) + x_2(s)$$

where x_1 is determined by the boundary conditions (BCs) and x_2 is independent of the BCs. As a result, we will have 6 terms to examine.

$$\langle x, \mathcal{P}x \rangle = \langle x_1, \mathcal{P}_1 x \rangle + \langle x_1, \mathcal{P}_2 x \rangle + \langle x_1, \mathcal{P}_3 x \rangle$$

$$+ \langle x_2, \mathcal{P}_1 x \rangle + \langle x_2, \mathcal{P}_2 x \rangle + \langle x_2, \mathcal{P}_3 x \rangle$$

We start with the non-BC term $\langle x_2, \mathcal{P}_1 x_{ss} \rangle$, $\langle x_2, \mathcal{P}_3 x_{ss} \rangle$, and $\langle x_2, \mathcal{P}_3 x_{ss} \rangle$.

$$\langle x_2, \mathcal{P}_1 x \rangle = \int_a^b \int_a^b (s-\theta)I(s-\theta)x_{ss}(\theta)^T M(s)x_{ss}(s)d\theta ds$$

$$= \int_a^b \int_a^b (\theta-s)I(\theta-s)x_{ss}(s)^T M(\theta)x_{ss}(\theta)d\theta ds$$

$$= \int_a^b \int_s^b x_{ss}(s)^T (\theta-s)M(\theta)x_{ss}(\theta)d\theta ds$$

$$= \left\langle x_{ss}, \mathcal{P}_{0,0,(\theta-s)M(\theta)}x_{ss} \right\rangle_{L_2} = \left\langle x_{ss}, \mathcal{P}_{0,0,T_{12}(s,\theta)}x_{ss} \right\rangle_{L_2}$$

where

$$T_{12}(s,\theta) = (\theta-s)M(\theta)$$

Next, we have

$$\langle x_2, \mathcal{P}_2 x \rangle = \int_a^b x(s)^T \int_a^b I(s-\theta)N_1(s,\theta)x_{ss}(\theta)d\theta ds$$

$$= \int_a^b \int_a^b \left(\int_a^b (s-\eta)I(s-\eta)x_{ss}(\eta)d\eta\right)^T I(s-\theta)N_1(s,\theta)x_{ss}(\theta)d\theta ds$$

$$= \int_a^b \int_a^b x_{ss}(\eta)^T \int_a^b \left((s-\eta)I(s-\eta)I(s-\theta)N_1(s,\theta)\right)ds\, x_{ss}(\theta)d\eta d\theta$$

Since

$$I(s - \eta)I(s - \theta) = I(\theta - \eta)I(s - \theta) + I(\eta - \theta)I(s - \eta)$$

$$\int_a^b \int_a^b x_{ss}(\eta)^T \int_a^b \left((s - \eta)I(s - \eta)I(s - \theta)N_1(s, \theta) \right) ds x_{ss}(\theta) d\eta d\theta$$

$$= \int_a^b \int_a^b I(\theta - \eta)x_{ss}(\eta)^T \int_a^b \left((s - \eta)I(s - \theta)N_1(s, \theta) \right) ds x_{ss}(\theta) d\eta d\theta$$

$$+ \int_a^b \int_a^b I(\eta - \theta)x_{ss}(\eta)^T \int_a^b \left((s - \eta)I(s - \eta)N_1(s, \theta) \right) ds x_{ss}(\theta) d\eta d\theta$$

$$= \int_a^b \int_a^b I(\theta - \eta)x_{ss}(\eta)^T \int_\theta^b \left((s - \eta)N_1(s, \theta) \right) ds x_{ss}(\theta) d\eta d\theta$$

$$+ \int_a^b \int_a^b I(\eta - \theta)x_{ss}(\eta)^T \int_\eta^b \left((s - \eta)N_1(s, \theta) \right) ds x_{ss}(\theta) d\eta d\theta$$

$$= \int_a^b \int_a^b I(\theta - s)x_{ss}(s)^T \int_\theta^b \left((\eta - s)N_1(\eta, \theta) \right) d\eta x_{ss}(\theta) d\eta d\theta$$

$$+ \int_a^b \int_a^b I(s - \theta)x_{ss}(s)^T \int_s^b \left((\eta - s)N_1(\eta, \theta) \right) d\eta x_{ss}(\theta) d\eta d\theta$$

$$= \langle x_{ss}, \mathcal{P}_{0, T_{21}, T_{22}} x_{ss} \rangle$$

where

$$T_{21}(s, \theta) = \int_s^b \left((\eta - s)N_1(\eta, \theta) \right) d\eta$$

$$T_{22}(s, \theta) = \int_\theta^b \left((\eta - s)N_1(\eta, \theta) \right) d\eta$$

The third term is slightly simpler:

$$\langle x_2, \mathcal{P}_3 x \rangle = \int_a^b x(s)^T \int_a^b I(\theta - s)N_2(s, \theta)x_{ss}(\theta) d\theta ds$$

$$= \int_a^b \int_a^b x_{ss}(\eta)^T \int_a^b \left((s - \eta)I(s - \eta)I(\theta - s)N_2(s, \theta) \right) ds x_{ss}(\theta) d\eta d\theta$$

$$= \int_a^b \int_a^b I(\theta - \eta)x_{ss}(\eta)^T \int_\eta^\theta \left((s - \eta)N_2(s, \theta) \right) ds x_{ss}(\theta) d\eta d\theta$$

$$= \int_a^b \int_a^b I(\theta - s)x_{ss}(s)^T \int_s^\theta \left((\eta - s)N_2(\eta, \theta) \right) d\eta x_{ss}(\theta) ds d\theta$$

$$= \langle x_{ss}, \mathcal{P}_{0, 0, T_{32}} x_{ss} \rangle$$

where

$$T_{32}(s, \theta) = \int_s^\theta \left((\eta - s)N_2(\eta, \theta) \right) d\eta$$

Combining these terms, we have

$$\langle x_2, \mathcal{P}x \rangle = \langle x_{ss}, \mathcal{P}_{0, E_1, E_2} x_{ss} \rangle$$

$$E_1(s, \theta) = T_{21}(s, \theta) = \int_s^b (\eta - s) N_1(\eta, \theta) d\eta$$

$$E_2(s, \theta) = T_{12}(s, \theta) + T_{22}(s, \theta) + T_{32}(s, \theta)$$

$$= \left((\theta - s) M(\theta) + \int_\theta^b (\eta - s) N_1(\eta, \theta) d\eta + \int_s^\theta (\eta - s) N_2(\eta, \theta) d\eta \right)$$

where we used

$$T_{12}(s, \theta) = (\theta - s) M(\theta)$$

$$T_{21}(s, \theta) = \int_s^b \left((\eta - s) N_1(\eta, \theta) \right) d\eta$$

$$T_{22}(s, \theta) = \int_\theta^b \left((\eta - s) N_1(\eta, \theta) \right) d\eta$$

$$T_{32}(s, \theta) = \int_s^\theta \left((\eta - s) N_2(\eta, \theta) \right) d\eta$$

Now we examine the 3 boundary-determined terms:

$$\langle x_1, \mathcal{P}x \rangle = \langle x_1, \mathcal{P}_1 x \rangle + \langle x_1, \mathcal{P}_2 x \rangle + \langle x_1, \mathcal{P}_3 x \rangle$$

Starting with $\langle x_1, \mathcal{P}_1 x \rangle$

$$\langle x_2, \mathcal{P}_1 x \rangle = \int_a^b x_2(s)^T M(s) x_{ss}(s) ds$$

$$= \int_a^b \left(\int_a^b B_a(s, \eta) x_{ss}(\eta) \right)^T M(s) x_{ss}(s) d\eta ds$$

$$= \int_a^b \int_a^b x_{ss}(\eta)^T B_a(s, \eta)^T M(s) x_{ss}(s) d\eta ds$$

$$= \int_a^b \int_a^b x_{ss}(s)^T B_a(\eta, s)^T M(\eta) x_{ss}(\eta) d\eta ds$$

$$= \langle x_{ss}, \mathcal{P}_{0, Q_1, Q_1} x_{ss} \rangle_{L_2}$$

where

$$Q_1(s, \eta) = B_a(\eta, s)^T M(\eta)$$

Next $\langle x_1, \mathcal{P}_2 x \rangle$:

$$\langle x_1, \mathcal{P}_2 x \rangle = \int_a^b x_1(s)^T \int_a^b I(s-\theta) N_1(s,\theta) x_{ss}(\theta) d\theta ds$$

$$= \int_a^b \int_a^b \left(\int_a^b B_a(s,\eta) x_{ss}(\eta) d\eta \right)^T I(s-\theta) N_1(s,\theta) x_{ss}(\theta) d\theta ds$$

$$= \int_a^b \int_a^b x_{ss}(\eta)^T \int_a^b \left(I(s-\theta) B_a(s,\eta)^T N_1(s,\theta) \right) ds\, x_{ss}(\theta) d\eta d\theta$$

$$= \int_a^b \int_a^b x_{ss}(\eta)^T \left(\int_\theta^b B_a(s,\eta)^T N_1(s,\theta) ds \right) x_{ss}(\theta) d\eta d\theta$$

$$= \int_a^b \int_a^b x_{ss}(s)^T \left(\int_\eta^b B_a(\theta,s)^T N_1(\theta,\eta) d\theta \right) x_{ss}(\eta) ds d\eta$$

$$= \langle x_{ss}, \mathcal{P}_{0,Q_2,Q_2} x_{ss} \rangle_{L_2}$$

$$Q_2(s,\eta) = \int_\eta^b B_a(\theta,s)^T N_1(\theta,\eta) d\theta$$

Finally

$$\langle x_1, \mathcal{P}_3 x \rangle = \int_a^b x_1(s)^T \int_a^b I(\theta-s) N_2(s,\theta) x_{ss}(\theta) d\theta ds$$

$$= \int_a^b \int_a^b x_{ss}(\eta)^T \left(\int_a^b \left(B_a(s,\eta)^T I(\theta-s) N_2(s,\theta) \right) ds \right) x_{ss}(\theta) d\eta d\theta$$

$$= \int_a^b \int_a^b x_{ss}(s)^T \left(\int_a^\eta \left(B_a(\theta,s)^T N_2(\theta,\eta) \right) d\theta \right) x_{ss}(\eta) ds d\eta$$

$$= \langle x_{ss}, \mathcal{P}_{0,Q_3,Q_3} x_{ss} \rangle$$

where

$$Q_3(s,\eta) = \int_a^\eta \left(B_a(\theta,s)^T N_2(\theta,\eta) \right) d\theta$$

Combining these terms, we have

$$\langle x_1, \mathcal{P} x \rangle = \langle x_{ss}, \mathcal{P}_{0,E_3,E_3} x_{ss} \rangle$$

$$E_3(s,\theta) = Q_1(s,\theta) + Q_2(s,\theta) + Q_3(s,\theta)$$

$$= B_a(\eta,s)^T M(\eta) + \int_\eta^b B_a(\theta,s)^T N_1(\theta,\eta) d\theta + \int_a^\eta B_a(\theta,s)^T N_2(\theta,\eta) d\theta$$

$$= Y_1(s,\theta)$$

where we used

$$Q_1(s, \eta) = B_a(\eta, s)^T M(\eta)$$

$$Q_2(s, \eta) = \int_\eta^b B_a(\theta, s)^T N_1(\theta, \eta) d\theta$$

$$Q_3(s, \eta) = \int_a^\eta \left(B_a(\theta, s)^T N_2(\theta, \eta) \right) d\theta$$

Combining the BC and non-BC terms yields the expression in the Lemma statement.

∎

II. Proof of Lemma 5

Lemma 2: Suppose $\mathbf{x}$ satisfies the conditions of Thm. 3. Then

$$\langle \mathbf{x}, \mathcal{P}_{\{M,N_1,N_2\}}\mathbf{x}_s \rangle = \langle \mathbf{x}_{ss}, \mathcal{P}_{\{0,Q_1,Q_2\}}\mathbf{x}_{ss} \rangle$$

where

$$Q_1(s,\theta) = F_1(s,\theta) + F_3(s,\theta)$$

$$Q_2(s,\theta) = F_2(s,\theta) + F_3(s,\theta)$$

$$F_1(s,\theta) = \int_s^b \left((\eta - s)F_4(\theta,\eta) + F_5(s,\eta) \right) d\eta$$

$$F_2(s,\theta) = \int_\theta^b \left((\eta - s)F_4(\theta,\eta) + F_5(s,\eta) \right) d\eta$$

$$F_3(s,\eta) = \int_a^b B_a(\zeta,s)^T Y_2(\zeta)B_b(\eta)d\zeta + \int_\eta^b Y_1(s,\zeta)d\zeta$$

$$+ \int_s^b (\zeta - s)Y_2(\zeta)d\zeta B_b(\eta)$$

$$F_4(\theta,\eta) = M(\eta) + \int_\theta^\eta N_1(\eta,\zeta)d\zeta$$

$$F_5(s,\eta) = \int_s^\eta (\zeta - s)N_2(\zeta,\eta)d\zeta.$$

Proof: The proof is similar to the proof of Lemma 4, but with more terms. As before, we decompose $\mathcal{P}_{\{M,N_1,N_2\}}$ as

$$\left(\mathcal{P}_{\{M,N_1,N_2\}}\psi\right)(s) = M(s)\psi(s) + \int_a^b I(s-\theta)N_1(s,\theta)\phi(\theta)d\theta + \int_a^b I(\theta - s)N_2(s,\theta)\phi(\theta)d\theta,$$

$$= (\mathcal{P}_1\psi)(s) + (\mathcal{P}_2\psi)(s) + (\mathcal{P}_3\psi)(s),$$

In this case, however, we have 2 decompositions:

$$x(s) = \int_a^b B_a(s,\eta)x_{ss}(\eta)d\eta + \int_a^s (s-\eta)x_{ss}(\eta)d\eta = \int_a^b \left(B_a(s,\eta) + I(s-\eta)(s-\eta)\right)x_{ss}(\eta)d\eta$$

$$= x_1(s) + x_2(s)$$

$$x_s(s) = \int_a^b B_b(\eta)x_{ss}(\eta)d\eta + \int_a^s x_{ss}(\eta)d\eta = \int_a^b \left(B_b(\eta) + I(s-\eta)\right)x_{ss}(\eta)d\eta$$

$$= x_3 + x_4(s)$$

where x_1 and x_3 are determined by the boundary conditions (BCs) and x_2 and x_4 is independent of the BCs. This means we are going to have 12 terms to manage.

$$\langle x, \mathcal{P}_1 x_s \rangle = \langle x_1, \mathcal{P}_1 x_3 \rangle + \langle x_2, \mathcal{P}_1 x_3 \rangle + \langle x_1, \mathcal{P}_1 x_4 \rangle + \langle x_2, \mathcal{P}_1 x_4 \rangle$$

$$+ \langle x_1, \mathcal{P}_2 x_3 \rangle + \langle x_2, \mathcal{P}_2 x_3 \rangle + \langle x_1, \mathcal{P}_2 x_4 \rangle + \langle x_2, \mathcal{P}_2 x_4 \rangle$$

$$+ \langle x_1, \mathcal{P}_3 x_3 \rangle + \langle x_2, \mathcal{P}_3 x_3 \rangle + \langle x_1, \mathcal{P}_3 x_4 \rangle + \langle x_2, \mathcal{P}_3 x_4 \rangle$$

As before, we start with the non-BC term $\langle x_2, \mathcal{P}_1 x_4 \rangle$, $\langle x_2, \mathcal{P}_3 x_4 \rangle$, and $\langle x_2, \mathcal{P}_3 x_4 \rangle$.

$$\langle x_2, \mathcal{P}_1 x_4 \rangle = \int_a^b \left(\int_a^b (s-\eta)I(s-\eta)x_{ss}(\eta)d\eta \right)^T \left(\int_a^b M(s)I(s-\zeta)x_{ss}(\zeta)d\zeta \right) ds$$

$$= \int_a^b \int_a^b x_{ss}(\eta)^T \left(\int_a^b (s-\eta)I(s-\eta)I(s-\zeta)M(s)ds \right) x_{ss}(\zeta)d\zeta d\eta$$

$$= \int_a^b \int_a^b x_{ss}(\eta)^T I(\zeta-\eta) \left(\int_\zeta^b (s-\eta)M(s)ds \right) x_{ss}(\zeta)d\zeta d\eta$$

$$+ \int_a^b \int_a^b x_{ss}(\eta)^T I(\eta-\zeta) \left(\int_\eta^b (s-\eta)M(s)ds \right) x_{ss}(\zeta)d\zeta d\eta$$

$$= \int_a^b \int_a^b x_{ss}(s)^T I(\theta-s) \left(\int_\theta^b (\eta-s)M(\eta)d\eta \right) x_{ss}(\theta)d\theta ds$$

$$+ \int_a^b \int_a^b x_{ss}(s)^T I(s-\theta) \left(\int_s^b (\eta-s)M(\eta)d\eta \right) x_{ss}(\theta)d\theta ds$$

$$= \langle x_{ss}, \mathcal{P}_{0,T_{11},T_{12}} x_{ss} \rangle_{L_2}$$

where

$$T_{11}(s,\theta) = \int_s^b (\eta-s)M(\eta)d\eta$$

$$T_{12}(s,\theta) = \int_\theta^b (\eta-s)M(\eta)d\eta$$

Next, we have

$$\langle x_2, \mathcal{P}_2 x_4 \rangle = \int_a^b x(s)^T \int_a^b I(s-\theta)N_1(s,\theta)x_s(\theta)d\theta ds$$

$$= \int_a^b \int_a^b \left(\int_a^b (s-\eta)I(s-\eta)x_{ss}(\eta)d\eta \right)^T I(s-\theta)N_1(s,\theta) \left(\int_a^b I(\theta-\zeta)x_{ss}(\zeta)d\zeta \right) d\theta ds$$

$$= \int_a^b \int_a^b x_{ss}(\eta)^T \left(\int_a^b \int_a^b ((s-\eta)N_1(s,\theta)I(s-\eta)I(s-\theta)I(\theta-\zeta)) \, d\theta ds \right) x_{ss}(\zeta)d\eta d\zeta$$

$$= \int_a^b \int_a^b x_{ss}(\eta)^T I(\eta-\zeta) \left(\int_\eta^b \int_\zeta^s ((s-\eta)N_1(s,\theta)) \, d\theta ds \right) x_{ss}(\zeta)d\eta d\zeta$$

$$+ \int_a^b \int_a^b x_{ss}(\eta)^T I(\zeta-\eta) \left(\int_\zeta^b \int_\zeta^s ((s-\eta)N_1(s,\theta)) \, d\theta ds \right) x_{ss}(\zeta)d\eta d\zeta$$

$$= \int_a^b \int_a^b x_{ss}(s)^T I(s-\theta) \left(\int_s^b \int_\theta^\eta ((\eta-s)N_1(\eta,\zeta)) \, d\zeta d\eta \right) x_{ss}(\theta)ds d\theta$$

$$+ \int_a^b \int_a^b x_{ss}(s)^T I(\theta-s) \left(\int_\theta^b \int_\theta^\eta ((\eta-s)N_1(\eta,\zeta)) \, d\zeta d\eta \right) x_{ss}(\theta)ds d\theta$$

$$= \langle x_{ss}, \mathcal{P}_{0,T_{21},T_{22}} x_{ss} \rangle_{L_2}$$

where

$$T_{21} = \int_s^b \int_\theta^\eta (\eta-s)N_1(\eta,\zeta)d\zeta d\eta$$

$$T_{22} = \int_\theta^b \int_\theta^\eta (\eta - s) N_1(\eta, \zeta) d\zeta d\eta$$

This holds since if $\zeta > \eta$,

$$I(s-\eta)I(s-\theta)I(\theta-\zeta) = I(s-\eta)I(s-\zeta)I(s-\theta)I(\theta-\zeta) + I(s-\eta)I(\zeta-s)I(s-\theta)I(\theta-\zeta) = I(s-\zeta)I(s-\theta)I(\theta-\zeta)$$

That is

$$I(s - \theta)I(\theta - \zeta) = \begin{cases} 1, & \text{if } s > \theta > \zeta \text{ and } s \geq \zeta \\ 0, & \text{otherwise.} \end{cases}$$

Finally, we have

$$\begin{aligned}
\langle x_2, \mathcal{P}_3 \S_4 \rangle &= \int_a^b x(s)^T \int_a^b I(\theta - s) N_2(s, \theta) x_s(\theta) d\theta ds \\
&= \int_a^b \int_a^b \left(\int_a^b (s - \eta) I(s - \eta) x_{ss}(\eta) d\eta \right)^T I(\theta - s) N_2(s, \theta) \left(\int_a^b I(\theta - \zeta) x_{ss}(\zeta) d\zeta \right) d\theta ds \\
&= \int_a^b \int_a^b x_{ss}(\eta)^T \left(\int_a^b \int_a^b ((s - \eta) N_2(s, \theta) I(s - \eta) I(\theta - s) I(\theta - \zeta)) d\theta ds \right) x_{ss}(\zeta) d\eta d\zeta \\
&= \int_a^b \int_a^b x_{ss}(\eta)^T I(\eta - \zeta) \left(\int_\eta^b \int_\eta^\theta ((s - \eta) N_2(s, \theta)) ds d\theta \right) x_{ss}(\zeta) d\eta d\zeta \\
&+ \int_a^b \int_a^b x_{ss}(\eta)^T I(\zeta - \eta) \left(\int_\zeta^b \int_\eta^\theta ((s - \eta) N_2(s, \theta)) ds d\theta \right) x_{ss}(\zeta) d\eta d\zeta \\
&= \int_a^b \int_a^b x_{ss}(s)^T I(s - \theta) \left(\int_s^b \int_s^\zeta ((\eta - s) N_2(\eta, \zeta)) d\eta d\zeta \right) x_{ss}(\theta) ds d\theta \\
&+ \int_a^b \int_a^b x_{ss}(s)^T I(\theta - s) \left(\int_\theta^b \int_s^\zeta ((\eta - s) N_2(\eta, \zeta)) d\eta d\zeta \right) x_{ss}(\theta) ds d\theta \\
&= \langle x_{ss}, \mathcal{P}_{0, T_{31}, T_{32}} x_{ss} \rangle_{L_2}
\end{aligned}$$

where

$$T_{31}(s, \theta) = \int_s^b \int_s^\zeta (\eta - s) N_2(\eta, \zeta) d\eta d\zeta$$

$$T_{32}(s, \theta) = \int_\theta^b \int_s^\zeta (\eta - s) N_2(\eta, \zeta) d\eta d\zeta$$

Combining these terms, we have

$$\langle x_2, \mathcal{P} x_4 \rangle = \langle x_{ss}, \mathcal{P}_{0, F_1, F_2} x_{ss} \rangle$$

$$F_1(s,\theta) = T_{11}(s,\theta) + T_{21}(s,\theta) + T_{31}(s,\theta)$$

$$= \int_s^b (\eta - s)M(\eta)d\eta + \int_s^b \int_\theta^\eta (\eta - s)N_1(\eta,\zeta)d\zeta d\eta + \int_s^b \int_s^\zeta (\eta - s)N_2(\eta,\zeta)d\eta d\zeta$$

$$= \int_s^b \left((\eta - s)\left(M(\eta) + \int_\theta^\eta N_1(\eta,\zeta)d\zeta \right) + \int_s^\eta (\zeta - s)N_2(\zeta,\eta)d\zeta \right) d\eta$$

$$= \int_s^b \left((\eta - s)F_4(\theta,\eta) + F_5(s,\eta) \right) d\eta$$

$$F_2(s,\theta) = T_{11}(s,\theta) + T_{21}(s,\theta) + T_{31}(s,\theta)$$

$$= \int_\theta^b (\eta - s)M(\eta)d\eta + \int_\theta^b \int_\theta^\eta (\eta - s)N_1(\eta,\zeta)d\zeta d\eta + \int_\theta^b \int_s^\zeta (\eta - s)N_2(\eta,\zeta)d\eta d\zeta$$

$$= \int_\theta^b \left((\eta - s)\left(M(\eta) + \int_\theta^\eta N_1(\eta,\zeta)d\zeta \right) + \int_s^\eta (\zeta - s)N_2(\zeta,\eta)d\zeta \right) d\eta$$

$$= \int_\theta^b \left((\eta - s)F_4(\theta,\eta) + F_5(s,\eta) \right) d\eta$$

where we used

$$T_{11}(s,\theta) = \int_s^b (\eta - s)M(\eta)d\eta$$

$$T_{12}(s,\theta) = \int_\theta^b (\eta - s)M(\eta)d\eta$$

$$T_{21} = \int_s^b \int_\theta^\eta (\eta - s)N_1(\eta,\zeta)d\zeta d\eta$$

$$T_{22} = \int_\theta^b \int_\theta^\eta (\eta - s)N_1(s,\zeta)d\zeta d\eta$$

$$T_{31}(s,\theta) = \int_s^b \int_s^\zeta (\eta - s)N_2(\eta,\zeta)d\eta d\zeta$$

$$T_{32}(s,\theta) = \int_\theta^b \int_s^\zeta (\eta - s)N_2(\eta,\zeta)d\eta d\zeta$$

We now deal with the remaining 9 BC-determined terms. We start with the 3 terms in

$$\langle x, \mathcal{P}_1 x_s \rangle = \langle x_1, \mathcal{P}_1 x_3 \rangle + \langle x_1, \mathcal{P}_2 x_3 \rangle + \langle x_1, \mathcal{P}_3 x_3 \rangle$$

First, we have

$$\langle x_1, \mathcal{P}_1 x_4 \rangle = \int_a^b x_1(s)^T M(s) x_4(s) ds$$

$$= \int_a^b \left(\int_a^b B_a(s,\eta) x_{ss}(\eta) d\eta \right)^T \left(\int_a^b M(s) I(s-\zeta) x_{ss}(\zeta) d\zeta \right) ds$$

$$= \int_a^b \int_a^b x_{ss}(\eta)^T \left(\int_a^b B_a(s,\eta)^T I(s-\zeta) M(s) ds \right) x_{ss}(\zeta) d\zeta d\eta$$

$$= \int_a^b \int_a^b x_{ss}(\eta)^T \left(\int_\zeta^b B_a(s,\eta)^T M(s) ds \right) x_{ss}(\zeta) d\zeta d\eta$$

$$= \int_a^b \int_a^b x_{ss}(s)^T \left(\int_\eta^b B_a(\zeta,s)^T M(\zeta) d\zeta \right) x_{ss}(\eta) d\eta ds$$

$$= \langle x_{ss}, \mathcal{P}_{0,Q_{11},Q_{11}} x_{ss} \rangle_{L_2}$$

$$Q_{11}(s,\eta) = \int_\eta^b B_a(\zeta,s)^T M(\zeta) d\zeta$$

$$\langle x_1, \mathcal{P}_2 x_4 \rangle = \int_a^b x_1(s)^T \int_a^b I(s-\theta) N_1(s,\theta) x_4(\theta) d\theta ds$$

$$= \int_a^b \int_a^b \left(\int_a^b B_a(s,\eta) x_{ss}(\eta) d\eta \right)^T I(s-\theta) N_1(s,\theta) \left(\int_a^b I(\theta-\zeta) x_{ss}(\zeta) d\zeta \right) d\theta ds$$

$$= \int_a^b \int_a^b x_{ss}(\eta)^T \left(\int_a^b \int_a^b \left(B_a(s,\eta)^T N_1(s,\theta) I(s-\theta) I(\theta-\zeta) \right) d\theta ds \right) x_{ss}(\zeta) d\eta d\zeta$$

$$= \int_a^b \int_a^b x_{ss}(\eta)^T \left(\int_\zeta^b \int_\theta^b \left(B_a(s,\eta)^T N_1(s,\theta) \right) ds d\theta \right) x_{ss}(\zeta) d\eta d\zeta$$

$$= \int_a^b \int_a^b x_{ss}(s)^T \left(\int_\eta^b \int_\zeta^b B_a(\theta,s)^T N_1(\theta,\zeta) d\theta d\zeta \right) x_{ss}(\eta) ds d\eta$$

$$= \langle x_{ss}, \mathcal{P}_{0,Q_{21},Q_{21}} x_{ss} \rangle_{L_2}$$

where

$$Q_{21}(s,\eta) = \int_\eta^b \int_\zeta^b B_a(\theta,s)^T N_1(\theta,\zeta) d\theta d\zeta$$

$$\langle x_1, \mathcal{P}_3 x_4 \rangle = \int_a^b x_1(s)^T \int_a^b I(\theta - s) N_2(s, \theta) x_4(\theta) d\theta ds$$

$$= \int_a^b \int_a^b \left(\int_a^b B_a(s, \eta) x_{ss}(\eta) d\eta \right)^T I(\theta - s) N_2(s, \theta) \left(\int_a^b I(\theta - \zeta) x_{ss}(\zeta) d\zeta \right) d\theta ds$$

$$= \int_a^b \int_a^b x_{ss}(\eta)^T \left(\int_a^b \int_a^b \left(B_a(s, \eta)^T N_2(s, \theta) I(\theta - s) I(\theta - \zeta) \right) d\theta ds \right) x_{ss}(\zeta) d\eta d\zeta$$

$$= \int_a^b \int_a^b x_{ss}(\eta)^T \left(\int_\zeta^b \int_a^\theta \left(B_a(s, \eta)^T N_2(s, \theta) \right) ds d\theta \right) x_{ss}(\zeta) d\eta d\zeta$$

$$= \int_a^b \int_a^b x_{ss}(s)^T \left(\int_\eta^b \int_a^\zeta B_a(\theta, s)^T N_2(\theta, \zeta) d\theta d\zeta \right) x_{ss}(\eta) ds d\eta$$

$$= \langle x_{ss}, \mathcal{P}_{0, Q_{31}, Q_{31}} x_{ss} \rangle_{L_2}$$

where

$$Q_{31}(s, \eta) = \int_\eta^b \int_a^\zeta B_a(\theta, s)^T N_2(\theta, \zeta) d\theta d\zeta$$

summing these terms, we have

$$\langle x, \mathcal{P}_{0, Q_{c1}, Q_{c1}} x \rangle_{L_2}$$

$$Q_{c1}(s, \eta) = Q_{11}(s, \eta) + Q_{21}(s, \eta) + Q_{31}(s, \eta)$$

$$= \int_\eta^b \left(B_a(\zeta, s)^T M(\zeta) + \int_\zeta^b B_a(\theta, s)^T N_1(\theta, \zeta) d\theta + \int_a^\zeta B_a(\theta, s)^T N_2(\theta, \zeta) d\theta \right) d\zeta$$

$$= \int_\eta^b Y_1(s, \zeta) d\zeta$$

$$\langle x_1, \mathcal{P}_1 x_3 \rangle + \langle x_1, \mathcal{P}_2 x_3 \rangle + \langle x_1, \mathcal{P}_3 x_3 \rangle$$

Next,

$$\langle x_1, \mathcal{P}_1 x_3 \rangle = \int_a^b x_1(s)^T M(s) x_3(s) ds$$

$$= \int_a^b \left(\int_a^b B_a(s, \eta) x_{ss}(\eta) d\eta \right)^T \left(\int_a^b M(s) B_b(\zeta) x_{ss}(\zeta) d\zeta \right) ds$$

$$= \int_a^b \int_a^b x_{ss}(\eta)^T \left(\int_a^b B_a(s, \eta)^T M(s) ds \right) B_b(\zeta) x_{ss}(\zeta) d\zeta d\eta$$

$$= \int_a^b \int_a^b x_{ss}(s)^T \left(\int_a^b B_a(\zeta, s)^T M(\zeta) d\zeta \right) B_b(\eta) x_{ss}(\eta) d\eta ds$$

$$= \langle x_{ss}, \mathcal{P}_{0, Q_{12}, Q_{12}} x_{ss} \rangle_{L_2}$$

$$Q_{12}(s, \eta) = \int_a^b B_a(\zeta, s)^T M(\zeta) d\zeta B_b(\eta)$$

$$\langle x_1, \mathcal{P}_2 x_3 \rangle = \int_a^b x_1(s)^T \int_a^b I(s-\theta) N_1(s,\theta) x_3(\theta) d\theta ds$$

$$= \int_a^b \int_a^b \left(\int_a^b B_a(s,\eta) x_{ss}(\eta) d\eta \right)^T I(s-\theta) N_1(s,\theta) \left(\int_a^b B_b(\zeta) x_{ss}(\zeta) d\zeta \right) d\theta ds$$

$$= \int_a^b \int_a^b x_{ss}(\eta)^T \left(\int_a^b \int_a^b \left(B_a(s,\eta)^T N_1(s,\theta) I(s-\theta) B_b(\zeta) \right) d\theta ds \right) x_{ss}(\zeta) d\eta d\zeta$$

$$= \int_a^b \int_a^b x_{ss}(\eta)^T \left(\int_a^b \int_a^s \left(B_a(s,\eta)^T N_1(s,\theta) B_b(\zeta) \right) d\theta ds \right) x_{ss}(\zeta) d\eta d\zeta$$

$$= \int_a^b \int_a^b x_{ss}(s)^T \left(\int_a^b \int_a^\zeta B_a(\zeta,s)^T N_1(\zeta,\theta) B_b(\eta) d\theta d\zeta \right) x_{ss}(\eta) ds d\eta$$

$$= \langle x_{ss}, \mathcal{P}_{0,Q_{22},Q_{22}} x_{ss} \rangle_{L_2}$$

$$Q_{22}(s,\eta) = \int_a^b \int_a^\zeta B_a(\zeta,s)^T N_1(\zeta,\theta) B_b(\eta) d\theta d\zeta$$

$$\langle x_1, \mathcal{P}_3 x_3 \rangle = \int_a^b x_1(s)^T \int_a^b I(\theta-s) N_2(s,\theta) x_3(\theta) d\theta ds$$

$$= \int_a^b \int_a^b \left(\int_a^b B_a(s,\eta) x_{ss}(\eta) d\eta \right)^T I(\theta-s) N_2(s,\theta) \left(\int_a^b B_b(\zeta) x_{ss}(\zeta) d\zeta \right) d\theta ds$$

$$= \int_a^b \int_a^b x_{ss}(\eta)^T \left(\int_a^b \int_a^b \left(B_a(s,\eta)^T N_2(s,\theta) I(\theta-s) B_b(\zeta) \right) d\theta ds \right) x_{ss}(\zeta) d\eta d\zeta$$

$$= \int_a^b \int_a^b x_{ss}(\eta)^T \left(\int_a^b \int_s^b \left(B_a(s,\eta)^T N_2(s,\theta) B_b(\zeta) \right) d\theta ds \right) x_{ss}(\zeta) d\eta d\zeta$$

$$= \int_a^b \int_a^b x_{ss}(s)^T \left(\int_a^b \int_\zeta^b B_a(\zeta,s)^T N_2(\zeta,\theta) B_b(\eta) d\theta d\zeta \right) x_{ss}(\eta) ds d\eta$$

$$= \langle x_{ss}, \mathcal{P}_{0,Q_{32},Q_{32}} x_{ss} \rangle_{L_2}$$

$$Q_{32}(s,\eta) = \int_a^b \int_\zeta^b B_a(\zeta,s)^T N_2(\zeta,\theta) B_b(\eta) d\theta d\zeta$$

summing these terms, we have

$$\langle x_{ss}, \mathcal{P}_{0,Q_{c2},Q_{c2}} x_{ss} \rangle_{L_2}$$

$$Q_{c2}(s,\eta) = Q_{12}(s,\eta) + Q_{22}(s,\eta) + Q_{32}(s,\eta)$$

$$= \int_a^b B_a(\zeta,s)^T \left(M(\zeta) + \int_a^\zeta N_1(\zeta,\theta) d\theta + \int_\zeta^b N_2(\zeta,\theta) d\theta \right) B_b(\eta) d\zeta$$

$$= \int_a^b B_a(\zeta,s)^T Y_2(\zeta) B_b(\eta) d\zeta$$

Finally, we have

$$\langle x_2, \mathcal{P}_1 x_3 \rangle + \langle x_2, \mathcal{P}_2 x_3 \rangle + \langle x_2, \mathcal{P}_3 x_3 \rangle$$

$$\langle x_2, \mathcal{P}_1 x_3 \rangle = \int_a^b x_2(s)^T M(s) x_3 ds$$

$$= \int_a^b \left(\int_a^b (s-\eta) I(s-\eta) x_{ss}(\eta) d\eta \right)^T \left(\int_a^b M(s) B_b(\zeta) x_{ss}(\zeta) d\zeta \right) ds$$

$$= \int_a^b \int_a^b x_{ss}(\eta)^T \left(\int_a^b (s-\eta) I(s-\eta) M(s) ds \right) B_b(\zeta) x_{ss}(\zeta) d\zeta d\eta$$

$$= \int_a^b \int_a^b x_{ss}(\eta)^T \left(\int_\eta^b (s-\eta) M(s) ds \right) B_b(\zeta) x_{ss}(\zeta) d\zeta d\eta$$

$$= \int_a^b \int_a^b x_{ss}(s)^T \left(\int_s^b (\zeta-s) M(\zeta) d\zeta \right) B_b(\eta) x_{ss}(\eta) d\eta ds$$

$$= \langle x_{ss}, \mathcal{P}_{0, Q_{13}, Q_{13}} x_{ss} \rangle_{L_2}$$

$$Q_{13}(s, \eta) = \int_s^b (\zeta-s) M(\zeta) d\zeta B_b(\eta) d\zeta$$

$$\langle x_2, \mathcal{P}_2 x_3 \rangle = \int_a^b x_2(s)^T \int_a^b I(s-\theta) N_1(s, \theta) x_3(\theta) d\theta ds$$

$$= \int_a^b \int_a^b \left(\int_a^b (s-\eta) I(s-\eta) x_{ss}(\eta) d\eta \right)^T I(s-\theta) N_1(s, \theta) \left(\int_a^b B_b(\zeta) x_{ss}(\zeta) d\zeta \right) d\theta ds$$

$$= \int_a^b \int_a^b x_{ss}(\eta)^T \left(\int_a^b \int_a^b ((s-\eta) N_1(s, \theta) I(s-\eta) I(s-\theta) B_b(\zeta)) d\theta ds \right) x_{ss}(\zeta) d\eta d\zeta$$

$$= \int_a^b \int_a^b x_{ss}(\eta)^T \left(\int_\eta^b \int_a^s ((s-\eta) N_1(s, \theta) B_b(\zeta)) d\theta ds \right) x_{ss}(\zeta) d\eta d\zeta$$

$$= \int_a^b \int_a^b x_{ss}(s)^T \left(\int_s^b \int_a^\zeta ((\zeta-s) N_1(\zeta, \theta) B_b(\eta)) d\theta d\zeta \right) x_{ss}(\zeta) ds d\eta$$

$$= \langle x_{ss}, \mathcal{P}_{0, Q_{23}, Q_{23}} x_{ss} \rangle_{L_2}$$

$$Q_{23}(s, \eta) = \int_s^b \int_a^\zeta (\zeta-s) N_1(\zeta, \theta) B_b(\eta) d\theta d\zeta$$

$$\langle x_2, \mathcal{P}_3 x_3 \rangle = \int_a^b x_2(s)^T \int_a^b I(\theta - s) N_2(s, \theta) x_3(\theta) d\theta ds$$

$$= \int_a^b \int_a^b \left(\int_a^b (s - \eta) I(s - \eta) x_{ss}(\eta) d\eta \right)^T I(\theta - s) N_2(s, \theta) \left(\int_a^b B_b(\zeta) x_{ss}(\zeta) d\zeta \right) d\theta ds$$

$$= \int_a^b \int_a^b x_{ss}(\eta)^T \left(\int_a^b \int_a^b ((s - \eta) N_2(s, \theta) I(s - \eta) I(\theta - s) B_b(\zeta)) \, d\theta ds \right) x_{ss}(\zeta) d\eta d\zeta$$

$$= \int_a^b \int_a^b x_{ss}(\eta)^T \left(\int_\eta^b \int_s^b ((s - \eta) N_2(s, \theta) B_b(\zeta)) \, d\theta ds \right) x_{ss}(\zeta) d\eta d\zeta$$

$$= \int_a^b \int_a^b x_{ss}(s)^T \left(\int_s^b \int_\zeta^b ((\zeta - s) N_2(\zeta, \theta) B_b(\eta)) \, d\theta d\zeta \right) x_{ss}(\zeta) ds d\eta$$

$$= \langle x_{ss}, \mathcal{P}_{0, Q_{33}, Q_{33}} x_{ss} \rangle_{L_2}$$

$$Q_{33}(s, \eta) = \int_s^b \int_\zeta^b (\zeta - s) N_2(\zeta, \theta) B_b(\eta) d\theta d\zeta$$

summing these terms, we have

$$\langle x, \mathcal{P}_{0, Q_{c3}, Q_{c3}} x \rangle_{L_2}$$

$$Q_{c3}(s, \eta) = Q_{13}(s, \eta) + Q_{23}(s, \eta) + Q_{33}(s, \eta)$$

$$= \int_s^b (\zeta - s) \left(M(\zeta) + \int_a^\zeta N_1(\zeta, \theta) d\theta + \int_\zeta^b N_2(\zeta, \theta) d\theta \right) B_b(\eta) d\zeta$$

$$= \int_s^b (\zeta - s) Y_2(\zeta) d\zeta B_b(\eta)$$

Combining all boundary terms, we have

$$\langle x_1, \mathcal{P}_1 x_3 \rangle + \langle x_2, \mathcal{P}_1 x_3 \rangle + \langle x_1, \mathcal{P}_1 x_4 \rangle$$

$$+ \langle x_1, \mathcal{P}_2 x_3 \rangle + \langle x_2, \mathcal{P}_2 x_3 \rangle + \langle x_1, \mathcal{P}_2 x_4 \rangle$$

$$+ \langle x_1, \mathcal{P}_3 x_3 \rangle + \langle x_2, \mathcal{P}_3 x_3 \rangle + \langle x_1, \mathcal{P}_3 x_4 \rangle$$

$$= \langle x_{ss}, \mathcal{P}_{0, F_3, F_3} x_{ss} \rangle_{L_2}$$

where

$$F_3(s, \eta) = Q_{c1}(s, \eta) + Q_{c2}(s, \eta) + Q_{c3}(s, \eta)$$

$$= \int_a^b B_a(\zeta, s)^T Y_2(\zeta) B_b(\eta) d\zeta + \int_\eta^b Y_1(s, \zeta) d\zeta + \int_s^b (\zeta - s) Y_2(\zeta) d\zeta B_b(\eta)$$

where we used

$$S_{c1}(s,\eta) = \int_\eta^b Y_1(s,\zeta)d\zeta$$

$$S_{2c}(s,\eta) = \int_a^b B_a(\zeta,s)^T Y_2(\zeta)B_b(\eta)d\zeta$$

$$S_{c3}(s,\eta) = \int_s^b (\zeta-s)Y_2(\zeta)d\zeta B_b(\eta)$$

Combining all terms, we have

$$\langle x, \mathcal{P}_1 x_s \rangle = \langle x_{ss}, \mathcal{P}_{0,F_1,F_2} x_{ss} \rangle_{L_2} + \langle x_{ss}, \mathcal{P}_{0,F_3,F_3} x_{ss} \rangle_{L_2}$$

where

$$F_1(s,\theta) = \int_s^b \left((\eta-s)F_4(\theta,\eta) + F_5(s,\eta) \right) d\eta$$

$$F_2(s,\theta) = \int_\theta^b \left((\eta-s)F_4(\theta,\eta) + F_5(s,\eta) \right) d\eta$$

and

$$F_3(s,\eta) = \int_a^b B_a(\zeta,s)^T Y_2(\zeta)B_b(\eta)d\zeta + \int_\eta^b Y_1(s,\zeta)d\zeta + \int_s^b (\zeta-s)Y_2(\zeta)d\zeta B_b(\eta)$$

∎

III. PROOF OF LEMMA 6

Lemma 3: Suppose $\mathbf{x}$ satisfies the conditions of Thm. 3. Then

$$\langle \mathbf{x}, \mathcal{P}_{\{M,N_1,N_2\}}\mathbf{x} \rangle = \langle \mathbf{x}_{ss}, \mathcal{P}_{\{0,T_1,T_2\}}\mathbf{x}_{ss} \rangle$$

where

$$T_1(s,\theta) = G_1(s,\theta) + G_3(s,\theta)$$

$$T_2(s,\theta) = G_2(s,\theta) + G_3(s,\theta)$$

$$G_1(s,\theta) = \int_s^b \left((\eta - s)G_4(\theta,\eta) + G_5(s,\theta,\eta) \right) d\eta$$

$$G_2(s,\theta) = \int_\theta^b \left((\eta - s)G_4(\theta,\eta) + G_5(s,\theta,\eta) \right) d\eta$$

$$G_3(s,\theta) = \int_a^b B_a(\eta,s)^T Y_3(\eta,\theta)d\eta$$

$$+ \int_\theta^b (\eta - \theta)Y_1(s,\eta)d\eta + \int_s^b (\eta - s)Y_3(\eta,\theta)d\eta$$

$$G_4(\theta,\eta) = (\eta - \theta)M(\eta) + \int_\theta^\eta (\zeta - \theta)N_1(\eta,\zeta)d\zeta$$

$$G_5(s,\theta,\eta) = \int_s^\eta (\zeta - s)(\eta - \theta)N_2(\zeta,\eta)d\zeta.$$

Proof:

First, we decompose $\mathcal{P}_{\{M,N_1,N_2\}}$ as

$$\left(\mathcal{P}_{\{M,N_1,N_2\}}\psi \right)(s) = M(s)\psi(s) + \int_a^b I(s - \theta)N_1(s,\theta)\phi(\theta)d\theta + \int_a^b I(\theta - s)N_2(s,\theta)\phi(\theta)d\theta,$$

$$= (\mathcal{P}_1\psi)(s) + (\mathcal{P}_2\psi)(s) + (\mathcal{P}_3\psi)(s),$$

As before, we decompose x as

$$x(s) = \int_a^b B_a(s,\eta)x_{ss}(\eta)d\eta + \int_a^s (s - \eta)x_{ss}(\eta)d\eta = \int_a^b \left(B_a(s,\eta) + I(s - \eta)(s - \eta) \right) x_{ss}(\eta)d\eta$$

$$= x_1(s) + x_2(s)$$

As a result, we will have the following 12 terms to examine.

$$\langle x, \mathcal{P}x \rangle = \langle x_1, \mathcal{P}_1 x_1 \rangle + \langle x_2, \mathcal{P}_1 x_1 \rangle + \langle x_1, \mathcal{P}_1 x_2 \rangle + \langle x_2, \mathcal{P}_1 x_2 \rangle$$

$$+ \langle x_1, \mathcal{P}_2 x_3 \rangle + \langle x_2, \mathcal{P}_2 x_1 \rangle + \langle x_1, \mathcal{P}_2 x_2 \rangle + \langle x_2, \mathcal{P}_2 x_2 \rangle$$

$$+ \langle x_1, \mathcal{P}_3 x_3 \rangle + \langle x_2, \mathcal{P}_3 x_1 \rangle + \langle x_1, \mathcal{P}_3 x_2 \rangle + \langle x_2, \mathcal{P}_3 x_2 \rangle$$

We start with the non-BC terms $\langle x_2, \mathcal{P}_1 x_2 \rangle$, $\langle x_2, \mathcal{P}_2 x_2 \rangle$, and $\langle x_2, \mathcal{P}_3 x_2 \rangle$.

$$\langle x_2, \mathcal{P}_1 x_2 \rangle = \int_a^b \left(\int_a^b (s-\eta) I(s-\eta) x_{ss}(\eta) d\eta \right)^T \left(\int_a^b M(s)(s-\zeta) I(s-\zeta) x_{ss}(\zeta) d\zeta \right) ds$$

$$= \int_a^b \int_a^b x_{ss}(\eta)^T \left(\int_a^b (s-\eta)(s-\zeta) I(s-\eta) I(s-\zeta) M(s) ds \right) x_{ss}(\zeta) d\zeta d\eta$$

$$= \int_a^b \int_a^b x_{ss}(\eta)^T I(\zeta-\eta) \left(\int_\zeta^b (s-\eta)(s-\zeta) M(s) ds \right) x_{ss}(\zeta) d\zeta d\eta$$

$$+ \int_a^b \int_a^b x_{ss}(\eta)^T I(\eta-\zeta) \left(\int_\eta^b (s-\eta)(s-\zeta) M(s) ds \right) x_{ss}(\zeta) d\zeta d\eta$$

$$= \int_a^b \int_a^b x_{ss}(s)^T I(\theta-s) \left(\int_\theta^b (\eta-s)(\eta-\theta) M(\eta) d\eta \right) x_{ss}(\theta) d\theta ds$$

$$+ \int_a^b \int_a^b x_{ss}(s)^T I(s-\theta) \left(\int_s^b (\eta-s)(\eta-\theta) M(\eta) d\eta \right) x_{ss}(\theta) d\theta ds$$

$$= \langle x_{ss}, \mathcal{P}_{0,T_{11},T_{12}} x_{ss} \rangle_{L_2}$$

$$T_{11}(s,\theta) = \int_s^b (\eta-s)(\eta-\theta) M(\eta) d\eta$$

$$T_{12}(s,\theta) = \int_\theta^b (\eta-s)(\eta-\theta) M(\eta) d\eta$$

For the $\langle x_2, \mathcal{P}_2 x_2 \rangle$ term,

$$\langle x_2, \mathcal{P}_2 x_2 \rangle = \int_a^b x(s)^T \int_a^b I(s-\theta) N_1(s,\theta) x(\theta) d\theta ds$$

$$= \int_a^b \int_a^b \left(\int_a^b (s-\eta) I(s-\eta) x_{ss}(\eta) d\eta \right)^T I(s-\theta) N_1(s,\theta) \left(\int_a^b (\theta-\zeta) I(\theta-\zeta) x_{ss}(\zeta) d\zeta \right) d\theta ds$$

$$= \int_a^b \int_a^b x_{ss}(\eta)^T \left(\int_a^b \int_a^b ((s-\eta)(\theta-\zeta) N_1(s,\theta) I(s-\eta) I(s-\theta) I(\theta-\zeta)) d\theta ds \right) x_{ss}(\zeta) d\eta d\zeta$$

$$= \int_a^b \int_a^b x_{ss}(\eta)^T I(\eta-\zeta) \left(\int_\eta^b \int_\zeta^s ((s-\eta)(\theta-\zeta) N_1(s,\theta)) d\theta ds \right) x_{ss}(\zeta) d\eta d\zeta$$

$$+ \int_a^b \int_a^b x_{ss}(\eta)^T I(\zeta-\eta) \left(\int_\zeta^b \int_\zeta^s ((s-\eta)(\theta-\zeta) N_1(s,\theta)) d\theta ds \right) x_{ss}(\zeta) d\eta d\zeta$$

$$= \int_a^b \int_a^b x_{ss}(s)^T I(s-\theta) \left(\int_s^b \int_\theta^\eta ((\eta-s)(\zeta-\theta) N_1(\eta,\zeta)) d\zeta d\eta \right) x_{ss}(\theta) ds d\theta$$

$$+ \int_a^b \int_a^b x_{ss}(\eta)^T I(\theta-\eta) \left(\int_\theta^b \int_\theta^\eta ((\eta-s)(\zeta-\theta) N_1(\eta,\zeta)) d\zeta d\eta \right) x_{ss}(\theta) d\eta d\theta$$

$$= \langle x_{ss}, \mathcal{P}_{0,T_{21},T_{22}} x_{ss} \rangle_{L_2}$$

where

$$T_{21} = \int_s^b \int_\theta^\eta (\eta-s)(\zeta-\theta) N_1(\eta,\zeta) d\zeta d\eta$$

$$T_{22} = \int_\theta^b \int_\theta^\eta (\eta - s)(\zeta - \theta)N_1(\eta, \zeta)d\zeta d\eta$$

$$\langle x_2, \mathcal{P}_3 x_2 \rangle = \int_a^b x(s)^T \int_a^b I(\theta - s)N_2(s, \theta)x_s(\theta)d\theta ds$$

$$= \int_a^b \int_a^b \left(\int_a^b (s - \eta)I(s - \eta)x_{ss}(\eta)d\eta \right)^T I(\theta - s)N_2(s, \theta) \left(\int_a^b (\theta - \zeta)I(\theta - \zeta)x_{ss}(\zeta)d\zeta \right) d\theta ds$$

$$= \int_a^b \int_a^b x_{ss}(\eta)^T \left(\int_a^b \int_a^b ((s - \eta)(\theta - \zeta)N_2(s, \theta)I(s - \eta)I(\theta - s)I(\theta - \zeta)) \, d\theta ds \right) x_{ss}(\zeta)d\eta d\zeta$$

$$= \int_a^b \int_a^b x_{ss}(\eta)^T I(\eta - \zeta) \left(\int_\eta^b \int_\eta^\theta ((s - \eta)(\theta - \zeta)N_2(s, \theta)) \, ds d\theta \right) x_{ss}(\zeta)d\eta d\zeta$$

$$+ \int_a^b \int_a^b x_{ss}(\eta)^T I(\zeta - \eta) \left(\int_\zeta^b \int_\eta^\theta ((s - \eta)(\theta - \zeta)N_2(s, \theta)) \, ds d\theta \right) x_{ss}(\zeta)d\eta d\zeta$$

$$= \int_a^b \int_a^b x_{ss}(s)^T I(s - \theta) \left(\int_s^b \int_s^\zeta ((\eta - s)(\zeta - \theta)N_2(\eta, \zeta)) \, d\eta d\zeta \right) x_{ss}(\theta)ds d\theta$$

$$+ \int_a^b \int_a^b x_{ss}(s)^T I(\theta - s) \left(\int_\theta^b \int_s^\zeta ((\eta - s)(\zeta - \theta)N_2(\eta, \zeta)) \, d\eta d\zeta \right) x_{ss}(\theta)ds d\theta$$

$$= \langle x_{ss}, \mathcal{P}_{0, T_{31}, T_{32}} x_{ss} \rangle_{L_2}$$

where

$$T_{31}(s, \theta) = \int_s^b \int_s^\zeta (\eta - s)(\zeta - \theta)N_2(\eta, \zeta)d\eta d\zeta$$

$$T_{32}(s, \theta) = \int_\theta^b \int_s^\zeta (\eta - s)(\zeta - \theta)N_2(\eta, \zeta)d\eta d\zeta$$

Combining these terms, we have

$$\langle x_2, \mathcal{P}\S_2 \rangle = \langle x_{ss}, \mathcal{P}_{0, G_1, G_2} x_{ss} \rangle$$

where

$$G_1(s,\theta) = T_{11}(s,\theta) + T_{21}(s,\theta) + T_{31}(s,\theta)$$

$$= \int_s^b (\eta-s)(\eta-\theta)M(\eta)d\eta + \int_s^b \int_\theta^\eta (\eta-s)(\zeta-\theta)N_1(\eta,\zeta)d\zeta d\eta + \int_s^b \int_s^\zeta (\eta-s)(\zeta-\theta)N_2(\eta,\zeta)d\eta d\zeta$$

$$= \int_s^b \left((\eta-s)\left((\eta-\theta)M(\eta) + \int_\theta^\eta (\zeta-\theta)N_1(\eta,\zeta)d\zeta \right) + \int_s^\eta (\zeta-s)(\eta-\theta)N_2(\zeta,\eta)d\zeta \right) d\eta$$

$$= \int_s^b \left((\eta-s)G_4(\theta,\eta) + G_5(s,\theta,\eta) \right) d\eta$$

$$G_2(s,\theta) = T_{12}(s,\theta) + T_{22}(s,\theta) + T_{32}(s,\theta)$$

$$= \int_\theta^b (\eta-s)(\eta-\theta)M(\eta)d\eta + \int_\theta^b \int_\theta^\eta (\eta-s)(\zeta-\theta)N_1(\eta,\zeta)d\zeta d\eta + \int_\theta^b \int_s^\zeta (\eta-s)(\zeta-\theta)N_2(\eta,\zeta)d\eta d\zeta$$

$$= \int_\theta^b \left((\eta-s)\left((\eta-\theta)M(\eta) + \int_\theta^\eta (\zeta-\theta)N_1(\eta,\zeta)d\zeta \right) + \int_s^\eta (\zeta-s)(\eta-\theta)N_2(\zeta,\eta)d\zeta \right) d\eta$$

$$= \int_\theta^b \left((\eta-s)G_4(\theta,\eta) + G_5(s,\theta,\eta) \right) d\eta$$

where we used

$$T_{11}(s,\theta) = \int_s^b (\eta-s)(\eta-\theta)M(\eta)d\eta$$

$$T_{12}(s,\theta) = \int_\theta^b (\eta-s)(\eta-\theta)M(\eta)d\eta$$

$$T_{21}(s,\theta) = \int_s^b \int_\theta^\eta (\eta-s)(\zeta-\theta)N_1(\eta,\zeta)d\zeta d\eta$$

$$T_{22}(s,\theta) = \int_\theta^b \int_\theta^\eta (\eta-s)(\zeta-\theta)N_1(s,\zeta)d\zeta d\eta$$

$$T_{31}(s,\theta) = \int_s^b \int_s^\zeta (\eta-s)(\zeta-\theta)N_2(\eta,\zeta)d\eta d\zeta$$

$$T_{32}(s,\theta) = \int_\theta^b \int_s^\zeta (\eta-s)(\zeta-\theta)N_2(\eta,\zeta)d\eta d\zeta$$

For the remaining BC terms, as before, we expand them separately

Next, we have

$$\langle x_1, \mathcal{P}_1 x_1 \rangle + \langle x_1, \mathcal{P}_2 x_1 \rangle + \langle x_1, \mathcal{P}_3 x_1 \rangle$$

$$\langle x_1, \mathcal{P}_1 x_1 \rangle = \int_a^b x_1(s)^T M(s) x_1(s) ds$$

$$= \int_a^b \left(\int_a^b B_a(s,\eta) x_{ss}(\eta) d\eta \right)^T M(s) \left(\int_a^b B_a(s,\zeta) x_{ss}(\zeta) d\zeta \right) ds$$

$$= \int_a^b \int_a^b x_{ss}(\eta)^T \left(\int_a^b B_a(s,\eta)^T M(s) B_a(s,\zeta) ds \right) x_{ss}(\zeta) d\zeta d\eta$$

$$= \int_a^b \int_a^b x_{ss}(s)^T \left(\int_a^b B_a(\zeta,s)^T M(\zeta) B_a(\zeta,\eta) d\zeta \right) x_{ss}(\eta) d\eta ds$$

$$= \langle x_{ss}, \mathcal{P}_{0,S_{11},S_{11}} x_{ss} \rangle_{L_2}$$

$$S_{11}(s,\eta) = \int_a^b B_a(\zeta,s)^T M(\zeta) B_a(\zeta,\eta) d\zeta$$

$$\langle x_1, \mathcal{P}_2 x_1 \rangle = \int_a^b x_1(s)^T \int_a^b I(s-\theta) N_1(s,\theta) x_1(\theta) d\theta ds$$

$$= \int_a^b \int_a^b \left(\int_a^b B_a(s,\eta) x_{ss}(\eta) d\eta \right)^T I(s-\theta) N_1(s,\theta) \left(\int_a^b B_a(\theta,\zeta) x_{ss}(\zeta) d\zeta \right) d\theta ds$$

$$= \int_a^b \int_a^b x_{ss}(\eta)^T \left(\int_a^b \int_a^b \left(B_a(s,\eta)^T N_1(s,\theta) I(s-\theta) B_a(\theta,\zeta) \right) d\theta ds \right) x_{ss}(\zeta) d\eta d\zeta$$

$$= \int_a^b \int_a^b x_{ss}(\eta)^T \left(\int_a^b \int_a^s \left(B_a(s,\eta)^T N_1(s,\theta) B_a(\theta,\zeta) \right) d\theta ds \right) x_{ss}(\zeta) d\eta d\zeta$$

$$= \int_a^b \int_a^b x_{ss}(s)^T \left(\int_a^b \int_a^\zeta \left(B_a(\zeta,s)^T N_1(\zeta,\theta) B_a(\theta,\eta) \right) d\theta d\zeta \right) x_{ss}(\eta) ds d\eta$$

$$= \langle x_{ss}, \mathcal{P}_{0,S_{21},S_{21}} x_{ss} \rangle_{L_2}$$

$$S_{21}(s,\eta) = \int_a^b \int_a^\zeta B_a(\zeta,s)^T N_1(\zeta,\theta) B_a(\theta,\eta) d\theta d\zeta$$

$$\langle x_1, \mathcal{P}_3 x_1 \rangle = \int_a^b x_1(s)^T \int_a^b I(\theta-s) N_2(s,\theta) x_1(\theta) d\theta ds$$

$$= \int_a^b \int_a^b \left(\int_a^b B_a(s,\eta) x_{ss}(\eta) d\eta \right)^T I(\theta-s) N_2(s,\theta) \left(\int_a^b B_a(\theta,\zeta) x_{ss}(\zeta) d\zeta \right) d\theta ds$$

$$= \int_a^b \int_a^b x_{ss}(\eta)^T \left(\int_a^b \int_a^b \left(B_a(s,\eta)^T N_2(s,\theta) I(\theta-s) B_a(\theta,\zeta) \right) d\theta ds \right) x_{ss}(\zeta) d\eta d\zeta$$

$$= \int_a^b \int_a^b x_{ss}(\eta)^T \left(\int_a^b \int_s^b \left(B_a(s,\eta)^T N_2(s,\theta) B_a(\theta,\zeta) \right) d\theta ds \right) x_{ss}(\zeta) d\eta d\zeta$$

$$= \int_a^b \int_a^b x_{ss}(s)^T \left(\int_a^b \int_\zeta^b \left(B_a(\zeta,s)^T N_2(\zeta,\theta) B_a(\theta,\eta) \right) d\theta d\zeta \right) x_{ss}(\eta) ds d\eta$$

$$= \langle x_{ss}, \mathcal{P}_{0,S_{31},S_{31}} x_{ss} \rangle_{L_2}$$

$$S_{31}(s,\eta) = \int_a^b \int_\zeta^b B_a(\zeta,s)^T N_2(\zeta,\theta) B_a(\theta,\eta) d\theta d\zeta$$

summing these BC terms, we have

$$\langle x_{ss}, \mathcal{P}_{0,S_{c1},S_{c1}} x_{ss} \rangle_{L_2}$$

$$S_{c1}(s,\eta) = S_{11}(s,\eta) + S_{21}(s,\eta) + S_{31}(s,\eta)$$

$$= \int_a^b B_a(\zeta,s)^T \left(M(\zeta) B_a(\zeta,\eta) + \int_a^\zeta N_1(\zeta,\theta) B_a(\theta,\eta) d\theta + \int_\zeta^b N_2(\zeta,\theta) B_a(\theta,\eta) d\theta \right) d\zeta$$

$$= \int_a^b B_a(\zeta,s)^T Y_3(\zeta,\eta) d\zeta$$

Next, we have

$$\langle x_1, \mathcal{P}_1 x_2 \rangle + \langle x_1, \mathcal{P}_2 x_2 \rangle + \langle x_1, \mathcal{P}_3 x_2 \rangle$$

$$\langle x_1, \mathcal{P}_1 x_2 \rangle = \int_a^b x_1(s)^T M(s) x_2(s) ds$$

$$= \int_a^b \left(\int_a^b B_a(s,\eta) x_{ss}(\eta) d\eta \right)^T \left(\int_a^b M(s)(s-\zeta) I(s-\zeta) x_{ss}(\zeta) d\zeta \right) ds$$

$$= \int_a^b \int_a^b x_{ss}(\eta)^T \left(\int_a^b B_a(s,\eta)^T (s-\zeta) I(s-\zeta) M(s) ds \right) x_{ss}(\zeta) d\zeta d\eta$$

$$= \int_a^b \int_a^b x_{ss}(\eta)^T \left(\int_\zeta^b B_a(s,\eta)^T (s-\zeta) M(s) ds \right) x_{ss}(\zeta) d\zeta d\eta$$

$$= \int_a^b \int_a^b x_{ss}(s)^T \left(\int_\eta^b B_a(\zeta,s)^T (\zeta-\eta) M(\zeta) d\zeta \right) x_{ss}(\eta) d\eta ds$$

$$= \langle x_{ss}, \mathcal{P}_{0,S_{12},S_{12}} x_{ss} \rangle_{L_2}$$

$$S_{12}(s,\eta) = \int_\eta^b B_a(\zeta,s)^T (\zeta-\eta) M(\zeta) d\zeta$$

$$\langle x_1, \mathcal{P}_2 x_2 \rangle = \int_a^b x_1(s)^T \int_a^b I(s-\theta)N_1(s,\theta)x_2(\theta)d\theta ds$$

$$= \int_a^b \int_a^b \left(\int_a^b B_a(s,\eta)x_{ss}(\eta)d\eta \right)^T I(s-\theta)N_1(s,\theta) \left(\int_a^b (\theta-\zeta)I(\theta-\zeta)x_{ss}(\zeta)d\zeta \right) d\theta ds$$

$$= \int_a^b \int_a^b x_{ss}(\eta)^T \left(\int_a^b \int_a^b \left(B_a(s,\eta)^T(\theta-\zeta)N_1(s,\theta)I(s-\theta)I(\theta-\zeta) \right) d\theta ds \right) x_{ss}(\zeta)d\eta d\zeta$$

$$= \int_a^b \int_a^b x_{ss}(\eta)^T \left(\int_\zeta^b \int_\theta^b \left(B_a(s,\eta)^T(\theta-\zeta)N_1(s,\theta) \right) ds d\theta \right) x_{ss}(\zeta)d\eta d\zeta$$

$$= \int_a^b \int_a^b x_{ss}(s)^T \left(\int_\eta^b \int_\zeta^b \left(B_a(\theta,s)^T(\zeta-\eta)N_1(\theta,\zeta) \right) d\theta d\zeta \right) x_{ss}(\eta)ds d\eta$$

$$= \langle x_{ss}, \mathcal{P}_{0,S_{22},S_{22}} x_{ss} \rangle_{L_2}$$

$$S_{22}(s,\eta) = \int_\eta^b \int_\zeta^b B_a(\theta,s)^T(\zeta-\eta)N_1(\theta,\zeta)d\theta d\zeta$$

$$\langle x_1, \mathcal{P}_3 x_2 \rangle = \int_a^b x_1(s)^T \int_a^b I(\theta-s)N_2(s,\theta)x_2(\theta)d\theta ds$$

$$= \int_a^b \int_a^b \left(\int_a^b B_a(s,\eta)x_{ss}(\eta)d\eta \right)^T I(\theta-s)N_2(s,\theta) \left(\int_a^b (\theta-\zeta)I(\theta-\zeta)x_{ss}(\zeta)d\zeta \right) d\theta ds$$

$$= \int_a^b \int_a^b x_{ss}(\eta)^T \left(\int_a^b \int_a^b \left(B_a(s,\eta)^T(\theta-\zeta)N_2(s,\theta)I(\theta-s)I(\theta-\zeta) \right) d\theta ds \right) x_{ss}(\zeta)d\eta d\zeta$$

$$= \int_a^b \int_a^b x_{ss}(\eta)^T \left(\int_\zeta^b \int_a^\theta \left(B_a(s,\eta)^T(\theta-\zeta)N_2(s,\theta) \right) ds d\theta \right) x_{ss}(\zeta)d\eta d\zeta$$

$$= \int_a^b \int_a^b x_{ss}(s)^T \left(\int_\eta^b \int_a^\zeta \left(B_a(\theta,s)^T(\zeta-\eta)N_2(\theta,\zeta) \right) d\theta d\zeta \right) x_{ss}(\eta)ds d\eta$$

$$= \langle x_{ss}, \mathcal{P}_{0,S_{32},S_{32}} x_{ss} \rangle_{L_2}$$

$$S_{32}(s,\eta) = \int_\eta^b \int_a^\zeta B_a(\theta,s)^T(\zeta-\eta)N_2(\theta,\zeta)d\theta d\zeta$$

summing these terms, we have

$$\langle x, \mathcal{P}_{0,S_{2c},S_{2c}} x \rangle_{L_2}$$

$$S_{2c}(s,\eta) = S_{12}(s,\eta) + S_{22}(s,\eta) + S_{32}(s,\eta)$$

$$= \int_\eta^b (\zeta-\eta) \left(B_a(\zeta,s)^T M(\zeta) + \int_\zeta^b B_a(\theta,s)^T N_1(\theta,\zeta)d\theta + \int_a^\zeta B_a(\theta,s)^T N_2(\theta,\zeta)d\theta \right) d\zeta$$

$$= \int_\eta^b (\zeta-\eta)Y_1(s,\zeta)d\zeta$$

$$\langle x_2, \mathcal{P}_1 x_1 \rangle + \langle x_2, \mathcal{P}_2 x_1 \rangle + \langle x_2, \mathcal{P}_3 x_1 \rangle$$

$$
\begin{aligned}
\langle x_2, \mathcal{P}_1 x_1 \rangle &= \int_a^b x_2(s)^T M(s) x_1(s) ds \\
&= \int_a^b \left(\int_a^b (s-\eta) I(s-\eta) x_{ss}(\eta) d\eta \right)^T \left(\int_a^b M(s) B_a(s,\zeta) x_{ss}(\zeta) d\zeta \right) ds \\
&= \int_a^b \int_a^b x_{ss}(\eta)^T \left(\int_a^b (s-\eta) I(s-\eta) M(s) B_a(s,\zeta) ds \right) x_{ss}(\zeta) d\zeta d\eta \\
&= \int_a^b \int_a^b x_{ss}(\eta)^T \left(\int_\eta^b (s-\eta) M(s) B_a(s,\zeta) ds \right) x_{ss}(\zeta) d\zeta d\eta \\
&= \int_a^b \int_a^b x_{ss}(s)^T \left(\int_s^b (\zeta-s) M(\zeta) B_a(\zeta,\eta) d\zeta \right) x_{ss}(\eta) d\eta ds \\
&= \langle x_{ss}, \mathcal{P}_{0,S_{13},S_{13}} x_{ss} \rangle_{L_2} \\
S_{13}(s,\eta) &= \int_s^b (\zeta-s) M(\zeta) B_a(\zeta,\eta) d\zeta
\end{aligned}
$$

$$
\begin{aligned}
\langle x_2, \mathcal{P}_2 x_1 \rangle &= \int_a^b x_2(s)^T \int_a^b I(s-\theta) N_1(s,\theta) x_1(\theta) d\theta ds \\
&= \int_a^b \int_a^b \left(\int_a^b (s-\eta) I(s-\eta) x_{ss}(\eta) d\eta \right)^T I(s-\theta) N_1(s,\theta) \left(\int_a^b B_a(\theta,\zeta) x_{ss}(\zeta) d\zeta \right) d\theta ds \\
&= \int_a^b \int_a^b x_{ss}(\eta)^T \left(\int_a^b \int_a^b ((s-\eta) N_1(s,\theta) I(s-\eta) I(s-\theta) B_a(\theta,\zeta)) d\theta ds \right) x_{ss}(\zeta) d\eta d\zeta \\
&= \int_a^b \int_a^b x_{ss}(\eta)^T \left(\int_\eta^b \int_a^s ((s-\eta) N_1(s,\theta) B_a(\theta,\zeta)) d\theta ds \right) x_{ss}(\zeta) d\eta d\zeta \\
&= \int_a^b \int_a^b x_{ss}(s)^T \left(\int_s^b \int_a^\zeta ((\zeta-s) N_1(\zeta,\theta) B_a(\theta,\eta)) d\theta d\zeta \right) x_{ss}(\eta) ds d\eta \\
&= \langle x_{ss}, \mathcal{P}_{0,S_{23},S_{23}} x_{ss} \rangle_{L_2} \\
S_{23}(s,\eta) &= \int_s^b \int_a^\zeta (\zeta-s) N_1(\zeta,\theta) B_a(\theta,\eta) d\theta d\zeta
\end{aligned}
$$

$$\langle x_2, \mathcal{P}_3 x_1 \rangle = \int_a^b x_2(s)^T \int_a^b I(\theta - s) N_2(s, \theta) x_1(\theta) d\theta ds$$

$$= \int_a^b \int_a^b \left(\int_a^b (s - \eta) I(s - \eta) x_{ss}(\eta) d\eta \right)^T I(\theta - s) N_2(s, \theta) \left(\int_a^b B_a(\theta, \zeta) x_{ss}(\zeta) d\zeta \right) d\theta ds$$

$$= \int_a^b \int_a^b x_{ss}(\eta)^T \left(\int_a^b \int_a^b ((s - \eta) N_2(s, \theta) I(s - \eta) I(\theta - s) B_a(\theta, \zeta)) \, d\theta ds \right) x_{ss}(\zeta) d\eta d\zeta$$

$$= \int_a^b \int_a^b x_{ss}(\eta)^T \left(\int_\eta^b \int_s^b ((s - \eta) N_2(s, \theta) B_a(\theta, \zeta)) \, d\theta ds \right) x_{ss}(\zeta) d\eta d\zeta$$

$$= \int_a^b \int_a^b x_{ss}(s)^T \left(\int_s^b \int_\zeta^b ((\zeta - s) N_2(\zeta, \theta) B_a(\theta, \eta)) \, d\theta d\zeta \right) x_{ss}(\eta) ds d\eta$$

$$= \langle x_{ss}, \mathcal{P}_{0, S_{33}, S_{33}} x_{ss} \rangle_{L_2}$$

$$S_{33}(s, \eta) = \int_s^b \int_\zeta^b (\zeta - s) N_2(\zeta, \theta) B_a(\theta, \eta) d\theta d\zeta$$

summing these terms, we have

$$\langle x, \mathcal{P}_{0, S_{c3}, S_{c3}} x \rangle_{L_2}$$

$$S_{c3}(s, \eta) = S_{13}(s, \eta) + S_{23}(s, \eta) + S_{33}(s, \eta)$$

$$= \int_s^b (\zeta - s) \left(M(\zeta) B_a(\zeta, \eta) + \int_a^\zeta N_1(\zeta, \theta) B_a(\theta, \eta) d\theta + \int_\zeta^b N_2(\zeta, \theta) B_a(\theta, \eta) d\theta \right) d\zeta$$

$$= \int_s^b (\zeta - s) Y_3(\zeta, \eta) d\zeta$$

Combining all boundary terms, we have

$$\langle x_1, \mathcal{P}_1 x_1 \rangle + \langle x_2, \mathcal{P}_1 x_1 \rangle + \langle x_1, \mathcal{P}_1 x_2 \rangle$$

$$+ \langle x_1, \mathcal{P}_2 x_3 \rangle + \langle x_2, \mathcal{P}_2 x_1 \rangle + \langle x_1, \mathcal{P}_2 x_2 \rangle$$

$$+ \langle x_1, \mathcal{P}_3 x_3 \rangle + \langle x_2, \mathcal{P}_3 x_1 \rangle + \langle x_1, \mathcal{P}_3 x_2 \rangle$$

$$= \langle x_{ss}, \mathcal{P}_{0, G_3, G_3} x_{ss} \rangle_{L_2}$$

where

$$G_3(s, \theta) = S_{c1}(s, \theta) + S_{c2}(s, \theta) + S_{c3}(s, \theta) \quad = \int_a^b B_a(\eta, s)^T Y_3(\eta, \theta) d\eta + \int_\theta^b (\eta - \theta) Y_1(s, \eta) d\eta + \int_s^b (\eta - s) Y_3(\eta, \theta) d\eta$$

where we used

$$S_{c1}(s,\eta) = \int_a^b B_a(\zeta,s)^T Y_3(\zeta,\eta)d\zeta$$

$$S_{2c}(s,\eta) = \int_\eta^b (\zeta - \eta)Y_1(s,\zeta)d\zeta$$

$$S_{c3}(s,\eta) = \int_s^b (\zeta - s)Y_3(\zeta,\eta)d\zeta$$

Concluding, we have

$$\begin{aligned}
\langle x, \mathcal{P}x \rangle &= \langle x_1, \mathcal{P}_1 x_1 \rangle + \langle x_2, \mathcal{P}_1 x_1 \rangle + \langle x_1, \mathcal{P}_1 x_2 \rangle + \langle x_2, \mathcal{P}_1 x_2 \rangle \\
&\quad + \langle x_1, \mathcal{P}_2 x_3 \rangle + \langle x_2, \mathcal{P}_2 x_1 \rangle + \langle x_1, \mathcal{P}_2 x_2 \rangle + \langle x_2, \mathcal{P}_2 x_2 \rangle \\
&\quad + \langle x_1, \mathcal{P}_3 x_3 \rangle + \langle x_2, \mathcal{P}_3 x_1 \rangle + \langle x_1, \mathcal{P}_3 x_2 \rangle + \langle x_2, \mathcal{P}_3 x_2 \rangle \\
&= \langle x_{ss}, \mathcal{P}_{0,G_1,G_2} x_{ss} \rangle_{L_2} + \langle x_{ss}, \mathcal{P}_{0,G_3,G_3} x_{ss} \rangle_{L_2}
\end{aligned}$$

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