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Distributed Localization of a Moving Target: Structural Observability-based Convergence Analysis

Osama Ennasr and Xiaobo Tan

Abstract— **Due to its many applications in wireless sensor networks, localization of a moving target has received increasing attention. One popular class of localization schemes uses time-difference-of-arrival (TDOA) of some beacon signal detected by multiple agents in the network. However, much of the work on TDOA-based source localization in the literature adopts a centralized approach, where all measurements are sent to a reference agent which produces an estimate of the target's location. In this work, we use first observability principles to show that it is impossible to estimate the target's position with an insufficient number of TDOA measurements. Then, we argue that, by averaging the estimates of neighboring agents, each agent in the network can successfully estimate the source location if and only if every agent is part of a network that has a sufficient number of TDOA measurements, even if each agent has access to an insufficient number of measurements. A numerical example is provided to illustrate these results.**

I. Introduction

Source localization using measurements from spatially separated sensors has many applications in wireless sensor networks (WSNs) [1] [2]. Various algorithms have been proposed in this area, among which time-difference-of-arrival (TDOA) localization methods are widely used for accurate localization of a target. Generally speaking, TDOA algorithms rely on a source emitting a signal periodically, which is detected by special receivers deployed either at fixed locations or on mobile robots. If multiple receivers detect the same signal, it is possible to infer the source's location using the detection times at these receivers.

Much of the work on TDOA-based source localization in the literature adopts a centralized approach, in which a reference node is chosen and the times of arrival (TOA) of the emitted signal for all other nodes in the network are subtracted from the reference node's TOA, generating TDOA measurements at the reference node. If the propagation speed of the signal is known, the TDOA measurements can be converted to range-difference measurements, which are then used to estimate the location of the target [3], [4]. This centralized approach has a long history and is widely used in aerospace systems [5]. Geometric treatment of the problem for a stationary source was considered in [6] and [7], where the authors infer the source location from the geometric relations imposed by TDOA measurements between a reference node and all other nodes in the network. If the source's location changes with time, dynamic approaches are generally used for localization, in which a filter is used to estimate the source's location. In [8], the authors employ an Extended Kalman Filter (EKF) for source localization, while in [9], the Unscented Kalman Filter and Particle Filtering are considered to estimate the target's location.

Due to power and bandwidth constraints in WSNs, centralized information processing may be infeasible, particularly for a large-scale and unreliable network. Moreover, some sensors cannot transmit their measurements to the reference node due to their limited communication ranges. These drawbacks motivated the investigation of distributed strategies for TDOA localization. In [10], distributed source localization in multihop networks was considered. A connected dominating set of nodes work as the network backbone to collect the measurements, and a leader node of that set is selected to estimate the target's location, essentially acting as a centralized estimator of the target's position. In [1], the authors alleviate the need for a common reference node by employing a network of paired sensors where all such pairs can communicate with one another. As we discuss later in Section III, this approach can be considered as semi-centralized, as each pair is required to have sufficient TDOA measurements to estimate the target's location independently.

In this work, we use a structural observability approach to investigate the network topology conditions for distributed localization of a moving target. We show that source localization is not possible (centrally, and therefore distributively) when the number of TDOA measurement is insufficient. Then, we demonstrate that it is possible to successfully estimate the target's position in a distributed manner if every agent is part of a network that collectively has a sufficient number of TDOA measurements, even if each agent has an insufficient number of measurements.

Our work revolves around distributed observability in WSNs; in particular, the notion of structural observability of systems is exploited. Structural analysis deals with system properties that do not depend on the numerical values of the parameters, but only on the underlying structure (zeros and non-zeros) of the system [11] [12] [13]. It turns out that if a structural property holds for one possible choice of non-zero elements as free parameters, it is true for almost all choices of non-zero elements and, therefore, is called a *generic property* of the system. Furthermore, it can be shown that those particular (non-admissible) choices for which the generic property does not hold, lie on some algebraic variety with zero Lebesgue measure [14]. This work is similar to [12] and [15] in that it employs structural analysis on the system matrices. However, the results reported in [12] and

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O. Ennasr and X. Tan are with the Department of Electrical and Computer Engineering, Michigan State University, East Lansing, MI 48824, USA. Email: {ennasros, xbtan}@egr.msu.edu

[15] cannot be used here, as the nonzero parameters of our system matrices after discretization of dynamics and linearizion of measurements lie on that algebraic variety. This is because the aforementioned work treats all nonzero elements as free parameters, which in turn disguises the importance of the number of TDOA measurements. In Section III, we discuss the relationship between the number of TDOA measurements and the observability of the system.

The remainder of this paper is organized as follows. In Section II we present the target movement model and the TDOA measurement model. Section III provides the main results of this paper on distributed estimation, while a numerical simulation example is given in Section IV. Finally, Section V provides concluding remarks and future research directions.

II. Problem Setup

We consider a moving target in the 3D space with $p(t) = \begin{bmatrix} p^x(t) & p^y(t) & p^z(t) \end{bmatrix}^T$ denoting its coordinates at time t. The target moves randomly in space according to the following model

$$\begin{bmatrix} \dot{p}(t) \\ \ddot{p}(t) \end{bmatrix} = \begin{bmatrix} 0 & I_3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p(t) \\ \dot{p}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ I_3 \end{bmatrix} w(t) \tag{1}$$

where I_3 is the 3×3 identity matrix and $w(t) \in \mathbb{R}^3$ is the process noise which is assumed to be zero-mean white Gaussian noise with covariance matrix Q. The target emits a signal periodically that gets detected by a group of N agents, or nodes, at different times depending on each agent's relative distance to the target.

At each detection, agent i records the signal's time-of-arrival (TOA) and acquires the TOAs of all other agents that can communicate their information to agent i. We call the set of agents that can send their information to agent i as *neighbors* of agent i, and denote them with $\mathcal{N}_i$. Each agent then subtracts the TOAs of its neighbors from its own TOA, generating a list of time-difference-of-arrival (TDOA) measurements. Assuming that the propagation speed of the signal is known, these TDOA measurements can be converted to a list of range differences, and the measurements available for each agent can be represented by

$$y_i(kT) = h_i(kT) + v_i(kT) \tag{2}$$

where

$$h_i(x(t)) = \begin{bmatrix} h_{i,1}(x(t)) \\ \vdots \\ h_{i,|\mathcal{N}_i|}(x(t)) \end{bmatrix} \tag{3}$$

with

$$h_{i,j}(x(t)) = \|p(t) - p_i(t)\| - \|p(t) - p_{i,j}(t)\| \tag{4}$$

Here, T is the period at which the signal is emitted, $v_i(t) \in \mathbb{R}^{|\mathcal{N}_i|}$ is the measurement noise, assumed to be zero-mean white Gaussian noise with covariance matrix R_i, $p_i(t)$ is the position of agent i, and $p_{i,j}(t)$ is the position of the j-th neighbor of agent i.

Denoting the target's state as $x(t) = \begin{bmatrix} p(t) & \dot{p}(t) \end{bmatrix}^T$, and descritizing the model in (1) with sampling time T, with slight abuse of notation, we can write the discrete-time model of the target as

$$x(k+1) = Ax(k) + Bw(k) \tag{5}$$

where

$$A = \begin{bmatrix} 1 & 0 & 0 & T & 0 & 0 \\ 0 & 1 & 0 & 0 & T & 0 \\ 0 & 0 & 1 & 0 & 0 & T \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{T^2}{2} & 0 & 0 \\ 0 & \frac{T^2}{2} & 0 \\ 0 & 0 & \frac{T^2}{2} \\ T & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & T \end{bmatrix} \tag{6}$$

are obtained by discretization of the system matrices shown in (1).

The time-varying measurement matrix $H_i(k)$ can be obtained from (2), where

$$H_i(k) = \begin{bmatrix} \frac{\partial h_{i,1}(k)}{\partial p^x(k)} & \frac{\partial h_{i,1}(k)}{\partial p^y(k)} & \frac{\partial h_{i,1}(k)}{\partial p^z(k)} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial h_{i,|\mathcal{N}_i|}(k)}{\partial p^x(k)} & \frac{\partial h_{i,|\mathcal{N}_i|}(k)}{\partial p^y(k)} & \frac{\partial h_{i,|\mathcal{N}_i|}(k)}{\partial p^z(k)} & 0 & 0 & 0 \end{bmatrix} \tag{7}$$

and

$$\frac{\partial h_{i,j}(k)}{\partial p^x(k)} = \frac{p^x(k) - p_i^x(k)}{\|p(k) - p_i(k)\|} - \frac{p^x(k) - p_{i,j}^x(k)}{\|p(k) - p_{i,j}(k)\|} \tag{8}$$

$$\frac{\partial h_{i,j}(k)}{\partial p^y(k)} = \frac{p^y(k) - p_i^y(k)}{\|p(k) - p_i(k)\|} - \frac{p^y(k) - p_{i,j}^y(k)}{\|p(k) - p_{i,j}(k)\|} \tag{9}$$

$$\frac{\partial h_{i,j}(k)}{\partial p^z(k)} = \frac{p^z(k) - p_i^z(k)}{\|p(k) - p_i(k)\|} - \frac{p^z(k) - p_{i,j}^z(k)}{\|p(k) - p_{i,j}(k)\|} \tag{10}$$

III. Distributed Estimation

The goal is for every agent to estimate the target's position without requiring a central node to collect all measurements and propagate an estimate to all agents in the network. To that end, we discuss two distributed estimation schemes.

A. Semi-centralized estimation

In this approach, each agent runs its own filter using its own TDOA measurements. Here, agents exchange only the TOA values to generate TDOA measurements, along with their respective positions, without exchanging any other pieces of information. Each node implements an Extended Kalman Filter (EKF) to estimate the system's state

$$\hat{x}_i(k|k-1) = A\hat{x}_i(k-1|k-1) \tag{11}$$

$$\hat{x}_i(k|k) = \hat{x}_i(k|k-1) + K_i(k)\big[y_i(k) - h_i(\hat{x}_i(k|k-1))\big] \tag{12}$$

where $\hat{x}_i(k|j)$ is the i-th node's estimate of the state at time k after the j-th measurement has been processed, and $K_i(k)$ is its filtering gain, which is computed according to

$$K_i(k) = P_i(k|k-1)H_i(k)^T \times \left[H_i(k)P_i(k|k-1)H_i(k)^T + R_i\right]^{-1} \tag{13}$$

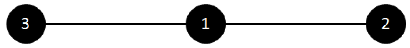

Fig. 1. Network of 3 agents monitoring the target with agent 1 as the reference node.

and

$$P_i(k|k-1) = AP_i(k-1|k-1)A^T + BQB^T \quad (14)$$

$$P_i(k|k) = \big[I - K_i(k)H_i(k)\big]P_i(k|k-1) \big[I - K_i(k)H_i(k)\big]^T + K_i(k)R_iK_i(k)^T \quad (15)$$

where $P_i(k|j)$ is the i-th agent's error covariance matrix at time k after the j-th measurement has been processed. It is assumed that the agents can effectively exchange and process these pieces of information between consecutive signal emissions.

It is well-known (see [16] and [17]) that the estimation error for agent i under this scheme

$$\tilde{x}_i(k+1) = A(I - K_i(k)H_i(k))\tilde{x}_i(k) + \eta_i(k) \quad (16)$$

is stable if and only if the pair $(A, H_i(k))$ is observable, where $\tilde{x}_i(k) \triangleq x(k) - \hat{x}_i(k|k)$ is the estimation error for agent i and the vector $\eta_i(k)$ collects the terms independent of $\tilde{x}_i(k)$. In the following, we will show that the pair $(A, H_i(k))$ is unobservable when agent i has less than 3 TDOA measurements. To avoid clutter, we will consider only agent 1 of the network, and drop the i subscript from the following analysis.

As discussed earlier, if a structural property is true for one *admissible* choice of non-zero elements, it is true for *almost all* choices of non-zero elements. Additionally, it can be shown that the choices of parameters for which the generic property does not hold, lie on a hypersurface (see Definition 3.1) in the free parameter space with zero Lebesgue measure [14]. Due to the fixed structure of our system matrix A in (6) and the time-varying measurement matrix $H(k)$ in (7), it is beneficial to utilize structural analysis when examining the observability of our system. In the following, we employ a structural approach to establish the minimum number of TDOA measurements needed to render the process generically observable.

Definition 3.1: Let $f = f(x_1, \dots, x_n)$ be a polynomial in the n variables $x_1, \dots, x_n$ with coefficients in $\mathbb{R}$, then the point $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n)$ in $\mathbb{R}^n$ is said to be a zero of f if $f(\bar{x}_1, \dots, \bar{x}_n) = 0$. The set of zeros of f is called the locus of f. A subset V of $\mathbb{R}^n$ is called a hypersurface in $\mathbb{R}^n$ if it is the locus of a nonconstant polynomial.

First, we consider the case where agent 1 only has two neighbors and, therefore, only two TDOA measurements as shown in Figure 1. The measurement matrix $H(k)$, in this case, admits the following structure

$$H_\lambda = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & 0 & 0 & 0 \\ \lambda_4 & \lambda_5 & \lambda_6 & 0 & 0 & 0 \end{bmatrix} \quad (17)$$

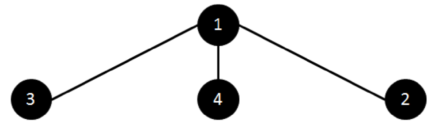

Fig. 2. Network of 4 agents monitoring the target with agent 1 as the reference node.

The system is said to be generically observable if the pair (A, H_λ) is observable for almost all values of $T, \lambda_1, \dots, \lambda_6$. In other words, the system is generically observable if and only if the observability matrix $\mathcal{O}$ is full rank for almost all values of $T, \lambda_1, \dots, \lambda_6$, where

$$\mathcal{O} = \begin{bmatrix} H_\lambda \\ H_\lambda A \\ H_\lambda A^2 \\ H_\lambda A^3 \\ H_\lambda A^4 \\ H_\lambda A^5 \end{bmatrix} \quad (18)$$

It is well known that rank$(\mathcal{O}) < 6$ if and only if all 6×6 minors of $\mathcal{O}$ are zero [14]. Using a program capable of processing symbolic math, such as *Wolfram Mathematica* and its command `Minors`, we can easily verify that all 6×6 minors of $\mathcal{O}$ in (18) are zero, regardless of the values of $T, \lambda_1, \dots, \lambda_6$. This implies that the process in (5) with measurement matrix (7) is unobservable when the node has two or less TDOA measurements[1].

Now we consider the case where agent 1 has three neighbors and, therefore, three TDOA measurements as shown in Figure 2. The measurement matrix $H(k)$, in this case, admits the following structure

$$H_\lambda = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & 0 & 0 & 0 \\ \lambda_4 & \lambda_5 & \lambda_6 & 0 & 0 & 0 \\ \lambda_7 & \lambda_8 & \lambda_9 & 0 & 0 & 0 \end{bmatrix} \quad (19)$$

Checking all 6×6 minors of $\mathcal{O}$, we observe that some minors of $\mathcal{O}$ are not identically zero and are all of the form

$$\alpha T^3 (\lambda_3 (\lambda_5\lambda_7 - \lambda_4\lambda_8) + \lambda_2 (\lambda_4\lambda_9 - \lambda_6\lambda_7) + \lambda_1 (\lambda_6\lambda_8 - \lambda_5\lambda_9))^2 \quad (20)$$

for some $\alpha \in \mathbb{R}$. Therefore, we conclude that rank$(\mathcal{O}) = 6$ for almost all values of T and $\lambda_1, \dots, \lambda_9$, and that the pair $(A, H(k))$ is generically observable if the agent has a minimum of 3 TDOA measurements[2]. Furthermore, the set of values that render the pair unobservable is a hypersurface in the free parameter space where the expression in (20) is zero. Interestingly, this means that the process is generically observable except when:

- The sampling time used for discretization of the system in (5) is 0.

[1] If the node has only one TDOA measurements, and therefore only one neighbor, then there is only one 6×6 minor of $\mathcal{O}$ and it is det$(\mathcal{O})$.

[2] If the node has more than 3 TDOA measurements, it can be verified that rank$(\mathcal{O}) = 6$ if rank$(H_\lambda) \geq 3$.

- All of the points which satisfy

$$\det\left(\begin{bmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ \lambda_4 & \lambda_5 & \lambda_6\\ \lambda_7 & \lambda_8 & \lambda_9\end{bmatrix}\right)=0$$

i.e., when the 3 TDOA measurements are linearly dependent.

For all agents in the network to be able to estimate the target's position, we would require that all such pairs $(A, H_1(k))$,$(A, H_2(k))$,...,$(A, H_N(k))$ to be observable, i.e., we would require the pair $(I_N \otimes A, D_H)$ to be observable, where $\otimes$ denotes the Kronecker product, and

$$D_H(k) \triangleq \begin{bmatrix}H_1(k) & & 0\\ & \ddots & \\ 0 & & H_N(k)\end{bmatrix} \tag{21}$$

Under this formulation, each agent can estimate the target's location when it has a minimum of 3 TDOA measurements, corresponding to each agent having a minimum of 3 neighbors. For that reason, we refer to this approach as *semi-centralized*, as each agent needs to be heavily connected such that the system is observable using its own measurements. Next, we discuss how the number of required communication links can be reduced, and argue that it is possible to estimate the target's location without the need for heavily connecting each agent.

B. Estimate exchange and distributed estimation

We begin this discussion by investigating the network-wide estimation error from all agents under the semi-centralized approach in the previous subsection. Let $\underline{\hat{x}}(k|k) = \begin{bmatrix}\hat{x}_1(k|k)^T & \ldots & \hat{x}_N(k|k)^T\end{bmatrix}^T$ denote the network-wide estimate of the network-wide state $\underline{x}(k) = \begin{bmatrix}x(k)^T & \ldots & x(k)^T\end{bmatrix}^T = 1_N \otimes x(k)$, where $1_N \in \mathbb{R}^N$ is the column vector whose entries are all 1. The dynamics of this network-wide state can be derived as follows

$$\begin{aligned}\underline{x}(k+1) =& 1_N \otimes (Ax(k) + Bw(k))\\ =& (I_N \otimes A)\,\underline{x}(k) + (I_N \otimes B)\underline{w}(k)\end{aligned} \tag{22}$$

with $\underline{w}(k) \triangleq 1_N \otimes w(k)$ representing the network-wide process noise. Denoting the i-th agent's estimation error by $\tilde{x}_i(k) \triangleq x(k) - \hat{x}_i(k|k)$, and the network-wide estimation error $\underline{\tilde{x}}(k) \triangleq \begin{bmatrix}\tilde{x}_1(k)^T & \ldots & \tilde{x}_N(k)^T\end{bmatrix}^T$, the dynamics of $\underline{\tilde{x}}(k)$ are given by

$$\begin{aligned}\underline{\tilde{x}}(k+1) =& (I_N \otimes A)\,(I_{6N} - K(k)D_H(k))\,\underline{\tilde{x}}(k)\\ &+ \underline{\eta}(k)\end{aligned} \tag{23}$$

where $K(k)$ is a block-diagonal matrix of the filtering gains $K_1(k) \ldots K_N(k)$, and the vector $\underline{\eta}(k)$ collects the terms independent of $\underline{\tilde{x}}(k)$. This network-wide estimation error, as discussed in the previous subsection, can be stabilized if and only if the pair $(I_N \otimes A, D_H(k))$ is generically observable, where each agent needs to have a sufficient number of neighbors to estimate the process using only its own TDOA measurements.

Now, let us reconsider the dynamical system in (22), and noting that for a stochastic matrix $W \in \mathbb{R}^{N\times N}$, $W1_N = 1_N$, we can rewrite (22) as

$$\begin{aligned}\underline{x}(k+1) =& 1_N \otimes (Ax(k) + Bw(k))\\ =& W1_N \otimes Ax(k) + 1_N \otimes Bw(k)\\ =& (W \otimes A)\,\underline{x}(k) + (I_N \otimes B)\underline{w}(k)\end{aligned} \tag{24}$$

For the dynamical system in (24), a filter can be designed with estimation error dynamics that can be expressed as

$$\begin{aligned}\underline{\tilde{x}}(k+1) =& (W \otimes A)\,(I_{6N} - K_c(k)D_H(k))\,\underline{\tilde{x}}(k)\\ &+ \underline{\eta}(k)\end{aligned} \tag{25}$$

where $K_c(k)$ is the filter gain, which can stabilize the error dynamics if and only if the pair $(W \otimes A, D_H(k))$ is generically observable. In the following, we will show that it is possible to obtain a network-wide estimation error with dynamics similar to (25) by averaging the estimates among neighboring agents.

Let $W \in \mathbb{R}^{N\times N}$ be a stochastic matrix with entries $w_{ij} > 0$ if $i = j$ or if agents i and j can exchange information; otherwise $w_{ij} = 0$. We assume here that the communication links are bidirectional, and that if agent j can send its information to agent i, then the reverse is also true and $w_{ij}, w_{ji} > 0$. Every agent in the network implements a filtering scheme similar to the semi-centralized case, but followed by updating its estimate by averaging the estimates from neighbors and itself. The filter implemented by each agent in the network is then given by

$$\hat{x}_i(k|k-1) = A\bar{x}_i(k-1) \tag{26}$$

$$\begin{aligned}\hat{x}_i(k|k) =& \hat{x}_i(k|k-1)\\ &+K_i(k)\big[y_i(k) - h_i(\hat{x}_i(k|k-1))\big]\end{aligned} \tag{27}$$

$$\bar{x}_i(k) = \sum_{j=1}^{N} w_{ij}\hat{x}_j(k|k) \tag{28}$$

Denoting $\hat{x}_i(k|k-1)$ by $\hat{x}_i(k)$, and substituting (28) and (27) into (26), a one-step formulation of agent i's estimate can be expressed as

$$\begin{aligned}\hat{x}_i(k+1) =& \sum_{j=1}^{N} w_{ij}\Big[A\hat{x}_j(k)+\\ & AK_j(k)\Big(y_j(k) - h_j(\hat{x}_j(k))\Big)\Big]\end{aligned} \tag{29}$$

The i-th agent's estimation error is then given by

$$\begin{aligned}\tilde{x}_i(k+1) =& \sum_{j=1}^{N} w_{ij}\left[A(I - K_j(k)H_j(k))\tilde{x}_j(k)\right.\\ &\left.+\ \eta_j(k)\right]\end{aligned} \tag{30}$$

Denoting the network-wide estimation error by $\underline{\tilde{x}}(k) \triangleq \begin{bmatrix}\tilde{x}_1(k)^T & \ldots & \tilde{x}_N(k)^T\end{bmatrix}^T$, then

$$\begin{aligned}\underline{\tilde{x}}(k+1) =& (W \otimes A)\,(I_{6N} - K(k)D_H(k))\,\underline{\tilde{x}}(k)\\ &+ \underline{\eta}(k)\end{aligned} \tag{31}$$

which is similar to (25), except that here the gain matrix $K(k)$ is restricted to be block-diagonal. As explained in [12], computing such a constrained gain is possible via an iterative cone-complementarity optimization algorithm if the pair $(W \otimes A, D_H)$ is observable; see [18] and [19] for details.

Following the previous discussion, we call the system *observable in a distributed sense* when the pair $(W \otimes A, D_H)$ is observable. We now investigate the conditions on the matrix W and, therefore, the topology of the undirected communication graph among agents, that would render the pair $(W \otimes A, D_H)$ observable. We first note the following property regarding the powers of the matrix W from [20].

Lemma 3.1: Let $[W^l]_{ij}$ denote the (i,j) element of the matrix W^l. Then, $[W^l]_{ij} > 0$ if there is a path between agents i and j of length less than or equal to l; otherwise $[W^k]_{ij} = 0$.

The pair $(W \otimes A, D_H)$ is said to be observable if and only if $\text{rank}(\mathcal{O}) = 6N$. Here,

$$\mathcal{O} = \begin{bmatrix} D_H(k) \\ D_H(k)(W \otimes A) \\ D_H(k)(W \otimes A)^2 \\ \vdots \\ D_H(k)(W \otimes A)^p \end{bmatrix} = \begin{bmatrix} D_H(k) \\ D_H(k)(W \otimes A) \\ D_H(k)(W^2 \otimes A^2) \\ \vdots \\ D_H(k)(W^p \otimes A^p) \end{bmatrix} \quad (32)$$

where $p = 6N - 1$. Equivalently, denoting $\mathcal{O}_i$ as the block column representing agent i's subsystem, we can write $\mathcal{O} = \begin{bmatrix} \mathcal{O}_1 \mid \dots \mid \mathcal{O}_N \end{bmatrix}$. From the structure of D_H, it is easy to see that $\text{rank}(\mathcal{O}) = \sum_{i=1}^{N} \text{rank}(\mathcal{O}_i)$.

To keep the analysis simple, we will first consider a network with only 4 agent ($N = 4$), and then extend the results for a network with more than 4 agents.

Theorem 3.1: For a network of 4 agents (i.e., $N = 4$), the system is generically observable in a distributed sense if and only if the undirected communication graph is connected.

Proof: Since it is always possible to renumber the agents, we will only consider the subsystem corresponding to agent 1, and write

$$\mathcal{O}_1 = \left[\begin{array}{c} H_1(k) \\ 0 \\ \vdots \\ 0 \\ \hline [W]_{11} H_1(k)A \\ \vdots \\ [W]_{41} H_4(k)A \\ \hline \vdots \\ \hline [W^{23}]_{11} H_1(k)A^{23} \\ \vdots \\ [W^{23}]_{41} H_4(k)A^{23} \end{array}\right] \quad (33)$$

This agent can be connected to the network in 4 possible ways that are shown in Fig. 3[3]. We will now examine each case individually and show that $\text{rank}(\mathcal{O}_1) = 6$ for all 4 cases.

[3]The graphs shown in Fig. 3 represent the minimum number of links required for each graph to be connected and it is possible to add more edges among agents in these graphs. However, adding more links will only help in terms of observability.

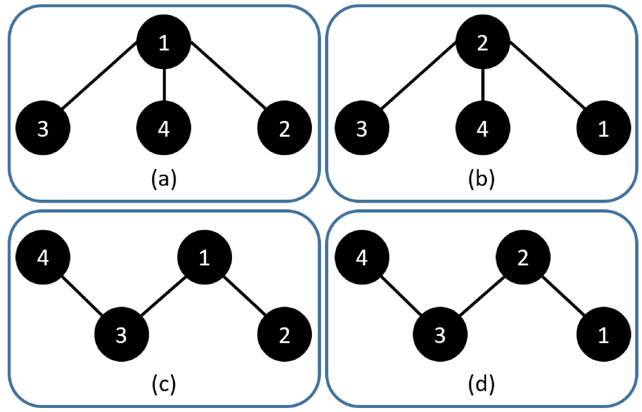

Fig. 3. The 4 possible configurations for node 1 to be connected to the network using the minimum number of edges.

Case (a): This case is the easiest to examine, as agent 1 has a sufficient number of neighbors. The measurement matrix $H_1(k)$ has the following structure

$$H_1(\lambda) = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & 0 & 0 & 0 \\ \lambda_4 & \lambda_5 & \lambda_6 & 0 & 0 & 0 \\ \lambda_7 & \lambda_8 & \lambda_9 & 0 & 0 & 0 \end{bmatrix} \quad (34)$$

and from the discussion on semi-centralized approach, the pair $(A, H_1(\lambda))$ is generically observable. Recalling that

$$\text{rank}\left(\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}\right) = \text{rank}\left(\begin{bmatrix} CA^k \\ CA^{k+1} \\ \vdots \\ CA^{k+n-1} \end{bmatrix}\right)$$

it immediately follows that $\text{rank}(\mathcal{O}_1) = 6$, since

$$\text{rank}\left(\mathcal{O}_1\right) = \text{rank}\left(\begin{bmatrix} [W]_{11} H_1(\lambda)A \\ [W^2]_{11} H_1(\lambda)A^2 \\ \vdots \\ [W^7]_{11} H_1(\lambda)A^7 \end{bmatrix}\right) = 6$$

for almost all values of T and λ.

Case (b): The structure of $H_i(k)$ for $i = 1, \dots, 4$ is

$$H_1(\lambda) = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & 0 & 0 & 0 \end{bmatrix}$$
$$H_2(\lambda) = \begin{bmatrix} -\lambda_1 & -\lambda_2 & -\lambda_3 & 0 & 0 & 0 \\ \lambda_4 & \lambda_5 & \lambda_6 & 0 & 0 & 0 \\ \lambda_7 & \lambda_8 & \lambda_9 & 0 & 0 & 0 \end{bmatrix}$$
$$H_3(\lambda) = \begin{bmatrix} -\lambda_4 & -\lambda_5 & -\lambda_6 & 0 & 0 & 0 \end{bmatrix}$$
$$H_4(\lambda) = \begin{bmatrix} -\lambda_7 & -\lambda_8 & -\lambda_9 & 0 & 0 & 0 \end{bmatrix}$$

We first note that the pair

$$\left(A, \begin{bmatrix} H_1(\lambda) \\ H_2(\lambda) \end{bmatrix}\right)$$

is generically observable, which can be shown following the same analysis in the discussion for the semi-centralized approach. This in turn, ensures that $\text{rank}(\mathcal{O}_1) = 6$.

Case (c): The structure of $H_i(k)$ for $i = 1, \ldots, 4$ follows

$$H_1(\lambda) = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & 0 & 0 & 0 \\ \lambda_4 & \lambda_5 & \lambda_6 & 0 & 0 & 0 \end{bmatrix}$$
$$H_2(\lambda) = \begin{bmatrix} -\lambda_1 & -\lambda_2 & -\lambda_3 & 0 & 0 & 0 \end{bmatrix}$$
$$H_3(\lambda) = \begin{bmatrix} -\lambda_4 & -\lambda_5 & -\lambda_6 & 0 & 0 & 0 \\ \lambda_7 & \lambda_8 & \lambda_9 & 0 & 0 & 0 \end{bmatrix}$$
$$H_4(\lambda) = \begin{bmatrix} -\lambda_7 & -\lambda_8 & -\lambda_9 & 0 & 0 & 0 \end{bmatrix}$$

Following procedures similar to those presented in the semi-centralized approach discussion, it can be shown that the pair

$$\left(A, \begin{bmatrix} H_1(\lambda) \\ H_3(\lambda) \\ H_4(\lambda) \end{bmatrix} \right)$$

is generically observable, and rank$(\mathcal{O}_1) = 6$ for almost all values of T and λ.

Case (d): The structure of $H_i(k)$ for $i = 1, \ldots, 4$ is as

$$H_1(\lambda) = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & 0 & 0 & 0 \end{bmatrix}$$
$$H_2(\lambda) = \begin{bmatrix} -\lambda_1 & -\lambda_2 & -\lambda_3 & 0 & 0 & 0 \\ \lambda_4 & \lambda_5 & \lambda_6 & 0 & 0 & 0 \end{bmatrix}$$
$$H_3(\lambda) = \begin{bmatrix} -\lambda_4 & -\lambda_5 & -\lambda_6 & 0 & 0 & 0 \\ \lambda_7 & \lambda_8 & \lambda_9 & 0 & 0 & 0 \end{bmatrix}$$
$$H_4(\lambda) = \begin{bmatrix} -\lambda_7 & -\lambda_8 & -\lambda_9 & 0 & 0 & 0 \end{bmatrix}$$

Once again, it can be shown that the pair

$$\left(A, \begin{bmatrix} H_1(\lambda) \\ H_3(\lambda) \\ H_4(\lambda) \end{bmatrix} \right)$$

is generically observable, and rank$(\mathcal{O}_1) = 6$ for almost all values of T and λ.

Since it is always possible to renumber the agents, the pair $(W \otimes A, D_H(k))$ is generically observable if and only if every agent is connected to the network of 4 agents in one of the possible configurations shown in Figure 3. Or, equivalently, for a network of 4 agents, the system is generically observable in a distributed sense if and only if the network is connected. ■

Corollary 3.1: For a networks with more than 4 agents, i.e. $N > 4$, the system is generically observable in a distributed sense if and only if every agent is part of a network that collectively has 3 TDOA measurements.

Proof: If part: If agent i is part of a network that collectively has 3 TDOA measurements, then by similar arguments to those in Theorem 3.1, rank$(\mathcal{O}_i) = 6$ for almost all values of T and λ. It immediately follows that if all agents in the network are part of network with 3 TDOA measurements, then rank$(\mathcal{O}) = 6N$ for almost all values of T and λ.

Only if part: If agent i is not part of a network that collectively has 3 TDOA measurements, then the agent is either disconnected from the network, or it is part of a network with less than 3 TDOA measurements. In the first case, agent i has no neighbors, and therefore no TDOA measurements to estimate the process on its own, or to obtain the estimates of other nodes in the network. In the second case, the number of TDOA measurements is insufficient, and the target's location cannot be estimated even under a centralized scheme, let alone a distributed one. ■

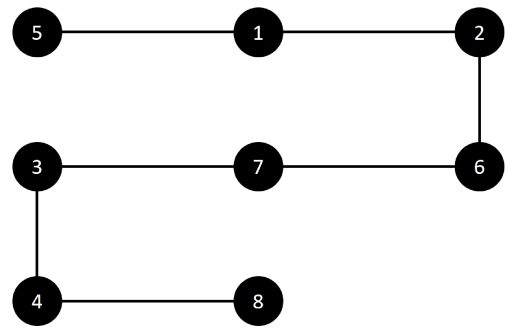

Fig. 4. Network topology showing communication links between 8 agents.

IV. Simulation Results

In this section, we simulate the proposed algorithm on a network of 8 agents estimating the position of a moving target in a 3D environment. The communication links between the agents are dictated by the network topology represented in Fig. 4. Spatially, the agents are located at $(-100, 100, 50)$, $(-100, -100, -50)$, $(100, -100, 50)$, $(100, 100, -50)$, $(150, 150, -20)$, $(-150, -150, 20)$, $(150, -150, -20)$, and $(150, 150, 20)$, respectively. For the moving target, the initial position is set to $(-75, -50, -5)$ with sampling period $T = 0.25$ seconds and process noise variance of 0.35. Finally, the measurement noise variance was set to 9, and initial estimates were initialized with $\hat{x}_0^i = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$.

The simulation environment is shown in Fig. 5, while Fig. 6 shows the norm of the estimation error of all agents decreasing as the network estimates the target's location. The network topology in Fig. 4 represents a worst case senario where no agent in the network has a sufficient number of neighbors to accurately estimate the target's position independently, while maintaining a connected network. Carefully inspecting the estimation error norms in Fig. 6 yields two key observations. First, and most importantly, we note that the distributed estimators successfully localize the moving target as time goes on. The other interesting observation is the spike that is apparent around 10s. We note that this spike is due to the network topology, which prohibits any agent from accurately localizing the target without exchanging information with its neighbors. This was verified through additional simulation, where a complete network (i.e. one where any two agents can exchange information) does not exhibit such a spike in error norms.

V. Conclusion and Future Work

In this paper, we discuss the problem of distributed localization of a moving target by a network of agents using TDOA measurements form first observability principles. We

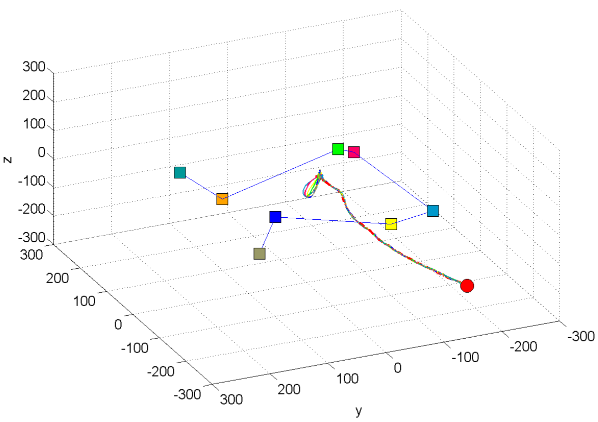

Fig. 5. Simulation of the proposed method with a network of 8 agents. The circle represents the location of the target, while the squares represent the positions of each agent. The straight thin lines between squares represent the communication links among agents. Plotted data show the path taken by the target over time, along with each agent's estimate of the target's location.

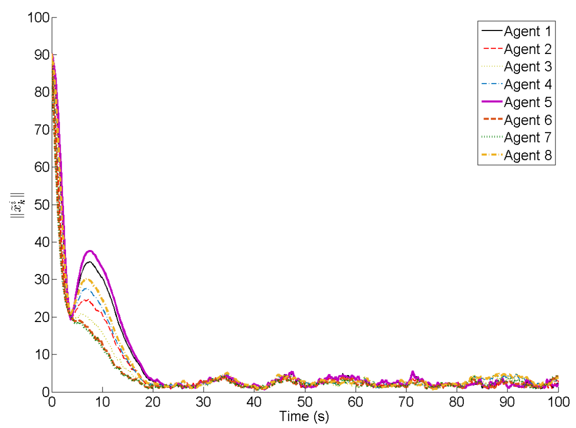

Fig. 6. Norm of estimation error for each agent in the network.

show that it is indeed possible to estimate the target's position by every agent in the network if every agent is connected to a network with a minimum of 3 TDOA measurements. While this work focuses on the case of fixed sensor locations, it is desirable to consider the case where the sensors will be mobile robots. Therefore, future work will expand analysis to the cases of intermittent observations and switching network topologies along with the development of a cooperative control law in order to localize and track a moving target.

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