

Soliton Hierarchies from Matrix Loop Algebras

Wen-Xiu Ma and Xing Lü

Abstract. Matrix loop algebras, both semisimple and non-semisimple, are used to generate soliton hierarchies. Hamiltonian structures to guarantee the Liouville integrability are determined by using the trace identity or the variational identity. An application example is presented from a perturbed Kaup–Newell matrix spectral problem associated with the three-dimensional real special linear algebra.

Mathematics Subject Classification (2010). Primary 37K10; Secondary 35Q53.

Keywords. Zero curvature equation, soliton hierarchy, Hamiltonian structure.

1. Introduction

Soliton hierarchies possessing Hamiltonian structures or bi-Hamiltonian structures provide examples of integrable systems. Within given matrix loop algebras, zero curvature equations associated with matrix spectral problems (or equivalently, Lax pairs) are essential objects in generating soliton hierarchies and their Hamiltonian structures (see, e.g., [1–7]).

Among celebrated examples are the Korteweg–de Vries hierarchy [8], the Ablowitz–Kaup–Newell–Segur hierarchy [9], the Dirac hierarchy [10], the Kaup–Newell hierarchy [11], the Wadati–Konno–Ichikawa hierarchy [12] and the Heisenberg hierarchy [13]. All those soliton hierarchies are generated from the three-dimensional real special linear algebra $\mathfrak{sl}(2, \mathbb{R})$. This Lie algebra is simple and has the basis

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad (1)$$

with the standard commutation relations:

$$[e_1, e_2] = 2e_2, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = 2e_3. \quad (2)$$

Its derived algebra is itself, and so, it is 3-dimensional, too. The only other three-dimensional real Lie algebras with a three-dimensional derived algebra is the spe-

cial orthogonal algebra $\mathfrak{so}(3, \mathbb{R})$, whose basis $\{e_1, e_2, e_3\}$ satisfying the circular commutation relations: $[e_1, e_2] = e_3$, $[e_2, e_3] = e_1$, $[e_3, e_1] = e_2$, is called a standard basis. Those two Lie algebras have been widely used in generating soliton hierarchies in integrable systems (see, e.g., [8–16] using $\mathfrak{sl}(2, \mathbb{R})$ and [17–21] using $\mathfrak{so}(3, \mathbb{R})$).

For a given matrix Lie algebra g , its loop algebra $\tilde{g}$ adopted in this paper is defined as

$$\tilde{g} = \left\{ \sum_{i \geq 0} M_i \lambda^{n-i} \mid M_i \in g, i \geq 0, n \in \mathbb{Z} \right\}, \quad (3)$$

that is, the space of all Laurent series in λ with coefficients in g and a finite regular part. Particular examples of a matrix loop algebra contain the linear combinations: $\lambda^m d_1 f_1 + \lambda^n d_2 f_2 + \lambda^l d_3 f_3$ with arbitrary integers m, n, l , real constants d_1, d_2, d_3 and elements f_1, f_2, f_3 in g . Matrix loop algebras provide a structural basis for our study of soliton hierarchies.

Let us also recall the Liouville integrability of PDEs (see, e.g., [14, 15, 21]). Let $x = (x^1, \dots, x^p)$ be the vector of spatial variables and $u = (u^1, \dots, u^q)^T$ the vector of dependent variables. A Hamiltonian system of evolutionary PDEs is

$$u_t = J \frac{\delta \mathcal{H}}{\delta u}, \quad u = u(x, t), \quad (4)$$

where $J = J(x, t, u)$ is a Hamiltonian operator and $\frac{\delta}{\delta u}$ stands for the variational derivative [22]. A conserved functional of a Hamiltonian system (4) is a functional $\mathcal{T} = \int T dx$ which determines a conservation law of (4): $D_t T + \text{Div } X = 0$, in which Div denotes spatial divergence. For a given differential function F , its corresponding one-form is given by

$$dF := \sum_{i=1}^p \frac{\partial F}{\partial x^i} dx^i + \frac{\partial F}{\partial t} dt + \sum_{\alpha=1}^q \sum_{\#L \geq 0} \frac{\partial F}{\partial u_L^\alpha} du_L^\alpha,$$

where if $\#L = 0$, then $u_L^\alpha = u^\alpha$, and if $\#L = k \geq 1$, then $u_L^\alpha = \frac{\partial^k u^\alpha}{\partial x^{l_1} \dots \partial x^{l_k}}$, for $L = (l_1, \dots, l_k)$, $1 \leq l_i \leq p$, $1 \leq i \leq k$, with $\#L = l_1 + \dots + l_k$.

Definition 1. Let I be a set of integers and $r \geq 1$ a natural number. We say that a set of r -tuples of differential functions $\{S_n = (S_n^1, \dots, S_n^r)^T \mid n \in I\}$ is independent, if all r -tuples of one-forms, $dS_n = (dS_n^1, \dots, dS_n^r)^T$, $n \in I$, are linearly independent at every point in the infinite jet space. A set of conserved functionals $\{\mathcal{H}_n \mid n \in I\}$ of a Hamiltonian system (4) is said to be independent, if all characteristics $\{J \frac{\delta \mathcal{H}_n}{\delta u} \mid n \in I\}$ of the associated Hamiltonian vector fields are independent.

By the differential order of an r -tuple S of differential functions, we mean the order of the highest-order derivative of u with respect to x in S . It is obvious to see that if a set of r -tuples of differential functions has distinct differential orders, then it is independent.

Definition 2. A Hamiltonian system of evolutionary PDEs, (4), is called to be Liouville integrable, if there exists infinitely many conserved functionals $\{\mathcal{H}_n\}_{n=0}^\infty$,

which are in involution with respect to the Poisson bracket:

$$\{\mathcal{H}_m, \mathcal{H}_n\}_J := \int \left(\frac{\delta \mathcal{H}_m}{\delta u} \right)^T J \frac{\delta \mathcal{H}_n}{\delta u} dx = 0, \quad m, n \geq 0, \quad (5)$$

and the characteristics of whose associated Hamiltonian vector fields

$$K_n := J \frac{\delta \mathcal{H}_n}{\delta u}, \quad n \geq 0, \quad (6)$$

are independent.

In this paper, we would like to focus on an application of the matrix loop algebra $\widetilde{\mathfrak{sl}}(2, \mathbb{R})$ within the zero curvature formulation. We will introduce a perturbed Kaup–Newell matrix spectral problem, based on $\widetilde{\mathfrak{sl}}(2, \mathbb{R})$, and construct its associated integrable Hamiltonian hierarchy through zero curvature equations. The corresponding Hamiltonian structures will be furnished by using the trace identity, and all systems in the resulting perturbed Kaup–Newell hierarchy will be shown to be Liouville integrable. A few concluding remarks will be given in the last section.

2. Zero curvature formulation

Lax proposed an operator pair approach for studying the Korteweg–de Vries equation [8], and such an involved pair is nowadays called a Lax pair. It is realized (see, e.g., [23, 24]) that a Lax pair presentation is generally equivalent to a zero curvature presentation. We say that an integrable system of PDEs possesses a zero curvature representation, if it can be generated from a zero curvature equation

$$U_t - V_x + [U, V] = 0, \quad (7)$$

where $x, t \in \mathbb{R}$, and the two matrices U and V , called a spectral matrix and a Lax matrix (or operator), are taken from a given matrix loop algebra [3, 25].

As soon as a spectral matrix U is well selected, in order to present a soliton hierarchy, we start to solve a stationary zero curvature equation

$$W_x = [U, W] \quad (8)$$

in $\tilde{\mathfrak{g}}$. Then, introduce a series of Lax matrices

$$V^{[m]} = (\lambda^m W)_+ + \Delta_m, \quad \Delta_m \in \tilde{\mathfrak{g}}, \quad m \geq 0, \quad (9)$$

where P_+ denotes the polynomial part of P in λ , such that the corresponding zero curvature equations

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0, \quad m \geq 0, \quad (10)$$

yield a hierarchy of soliton equations

$$u_{t_m} = K_m, \quad m \geq 0. \quad (11)$$

The structure of W often tells how to determine the modification terms Δ_m , $m \geq 0$. The associated Lax pairs are starting points to find soliton solutions by the inverse scattering transform [1, 2].

One of our tasks in the study of integrable systems is to construct Hamiltonian structures or bi-Hamiltonian structures [26],

$$u_{t_m} = K_m = J \frac{\delta \mathcal{H}_m}{\delta u} = M \frac{\delta \mathcal{H}_{m-1}}{\delta u}, \quad m \geq 1, \quad (12)$$

which naturally generate a hereditary recursion operator $\Phi = MJ^{-1}$, and thus, infinitely many commuting conserved functionals and symmetries [27, 28]. The basic tool for constructing Hamiltonian functionals is the trace identity in the semisimple case [14]:

$$\frac{\delta}{\delta u} \int \text{tr} \left(\frac{\partial U}{\partial \lambda} W \right) dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma \text{tr} \left(\frac{\partial U}{\partial u} W \right), \quad \gamma = -\frac{\lambda}{2} \frac{d}{d\lambda} \ln |\text{tr}(W^2)|, \quad (13)$$

or generally, the variational identity in the non-semisimple case [29]:

$$\frac{\delta}{\delta u} \int \left\langle \frac{\partial U}{\partial \lambda}, W \right\rangle dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma \left\langle \frac{\partial U}{\partial u}, W \right\rangle, \quad \gamma = -\frac{\lambda}{2} \frac{d}{d\lambda} \ln |\langle W, W \rangle|, \quad (14)$$

where $\langle \cdot, \cdot \rangle$ is a symmetric, non-degenerate and ad-invariant bilinear form over the matrix loop algebra $\tilde{g}$.

3. An example: a perturbed integrable Kaup–Newell hierarchy

3.1. A perturbed Kaup–Newell hierarchy

We apply the zero curvature formulation to present a perturbed integrable Kaup–Newell hierarchy. We start with a new 2×2 matrix spectral problem:

$$\phi_x = U\phi = U(u, \lambda)\phi, \quad U = \begin{bmatrix} \lambda + \alpha p & \lambda p \\ q & -\lambda - \alpha p \end{bmatrix}, \quad u = \begin{bmatrix} p \\ q \end{bmatrix}, \quad (15)$$

where λ is the spectral parameter and α is a fixed constant. If $\alpha = 0$, then (15) reduces to the standard Kaup–Newell spectral problem [11], and thus, (15) is called a perturbed Kaup–Newell spectral problem and the corresponding soliton hierarchy is called a perturbed Kaup–Newell hierarchy.

Once a matrix spectral problem is chosen, it is inherently feasible to calculate the corresponding soliton hierarchy. First, we solve the stationary zero curvature equation (8) for $W \in \tilde{\text{sl}}(2, \mathbb{R})$. When W is assumed to be

$$W = \begin{bmatrix} a & b \\ \lambda^{-1}c & -a \end{bmatrix}, \quad (16)$$

the stationary zero curvature equation (8) becomes

$$a_x = pc - qb, \quad b_x = 2\lambda b - 2\lambda pa + 2\alpha pb, \quad c_x = -2\lambda c + 2\lambda qa - 2\alpha pc. \quad (17)$$

This leads to

$$pc_x + qb_x = -2\lambda(pc - qb) - 2\alpha p^2c + 2\alpha pqb. \quad (18)$$

Further, expand a, b and c as the Laurent series in λ :

$$a = \sum_{i \geq 0} a_i \lambda^{-i}, \quad b = \sum_{i \geq 0} b_i \lambda^{-i}, \quad c = \sum_{i \geq 0} c_i \lambda^{-i}, \quad (19)$$

and take the initial data

$$a_0 = 1, \quad b_0 = p, \quad c_0 = q, \quad (20)$$

to fix a solution to the equations from the highest powers of λ in (17):

$$a_{0,x} = pc_0 - qb_0, \quad b_0 = pa_0, \quad c_0 = qa_0.$$

Then, based on (18), we see that the system (17) gives rise to

$$\begin{cases} a_{i+1,x} = -\frac{1}{2}(pc_{i,x} + qb_{i,x}) - \alpha p^2 c_i + \alpha p q b_i, \\ b_{i+1} = \frac{1}{2}b_{i,x} + pa_{i+1} - \alpha p b_i, \\ c_{i+1} = -\frac{1}{2}c_{i,x} + qa_{i+1} - \alpha p c_i, \end{cases} \quad i \geq 0. \quad (21)$$

While using the above recursion relations, we impose the condition that the constants of integration take the value of zero:

$$a_i|_{u=0} = b_i|_{u=0} = c_i|_{u=0} = 0, \quad i \geq 1, \quad (22)$$

to uniquely determine the sequence of $\{a_i, b_i, c_i | i \geq 1\}$. This way, the first two sets can be worked out:

$$\begin{aligned} a_1 &= -\frac{pq}{2}, \quad b_1 = \frac{1}{2}(p_x - 2\alpha p^2 - p^2 q), \quad c_1 = -\frac{1}{2}(q_x + 2\alpha p q + p q^2); \\ a_2 &= -\frac{1}{4}(qp_x - p q_x) + \alpha p^2 q + \frac{3}{8}p^2 q^2, \\ b_2 &= \frac{1}{4}p_{xx} - \frac{3}{4}qp p_x - \frac{3}{2}\alpha p p_x + p^3 \left(\alpha^2 + \frac{3}{2}\alpha q + \frac{3}{8}q^2 \right), \\ c_2 &= \frac{1}{4}q_{xx} + \frac{3}{4}qp q_x + \frac{1}{2}\alpha q p_x + \alpha p q_x + p^2 q \left(\alpha^2 + \frac{3}{2}\alpha q + \frac{3}{8}q^2 \right). \end{aligned}$$

We saw above the localness of the first three sets of $\{a_i, b_i, c_i | i \geq 1\}$. This is not an accident, and the functions a_i, b_i, c_i , $i \geq 1$, are all local, indeed. We can verify this fact as follows. First from $W_x = [U, W]$, we get

$$\frac{d}{dx} \text{tr}(W^2) = 2 \text{tr}(W W_x) = 2 \text{tr}(W[U, W]) = 0,$$

and so, due to $\text{tr}(W^2) = 2(a^2 + \lambda^{-1}bc)$, we can compute that

$$a^2 + \lambda^{-1}bc = (a^2 + \lambda^{-1}bc)|_{u=0} = 1, \quad (23)$$

the second step of which follows from the initial data in (20) and the recursion relations in (21). Then, by using the Laurent expansions in (19) and noting the

initial data in (20) again, balancing the coefficients of λ^i in (23) for each $i \geq 1$ yields

$$a_i = -\frac{1}{2} \left(\sum_{k+l=i, k, l \geq 1} a_k a_l + \sum_{k+l=i-1, k, l \geq 0} b_k c_l \right), \quad i \geq 1. \quad (24)$$

Based on this recursion relation (24) and the last two recursion relations in (21), applying the mathematical induction finally tells that all functions a_i, b_i, c_i , $i \geq 1$, are differential polynomials in p and q , i.e., they are all local; and that for each $i \geq 2$, the differential orders of the differential functions a_i, b_i and c_i are $i-2, i-1$ and $i-1$, respectively.

Now as usual, we introduce

$$V^{[m]} = \lambda(\lambda^m W)_+ + \delta_m e_1, \quad m \geq 0, \quad (25)$$

where δ_m are differential functions to be determined later. A direct computation shows that $V_x^{[m]} - [U, V^{[m]}]$ is equal to

$$\begin{bmatrix} \delta_{m,x} & 2\lambda(b_{m+1} - pa_{m+1} + p\delta_m) \\ 2(-c_{m+1} + qa_{m+1} - q\delta_m) & -\delta_{m,x} \end{bmatrix}. \quad (26)$$

Therefore, the corresponding zero curvature equations (10) precisely present

$$\begin{cases} \alpha p_{t_m} = \delta_{m,x}, \quad p_{t_m} = 2(b_{m+1} - pa_{m+1} + p\delta_m), \\ q_{t_m} = 2(-c_{m+1} + qa_{m+1} - q\delta_m), \end{cases} \quad m \geq 0. \quad (27)$$

To satisfy the above third equation, we choose, based on (21), that

$$\delta_m = \alpha b_m, \quad m \geq 0, \quad (28)$$

and then, all the systems in (27) determine a soliton hierarchy

$$u_{t_m} = K_m = \begin{bmatrix} p \\ q \end{bmatrix}_{t_m} = \begin{bmatrix} b_{m,x} \\ c_{m,x} + 2\alpha p c_m - 2\alpha q b_m \end{bmatrix}, \quad m \geq 0, \quad (29)$$

which is the required perturbed Kaup–Newell hierarchy. The first nonlinear system in this perturbed hierarchy is given by

$$u_{t_1} = \begin{bmatrix} p \\ q \end{bmatrix}_{t_1} = \begin{bmatrix} \frac{1}{2}(p_{xx} - 2pp_xq - p^2q_x - 4\alpha pp_x) \\ -\frac{1}{2}q_{xx} - \frac{1}{2}p_xq^2 - pqq_x - 2\alpha(p_xq + pq_x) \end{bmatrix}. \quad (30)$$

3.2. Hamiltonian structures and Liouville integrability

We shall show that all systems in the perturbed Kaup–Newell hierarchy (29) are Liouville integrable. Towards this end, let us first establish Hamiltonian structures for the perturbed hierarchy (29) by using the trace identity (13).

In the perturbed Kaup–Newell case discussed above, the trace identity (13) reads

$$\frac{\delta}{\delta u} \int (2a + \lambda^{-1}pc) dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma \begin{bmatrix} 2\alpha a + c \\ b \end{bmatrix}. \quad (31)$$

Balancing the coefficients of λ^{-m-1} for each $m \geq 0$ in this equality tells that $\gamma = 0$ and that

$$\frac{\delta}{\delta u} \mathcal{H}_m = \begin{bmatrix} 2\alpha a_m + c_m \\ b_m \end{bmatrix}, \quad m \geq 0, \quad (32)$$

with the Hamiltonian functionals being defined by

$$\mathcal{H}_0 = \int (2\alpha p + pq) dx, \quad \mathcal{H}_m = \int \left(-\frac{2a_{m+1} + pc_m}{m} \right) dx, \quad m \geq 1. \quad (33)$$

It follows now that the hierarchy (29) has the Hamiltonian structures:

$$u_{t_m} = K_m = J \frac{\delta \mathcal{H}_m}{\delta u}, \quad J = \begin{bmatrix} 0 & \partial \\ \partial & 0 \end{bmatrix}, \quad m \geq 0. \quad (34)$$

From the recursion relations in (21), we can obtain

$$K_{m+1} = \Phi K_m, \quad m \geq 0, \quad (35)$$

where Φ is the recursion operator

$$\Phi = \begin{bmatrix} \frac{1}{2}\partial - \frac{1}{2}\partial p \partial^{-1} q - \alpha \partial p \partial^{-1} & -\frac{1}{2}\partial p \partial^{-1} p \\ -\frac{1}{2}\partial q \partial^{-1} q - \alpha \partial q \partial^{-1} - \alpha q & -\frac{1}{2}\partial - \frac{1}{2}\partial q \partial^{-1} p - \alpha p \end{bmatrix}. \quad (36)$$

We readily check that $J\Psi = \Phi J$, where Ψ is the adjoint operator of Φ , and thus, all systems, except the first one, in the perturbed Kaup–Newell hierarchy (29) are bi-Hamiltonian:

$$u_{t_m} = K_m = J \frac{\delta \mathcal{H}_m}{\delta u} = M \frac{\delta \mathcal{H}_{m-1}}{\delta u}, \quad m \geq 1, \quad (37)$$

where the second Hamiltonian operator is defined by

$$M = \Phi J = \begin{bmatrix} -\frac{1}{2}\partial p \partial^{-1} p \partial & \frac{1}{2}\partial^2 - \frac{1}{2}\partial p \partial^{-1} q \partial - \alpha \partial p \\ -\frac{1}{2}\partial^2 - \frac{1}{2}\partial q \partial^{-1} p \partial - \alpha p \partial & -\frac{1}{2}\partial q \partial^{-1} q \partial - \alpha \partial q - \alpha q \partial \end{bmatrix}.$$

Now from an observation of the Hamiltonian structures presented in (34) and the differential orders of the sequence $\{a_i, b_i, c_i | i \geq 1\}$ shown in the last subsection, it follows that the perturbed Kaup–Newell hierarchy (29) is Liouville integrable. Namely, every system in the perturbed hierarchy (29) possesses infinitely many independent commuting conserved functionals:

$$\{\mathcal{H}_k, \mathcal{H}_l\}_J := \int \left(\frac{\delta \mathcal{H}_k}{\delta u} \right)^T J \frac{\delta \mathcal{H}_l}{\delta u} dx = 0, \quad k, l \geq 0, \quad (38)$$

and infinitely many independent commuting symmetries:

$$[K_k, K_l] := K'_k(u)[K_l] - K'_l(u)[K_k] = J \frac{\delta}{\delta u} \{\mathcal{H}_k, \mathcal{H}_l\}_J = 0, \quad k, l \geq 0, \quad (39)$$

where K' is the Gateaux derivative. These commuting relations are also consequences of the Virasoro algebra of Lax matrices (see, e.g., [30] for details).

4. Concluding remarks

Matrix loop algebras were used to search for integrable Hamiltonian equations, which come in hierarchies. Within the matrix loop algebra $\widetilde{\mathfrak{sl}}(2, \mathbb{R})$, the Kaup–Newell spectral problem was perturbed by linear perturbation, and a hierarchy of associated integrable bi-Hamiltonian equations was successfully generated. Their Hamiltonian structures and Liouville integrability were established by the trace identity.

The spectral problem (15) is a special reduction of general matrix Lax pairs associated with semisimple Lie algebras (see, e.g., [3]–[7]). However, determination of all integrable reductions within the category of semisimple Lie algebras is one of the most important problems in the theory of integrable system [2], and it is still very interesting to see concrete examples of soliton hierarchies of integrable Hamiltonian equations. Among typical discussed spectral matrices associated with $\widetilde{\mathfrak{sl}}(2, \mathbb{R})$ and $\widetilde{\mathfrak{so}}(3, \mathbb{R})$ are the following three cases:

$$U(u, \lambda) = \lambda e_1 + p e_2 + q e_3, \quad \lambda^2 e_1 + \lambda p e_2 + \lambda q e_3, \quad \lambda e_1 + \lambda p e_2 + \lambda q e_3,$$

where $u = (p, q)^T$ and e_1, e_2, e_3 are three matrices in a standard basis. These correspond to the Ablowitz–Kaup–Newell–Segur type hierarchy [17], the Kaup–Newell type hierarchy [18] and the Wadati–Konno–Ichikawa type hierarchy [19], when the underlying matrix loop algebra is $\widetilde{\mathfrak{so}}(3, \mathbb{R})$. In those three examples, the vector u consists of only two dependent variables, p and q . There are various examples of soliton hierarchies with three or more dependent variables (see, e.g., [22, 25, 31]).

We also point out that given initial matrix loop algebras, it still requires a considerable amount of time to compute soliton hierarchies within the zero curvature formulation, and it is much more complicated in the case of higher spatial dimensions. The study of integrable couplings [22], associated with non-semisimple matrix loop algebras, provides specific examples of soliton hierarchies generated from higher-order matrix spectral problems. The resulting soliton hierarchies can be solved by applying Darboux transformations associated with the underlying matrix spectral problems (see, e.g., [32, 33]), possibly yielding lump solutions [34, 35]. It is, however, known that the variational identity [25, 29] does not present Hamiltonian structures for the bi-integrable couplings:

$$u_t = K(u), \quad v_t = K'(u)[v], \quad w_t = K'(u)[w],$$

where K' stands for the Gateaux derivative. It remains open how to generalize the variational identity such that we can furnish Hamiltonian structures for such integrable couplings.

Acknowledgment. The work was supported in part by the Open Fund of IPOC (BUPT) under grant no. IPOC2016B008, Natural Science Fund for Colleges and Universities of Jiangsu Province of China under the grant 17KJB110020, the National Natural Science Foundation of China under the grants 11371326, 11271008, 11301454 and 11271168, NSF under the grant DMS-1664561, and the distinguished professorships of the Shanghai University of Electric Power and the Shanghai Second Polytechnic University.

References

- [1] M.J. Ablowitz and P.A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*. Cambridge University Press, Cambridge, 1991.
- [2] S. Novikov, S.V. Manakov, L.P. Pitaevskii and V.E. Zakharov, *Theory of Solitons – The Inverse Scattering Method*. Consultants Bureau/A Division of Plenum Publishing Corporation, New York, 1984.
- [3] V.G. Drinfel'd and V.V. Sokolov, *Equations of Korteweg-de Vries type and simple Lie algebras*. Soviet Math. Dokl. **23** (1981), 457–462.
- [4] G. Wilson, *The modified Lax and two-dimensional Toda lattice equations associated with simple Lie algebras*. Erg. Theory Dynam. Sys. **1** (1981), 361–380.
- [5] L.A. Dickey, *Soliton Equations and Hamiltonian Systems*. World Scientific, Singapore, 2003.
- [6] M.F. de Groot, T.J. Hollowood and J.L. Miramontes, *Generalized Drinfel'd–Sokolov hierarchies*. Comm. Math. Phys. **145** (1992), 57–84.
- [7] N.J. Burroughs, M.F. de Groot, T.J. Hollowood and J.L. Miramontes, *Generalized Drinfel'd–Sokolov hierarchies II – The Hamiltonian structures*. Comm. Math. Phys. **153** (1993), 187–215.
- [8] P.D. Lax, *Integrals of nonlinear equations of evolution and solitary waves*. Comm. Pure Appl. Math. **21** (1968), 467–490.
- [9] M.J. Ablowitz, D.J. Kaup, A.C. Newell and H. Segur, *The inverse scattering transform–Fourier analysis for nonlinear problems*. Stud. Appl. Math. **53** (1974), 249–315.
- [10] H. Grosse, *New solitons connected to the Dirac equation*. Phys. Rep. **134** (1986), 297–304.
- [11] D.J. Kaup and A.C. Newell, *An exact solution for a derivative nonlinear Schrödinger equation*. J. Math. Phys. **19** (1978), 798–801.
- [12] M. Wadati, K. Konno and Y.H. Ichikawa, *New integrable nonlinear evolution equations*. J. Phys. Soc. Jpn. **47** (1979), 1698–1700.
- [13] D.Y. Chen and Y.S. Li, *The transformation operator V of nonlinear evolution equation*. Acta Math. Sin. **2** (1986), 343–356.
- [14] G.Z. Tu, *On Liouville integrability of zero-curvature equations and the Yang hierarchy*. J. Phys. A: Math. Gen. **22** (1989), 2375–2392.
- [15] W.X. Ma, *A new hierarchy of Liouville integrable generalized Hamiltonian equations and its reduction*. Chin. Ann. Math. A **13** (1992), 115–123; Chin. J. Contemp. Math. **13** (1992), 79–89.
- [16] L.Y. Jiang, Y.Y. Jin, W.X. Ma, S.F. Shen and H.Y. Zhu, *Novel soliton hierarchies of Levi type and their bi-Hamiltonian structures*. Commun. Nonlinear Sci. Numer. Simul. **23** (2015), 388–396.
- [17] W.X. Ma, *A soliton hierarchy associated with $so(3, \mathbb{R})$* . Appl. Math. Comput. **220** (2013), 117–122.
- [18] W.X. Ma, *A spectral problem based on $so(3, \mathbb{R})$ and its associated commuting soliton equation*. J. Math. Phys. **54** (2013), 103509, 8 pp.
- [19] W.X. Ma, S. Manukure and H.C. Zheng, *A counterpart of the Wadati–Konno–Ichikawa soliton hierarchy associated with $so(3, \mathbb{R})$* . Z. Naturforsch. A **69** (2014), 411–419.

- [20] W.X. Ma, S.F. Shen, S.M. Yu, H.Q. Zhang and W.Y. Zhang, *An integrable $SO(3, R)$ -counterpart of the Heisenberg soliton hierarchy*. Rep. Math. Phys. **74** (2014), 283–299.
- [21] W.X. Ma, *An integrable counterpart of the D-AKNS soliton hierarchy from $so(3, \mathbb{R})$* . Phys. Lett. A **378** (2014), 1717–1720.
- [22] W.X. Ma and B. Fuchssteiner, *Integrable theory of the perturbation equations*. Chaos, Solitons Fractals **7** (1996), 1227–1250.
- [23] B.A. Kupershmidt and G. Wilson, *Conservation laws and symmetries of generalized sine-Gordon equations*. Comm. Math. Phys. **81** (1981), 189–202.
- [24] W.X. Ma, *Lax representations of hierarchies of evolution equations with Lienard recursive structures*. Acta Math. Sci. **14** (1994), 409–418.
- [25] W.X. Ma, *Integrable couplings and matrix loop algebras*. In: W.X. Ma and D. Kaup (eds.), *Proceedings of the 2nd International Workshop on Nonlinear and Modern Mathematical Physics*, AIP Conference Proceedings **1562**, American Institute of Physics, Melville, NY, 2013, pp. 105–122.
- [26] F. Magri, *A simple model of the integrable Hamiltonian equation*. J. Math. Phys. **19** (1978), 1156–1162.
- [27] P.J. Olver, *Evolution equations possessing infinitely many symmetries*. J. Math. Phys. **18** (1977), 1212–1215.
- [28] B. Fuchssteiner and A.S. Fokas, *Symplectic structures, their Bäcklund transformations and hereditary symmetries*. Phys. D. **4** (1981), 47–66.
- [29] W.X. Ma and M. Chen, *Hamiltonian and quasi-Hamiltonian structures associated with semi-direct sums of Lie algebras*. J. Phys. A: Math. Gen. **39** (2006), 10787–10801.
- [30] W.X. Ma, *The algebraic structures of isospectral Lax operators and applications to integrable equations*. J. Phys. A: Math. Gen. **25** (1992), 5329–5343.
- [31] Y.F. Zhang and H.W. Tam, *A few new higher-dimensional Lie algebras and two types of coupling integrable couplings of the AKNS hierarchy and the KN hierarchy*. Commun. Nonlinear Sci. Numer. Simul. **16** (2011), 76–85.
- [32] V.B. Matveev and M.A. Salle, *Darboux Transformations and Solitons*. Springer-Verlag, Berlin, 1991.
- [33] W.X. Ma, *Darboux transformations for a Lax integrable system in $2n$ -dimensions*. Lett. Math. Phys. **39** (1997), 33–49.
- [34] K. Imai and K. Nozaki, *Lump solutions of the Ishimori-II equation*. Progr. Theoret. Phys. **96** (1996), 521–526.
- [35] X. Lü and W.X. Ma, *Study of lump dynamics based on a dimensionally reduced Hirota bilinear equation*. Nonlinear Dynam. **85** (2016), 1217–1222.

Wen-Xiu Ma
 Department of Mathematics and Statistics
 University of South Florida
 Tampa, Florida 33620-5700, USA
 e-mail: mawx@cas.usf.edu

Xing Lü
 School of Computer and
 Communication Engineering
 Univ. of Science and Technology Beijing
 Beijing 100083, China
 e-mail: xinglv655@aliyun.com