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Quantum hydrodynamic approximations to the finite temperature trapped Bose gases

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HIGHLIGHTS

- New Euler limit for systems of finite temperature Bose gases.
- The Euler limit is better than the previous one: the two fluids are coupled.
- New Navier–Stokes approximation for finite temperature Bose gases.
- The hydrodynamics approximations agree with the Landau two fluid theory.
- New scaling technique for the Chapman–Enskog expansion.

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ABSTRACT

For the quantum kinetic system modeling the Bose–Einstein Condensate that accounts for interactions between condensate and excited atoms, we use the Chapman–Enskog expansion to derive its hydrodynamic approximations, include both Euler and Navier–Stokes approximations. The hydrodynamic approximations describe not only the macroscopic behavior of the BEC but also its coupling with the non-condensates, which agrees with the Landau two-fluid theory.

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1. Introduction

After the realization of Bose–Einstein condensations (BECs) in trapped atomic vapors of ^{87}Rb, ^{7}Li, and ^{23}Na [1,2], a new period of intense experimental and theoretical research has been initiated. The equilibrium properties of these novel systems have been quite well understood, but there are still several open questions concerning their nonequilibrium behavior. One of the most important questions concerns the behavior of the condensate after cooling a nondegenerate trapped Bose gas to a temperature below the BEC critical temperature. While the experimental research has, up to now, concentrated mainly on the initial formation of BECs, their theoretical behavior at finite temperatures is a frontier of many-body physics. The theoretical description of BECs has to take into account the coupled nonequilibrium dynamics of both the condensed and noncondensed components of the gas under investigation, and has to involve the collisional processes of atoms between the two components. Such a quantum kinetic theory was initiated by Kirkpatrick and Dorfman [3,4], based on the rich body of research carried out in the period 1940–67 by Bogoliubov, Lee and Yang, Beliaev, Pitaevskii, Hugenholtz and Pines, Hohenberg and Martin, Gavoret and Nozi'eres, Kane and Kadanoff and many others. The terminology "Quantum Kinetic Theory" has been later introduced in a series of papers by Gardinier, Zoller and collaborators [5–8]. After that, there has been an explosion of research on quantum kinetic theory (see [3–24], and references therein). We refer to the review paper [25] and the books [26,27], for more discussions and a complete list of references on this rapidly expanding topic.

The current paper is devoted to the study of the hydrodynamic approximations of such a quantum kinetic system. The system

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contains two equations: a quantum Boltzmann equation describing the non-condensate atoms (with two types of collisions, one between excited atoms and one between condensate atoms and excited atoms), and a nonlinear Schrödinger (or Gross–Pitaevskii) equation for the condensate. The hydrodynamic limits of the system is an interesting mathematical question, first studied in [28], where an Euler limit has been derived. This derivation relies on the assumption that, in the considered trapped Bose gas, the non-condensate and condensate share the same local equilibrium. It is known (cf. [3,4]) that the condition of complete local equilibrium between the condensate and the thermal cloud requires the energy of a condensate atom in the local rest frame of the thermal cloud to be equal to the local thermal cloud chemical potential. When the condition is satisfied, there is no exchange of particles between the condensate and the thermal cloud (cf. [14]). As a consequence, in the derived fluid system, the mass of each component – condensate and non-condensate – does not exchange. Note that the two-fluid low-frequency dynamics of superfluid ^{4}He was first developed by Tisza and Landau [29]. Their description accounts for the characteristic features associated with superfluidity in terms of the relative motion of superfluid and normal fluid degrees of freedom, and was shown to be a consequence of a Bose broken symmetry (cf. [30]). In the Landau two-fluid theory, the two components superfluid and normal fluid exchange mass (cf. [28–30]). In this paper, we revisit the derivation of the Euler hydrodynamic limit of the system by a different point of view: following [3,4,14], we assume that even if the thermal cloud atoms are in equilibrium among themselves, the noncondensate and condensate parts may not be in local equilibrium with each other. Moreover, the derivation of the Navier–Stokes approximation of the system is also provided via the classical Chapman–Enskog expansion (cf. [31]). In such circumstance, the Euler limit includes the mass exchange between the condensate and the non-condensate. Our Euler and Navier–Stokes approximations agree with the Landau two-fluid theory (cf. [29,30]).

As an attempt to build a rigorous theory for quantum kinetic equations, some mathematical results have been obtained in [32–41]). Note that quantum kinetic equations have very similar formulations with the so-called wave turbulence kinetic equations. We refer to [42–50] for more recent advances on the rigorous theory of weak turbulence.

The plan of the paper is as follows. In Section 2 we introduce the quantum kinetic system and the scalings that will lead to the hydrodynamic approximation. In Section 3, we list the most important features of the two collision operators C_{12} and C_{22}. The two-fluid Euler and Navier–Stokes limits are then derived in two Sections 4 and 5 respectively.

2. The quantum kinetic system and scalings

2.1. The quantum kinetic system

Let us consider a trap Bose gas, whose temperature T is smaller than the Bose–Einstein transition temperature T_{BEC} and strictly greater than 0 K or -273.15°C. Denote $f(t,r,p)$ to be the density function of the normal fluid at time t, position r and momentum p and $\Phi(t,r)$ be the wave function of the condensated (or superfluid) phase. Employing the short-handed notation $f_i = f(t,r,p_i)$, $i = 1,2,3,4$, we first recall the quantum kinetic-Schrödinger system describing the dynamics of a BEC and its thermal cloud. The Schrödinger (or the Gross–Pitaevskii) equation for the condensates reads (cf. [12]):

$$\begin{aligned}
i\hbar\partial_t\Phi(t,r) &= \Big(-\frac{\hbar^2\Delta_r}{2m} + g[n_c(t,r)+2n_n(t,r)] \\
&\quad - i\Lambda_{12}[f](t,r) + V(r)\Big)\Phi(t,r), \quad (t,r)\in\mathbb{R}_+\times\mathbb{R}^3, \\
\Lambda_{12}[f](t,r) &= \frac{\hbar}{2n_c}\Gamma_{12}[f](t,r), \\
\Gamma_{12}[f](t,r) &= \int_{\mathbb{R}^3} C_{12}[f](t,r,p)\frac{dp}{(2\pi\hbar)^3}, \\
n_n(t,r) &= \int_{\mathbb{R}^3} f(t,r,p)dp, \\
\Phi(0,r) &= \Phi_0(r), \forall r\in\mathbb{R}^3,
\end{aligned} \tag{2.1}$$

where $n_c(t,r) = |\Phi|^2(t,r)$ is the condensate density, $\hbar$ is the Planck constant, g is the interaction coupling constant proportional to the s-wave scattering length a, $V(r)$ is the confinement potential, and the operator C_{12} can be found in the quantum Boltzmann equation for the non-condensate atoms (cf. [12]), written below:

$$\begin{aligned}
\partial_t f(t,r,p) &+ \frac{p}{m}\cdot\nabla_r f(t,r,p) - \nabla_r U(t,r)\cdot\nabla_p f(t,r,p) \qquad (2.2)\\
&= Q[f](t,r,p) := C_{12}[f](t,r,p) \\
&\quad + C_{22}[f](t,r,p), (t,r,p)\in\mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}^3, \\
C_{12}[f](t,r,p_1) &:= \lambda_1 n_c(t,r)\iint_{\mathbb{R}^3\times\mathbb{R}^3}\delta(mv_c+p_1-p_2-p_3) \\
&\quad\times\delta(\mathcal{E}_c+\mathcal{E}_{p_1}-\mathcal{E}_{p_2}-\mathcal{E}_{p_3}) \\
&\quad\times[(1+f_1)f_2f_3 - f_1(1+f_2)(1+f_3)]dp_2dp_3 \qquad (2.3)\\
&\quad -2\lambda_1 n_c(t,r)\iint_{\mathbb{R}^3\times\mathbb{R}^3}\delta(mv_c+p_2-p_1-p_3) \\
&\quad\times\delta(\mathcal{E}_c+\mathcal{E}_{p_2}-\mathcal{E}_{p_1}-\mathcal{E}_{p_3}) \\
&\quad\times[(1+f_2)f_1f_3 - f_2(1+f_1)(1+f_3)]dp_2dp_3, \\
C_{22}[f](t,r,p_1) &:= \lambda_2\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3}\delta(p_1+p_2-p_3-p_4) \\
&\quad\times\delta(\mathcal{E}_{p_1}+\mathcal{E}_{p_2}-\mathcal{E}_{p_3}-\mathcal{E}_{p_4})\times \\
&\quad\times[(1+f_1)(1+f_2)f_3f_4 \\
&\quad -f_1f_2(1+f_3)(1+f_4)]dp_2dp_3dp_4, \qquad (2.4)\\
f(0,r,p) &= f_0(r,p), (r,p)\in\mathbb{R}^3\times\mathbb{R}^3,
\end{aligned}$$

where $\lambda_1 = \frac{2g^2}{(2\pi)^2\hbar^4}$, $\lambda_2 = \frac{2g^2}{(2\pi)^5\hbar^7}$, m is the mass of the particles, $\mathcal{E}_p$ is the Hartree–Fock energy (cf. [12])

$$\mathcal{E}_p = \mathcal{E}(p) = \frac{|p|^2}{2m} + U(t,r). \tag{2.5}$$

Notice that C_{22} is the Boltzmann–Norheim (Uehling–Ulenbeck) quantum Boltzmann collision operator. If one writes

$$\Phi = |\Phi(t,r)|e^{i\phi(t,r)}, \tag{2.6}$$

the condensate velocity can be defined as

$$v_c(t,r) = \frac{\hbar}{m}\nabla\phi(t,r), \tag{2.7}$$

and the condensate chemical potential is then

$$\mu_c = \frac{1}{\sqrt{n_c}}\left(-\frac{\hbar^2\Delta_r}{2m} + V + g[2n_n+n_c]\right)\sqrt{n_c}. \tag{2.8}$$

When $V = 0$, the following system for the super-fluid of the condensate can be obtained

$$\begin{aligned}
\partial_t n_c &+ \nabla_r\cdot(n_c v_c) = -\Gamma_{12}[f] \\
\partial_t v_c &+ \frac{\nabla_r v_c^2}{2} = -\nabla_r\mu_c.
\end{aligned} \tag{2.9}$$

The potential U and the condensate energy $\mathcal{E}_c$ are written as follows

$$U(t,r) = V(r) + 2g[n_c(t,r) + n_n(t,r)], \tag{2.10}$$

and

$$\mathcal{E}_c(t,r) = \mu_c(t,r) + \frac{mv_c^2(t,r)}{2}. \tag{2.11}$$

For the sake of simplicity, we suppose that $V \equiv 0$ and define the differential quantity

$$\bar{dp} = \frac{dp}{(2\pi\hbar)^3}. \tag{2.12}$$

Notice that (2.3) describes collisions between the condensate and the non-condensate atoms (condensate growth term) and (2.4) describes collisions between non-condensate atoms.

Remark 2.1. At temperature T, bosons of mass m can be regarded as quantum-mechanical wavepackets which have an extent on the order of a thermal de Broglie wavelength $\lambda_{dB} = \left(\frac{2\pi\hbar^2}{mk_BT}\right)^{\frac{1}{2}}$, where k_B is the Boltzmann constant. The de Broglie wavelength λ_{dB} describes the position uncertainty associated with the thermal momentum distribution. When the gas temperature is high $T > T_{BEC}$, λ_{dB} is very small and the weakly interacting gas can be treated as a system of "billiard balls" (cf. [51,52]). The dynamics of the gas is described by the Boltzmann–Norheim (Uehling–Ulenbeck) equation, whose operator sometimes reads (cf. [53])

$$\begin{aligned}\mathcal{C}_{22}[f](t,r,p_1) = &\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} \delta(p_1+p_2-p_3-p_4)\\ &\times \delta(\mathcal{E}_{p_1}+\mathcal{E}_{p_2}-\mathcal{E}_{p_3}-\mathcal{E}_{p_4})\times\\ &\times [(1+\vartheta f_1)(1+\vartheta f_2)f_3f_4\\ &- f_1f_2(1+\vartheta f_3)(1+\vartheta f_4)]dp_2dp_3dp_4,\end{aligned} \tag{2.13}$$

where ϑ is proportional to $\hbar^3$. In the semiclassical limit, as ϑ tends to 0, the quantum Boltzmann collision operator becomes the classical one. This means at high temperature, the behavior of the "billiard balls" Bose gas is, in some sense, still very similar to classical gases.

At the BEC transition temperature, λ_{dB} becomes comparable to the distance between atoms. As a result, the atomic wavepackets "overlap" and the indistinguishability of atoms becomes important. At this temperature, bosons undergo a quantum-mechanical phase transition and the Bose–Einstein condensate is formed (cf. [51,52]). When the temperature of the gas is finite $T_{BEC} > T >$ 0K, the trapped Bose gas is composed of two distinct components: the high-density condensate, being localized at the center of the trapping potential, and the low-density cloud of thermally excited atoms, spreading over a much wider region. The dynamics of the thermal cloud atoms is described by the kinetic equation (2.2). At this low temperature, the de Broglie wavelength of the excited atoms is very large, in comparison with the high temperature boson de Broglie wavelength. As a consequence, the thermal cloud kinetic equation cannot be treated as a system of "billiard balls" anymore. This explains the difference between the forms of the two collision operators C_{22} and $\mathcal{C}_{22}$.

Note that, different from classical Boltzmann collision operators, where the collision kernels are functions depending on the types of particles considered, the derived collision kernel for the quantum Boltzmann collision operator for bosons is 1 (cf. [54]) when $T > T_{BEC}$.

2.2. Scalings

Different from the thesis [28], in which the two collision operators C_{12} and C_{22} are assumed to have the same equilibrium distribution function, we follow [14] to consider the most general regime, where excited atoms in the condensate need not be in local equilibrium with the condensate atoms. As a consequence, C_{12} and C_{22} in general do not share the same equilibrium distribution. A comparison between our results and the result of [28] will be discussed in details in Section 4.2. Relying on these physical assumptions, we propose a new approach to obtain new Euler and Navier–Stokes approximations of the system.

It is known that the dynamics of the trapped Bose gases depends on its temperature T. Let us restrict our attention to the case where T is smaller but very close to the Bose–Einstein critical temperate T_{BEC}. At this temperature regime, the collisions between excited atoms are rapid to establish a local equilibrium within the non-condensate component. As a consequence, the collision operator C_{22} can be assumed to be stronger than the collision operator C_{12}. This regime is often called *the state of partial local equilibrium* which arises near T_{BEC} when the density of the condensate is small.

Following [14], we define the *static equilibrium* of the system

$$\mathcal{F}_0(p) = \frac{1}{e^{\beta_0[(p-mv_{n0})^2/(2m)+U_0-\mu_0]}-1}, \tag{2.14}$$

where β_0 is the static temperature parameter, v_{n0} is the static fluid velocity, μ_0 is the static chemical potential, U_0 is the static mean field. We also set the static density to be

$$n_{n0} = \int_{\mathbb{R}^3} \mathcal{F}_0(p)\bar{dp}. \tag{2.15}$$

Note that when T is sufficiently close to T_{BEC}, the bosons are in the *particle-like regime*, i.e. they behave like particles. Let us also mention that when temperature T is very close to 0, the bosons will be in the phonon-like regime (cf. [55]). Since we are interested in the behavior of the particles when T is close to T_{BEC}, let us define the collision frequency with respect to C_{12}

$$\begin{aligned}\nu_{12}(p_1) = &\frac{\lambda_1 n_c}{m^2}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\delta(mv_c+p_1-p_2-p_3)\\ &\times \delta(\mathcal{E}_c+\mathcal{E}_{p_1}-\mathcal{E}_{p_2}-\mathcal{E}_{p_3})\times\\ &\times [\mathcal{F}_0(p_2)+\mathcal{F}_0(p_3)+1]dp_2dp_3+\\ &+2\frac{\lambda_1 n_c}{m^2}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\delta(mv_c+p_2-p_1-p_3)\\ &\times \delta(\mathcal{E}_c+\mathcal{E}_{p_2}-\mathcal{E}_{p_1}-\mathcal{E}_{p_3})\mathcal{F}_0(p_3)dp_2dp_3,\end{aligned} \tag{2.16}$$

as well as the associated mean collision frequency:

$$\bar{\nu}_{12} = \frac{1}{n_{n_0}m}\int_{\mathbb{R}^3}\nu_{12}(p)\mathcal{F}_0(p)\bar{dp}. \tag{2.17}$$

The inverse of $\nu_{12}(p)$ and $\bar{\nu}_{12}$ is defined to be, respectively, the free time $\tau_{12}(p)$ and the mean field time $\bar{\tau}_{12}$:

$$\tau_{12}(p) = \frac{1}{\nu_{12}(p)}, \quad \bar{\tau}_{12} = \frac{1}{\bar{\nu}_{12}}. \tag{2.18}$$

We now determine the average speed of the particles

$$\bar{c} = \frac{1}{n_{n0}m}\int_{\mathbb{R}^3}\sqrt{p^2}\mathcal{F}_0(p)\bar{dp}, \tag{2.19}$$

and the mean free path

$$l_{12} = \bar{c}\bar{\tau}_{12}. \tag{2.20}$$

Similarly, the collision frequency and the mean collision frequency associated to C_{22} can be defined

$$\begin{aligned}\nu_{22}(p_1) \;=\; & \frac{\lambda_2}{n_{n0}m}\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3}\delta(p_1+p_2-p_3-p_4)\\ & \times\ \delta(\mathcal{E}_{p_1}+\mathcal{E}_{p_2}-\mathcal{E}_{p_3}-\mathcal{E}_{p_4})\times\\ & \times\ \mathcal{F}_0(p_2)(1+\mathcal{F}_0(p_3))(1+\mathcal{F}_0(p_4))dp_2dp_3dp_4,\end{aligned} \tag{2.21}$$

and

$$\bar{\nu}_{22}=\frac{1}{n_{n0}m}\int_{\mathbb{R}^3}\nu_{22}(p)\mathcal{F}_0(p)\bar{d}p. \tag{2.22}$$

We also define the free time $\tau_{22}(p)$, the mean field time $\bar{\tau}_{22}$ and the mean free path l_{22}

$$\tau_{22}(p)\;=\;\frac{1}{\nu_{22}(p)},\quad \bar{\tau}_{22}=\frac{1}{\bar{\nu}_{22}},\quad l_{22}=\bar{c}\bar{\tau}_{22}. \tag{2.23}$$

Let L and θ be the reference length and time, respectively. Following [56,57], we introduce the rescaled variables

$$\tilde{r}=\frac{r}{L},\quad \tilde{t}=\frac{t}{\theta},\quad \tilde{p}=\frac{p}{P},P=m\bar{c},\quad \tilde{v}_c=\frac{v_c}{\bar{c}}. \tag{2.24}$$

Note that under this scaling,

$$n_n(t,r)\;=\;\int_{\mathbb{R}^3}f(t,r,p)dp\;=\;P^3\int_{\mathbb{R}^3}f(t,r,\tilde{p})d\tilde{p}. \tag{2.25}$$

We also rescale U as $\tilde{U}\;=\;U/U_0$, where U_0 is the reference potential field. Define

$$\begin{aligned}&\tilde{C}_{12}[f](t,r,\tilde{p}_1)\\ &:=\tilde{\lambda}_1 n_c(t,r)\iiint_{\mathbb{R}^3\times\mathbb{R}^3}\delta(\tilde{v}_c+\tilde{p}_1-\tilde{p}_2-\tilde{p}_3)\\ &\quad\times\delta(\mathcal{E}_c+\mathcal{E}_{\tilde{p}_1}-\mathcal{E}_{\tilde{p}_2}-\mathcal{E}_{\tilde{p}_3})\\ &\quad\times[(1+f_1)f_2f_3-f_1(1+f_2)(1+f_3)]d\tilde{p}_2d\tilde{p}_3\\ &\quad-2\tilde{\lambda}_1 n_c(t,r)\iiint_{\mathbb{R}^3\times\mathbb{R}^3}\delta(\tilde{v}_c+\tilde{p}_2-\tilde{p}_1-\tilde{p}_3)\\ &\quad\times\delta(\mathcal{E}_c+\mathcal{E}_{\tilde{p}_2}-\mathcal{E}_{\tilde{p}_1}-\mathcal{E}_{\tilde{p}_3})\\ &\quad\times[(1+f_2)f_1f_3-f_2(1+f_1)(1+f_3)]d\tilde{p}_2d\tilde{p}_3,\end{aligned} \tag{2.26}$$

$$\begin{aligned}&\tilde{C}_{22}[f](t,r,\tilde{p}_1)\\ &:=\tilde{\lambda}_2\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3}\delta(\tilde{p}_1+\tilde{p}_2-\tilde{p}_3-\tilde{p}_4)\\ &\quad\times\delta(\mathcal{E}_{\tilde{p}_1}+\mathcal{E}_{\tilde{p}_2}-\mathcal{E}_{\tilde{p}_3}-\mathcal{E}_{\tilde{p}_4})\times\\ &\quad\times[(1+f_1)(1+f_2)f_3f_4-f_1f_2(1+f_3)(1+f_4)]d\tilde{p}_2d\tilde{p}_3d\tilde{p}_4,\end{aligned} \tag{2.27}$$

where

$$\tilde{\lambda}_1=P^2\lambda_1/\bar{c}, \tag{2.28}$$

and

$$\tilde{\lambda}_2=P^5\lambda_2/\bar{c}. \tag{2.29}$$

As a consequence, we can define the rescaled mean free paths and the rescaled mean field times to be

$$\tilde{l}_{22}=\frac{l_{22}}{P^5},\quad \tilde{\tau}_{22}=\frac{\bar{\tau}_{22}}{P^5},$$

and

$$\tilde{l}_{12}=\frac{l_{12}}{P^2},\quad \tilde{\tau}_{12}=\frac{\bar{\tau}_{12}}{P^2}.$$

We also set

$$\hat{C}_{12}[f]:=\tilde{l}_{12}\tilde{C}_{12}[f],\quad \hat{C}_{22}[f]:=\tilde{l}_{22}\tilde{C}_{22}[f]. \tag{2.30}$$

The following rescaled version of (2.2) then follows:

$$\begin{aligned}&\frac{\sqrt{\tilde{l}_{12}\tilde{l}_{22}}}{\theta\bar{c}}\partial_{\tilde{t}}f+\frac{\sqrt{\tilde{l}_{12}\tilde{l}_{22}}}{L}\frac{P}{m\bar{c}}\tilde{p}\cdot\nabla_{\tilde{r}}f-\frac{\sqrt{\tilde{l}_{12}\tilde{l}_{22}}}{L}\frac{U_0}{P\bar{c}}\nabla_{\tilde{r}}\tilde{U}\cdot\nabla_{\tilde{p}}f\\ &=\sqrt{\frac{\tilde{l}_{22}}{\tilde{l}_{12}}}\hat{C}_{12}[f]+\sqrt{\frac{\tilde{l}_{12}}{\tilde{l}_{22}}}\hat{C}_{22}[f].\end{aligned} \tag{2.31}$$

Notice that $\frac{\tilde{\tau}_{22}}{\tilde{\tau}_{12}}=\frac{\tilde{l}_{22}}{\tilde{l}_{12}}$ is a dimensionless parameter and is proportional to $\frac{\tilde{\lambda}_1}{\tilde{\lambda}_2}$.

In this paper, we will consider two hydrodynamic approximations: Euler and Navier–Stokes.

- The Euler approximation is quite general and valid under a general physical situation. The collisions between excited atoms are fast to establish a local equilibrium within the non-condensate component, and the quantity $\tilde{\tau}_{22}$ is smaller than $\tilde{\tau}_{12}$ but the ratio between $\tilde{\tau}_{22}$ and $\tilde{\tau}_{12}$ is not necessarily very small.
- The Navier–Stokes approximation is valid under the physical assumption that the collisions between excited atoms are extremely rapid to establish a local equilibrium within the non-condensate component and $\tilde{\tau}_{22}\ll\tilde{\tau}_{12}$.

We suppose $\frac{\tilde{\tau}_{22}}{\tilde{\tau}_{12}}=\epsilon^2$. *The Euler approximation is valid in any physical assumption and we do not need to impose the assumption that ϵ is small, then ϵ is just a parameter. In the Navier–Stokes approximation, we need to impose the assumption that ϵ is small and then we will use it as the small parameter in the usual Chapman–Enskog expansion process.*

The constants $\frac{\sqrt{\tilde{l}_{12}\tilde{l}_{22}}}{\theta\bar{c}}$, $\frac{\sqrt{\tilde{l}_{12}\tilde{l}_{22}}}{L}$ can be set to be 1 by rescaling again the space and time variables $\tilde{t}\to\frac{\sqrt{\tilde{l}_{12}\tilde{l}_{22}}}{\theta\bar{c}}\tilde{t}$, $\tilde{r}\to\frac{\sqrt{\tilde{l}_{12}\tilde{l}_{22}}}{L}\tilde{r}$, and note that $\frac{P}{m\bar{c}}=1$, we obtain the following equation

$$\partial_{\tilde{t}}f+\tilde{p}\cdot\nabla_{\tilde{r}}f-\frac{U_0}{m\bar{c}^2}\nabla_{\tilde{r}}\tilde{U}\cdot\nabla_{\tilde{p}}f=\epsilon\hat{C}_{12}[f]+\frac{1}{\epsilon}\hat{C}_{22}[f]. \tag{2.32}$$

Notice that g is also the principle small parameter used in the derivation of the system (2.1)–(2.2). Indeed, the derivation starts with the usual Heisenberg equation of motion for the quantum field operator. The equation for the condensate wavefunction follows by averaging the Heisenberg equation with respect to a broken-symmetry nonequilibrium ensemble. Taking the difference between the Heisenberg equation and the equation for the condensate wavefunction and keeping only the terms of low orders with respect to g, we obtain the equation of the noncondensate field operator, which, by a Wigner transform, leads to the quantum Boltzmann equation. In this process, one computes the collision integrals C_{12}, C_{22} to second order $O(g^2)$ in g and keeps interaction effects in the excitation energies and chemical potential only to first order $O(g)$. For a more detailed explanation of this procedure, we refer to, for instance, Sections 3.1, 3.2 and 5.3 of the book [14]. *Since U_0 has to be chosen proportional to g, the dimensionless parameter $\frac{U_0}{m\bar{c}^2}$ might be considered to be small and set it to be $\tilde{g}=\epsilon^{\delta_0}$, $0<\delta_0<1$ in* Section 5, *where the Chapman–Enskog expansion is used.*

The equation then follows, as a result of the previous scaling

$$\partial_{\tilde{t}}f+\tilde{p}\cdot\nabla_{\tilde{r}}f-\tilde{g}\nabla_{\tilde{r}}\tilde{U}\cdot\nabla_{\tilde{p}}f=\epsilon\hat{C}_{12}[f]+\frac{1}{\epsilon}\hat{C}_{22}[f]. \tag{2.33}$$

Under this scaling, the Gross–Pitaevskii equation also becomes

$$\begin{aligned}&i\frac{\hbar}{\theta}\partial_{\tilde{t}}\Phi(t,r)\\ &=\Big(-\frac{\hbar^2\Delta_{\tilde{r}}}{2mL^2}+g[n_c(t,r)+2n_n(t,r)]-\frac{i}{\tilde{\tau}_{12}}\tilde{\Lambda}_{12}[f](t,r)\Big)\Phi(t,r),\end{aligned} \tag{2.34}$$

where

$$\tilde{\Lambda}_{12}[f] = \frac{\hbar}{2n_c}\int_{\mathbb{R}^3} \hat{C}_{12}[f]\bar{dp}.$$

By the same argument as above, we also obtain

$$\begin{aligned} i\frac{\sqrt{\tilde{l}_{12}\tilde{l}_{22}}}{\theta\bar{c}}\partial_{\tilde{t}}\Phi(t,r) = \Big(&-\frac{\hbar}{mL}\frac{\sqrt{\tilde{l}_{12}\tilde{l}_{22}}\Delta_{\tilde{r}}}{2L\bar{c}} \\ &+\frac{\sqrt{\tilde{l}_{12}\tilde{l}_{22}}U_0}{\hbar\bar{c}}\tilde{U}_*(t,r) - \frac{i\sqrt{\tilde{l}_{12}\tilde{l}_{22}}}{\tilde{l}_{12}}\tilde{\Lambda}_{12}[f](t,r)\Big)\Phi(t,r), \end{aligned} \tag{2.35}$$

where $\tilde{U}_* = U_*/U_0$ and $U_*(t,r) = g[n_c(t,r) + 2n_n(t,r)]$, since $U_*(t,r)$ has the dimension of $U(t,r)$.

Notice that $\frac{\hbar}{m\bar{c}}$ has the dimensions of a length (Compton wavelength) and $\frac{\hbar}{mL\bar{c}}$ is dimensionless; hence the quantity $\frac{\hbar}{mL}\frac{\sqrt{\tilde{l}_{12}\tilde{l}_{22}}}{2L\bar{c}}$ is dimensionless. Moreover, $\frac{\sqrt{\tilde{l}_{12}\tilde{l}_{22}}U_0}{\hbar\bar{c}}$ is the product of the three dimensionless parameters $\frac{\sqrt{\tilde{l}_{12}\tilde{l}_{22}}}{L}$, $\frac{mL\bar{c}}{\hbar}$ and $\frac{U_0}{m\bar{c}^2} = \tilde{g}$. Setting all of the dimensionless parameter to be 1 by the same rescaling argument used for (2.32) and dropping the tilde and hat signs

$$i\partial_t\Phi = \Big(-\frac{\Delta_r}{2} + gU_* - i\epsilon\Lambda_{12}[f]\Big)\Phi, \tag{2.36}$$

where g stands for the dimensionless parameter $\tilde{g} = \epsilon^{\delta_0}$. When $V = 0$, the following system for the super-fluid of the condensate can be deduced

$$\begin{aligned} \partial_t n_c + \nabla_r\cdot(n_c v_c) &= -\epsilon\Gamma_{12}[f] \\ \partial_t v_c + \frac{\nabla_r v_c^2}{2} &= -\nabla_r\mu_c. \end{aligned} \tag{2.37}$$

we then obtain the system

$$\begin{aligned} \partial_t f + p\cdot\nabla_r f - g\nabla_r U\cdot\nabla_p f &= \epsilon C_{12}[f] + \frac{1}{\epsilon}C_{22}[f], (0<\delta_0<1), \\ i\partial_t\Phi &= \Big(-\frac{\Delta_r}{2} + g[n_c+2n_n] - i\epsilon\Lambda_{12}[g]\Big)\Phi. \end{aligned} \tag{2.38}$$

We recall below the formulas for C_{12}, C_{22} and Λ_{12}

$$\begin{aligned} C_{12}[f](t,r,p_1) = {} & n_c(t,r)\iint_{\mathbb{R}^3\times\mathbb{R}^3}\delta(v_c+p_1-p_2-p_3) \\ &\times\delta(\mathcal{E}_c+\mathcal{E}_{p_1}-\mathcal{E}_{p_2}-\mathcal{E}_{p_3}) \\ &\times[(1+f_1)f_2f_3 - f_1(1+f_2)(1+f_3)]dp_2dp_3 \\ &-2n_c(t,r)\iint_{\mathbb{R}^3\times\mathbb{R}^3}\delta(v_c+p_2-p_1-p_3) \\ &\times\delta(\mathcal{E}_c+\mathcal{E}_{p_2}-\mathcal{E}_{p_1}-\mathcal{E}_{p_3}) \\ &\times[(1+f_2)f_1f_3 - f_2(1+f_1)(1+f_3)]dp_2dp_3, \end{aligned} \tag{2.39}$$

$$\begin{aligned} C_{22}[f](t,r,p_1) = {} & \iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3}\delta(p_1+p_2-p_3-p_4) \\ &\times\delta(\mathcal{E}_{p_1}+\mathcal{E}_{p_2}-\mathcal{E}_{p_3}-\mathcal{E}_{p_4})\times \\ &\times[(1+f_1)(1+f_2)f_3f_4 - f_1f_2(1+f_3) \\ &\times(1+f_4)]dp_2dp_3dp_4, \end{aligned} \tag{2.40}$$

$$\Lambda_{12}[f](t,r) = \frac{1}{n_c(t,r)}\int_{\mathbb{R}^3} C_{12}[f](t,r,p)dp. \tag{2.41}$$

We also define the differential operators

$$\mathcal{D}f = \partial_t f + p\cdot\nabla_r f - g\nabla_r U\cdot\nabla_p f - \epsilon C_{12}[f], \tag{2.42}$$

$$\mathbb{D}f = \partial_t f + p\cdot\nabla_r f - g\nabla_r U\cdot\nabla_p f, \tag{2.43}$$

$$\Pi f = \partial_t f + p\cdot\nabla_r f, \tag{2.44}$$

and then get

$$\mathbb{D}f = \epsilon C_{12}[f] + \frac{1}{\epsilon}C_{22}[f], (0<\delta_0<1). \tag{2.45}$$

The new constant ϵ is the small parameter that we will use in the usual Chapman–Enskog expansion process in Section 5.

3. Properties of the collision operators

In this section, we study the main properties of the two collision operators C_{12} and C_{22}.

3.1. Collision invariants and equilibrium of C_{22}

Let us start with C_{22}, which can be represented as:

$$C_{22}[f] = B_1[f,f] + B_2[f,f,f], \tag{3.1}$$

in which

$$\begin{aligned} B_1[f,g] = {} & \frac{1}{2}\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3}\delta(p_1+p_2-p_3-p_4) \\ &\times\delta(\mathcal{E}_{p_1}+\mathcal{E}_{p_2}-\mathcal{E}_{p_3}-\mathcal{E}_{p_4})\times \\ &\times[f_3g_4+f_4g_3-f_1g_2-f_2g_1]dp_2dp_3dp_4, \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} B_2[f,g,h] = {} & \frac{1}{6}\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3}\delta(p_1+p_2-p_3-p_4) \\ &\times\delta(\mathcal{E}_{p_1}+\mathcal{E}_{p_2}-\mathcal{E}_{p_3}-\mathcal{E}_{p_4})\times \\ &\times[f_3g_4h_1+f_4g_3h_1+f_3g_4h_2+f_4g_3h_2 \\ &+f_1g_4h_3+f_1g_3h_4+f_2g_4h_3+f_3g_3h_4 \\ &+f_4g_1h_3+f_3g_1h_4+f_4g_2h_3+f_3g_2h_4 \\ &-f_1g_2h_3-f_2g_1h_3-f_1g_2h_4-f_2g_1h_4 \\ &-f_3g_1h_2-f_3g_2h_1-f_4g_1h_2-f_4g_2h_1 \\ &-f_1g_3h_2-f_2g_3h_1-f_1g_4h_2-f_2g_4h_1]dp_2dp_3dp_4, \end{aligned} \tag{3.3}$$

where we have used the same notations $f_1, f_2, f_3, f_4, g_1, g_2, g_3, g_4, h_1, h_2, h_3, h_4$ with the ones used in (2.2).

The operator C_{22} shares some important features with the classical Boltzmann collision operator. Among these features, the following can be proved by switching the variables $(p_1,p_2)\leftrightarrow(p_2,p_1)$, $(p_1,p_2)\leftrightarrow(p_3,p_4)$, in the integrals of B_1 as in the classical case (cf. [58]):

$$\int_{\mathbb{R}^3}\Psi_i(p)B_1[f,g](p)dp = 0, \quad i = 0,1,2,3,4, \tag{3.4}$$

where

$$\Psi_0(p) = 1, \quad \Psi_i(p) = p^i, \quad (i=1,2,3), \quad \Psi_4(p) = |p|^2, \tag{3.5}$$

are the collision invariants and p^i is the ith component of the vector $p = (p^1,p^2,p^3)$.

Moreover, we also have

$$\int_{\mathbb{R}^3}\Psi_i(p)B_2[f,g,h](p)dp = 0, \quad i = 0,1,2,3,4. \tag{3.6}$$

Similar as the classical Boltzmann collision operator, C_{22} also has a local equilibrium of the form

$$\mathcal{F}(t,r,p) = \frac{1}{e^{\beta[(p-v_n)^2/2+U-\mu]}-1}, \tag{3.7}$$

where $\beta(t,r)$ is the temperature parameter, $v_n(t,r)$ is the local fluid velocity, $\mu(t,r)$ is the local chemical potential (which is different from the condensate chemical potential $\mu_c(t,r)$ defined in (2.8)), $U(t,r)$ is the mean field. Then

$$C_{22}[\mathcal{F}] = 0.$$

Let us now define the following Gaussian

$$\mathcal{M}(t,r,p) = \gamma(t,r)e^{-\frac{|p-u(t,r)|^2}{2\tau(t,r)}}, \tag{3.8}$$

where

$$\gamma(t,r) = e^{\beta(U(t,r)-\mu(t,r))}, \quad u(t,r) = v_n(t,r),$$
$$\tau(t,r) = \frac{1}{\beta(t,r)}. \tag{3.9}$$

The local equilibrium $\mathcal{F}$ can be expressed in terms of $\mathcal{M}$ as

$$\mathcal{F}(t,r,p) = \frac{\mathcal{M}(t,r,p)}{1-\mathcal{M}(t,r,p)}. \tag{3.10}$$

Note that u is a vector $u = (u_1, u_2, u_3)$.

3.2. Linearized operator of C_{22}

Let $L^2(\mathbb{R}^3)$ be the space of real, measurable functions, whose second power is integrable on $\mathbb{R}^3$, with the norm $\|\cdot\|_{L^2}$ and inner product $(,)_{L^2}$. We consider the linearized operator of C_{22} around a fixed equilibrium $\mathcal{F}(t,r,p)$, which, by a classical process can be defined as

$$\mathcal{L} := 2B_1(\mathcal{F},\cdot) + 3B_2(\mathcal{F},\mathcal{F},\cdot), \tag{3.11}$$

or equivalently

$$\begin{aligned}
&\mathcal{L}(\mathcal{F}f)(t,r,p_1)\\
&= \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} \delta(p_1+p_2-p_3-p_4)\delta(\mathcal{E}_{p_1}+\mathcal{E}_{p_2}-\mathcal{E}_{p_3}-\mathcal{E}_{p_4})\\
&\times \frac{\mathcal{M}_1\mathcal{M}_2}{(1-\mathcal{M}_1)(1-\mathcal{M}_2)(1-\mathcal{M}_3)(1-\mathcal{M}_4)}\times\\
&\times \Big[(1-\mathcal{M}_3)f(p_3) + (1-\mathcal{M}_4)f(p_4) - (1-\mathcal{M}_2)f(p_2)\\
&- (1-\mathcal{M}_1)f(p_1)\Big]dp_2dp_3dp_4,
\end{aligned} \tag{3.12}$$

for some function $f(p)$ and fixed values $(t,r) \in \mathbb{R}_+ \times \mathbb{R}^3$ and we employ the shorthand notations $\mathcal{M}_i = \mathcal{M}(t,r,p_i)$, $i = 1,2,3,4$. Now, let us consider the inner product between the above linearized operator and some test function φ. The classical argument (cf. [58]) for the classical linearized Boltzmann collision operator can be applied and gives:

$$\begin{aligned}
&\left(\frac{\mathcal{M}}{\mathcal{F}}\varphi, \mathcal{L}(\mathcal{F}f)\right)_{L^2}\\
&= -\frac{1}{4}\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} \delta(p_1+p_2-p_3-p_4)\\
&\times \delta(\mathcal{E}_{p_1}+\mathcal{E}_{p_2}-\mathcal{E}_{p_3}-\mathcal{E}_{p_4})\\
&\times \frac{\mathcal{M}_1\mathcal{M}_2}{(1-\mathcal{M}_1)(1-\mathcal{M}_2)(1-\mathcal{M}_3)(1-\mathcal{M}_4)}\times\\
&\times \Big[(1-\mathcal{M}_3)f(p_3) + (1-\mathcal{M}_4)f(p_4) - (1-\mathcal{M}_2)f(p_2)\\
&- (1-\mathcal{M}_1)f(p_1)\Big]\Big[(1-\mathcal{M}_3)\varphi(p_3) + (1-\mathcal{M}_4)\varphi(p_4)\\
&- (1-\mathcal{M}_2)\varphi(p_2) - (1-\mathcal{M}_1)\varphi(p_1)\Big]dp_1dp_2dp_3dp_4,
\end{aligned}$$

which implies

$$\left(\frac{\mathcal{M}}{\mathcal{F}}f, \mathcal{L}(\mathcal{F}f)\right)_{L^2} \le 0, \tag{3.13}$$

and

$$\left(\frac{\mathcal{M}}{\mathcal{F}}\varphi, \mathcal{L}(\mathcal{F}f)\right)_{L^2} = \left(\frac{\mathcal{M}}{\mathcal{F}}f, \mathcal{L}(\mathcal{F}\varphi)\right)_{L^2},$$

for all function φ and f such that the integrals are well-defined. The equality in (3.13) holds true if and only if $\frac{\mathcal{M}}{\mathcal{F}}f$ is identical to one of the five functions defined in (3.5).

From the above observation, we are now able to define the kernel of the linearized collision operator $\mathcal{L}$ of C_{22}:

$$\mathcal{N} := \ker\mathcal{L} = \operatorname{span}\left\{\frac{\mathcal{F}^2}{\mathcal{M}}\Psi_i : i = 0,\dots,4\right\},$$

and its orthogonal space:

$$\mathcal{R} := \mathcal{N}^\perp = \left\{G \in L^2(\mathbb{R}^3) : \left(G, \frac{\mathcal{F}^2}{\mathcal{M}}\Psi_i\right)_{L^2} = 0,\ i = 0,\dots,4\right\}.$$

On $L^2(\mathbb{R}^3)$, we also define the orthogonal projection operators $\mathbb{P}$ and $\mathbb{P}^\perp = 1-\mathbb{P}$ on to $\mathcal{N}$ and $\mathcal{R}$. By normalizing $\{\Psi_i\}_{i=0,\dots,4}$, we obtain the following orthonormal basis of the space $\mathcal{N}$

$$\left\{\frac{\psi_i}{\sqrt{\omega_i}}\frac{\mathcal{F}^2}{\mathcal{M}} : \ i = 0,\dots,4\right\}, \tag{3.14}$$

with

$$\psi_0 = 1; \ \ \psi_i = p^i - u_i, \ \ i = 1,2,3; \ \ \psi_4 = |p-u|^2 - 6\tau\frac{\Omega_1(\gamma)}{\Omega_0(\gamma)},$$

$$\begin{aligned}
\omega_0 &= \int_{\mathbb{R}^3}\frac{\mathcal{F}^2}{\mathcal{M}}dp = 2^{3/2}\pi\tau^{3/2}\gamma\Omega_0(\gamma);\\
\omega_i &= \int_{\mathbb{R}^3}\frac{\mathcal{F}^2}{\mathcal{M}}|\psi_i(p)|^2dp = 2^{5/2}\pi\tau^{5/2}\gamma\Omega_1(\gamma), \ \ i = 1,2,3;\\
\omega_4 &= \int_{\mathbb{R}^3}\frac{\mathcal{F}^2}{\mathcal{M}}|\psi_4(p)|^2dp = 2^{7/2}\pi\tau^{7/2}\gamma\,\Sigma(\Omega_0(\gamma),\Omega_1(\gamma),\Omega_2(\gamma));
\end{aligned}$$

where

$$\Sigma(x,y,z) = \frac{5xz-9y^2}{x},$$

and

$$\Omega_k(\gamma) = \int_0^\infty \frac{y^{k-1/2}}{e^y+\gamma}dy, \quad k > -1/2. \tag{3.15}$$

3.3. Hydrodynamics quantities

In order to study the hydrodynamics limit of the system, let us define the following moments of the function $f(t,r,p)$:

$$n_n[f](t,r) = \int_{\mathbb{R}^3} f(t,r,p)dp, \tag{3.16}$$

$$u[f](t,r) = v_n(t,r)[f](t,r) = \frac{1}{n_n[f](t,r)}\int_{\mathbb{R}^3} pf(t,r,p)dp, \tag{3.17}$$

$$\mathbb{E}_n[f](t,r) = \frac{1}{2}\int_{\mathbb{R}^3} f(t,r,p)|p-v_n[f](t,r)|^2dp, \tag{3.18}$$

$$\tilde{\mathbb{E}}_n[f](t,r) = \frac{2\mathbb{E}_n[f](t,r)}{3}, \quad e_n[f](t,r) = \frac{\tilde{\mathbb{E}}_n[f](t,r)}{n_n[f](t,r)}. \tag{3.19}$$

Replacing f by $\mathcal{F}$, we obtain

$$\begin{aligned}
n_n[\mathcal{F}] &= 2^{5/2}\pi\tau^{3/2}\Omega_1(\gamma),\\
\mathbb{E}_n[\mathcal{F}] &= 2^{5/2}\pi\tau^{5/2}\gamma\Omega_2(\gamma),
\end{aligned} \tag{3.20}$$

where Ω_1, Ω_2 are defined in (3.15). For the sake of simplicity, we denote $n_n[\mathcal{F}]$, $v_n[\mathcal{F}]$, $u[\mathcal{F}]$, $\mathbb{E}_n[\mathcal{F}]$, $\tilde{\mathbb{E}}_n[\mathcal{F}]$, $e_n[\mathcal{F}]$ by n_n, v_n, u, $\mathbb{E}_n$, $\tilde{\mathbb{E}}_n$, and e_n.

We indeed can compute γ and τ as

$$\gamma = \left(\frac{\mathrm{Id}\Omega_2}{\Omega_1^{5/3}}\right)^{-1}\left(\frac{2^{5/3}\pi^{2/3}\mathbb{E}_n}{{n_n}^{5/3}}\right), \tag{3.21}$$

and

$$\tau = \left(\frac{n_n}{2^{5/2}\pi\,\Omega_1 \left(\left(\frac{\mathrm{Id}\Omega_2}{\Omega_1^{5/3}} \right)^{-1} \left(\frac{2^{5/3}\pi^{2/3}\mathbb{E}_n}{n_n{}^{5/3}} \right) \right)} \right)^{2/3}. \tag{3.22}$$

3.4. Computing $\Gamma_{12}[\mathcal{F}]$

Now, let us consider the collision operator C_{12}. This operator also has the collision invariant property:

$$\begin{aligned} &\int_{\mathbb{R}^3} (\Psi_i(p) - v_{ci}) C_{12}[f]dp \\ &= \int_{\mathbb{R}^3} \left(\Psi_4(p) + 2U - 2\mu_c - v_c^2 \right) C_{12}[f]dp = 0, \quad i = 1, 2, 3. \end{aligned} \tag{3.23}$$

An important property of C_{12} is that $\mathcal{F}$ is not an equilibrium of C_{12}. We have:

$$\begin{aligned} &\Gamma_{12}[\mathcal{F}] := \int_{\mathbb{R}^3} C_{12}[\mathcal{F}]dp \\ &= -n_c[1 - e^{-\beta(\mu-\mu_c-(v_n-v_c)^2/2)}] \\ &\times \iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} \delta(v_c + p_1 - p_2 - p_3)\times \\ &\times \delta(\mathcal{E}_c + \mathcal{E}_{p_1} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3})(1 + \mathcal{F}(t, r, p_1)) \\ &\times \mathcal{F}(t, r, p_2)\mathcal{F}(t, r, p_3)dp_1dp_2dp_3. \end{aligned} \tag{3.24}$$

Expanding $\mathcal{F}$ into Taylor series of $\mathcal{M}$, we can simplify the above integral as

$$\begin{aligned} &\Gamma_{12}[\mathcal{F}] \\ &= -n_c[1 - e^{-\beta(\mu-\mu_c-(u-v_c)^2/2)}] \\ &\times \sum_{k_2,k_3\in\mathbb{N}\cup\{0\},k_1\in\mathbb{N}} \gamma^3 e^{-\frac{|v_c-u|^2(k_1+k_2+k_3)}{2\tau}} \times \\ &\times e^{\frac{(-2\mathcal{E}_c+2U+v_c^2)k_1}{2\tau}} \\ &\times \int_{x\cdot y=\frac{v_c^2}{2}+U-\mathcal{E}_c} e^{-(k_1+k_2)[|x|^2+x\cdot(v_c-u)]-(k_1+k_3)[|y|^2+y\cdot(v_c-u)]/(2\tau)}dxdy, \end{aligned} \tag{3.25}$$

with the notice that from (3.21) and (3.22), γ and τ are functions of n_n and $\mathbb{E}_n$.

4. The two-fluid Euler quantum hydrodynamic limit

In this section, we will derive a two-fluid Euler quantum hydrodynamic limit from (2.9)–(2.45). In this case, ϵ is a constant, so we will set it to be $\epsilon = 1$. Choose $\tilde{\epsilon}$ to be any small parameter. In order to obtain the Euler hydrodynamics limit, let us start with the following Hilbert expansion using $\tilde{\epsilon}$ as the small parameter (cf. [59]):

$$f = \sum_{i=0}^{n} \tilde{\epsilon}^i f^{(i)} + \tilde{\epsilon}^l \varsigma, \tag{4.1}$$

in which n and l are positive integers. As a consequence, we can replace f by its Hilbert expansion into

$$\mathcal{D}f = C_{22}[f],$$

to get a linear system of equations and a weakly nonlinear equation for the remainder ς, which reads as:

$$B_1(f^{(0)}, f^{(0)}) + B_2(f^{(0)}, f^{(0)}, f^{(0)}) = 0, \tag{4.2}$$

$$2B_1(f^{(0)}, f^{(1)}) + 3B_2(f^{(0)}, f^{(0)}, f^{(1)}) = \mathcal{D}f^{(0)}, \tag{4.3}$$

$$\begin{aligned} &2B_1(f^{(0)}, f^{(i)}) + 3B_2(f^{(0)}, f^{(0)}, f^{(i)}) \\ &= \mathcal{D}f^{(i-1)} - \sum_{j=1}^{i-1} B_1(f^{(i)}, f^{(i-j)}) \\ &\quad - \sum_{j,k=1,0<j+k<i}^{i-1} B_2(f^{(i)}, f^{(k)}, f^{(i-j-k)}), \end{aligned} \tag{4.4}$$

for $i = 2, 3, \ldots, n$.

The equation for the remainder r is as follows:

$$\begin{aligned} \mathcal{D}\varsigma &= \frac{1}{\tilde{\epsilon}}\mathcal{L}\varsigma + 2\sum_{i=1}^{n} \tilde{\epsilon}^{i-1} B_1(f^{(i)}, \varsigma) \\ &+ \tilde{\epsilon}^{l-1} B_1(\varsigma, \varsigma) + 3\sum_{i=1}^{n} B_2(\mathcal{F}, f^{(i)}, \varsigma) \\ &+ 3\sum_{i,j=1}^{n} \tilde{\epsilon}^{i+j-1} B_2(f^{(i)}, f^{(j)}, \varsigma) + 3\tilde{\epsilon}^{(l-1)} B_2(\mathcal{F}, \varsigma, \varsigma) \\ &+ 3\tilde{\epsilon}^{l-1}\sum_{i=1}^{n} \tilde{\epsilon}^i B_2(f^{(i)}, \varsigma, \varsigma) \\ &+ \tilde{\epsilon}^{2l-1} B_2(\varsigma, \varsigma, \varsigma) + \tilde{\epsilon}^{n-1}\mathfrak{Q}, \end{aligned} \tag{4.5}$$

where $\mathfrak{Q}$ is an operator of $\mathcal{F}, f^{(1)}, \ldots, f^{(n)}$.

Let us now consider each equation in the above system. From the first equation (4.2), we deduce that $f^{(0)}$ has to be a Bose–Einstein distribution:

$$f^{(0)} = \mathcal{F}. \tag{4.6}$$

Eqs. (4.3) and (4.4) lead to linear integral equations for $f^{(1)}, \ldots, f^{(i)}$. Thanks to Fredholm's theory, these linear integral equations are solvable if the right hand sides are orthogonal to $\mathcal{N}$ in $L^2(\mathbb{R}^3)$. As a consequence, $f^{(1)}$ can be solved from (4.3), if the following condition is satisfied

$$\mathbb{P}\mathcal{D}\mathcal{F} = 0. \tag{4.7}$$

We recall that $\mathbb{P}$ and $\mathbb{P}^\perp = 1 - \mathbb{P}$ are the orthogonal projection operators onto $\mathcal{N}$ and $\mathcal{R}$ in $L^2(\mathbb{R}^3)$.

4.1. The Euler quantum hydrodynamic limit of the thermal cloud kinetic equation

Integrating Eq. (2.2) in p, we obtain

$$\begin{aligned} &\partial_t \int_{\mathbb{R}^3} f(t, r, p)dp + \nabla_r \cdot \int_{\mathbb{R}^3} pf(t, r, p)dp \\ &- \int_{\mathbb{R}^3} \nabla_r U(t, r, p) \cdot \nabla_p f(t, r, p)dp \\ &= \int_{\mathbb{R}^3} C_{12}[f](t, r, p)dp + \int_{\mathbb{R}^3} C_{22}[f](t, r, p)dp. \end{aligned} \tag{4.8}$$

Using the fact that

$$\int_{\mathbb{R}^3} \nabla_r U(t, r) \cdot \nabla_p f(t, r, p)dp = \int_{\mathbb{R}^3} C_{22}[f](t, r, p)dp = 0,$$

we get

$$\partial_t \int_{\mathbb{R}^3} fdp + \nabla_r \cdot \int_{\mathbb{R}^3} pfdp = \Gamma_{12}[f]. \tag{4.9}$$

Eq. (4.9) can be rewritten as

$$\partial_t n_n + \nabla_r \cdot (n_n v_n) = \Gamma_{12}[f]. \tag{4.10}$$

For an arbitrary momentum vector $p = (p_1, p_2, p_3)$, we choose p_j, $j \in \{1, 2, 3\}$ as a test function for (2.2) and obtain

$$\begin{aligned} &\partial_t \int_{\mathbb{R}^3} f(t,r,p)p_j dp + \nabla_r \cdot \int_{\mathbb{R}^3} pf(t,r,p)p_j dp \\ &- \int_{\mathbb{R}^3} \nabla_r U(t,r) \cdot \nabla_p f(t,r,p)p_j dp \\ &= \int_{\mathbb{R}^3} p_j C_{12}[f](t,r,p)dp + \int_{\mathbb{R}^3} p_j C_{22}[f](t,r,p)dp. \end{aligned} \tag{4.11}$$

Due to the conservation of momentum for C_{12} and C_{22},

$$\int_{\mathbb{R}^3} (p_j - v_{cj}) C_{12}[f]dp = \int_{\mathbb{R}^3} p_j C_{22}[f]dp = 0,$$

we get

$$\begin{aligned} &\partial_t \int_{\mathbb{R}^3} f(t,r,p)p_j dp + \nabla_r \cdot \int_{\mathbb{R}^3} pf(t,r,p)p_j dp \\ &- \int_{\mathbb{R}^3} \nabla_r U(t,r) \cdot \nabla_p f(t,r,p)p_j dp \\ &= \int_{\mathbb{R}^3} v_{cj}(t,r) C_{12}[f](t,r,p)dp \\ &= v_{cj}(t,r)\Gamma_{12}[f](t,r). \end{aligned} \tag{4.12}$$

Let us look at the first term on the left hand side of (4.12)

$$\partial_t \int_{\mathbb{R}^3} fp_j dp = \partial_t (n_n v_{nj}) = \partial_t n_n v_{nj} + n_n \partial_t v_{nj}, \tag{4.13}$$

in which v_{nj} is the component of $v_n = (v_{n1}, v_{n2}, v_{n3})$.

By using (4.10), we can deduce from (4.13) that

$$\partial_t \int_{\mathbb{R}^3} fp_j dp = \Gamma_{12}[f]v_{nj} - v_{nj}\nabla_r \cdot (n_n v_n) + n_n \partial_t v_{nj}, \tag{4.14}$$

Now, let us look at the second term on the left hand side of (4.12),

$$\begin{aligned} \nabla_r \cdot \int_{\mathbb{R}^3} pfp_j dp &= \sum_{i=1}^3 \partial_{r_i} \int_{\mathbb{R}^3} p_i p_j f dp \\ &= \sum_{i=1}^3 \partial_{r_i} \int_{\mathbb{R}^3} [(p_i - v_{ni})(p_j - v_{nj}) + p_i v_{nj} \\ &\quad + p_j v_{ni} - v_{ni} v_{nj}] f dp \\ &= \sum_{i=1}^3 \partial_{r_i} \int_{\mathbb{R}^3} (p_i - v_{ni})(p_j - v_{nj}) f dp \\ &\quad + \sum_{i=1}^3 \partial_{r_i} \int_{\mathbb{R}^3} (p_i v_{nj} + p_j v_{ni}) f dp \\ &\quad - \sum_{i=1}^3 \partial_{r_i} \int_{\mathbb{R}^3} v_{ni} v_{nj} f dp. \end{aligned} \tag{4.15}$$

By observing that

$$\begin{aligned} \int_{\mathbb{R}^3} (p_i v_{nj} + p_j v_{ni}) f(t,r,p)dp &= 2v_{nj}(t,r)v_{ni}(t,r)n_n(t,r) \\ \int_{\mathbb{R}^3} v_{ni} v_{nj} f(t,r,p)dp &= v_{nj}(t,r)v_{ni}(t,r)n_n(t,r), \end{aligned} \tag{4.16}$$

we infer from Identity (4.15)

$$\begin{aligned} &\nabla_r \cdot \int_{\mathbb{R}^3} pfp_j dp \\ &= \sum_{i=1}^3 \partial_{r_i} \int_{\mathbb{R}^3} (p_i - v_{ni})(p_j - v_{nj}) f dp + \sum_{i=1}^3 \partial_{r_i} [v_{nj} v_{ni} n_n]. \end{aligned} \tag{4.17}$$

The last term on the left hand side of (4.12) can be rewritten in the following form, by integration by parts and the definition of n_n

$$-\int_{\mathbb{R}^3} \nabla_r U \cdot \nabla_p f p_j dp = \int_{\mathbb{R}^3} \partial_{r_j} U f dp = n_n \partial_{r_j} U. \tag{4.18}$$

Putting the three terms (4.14), (4.17) and (4.18) together, we find

$$n_n (\partial_t + v_n \cdot \nabla) v_{nj} = -\sum_{i=1}^3 \partial_{r_j} \mathcal{P}[f]_{ij} - n_n \partial_{r_j} U - (v_{nj} - v_{cj})\Gamma_{12}[f], \tag{4.19}$$

where

$$\mathcal{P}[f]_{ij} = \int_{\mathbb{R}^3} (p_i - v_{ni}(t,r)) \left(p_j - v_{nj}(t,r)\right) f(t,r,p)dp. \tag{4.20}$$

Choosing $|p|^2$, as a test function for (2.2) yields

$$\begin{aligned} &\partial_t \int_{\mathbb{R}^3} f(t,r,p)|p|^2 dp + \nabla_r \cdot \int_{\mathbb{R}^3} |p|^2 pf(t,r,p)dp \\ &- \int_{\mathbb{R}^3} \nabla_r U(t,r) \cdot \nabla_p f(t,r,p)|p|^2 dp \\ &= \int_{\mathbb{R}^3} |p|^2 C_{12}[f](t,r,p)dp + \int_{\mathbb{R}^3} |p|^2 C_{22}[f](t,r,p)dp. \end{aligned} \tag{4.21}$$

Let us recall the conservation of energy for C_{12} and C_{22}

$$\int_{\mathbb{R}^3} |p|^2 C_{22}[f]dp = 0,$$

and

$$\begin{aligned} 0 &= 2\int_{\mathbb{R}^3} \left(\mathcal{E}_p - \mathcal{E}_c\right) C_{12}[f]dp \\ &= \int_{\mathbb{R}^3} \left(|p|^2 + 2U - 2\mu_c - v_c^2\right) C_{12}[f]dp, \end{aligned}$$

which leads to

$$\begin{aligned} &\partial_t \int_{\mathbb{R}^3} f(t,r,p)|p|^2 dp + \nabla_r \cdot \int_{\mathbb{R}^3} |p|^2 pf(t,r,p)dp \\ &- \int_{\mathbb{R}^3} \nabla_r U(t,r) \cdot \nabla_p f(t,r,p)|p|^2 dp \\ &= \left(-2U + 2\mu_c + v_c^2\right) \Gamma_{12}[f](t,r). \end{aligned} \tag{4.22}$$

Similar as above, we consider each term on the right and left hand sides of (4.22). Let us start with the first term on the left hand side

$$\begin{aligned} \partial_t \int_{\mathbb{R}^3} f|p|^2 dp &= \partial_t \left(\int_{\mathbb{R}^3} f|p - v_n|^2 dp\right) \\ &\quad + \partial_t \left(\int_{\mathbb{R}^3} f 2p \cdot v_n dp\right) - \partial_t \left(\int_{\mathbb{R}^3} f|v_n|^2 dp\right), \end{aligned} \tag{4.23}$$

where we have used the identity

$$|p - v_n|^2 + 2p \cdot v_n - |v_n|^2 = |p|^2. \tag{4.24}$$

Since

$$\left(\int_{\mathbb{R}^3} fp \cdot v_n dp\right) = |v_n|^2 n_n = \left(\int_{\mathbb{R}^3} f|v_n|^2 dp\right),$$

we obtain from (4.23) that

$$\begin{aligned} \partial_t \int_{\mathbb{R}^3} f|p|^2 dp &= \partial_t \left(\int_{\mathbb{R}^3} f|p - v_n|^2 dp\right) + \partial_t \left(|v_n|^2 n_n\right) \\ &= 2\partial_t \mathbb{E} + \partial_t \left(|v_n|^2 n_n\right). \end{aligned} \tag{4.25}$$

Expanding the second term on the right hand side of (4.25) gives us

$$\partial_t \int_{\mathbb{R}^3} f|p|^2 dp = 2\partial_t \mathbb{E} + 2n_n v_n \cdot \partial_t v_n + |v_n|^2 \partial_t n_n, \tag{4.26}$$

which, by (4.10) and (4.19), can be rewritten as

$$\begin{aligned}&\partial_t \int_{\mathbb{R}^3} f|p|^2 dp\\ &= 2\partial_t \mathbb{E} + \sum_{j=1}^3 2v_{nj}\left[-\sum_{i=1}^3 \partial_{r_j}\mathcal{P}[f]_{ij} - n_n\partial_{r_j}U\right.\\ &\quad \left. - (v_{nj}-v_{cj})\Gamma_{12}[f] - n_n v_n\cdot\nabla_r v_{nj}\right]\\ &\quad + |v_n|^2[\Gamma_{12}[f] - \nabla_r\cdot(n_n v_n)].\end{aligned} \tag{4.27}$$

Now, for the second term on the left hand side of (4.22), it is straightforward that

$$\begin{aligned}&\nabla_r\cdot\int_{\mathbb{R}^3}|p|^2 pf dp\\ &= \nabla_r\cdot\left(\int_{\mathbb{R}^3}|p-v_n|^2(p-v_n)f dp\right)\\ &\quad + \nabla_r\cdot\left(\int_{\mathbb{R}^3}|v_n|^2 v_n f dp\right)\\ &\quad - 3\nabla_r\cdot\left(\int_{\mathbb{R}^3}|v_n|^2 p f dp\right) + 3\nabla_r\cdot\left(\int_{\mathbb{R}^3}|p|^2 v_n f dp\right),\end{aligned} \tag{4.28}$$

which, as a view of the identity

$$\int_{\mathbb{R}^3}|v_n|^2 v_n f dp = \int_{\mathbb{R}^3}|v_n|^2 p f dp = |v_n|^2 v_n n_n,$$

can be expressed as

$$\begin{aligned}&\nabla_r\cdot\int_{\mathbb{R}^3}|p|^2 pf dp\\ &= \nabla_r\cdot\left(\int_{\mathbb{R}^3}|p-v_n|^2(p-v_n)f dp\right) - 2\nabla_r\cdot\left(|v_n|^2 v_n n_n\right)\\ &\quad + 3\nabla_r\cdot\left(\int_{\mathbb{R}^3}|p|^2 v_n f dp\right).\end{aligned} \tag{4.29}$$

Using (4.24), we can rewrite (4.29) as

$$\begin{aligned}&\nabla_r\cdot\int_{\mathbb{R}^3}|p|^2 pf dp\\ &= \nabla_r\cdot\left(\int_{\mathbb{R}^3}|p-v_n|^2(p-v_n)f dp\right) - 2\nabla_r\cdot\left(|v_n|^2 v_n n_n\right)\\ &\quad + 3\nabla_r\cdot\left(\int_{\mathbb{R}^3}|p-v_n|^2 v_n f dp\right.\\ &\quad \left. + 2|v_n|^2\int_{\mathbb{R}^3} p f dp - |v_n|^2 v_n\int_{\mathbb{R}^3} f dp\right),\end{aligned} \tag{4.30}$$

which can be reduced to

$$\begin{aligned}&\nabla_r\cdot\int_{\mathbb{R}^3}|p|^2 pf dp\\ &= \nabla_r\cdot\left(\int_{\mathbb{R}^3}|p-v_n|^2(p-v_n)f dp\right) - 2\nabla_r\cdot\left(|v_n|^2 v_n n_n\right)\\ &\quad + 3\nabla_r\cdot\left(\int_{\mathbb{R}^3}|p-v_n|^2 v_n f dp + |v_n|^2 v_n n_n\right)\\ &= \nabla_r\cdot\left(\int_{\mathbb{R}^3}|p-v_n|^2(p-v_n)f dp\right) + \nabla_r\cdot\left(|v_n|^2 v_n n_n\right)\\ &\quad + 3\nabla_r\cdot\left(v_n\int_{\mathbb{R}^3}|p-v_n|^2 f dp\right),\end{aligned} \tag{4.31}$$

The last term on the left hand side of (4.22) can be rewritten in a straightforward manner as follows:

$$\int_{\mathbb{R}^3}|p|^2\nabla_r U\cdot\nabla_p f dp = -2\nabla_r U\cdot\int_{\mathbb{R}^3} p f dp. \tag{4.32}$$

Notice that the right hand side of (4.32) can be expressed in terms of n_n and v_n as

$$-2\nabla_r U\cdot\int_{\mathbb{R}^3} p f dp = -2\nabla_r U\cdot(n_n v_n). \tag{4.33}$$

As a consequence, we find

$$\int_{\mathbb{R}^3}|p|^2\nabla_r U\cdot\nabla_p f dp = -2\nabla_r U\cdot(n_n v_n). \tag{4.34}$$

Combining (4.22), (4.27), (4.31) and (4.34), yields

$$\begin{aligned}&2\partial_t\mathbb{E} + \sum_{j=1}^3 2v_{nj}\left[-\sum_{i=1}^3\partial_{r_j}\mathcal{P}[f]_{ij} - n_n\partial_{r_j}U\right.\\ &\quad \left. - (v_{nj}-v_{cj})\Gamma_{12}[f] - n_n v_n\cdot\nabla_r v_{nj}\right]\\ &\quad + |v_n|^2[\Gamma_{12}[f] - \nabla_r\cdot(n_n v_n)]\\ &\quad + \nabla_r\cdot\left(\int_{\mathbb{R}^3}|p-v_n|^2(p-v_n)f dp\right) + \nabla_r\cdot\left(|v_n|^2 v_n n_n\right)\\ &\quad + 3\nabla_r\cdot\left(v_n\int_{\mathbb{R}^3}|p-v_n|^2 f dp\right) - 2\nabla_r U\cdot(n_n v_n)\\ &= \left(-2U + 2\mu_c + v_c^2\right)\Gamma_{12}[f],\end{aligned} \tag{4.35}$$

which leads to

$$\begin{aligned}&\partial_t\mathbb{E} + \nabla_r\cdot(\mathbb{E}v_n)\\ &= -\nabla_r\cdot\mathcal{R}[f] - \sum_{i,j=1}^3\frac{1}{2}\left(v_{ni}\partial_{r_j} + v_{nj}\partial_{r_i}\right)P_{ij}\\ &\quad + \left(\frac{(v_n-v_c)^2}{2} + \mu_c - U\right)\Gamma_{12}[f],\end{aligned} \tag{4.36}$$

where

$$\mathcal{R}[f](t,r) = \int_{\mathbb{R}^3}\frac{|p-v_n|^2(p-v_n)}{2} f(t,r,p)dp. \tag{4.37}$$

The three equations (4.10), (4.19) and (4.36) lead to the following system of moment equations for the kinetic equation of the thermal cloud:

$$\begin{aligned}&\partial_t n_n + \nabla_r\cdot(n_n v_n) = \Gamma_{12}[f],\\ &n_n\left(\partial_t + v_n\cdot\nabla\right)v_{nj} = -\sum_{i=1}^3\partial_{r_j}\mathcal{P}[f]_{ij} - n_n\partial_{r_j}U - (v_{nj}-v_{cj})\Gamma_{12}[f],\\ &\partial_t\mathbb{E} + \nabla_r\cdot(\mathbb{E}v_n) = -\nabla_r\cdot\mathcal{R}[f] - \sum_{i,j=1}^3\frac{1}{2}\left(v_{ni}\partial_{r_j} + v_{nj}\partial_{r_i}\right)P_{ij}\\ &\qquad + \left(\frac{(v_n-v_c)^2}{2} + \mu_c - U\right)\Gamma_{12}[f].\end{aligned} \tag{4.38}$$

Replacing $\mathcal{F}$ into (4.38), we obtain $R[\mathcal{F}] = 0$,

$$P_{ij}(t,r) = \delta_{ij}\tilde{\mathbb{E}}(t,r) \equiv \delta_{ij}\int_{\mathbb{R}^3}\frac{|p|^2}{3} f_\infty(t,r,p)dp. \tag{4.39}$$

Moreover, we note that

$$\mathbb{E}(t,r) = \frac{3}{2}\tilde{\mathbb{E}}(t,r). \tag{4.40}$$

As a consequence, we can close the system (4.38) to obtain

$$\begin{aligned}&\partial_t n_c + \nabla_r\cdot(n_c v_c) = -\Gamma_{12}[\mathcal{F}],\\ &n_n\left(\partial_t + v_n\cdot\nabla\right)v_n = -\nabla_r\tilde{\mathbb{E}}_n - n_n\nabla_r U - (v_n - v_c)\Gamma_{12}[\mathcal{F}],\\ &\partial_t\tilde{\mathbb{E}}_n + \nabla_r\cdot(\tilde{\mathbb{E}}_n v_n) = -\frac{2}{3}\tilde{\mathbb{E}}_n\nabla_r\cdot v_n\\ &\qquad + \frac{2}{3}\left(\frac{(v_n-v_c)^2}{2} + \mu_c - U\right)\Gamma_{12}[\mathcal{F}].\end{aligned} \tag{4.41}$$

4.2. Comparison with a previous result

Putting the two systems (2.9) and (4.41) together, one finds the following two-fluid Euler quantum hydrodynamics

$$\begin{aligned}
&\partial_t n_c + \nabla_r \cdot (n_c v_c) = -\Gamma_{12}[\mathcal{F}],\\
&\partial_t v_c + \frac{\nabla_r v_c^2}{2} = -\nabla_r \mu_c,\\
&\partial_t n_n + \nabla_r \cdot (n_n v_n) = \Gamma_{12}[\mathcal{F}],\\
&n_n\,(\partial_t + v_n \cdot \nabla)\, v_{nj} = -\nabla_r \tilde{\mathbb{E}}_n - n_n \nabla_r U - (v_{nj} - v_{cj})\Gamma_{12}[\mathcal{F}],\\
&\partial_t \tilde{\mathbb{E}}_n + \nabla_r \cdot (\tilde{\mathbb{E}}_n v_n) = -\frac{2}{3}\tilde{\mathbb{E}}_n \nabla_r \cdot v_n\\
&\qquad\qquad + \frac{2}{3}\left(\frac{(v_n - v_c)^2}{2} + \mu_c - U\right)\Gamma_{12}[\mathcal{F}].
\end{aligned} \tag{4.42}$$

In (4.42), the condensate and non-condensate parts are coupled through both μ_c and Γ_{12}. Notice that Γ_{12} is already computed in (3.24) and (3.25).

In the thesis [28], the author has derived the following hydrodynamic limit:

$$\begin{aligned}
&\partial_t n_c + \nabla_r \cdot (n_c v_c) = 0,\\
&\partial_t n_n + \nabla_r \cdot (n_n v_n) = 0,\\
&\partial_t (n_n v_n) + \nabla_r \cdot (n_n v_n \otimes v_n + \mathbb{E}_n I_3) = -g n_n \nabla_r \cdot (2n_n + n_c),\\
&\partial_t (n_c v_c) + \nabla_r \cdot (n_c v_c \otimes v_c) = -\frac{g}{2} n_c \nabla_r \cdot (2n_n + n_c),\\
&\partial_t \Big(\frac{1}{2} n_n |v_n|^2 + \frac{1}{2} n_c |v_c|^2 + \frac{3}{2}\mathbb{E}_n + \frac{g}{4}(2n_n + n_c)^2\Big)\\
&\quad + \nabla_r \cdot \Big(\frac{1}{2} n_n |v_n|^2 v_n + \frac{1}{2} n_c |v_c|^2 v_c + \frac{5}{2}\mathbb{E}_n v_n\\
&\quad + \frac{g}{2}(2n_n + n_c)(2n_n v_n + n_c v_c)\Big) = 0,
\end{aligned} \tag{4.43}$$

where I_3 is the identity 3×3 matrix. It is also mentioned [28] that this is a two-phases Euler system, the second fluid (the superfluid) being pressureless and they do not exchange mass, contrary to what occurs in the Landau two-fluid theory [29,30].

On the contrary, our limit (4.42) agrees with the Landau two-fluid theory [29,30]. The main reason is that, following [14], in general excited atoms in the condensate need not be in local equilibrium with the condensate atoms. As a consequence, C_{12} and C_{22}, in most of the cases, do not share the same equilibrium distribution. Our equilibrium distribution $\mathcal{F}$ is the natural equilibrium used in most physical contexts [14] and $C_{22}[\mathcal{F}] = 0$ but $C_{12}[\mathcal{F}] \neq 0$. Therefore, the two fluids are coupled.

In [28], the author considers a very special choice of $\mathcal{F}$

$$\mathcal{F}(t, r, p) = \frac{1}{e^{\beta[(p - v_n)^2/2 - |v_c - v_n|^2/2 - U/2]} - 1},$$

where the temperature parameter β is a constant, instead of being a function of (t, r). Moreover, the effect of the chemical potential $\mu(t, r)$ is also ignored. This special choice of the distribution $\mathcal{F}$ implies $C_{22}[\mathcal{F}] = C_{12}[\mathcal{F}] = 0$. The two fluids are then decoupled, that is in contradiction with the Landau two-fluid theory [29,30], as the author pointed out.

5. The two-fluid Navier–Stokes quantum hydrodynamic approximations

This section is devoted to the derivation of the Navier–Stokes approximation of the system (2.37)–(2.45) through the Chapman–Enskog expansion, under the assumption $g = \epsilon^{\delta_0}$. Similar as in Section 4.1, we also have the expansion:

$$f = \sum_{i=0}^{n} \epsilon^i f^{(i)} + \epsilon^l \mathcal{S}, \tag{5.1}$$

in which n and l are positive integers.

Arguing similarly as above, we deduce that $f^{(0)}$ has to be a Bose–Einstein distribution:

$$f^{(0)} = \mathcal{F}. \tag{5.2}$$

Decompose $f^{(i)}$ into two parts

$$f^{(i)} = h^{(i)} + k^{(i)}, \tag{5.3}$$

where

$$h^{(i)} \in \mathcal{R}, \quad k^{(i)} \in \mathcal{N}.$$

From (4.3), one has

$$h^{(1)} = \mathcal{L}^{-1}\mathcal{D}\mathcal{F}. \tag{5.4}$$

Adopting the same techniques used in [60,61], we decompose $h^{(1)}$ into the sum of h' and h'':

$$h^{(1)} = h' + h'',$$

where h' and h'' satisfy the following system of equations:

$$\mathcal{L}h' = \mathbb{P}^{\perp}\mathcal{D}\mathcal{F}, \tag{5.5}$$

$$\mathbb{P}\mathcal{D}\mathcal{F} = -\epsilon \mathbb{P}\mathcal{D}h', \tag{5.6}$$

$$\mathcal{L}h'' = \epsilon \mathbb{P}^{\perp}\mathcal{D}k^{(1)}, \tag{5.7}$$

$$\mathbb{P}\mathcal{D}k^{(1)} = -\mathbb{P}\mathcal{D}h'', \tag{5.8}$$

and

$$\begin{aligned}
\mathcal{L}h^{(i)} = {} & \epsilon \mathbb{P}^{\perp}\mathcal{D}k^{(i)} + \mathbb{P}^{\perp}\mathcal{D}h^{(i-1)} - \sum_{j=1}^{i-1} Q_1(f^{(j)}, f^{(i-j)})\\
& - \sum_{j,k=0,0<j+k<i}^{i-1} Q_2(f^{(j)}, f^{(k)}, f^{(i-j-k)}),
\end{aligned} \tag{5.9}$$

$$\mathbb{P}\mathcal{D}k^{(i)} = -\mathbb{P}\mathcal{D}h^{(i)}. \tag{5.10}$$

By the Fredholm theory, the system (5.5)–(5.8) can be solved in $L^2(\mathbb{R}^3)$, if

$$\begin{aligned}
&h' = \mathcal{L}^{-1}(\mathbb{P}^{\perp}\mathcal{D}\mathcal{F}),\\
&h'' = \mathcal{L}^{-1}(\epsilon \mathbb{P}^{\perp}\mathcal{D}k^{(1)})
\end{aligned} \tag{5.11}$$

and

$$\begin{aligned}
h^{(i)} = {} & \mathcal{L}^{-1}\Big(\epsilon \mathbb{P}^{\perp}\mathcal{D}k^{(i)} + \mathbb{P}^{\perp}\mathcal{D}h^{(i-1)} - \sum_{j=1}^{i-1} B_1(f^{(j)}, f^{(i-j)})\\
& - \sum_{j,k=0;0<j+k<i}^{i-1} B_2(f^{(j)}, f^{(k)}, f^{(i-j-k)})\Big),
\end{aligned} \tag{5.12}$$

for $i = 2, 3, \ldots$

Eq. (5.11) yields

$$\mathbb{P}\mathcal{D}\mathcal{F} = -\epsilon \mathbb{P}\mathcal{D}h' = -\epsilon \mathbb{P}\mathcal{D}\mathcal{L}^{-1}\mathbb{P}^{\perp}\mathcal{D}\mathcal{F}. \tag{5.13}$$

Eq. (5.13) leads to the Navier–Stokes approximation, which will be computed in Section 5.2.

5.1. Inversion of the linearized operator of C_{22}

Define

$$\mathcal{A}(p) = p \otimes p - \frac{1}{3}|p|^2 Id, \quad \mathcal{B}(p) = \frac{1}{2}p(|p|^2 - 5), \tag{5.14}$$

clearly,

$$\mathcal{A}_{jk} \perp \ker\mathcal{L}, \quad \mathcal{B}_l \perp \ker\mathcal{L}, \quad \mathcal{B}_l \perp \mathcal{A}_{jk}, \quad j,k,l = 1,2,3. \tag{5.15}$$

By the same algebraic argument as the one used for the classical Boltzmann collision operator (cf. pp. 64–65 [57]), one can deduce that there exist scalar-valued functions $\alpha_0(|p|), \alpha_1(|p|)$ such that

$$\begin{aligned} \mathcal{L}^{-1}\left(\frac{\mathcal{F}^2(p)}{\mathcal{M}(p)}\mathcal{A}(p)\right) &= \alpha_0(|p|)\frac{\mathcal{F}^2(p)}{\mathcal{M}(p)}\mathcal{A}(p), \\ \mathcal{L}^{-1}\left(\frac{\mathcal{F}^2(p)}{\mathcal{M}(p)}\mathcal{B}(p)\right) &= \alpha_1(|p|)\frac{\mathcal{F}^2(p)}{\mathcal{M}(p)}\mathcal{B}(p). \end{aligned} \tag{5.16}$$

A direct consequence of (5.16) is the existence of scalar-valued functions $\beta_0(|p|)$ and $\beta_1(|p|)$ such that

$$\begin{aligned} \mathcal{L}^{-1}\left(\frac{\mathcal{F}^2(p)}{\mathcal{M}(p)}p^i p^j\right) &= \beta_0(|p|)\frac{\mathcal{F}^2(p)}{\mathcal{M}(p)}\mathcal{A}_{ij}(p), \\ \mathcal{L}^{-1}\left(\frac{\mathcal{F}^2(p)}{\mathcal{M}(p)}\left(|p|^2 - \frac{10\tau\Omega_2(\gamma)}{\Omega_1(\gamma)}\right)p^i\right) &= \beta_1(|p|)\frac{\mathcal{F}^2(p)}{\mathcal{M}(p)}\mathcal{B}_i(p), \end{aligned} \tag{5.17}$$

where p^i, $\mathcal{B}_i(p)$ are the ith component of the vectors p and $\mathcal{B}(p)$ respectively. In addition, $\mathcal{A}_{ij}(p)$ is the (i,j)th element of the matrix $\mathcal{A}(p)$. Note that these symmetry invariances are very similar to the ones obtained in the context of the classical Boltzmann collision operator (cf. Equation (2.100), pp. 64–65 [57]); we then denote

$$\mathfrak{C}_{ij}(p) := \beta_0(|p|)\frac{\mathcal{F}^2(p)}{\mathcal{M}(p)}\mathcal{A}_{ij}(p), \quad \mathfrak{C}_i(p) := \beta_1(|p|)\frac{\mathcal{F}^2(p)}{\mathcal{M}(p)}\mathcal{B}_i(p). \tag{5.18}$$

5.2. Navier–Stokes quantum hydrodynamic approximation of the thermal cloud

In this subsection, we will derive the Navier–Stokes system resulting from (5.13). First, observe that

$$\begin{aligned} &\mathbb{P}^{\perp}\Pi\mathcal{F} \\ &= \frac{\mathcal{F}^2}{\mathcal{M}}\sum_{i,j=1}^{3}\left\{(p^i - v_{ni})(p^j - v_{nj}) - \frac{1}{3}|p - v_n|^2\delta_{i,j}\right\}\frac{1}{\tau}\frac{\partial v_{nj}}{\partial x_i} \\ &\quad + \frac{\mathcal{F}^2}{\mathcal{M}}\left\{|p - v_n|^2 - \frac{10\tau\Omega_2(\gamma)}{3\Omega_1(\gamma)}\right\}\sum_{i=1}^{3}(p^i - v_{ni})\frac{1}{2\tau^2}\frac{\partial\tau}{\partial r_i}. \end{aligned} \tag{5.19}$$

Classical techniques for the classical Boltzmann collision operator can be applied (cf. [31]—pp. 456–457 and [60,61]), to get

$$\begin{aligned} &-\mathbb{P}\Pi\mathcal{L}^{-1}\mathbb{P}^{\perp}\Pi\mathcal{F} \\ &= \frac{\mathcal{F}^2}{\mathcal{M}}\left(\sum_{k=1}^{3}\frac{\psi_k}{\omega_k}\left(\sum_{i=1}^{3}\frac{\partial}{\partial r_i}\left(\varpi(\gamma,\tau)\left(\frac{\partial v_{nk}}{\partial r_i} + \frac{\partial v_{ni}}{\partial r_k}\right)\right)\right)\right. \\ &\quad \left. - \frac{2}{3}\frac{\partial}{\partial r_k}\left(\varpi(\gamma,\tau)\sum_{i=1}^{3}\frac{\partial v_{ni}}{\partial r_i}\right)\right) \\ &\quad + 2\frac{\psi_4}{\omega_4}\left(\sum_{i=1}^{3}\frac{\partial}{\partial r_i}\left(\varrho(\gamma,\tau)\frac{\partial\tau}{\partial r_i}\right)\right. \\ &\quad - \frac{2}{3}\varrho(\gamma,\tau)\left(\sum_{i=1}^{3}\frac{\partial v_{ni}}{\partial r_i}\right)^2 \\ &\quad \left. + \varpi(\gamma,\tau)\sum_{i,k=1}^{3}\frac{\partial v_{nk}}{\partial r_i}\left(\frac{\partial v_{nk}}{\partial r_i} + \frac{\partial v_{ni}}{\partial r_k}\right)\right)\Bigg), \end{aligned} \tag{5.20}$$

where

$$\varpi(\gamma,\tau) = -\frac{1}{\tau}\int_{\mathbb{R}^3}\xi_1\xi_2\mathfrak{C}_{12}(\xi)d\xi, \tag{5.21}$$

$$\varrho(\gamma,\tau) = -\frac{1}{4\tau^2}\int_{\mathbb{R}^3}|\xi|^2\xi_1\mathfrak{C}_1(\xi)d\xi, \tag{5.22}$$

with ξ_1, ξ_2 are the components of the vectors $\xi = (\xi_1,\xi_2,\xi_3)$ and $\mathfrak{C}_1, \mathfrak{C}_{12}$ are defined in (5.18).

Notice that

$$\mathcal{D}\mathcal{F} = \Pi\mathcal{F} + O(\epsilon^{\delta_0}).$$

The first order approximation in terms of ϵ of the quantity $\epsilon\mathbb{P}\mathcal{D}\mathcal{L}^{-1}\mathbb{P}^{\perp}\mathcal{D}\mathcal{F}$ is then $\epsilon\mathbb{P}\Pi\mathcal{L}^{-1}\mathbb{P}^{\perp}\Pi\mathcal{F}$. The Navier–Stokes system (5.13) becomes

$$\mathbb{P}\mathcal{D}\mathcal{F} = -\epsilon\mathbb{P}\Pi\mathcal{L}^{-1}\mathbb{P}^{\perp}\Pi\mathcal{F}, \tag{5.23}$$

which, thanks to the identity (5.20), leads to

$$\begin{aligned} &\partial_t n_n + \nabla_r\cdot(n_n v_n) = \epsilon\Gamma_{12}[\mathcal{F}], \\ &n_n\left(\partial_t + v_n\cdot\nabla\right)v_{nj} + \partial_{r_j}(n_n e_n) \\ &\quad = -n_n\nabla_r\epsilon^{\delta_0}U - (v_{nj} - v_{cj})\epsilon\Gamma_{12}[\mathcal{F}] \\ &\qquad + \epsilon\left[\sum_{i=1}^{3}\frac{\partial}{\partial r_i}\left(\bar{\varpi}(n_n,e_n)\left(\frac{\partial v_{nj}}{r_i} + \frac{\partial v_{ni}}{r_j}\right)\right)\right. \\ &\qquad \left. - \frac{2}{3}\frac{\partial}{\partial r_j}\left(\bar{\varpi}(n_n,e_n)\sum_{i=1}^{3}\frac{\partial v_{ni}}{r_i}\right)\right], \\ &\partial_t e_n + \nabla_r\cdot(e_n v_n) + \frac{2}{3}e_n\nabla_r\cdot v_n \\ &\quad = \frac{1}{n_n}\left[\frac{2}{3}\left(\frac{(v_n - v_c)^2}{2} + \mu_c - \epsilon^{\delta_0}U + e_n\right)\epsilon\Gamma_{12}[\mathcal{F}]\right] \\ &\qquad + \frac{\epsilon}{\mathcal{G}(n_n,e_n)}\left[\sum_{i=1}^{3}\frac{\partial}{\partial r_i}\left(\varrho_1(n_n,e_n)\frac{\partial e_n}{\partial r_i} + \varrho_2(n_n,e_n)\frac{\partial n_n}{\partial r_i}\right)\right. \\ &\qquad + \bar{\varpi}(n_n,e_n)\sum_{i,k=1}^{3}\frac{\partial v_{nk}}{\partial x_i}\left(\frac{\partial v_{nk}}{\partial x_i} + \frac{\partial v_{ni}}{\partial x_k}\right) \\ &\qquad \left. - \frac{2}{3}\varpi(n_n,e_n)\left(\sum_{i=1}^{3}\frac{\partial v_{ni}}{\partial x_i}\right)^2\right], \end{aligned} \tag{5.24}$$

where

$$\begin{aligned} \bar{\varpi}(n_n,e_n) &= \varpi(\gamma,\tau), \\ \mathcal{G}(n_n,e_n) &= 2^{5/2}\pi\tau^{3/2}\gamma\Omega_1(\gamma), \\ \varrho_1(n_n,e_n) &= \varrho(\gamma,\tau)\frac{\partial\tau}{\partial e_n}, \\ \varrho_2(n_n,e_n) &= \varrho(\gamma,\tau)\frac{\partial\tau}{\partial n_n}. \end{aligned} \tag{5.25}$$

Combining (2.9) and (5.24), we get the "closed system".

Moreover, the Navier–Stokes system of the excitations is very different from the Navier–Stokes system obtained from the classical Boltzmann equation (cf. [31]) in several points:

- First, in the classical Navier–Stokes system, the viscosity coefficient $\bar{\varpi}$ and the heat conduction coefficient ϱ_1 depend only on e_n. In the above quantum Boltzmann system, they depend on both e_n and n_n.
- Second, different from the classical Navier–Stokes system, the second derivatives of n_n also appear in the system.
- Third, the Navier–Stokes system of the excitations is coupled with the system of the BEC super fluid via the quantity $\epsilon\Gamma_{12}[\mathcal{F}]$, computed in (3.25).

The Navier–Stokes system for the excitations therefore has a completely different nature in comparison with the classical Navier–Stokes equation. And, hence, one could expect more complicated behaviors, that would be a subject of our future studies.

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