# Extensions of reverse volume difference inequalities

Alexander Koldobsky and Denghui Wu

**Abstract**

Volume difference inequalities are designed to estimate the difference between volumes of two bodies in terms of the maximal or minimal difference between areas of sections of these bodies. In this note we extend two such inequalities established in [11] and [4] from the hyperplane case to the case of sections of arbitrary dimensions.

**MSC (2010):** 52A20.

**Keywords:** sections, convex bodies, intersection bodies, Fourier transform.

## 1 Introduction

The following volume difference inequality was proved in [10] for $k = 1$, and in [13] for $1 < k < n$. Let $1 \leq k < n$, let $K$ be a generalized $k$-intersection body in $\mathbb{R}^n$ (we write $K \in \mathcal{BP}_k^n$; see definition in Section 2), and let $L$ be any origin-symmetric star body $L$ in $\mathbb{R}^n$ so that

$$\max_{F \in G_{n,n-k}} (|K \cap F| - |L \cap F|) > 0,$$

where $G_{n,n-k}$ is the Grassmannian of $(n-k)$-dimensional subspaces of $\mathbb{R}^n$, and $|\cdot|$ stands for volume of appropriate dimension. Then

$$|K|^{\frac{n-k}{n}} - |L|^{\frac{n-k}{n}} \leq c_{n,k}^k \max_{F \in G_{n,n-k}} (|K \cap F| - |L \cap F|)\,, \tag{1.1}$$

where

$$c_{n,k}^k = \frac{\omega_n^{\frac{n-k}{n}}}{\omega_{n-k}},$$

and $\omega_n = |B_2^n|$ is the volume of the unit Euclidean ball in $\mathbb{R}^n$. Note that $c_{n,k} \in (1/\sqrt{e}, 1)$; see, for example, [12, Lemma 2.1].

A volume difference inequality of a different kind was proved in [3] for $k = 1$, and in [4] for $1 \leq k < n$. Let $1 \leq k < n$, and let $L$ be a generalized $k$-intersection body in $\mathbb{R}^n$. Then, for any origin-symmetric star body $K$ in $\mathbb{R}^n$ such that $|L \cap F| \leq |K \cap F|$ for all $F \in G_{n,n-k}$,

$$|K|^{\frac{n-k}{n}} - |L|^{\frac{n-k}{n}} \geq c^k \frac{1}{(\sqrt{n} M(\bar{L}))^k} \min_{F \in G_{n,n-k}} (|K \cap F| - |L \cap F|)\,, \tag{1.2}$$

where $c > 0$ is an absolute constant, $\bar{L} = L/|L|^{1/n}$, $M(L) = \int_{S^{n-1}} ||\theta|| d\sigma(\theta)$, and $\sigma$ is the normalized Lebesgue measure on the sphere. Note that when $L$ is convex, has volume 1 and is in the minimal mean width position, then we have

$$\frac{1}{M(L)} \geq c \frac{\sqrt{n}}{\log(e+n)}, \tag{1.3}$$

where $c$ is an absolute constant; see [1].

Volume difference inequalities estimate the error in computations of volume of a body out of areas of its sections. They are closely related to Bourgain's slicing problem and to the Busemann-Petty problem; see [4] for details. We also refer to [4] for different extensions of (1.1) and (1.2) to arbitrary star and convex bodies and to arbitrary measures in place of volume. The paper [4] also provides volume difference inequalities for projections.

In the hyperplane case ($k = 1$), inequalities going in the opposite directions to (1.1) and (1.2) were proved in [4, 11]. It was shown in [4, Theorem 1.9] that for any $n \geq 5$ there exist origin-symmetric convex bodies $K, L$ in $\mathbb{R}^n$ such that $L \subset K$ and

$$|K|^{\frac{n-1}{n}} - |L|^{\frac{n-1}{n}} > c \frac{1}{\sqrt{n} M(\bar{L})} \max_{\xi \in S^{n-1}} \left( |K \cap \xi^\perp| - |L \cap \xi^\perp| \right), \tag{1.4}$$

where $c > 0$ is an absolute constant, and $\xi^\perp$ is the central hyperplane in $\mathbb{R}^n$ perpendicular to $\xi$. The result in [11, Section 6] shows that for $n \geq 5$ there exist origin-symmetric convex bodies $K, L$ in $\mathbb{R}^n$ such that $L \subset K$ and

$$|K|^{\frac{n-1}{n}} - |L|^{\frac{n-1}{n}} < c_{n,1} \min_{\xi \in S^{n-1}} \left( |K \cap \xi^\perp| - |L \cap \xi^\perp| \right). \tag{1.5}$$

We call (1.5) and (1.4) reverse volume difference inequalities. Combined with (1.3), these inequalities show that in the case $k = 1$ volume difference inequalities (1.1) and (1.2) are optimal up to a logarithmic term. In this note we extend the reverse volume difference inequalities to sections of codimensions $1 < k < n - 3$.

## 2 Preliminaries

Let $i \in \mathbb{N} \cup \{0\}$. Let $C^i(S^{n-1})$ be the space of all $i$-smooth functions on the unit sphere $S^{n-1}$, and let $C^i_e(S^{n-1})$ be the subspace of even functions in $C^i(S^{n-1})$. We denote by $C^i(G_{n,n-k})$ the space of all $i$-smooth functions on the Grassmann manifold $G_{n,n-k}$ of $(n-k)$-dimensional subspaces of $\mathbb{R}^n$.

A compact set $K \subset \mathbb{R}^n$ is called *star-shaped* if it contains the origin as its interior point, and $\alpha K \subset K$ for any $\alpha \in (0, 1)$. We say that $K \subset \mathbb{R}^n$ is a *star body* if it is star-shaped and its *Minkowski functional*

$$||x||_K := \min\{a \geq 0 : x \in aK\}$$

is a continuous function on $\mathbb{R}^n$. If $K$ is an origin-symmetric convex body, then the Minkowski functional $|| \cdot ||_K$ is a norm on $\mathbb{R}^n$. We say that a star body $K$ is *i-smooth*,

$i \in \mathbb{N} \cup \{0\}$, if the restriction of $||\cdot||_K$ to the unit sphere $S^{n-1}$ belongs to the class $C^i(S^{n-1})$. If $||\cdot||_K \in C^i(S^{n-1})$ for all $k \in \mathbb{N}$, then the body $K$ is called *infinitely smooth.*

The *radial function* of a star body $K$ is defined by

$$\rho_K(x) = ||x||_K^{-1}, \quad \text{for} \quad x \in \mathbb{R}^n, x \neq 0.$$

If $x \in S^{n-1}$, then $\rho_K(x)$ is the radius of $K$ in the direction of $x$.

Using polar coordinates, one can write volume of a star body $K$ as

$$|K| = \frac{1}{n} \int_{S^{n-1}} ||\theta||_K^{-n} d\theta, \tag{2.1}$$

where $d\theta$ denotes the uniform measure on the sphere with density 1.

The main tool used in this paper is the Fourier transform of distribution; we refer the reader to [2] for details. Denote by $\mathcal{S}(\mathbb{R}^n)$ the *Schwartz space* of rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$ (also referred to as *test functions*), and by $\mathcal{S}'(\mathbb{R}^n)$ the space of *distributions* on $\mathbb{R}^n$, the space of continuous linear functionals on $\mathcal{S}(\mathbb{R}^n)$. The Fourier transform $\hat{f}$ of a distribution $f$ is defined by $\langle \hat{f}, \varphi \rangle = \langle f, \hat{\varphi} \rangle$ for every test function $\varphi$. The Fourier transform is self-invertible up to a constant factor in the sense that $(\varphi^\wedge)^\wedge = (2\pi)^n \varphi$ for any even test function $\varphi$,

A distribution $f$ on $\mathbb{R}^n$ is called *even homogeneous of degree* $p \in \mathbb{R}$, if

$$\left\langle f(x), \varphi\left(\frac{x}{\alpha}\right) \right\rangle = |\alpha|^{n+p} \langle f, \varphi \rangle$$

for every test function $\varphi$ and every $\alpha \in \mathbb{R}, \alpha \neq 0$. The Fourier transform of an even homogeneous distribution of degree $p$ is an even homogeneous distribution of degree $-n-p$. We call a distribution $f$ *positive definite*, if for every test function

$$\langle f(x), \varphi * \overline{\varphi(-x)} \rangle \geq 0.$$

By Schwartz's generalization of Bochner's theorem, a distribution $f$ is positive definite if and only if $\hat{f}$ is a positive distribution, i.e. $\langle \hat{f}, \varphi \rangle \geq 0$ for every non-negative test function $\varphi$ (see [9, Section 2.5] for details).

For an infinitely smooth star body $K$, the Fourier transform in the sense of distributions of $\|x\|_K^p$, $-n < p < 0$ is an infinitely smooth function $g$ on $S^{n-1}$ extended to a homogeneous function of the order $-n-p$ on the whole of $\mathbb{R}^n$; see [9, Lemma 3.16]. When we write $(\|x\|_K^p)^\wedge(\theta)$, we mean $g(\theta)$, for $\theta \in S^{n-1}$.

We use the following version of Parseval's formula on the sphere.

**Proposition 2.1.** *([9, Theorem 3.22]) Let $K$ and $L$ be two infinitely smooth origin-symmetric star bodies in $\mathbb{R}^n$ and $0 < p < n$. Then*

$$\int_{S^{n-1}} \left(||\cdot||_K^{-p}\right)^\wedge(\theta) \left(||\cdot||_L^{-n+p}\right)^\wedge(\theta) d\theta = (2\pi)^n \int_{S^{n-1}} ||\theta||_K^{-p} ||\theta||_L^{-n+p} d\theta. \tag{2.2}$$

We also use a connection between the Fourier and Radon transforms.

**Proposition 2.2.** *([9, Theorem 3.24]) Let $1 \le k < n$, let $\phi$ be an even continuous integrable function on $\mathbb{R}^n$, and let $H$ be an $(n-k)$-dimensional subspace of $\mathbb{R}^n$. Suppose that $\phi$ is integrable on all translations of $H$ and that the Fourier transform $\hat{\phi}$ is integrable on $H^\perp$. Then*

$$(2\pi)^k \int_H \phi(x)dx = \int_{H^\perp} \hat{\phi}(x)dx. \tag{2.3}$$

We also need formulas expressing volume of central sections of star bodies in terms of the Fourier transform. It was proved in [7] (see also [9, Lemma 3.8]) that for every origin-symmetric star body $K$ and for every $\xi \in S^{n-1}$

$$|K \cap \xi^\perp| = \frac{1}{\pi(n-1)}(\|\cdot\|_K^{-n+1})^\wedge(\xi).$$

A lower dimensional version of the latter formula is as follows.

**Proposition 2.3.** *([9, Theorem 3.25]) Let $1 \le k < n$, and let $K$ be an origin-symmetric star body in $\mathbb{R}^n$ such that $K$ is $(k-1)$-smooth if $k$ is odd and $k$-smooth if $k$ is even. Suppose that $H$ is an $(n-k)$-dimensional subspace of $\mathbb{R}^n$. Then*

$$|K \cap H| = \frac{1}{(2\pi)^k(n-k)} \int_{S^{n-1}\cap H^\perp} (\|\cdot\|_K^{-n+k})^\wedge(\theta)d\theta. \tag{2.4}$$

In particular, if $K = B_2^n$ is the unit Euclidean ball in $\mathbb{R}^n$, then for every $H \in G_{n,n-k}$ and every $\xi \in S^{n-1}$,

$$|B_2^n \cap H| = \frac{k\omega_k}{(2\pi)^k(n-k)}(\|\cdot\|_2^{-n+k})^\wedge(\xi), \tag{2.5}$$

where

$$\omega_n = |B_2^n| = \frac{\pi^{n/2}}{\Gamma(1+\frac{n}{2})}.$$

The concept of an intersection body was introduced by Lutwak [14] (for $k = 1$) and generalized to $k > 1$ by Zhang [15]. We formulate the definition in an equivalent form established by Goodey-Weil [5] and Grinberg-Zhang [6]. Let $1 \le k \le n-1$. The class $\mathcal{BP}_k^n$ of *generalized $k$-intersection bodies* in $\mathbb{R}^n$ is the closure in the radial metric of radial $k$-sums of finite collections of origin-symmetric ellipsoids in $\mathbb{R}^n$. When $k = 1$, $\mathcal{BP}_1^n = \mathcal{I}_n$ is the original class of intersection bodies introduced by Lutwak.

Recall that the radial $k$-sum of star bodies $K$ and $L$ in $\mathbb{R}^n$ is a new star body $K +_k L$ whose radius in every direction $\xi \in S^{n-1}$ is given by

$$\rho_{K+_kL}^k(\xi) = \rho_K^k(\xi) + \rho_L^k(\xi), \qquad \forall \xi \in S^{n-1}.$$

The radial metric in the class of origin-symmetric star bodies is defined by

$$\rho(K, L) = \sup_{\xi \in S^{n-1}} |\rho_K(\xi) - \rho_L(\xi)|.$$

Another generalization of the concept of an intersection body was introduced in [8]. For an integer $k$, $1 \le k < n$ and star bodies $D, L$ in $\mathbb{R}^n$, we say that $D$ is the $k$-intersection body of $L$ if

$$|D \cap H^\perp| = |L \cap H|, \qquad \forall H \in Gr_{n-k}.$$

Taking the closure in the radial metric of the class of $k$-intersection bodies of star bodies, we define the class of $k$-intersection bodies $\mathcal{I}_k^n$. If $k = 1$, we get the original class of intersection bodies $\mathcal{I}_n$.

The following Fourier analytic characterization of $k$-intersection bodies was proved in [8].

**Proposition 2.4.** *([9, Theorem 4.8]) Let $1 \le k < n$. An origin-symmetric star body $D$ in $\mathbb{R}^n$ is a $k$-intersection body if and only if the function $||\cdot||^{-k}$ represents a positive definite distribution on $\mathbb{R}^n$.*

By [9, Theorem 4.13, Lemma 4.10], for every $1 \le k < n-3$ there exist infinitely smooth convex bodies with positive curvature in $\mathbb{R}^n$ which are not $k$-intersection bodies. We will use these bodies to prove the reverse volume difference inequalities for $1 \le k < n-3$. Note that by [9, Corollary 4.9] every origin-symmetric convex body in $\mathbb{R}^n$ is a $k$-intersection body for $k = n-3, n-2$. The authors do not know whether it is possible to extend the reverse inequalities to these values of $k$.

# 3 Main results

The following theorem extends the reverse volume difference inequality (1.5) to the case of lower dimensional sections.

**Theorem 3.1.** *Let $1 \le k < n-3$, and let $K$ be an infinitely smooth origin-symmetric convex body in $\mathbb{R}^n$, with strictly positive curvature, that is not a $k$-intersection body. Then there exists an origin-symmetric convex body $L$ in $\mathbb{R}^n$ such that $L \subset K$ and*

$$|K|^{\frac{n-k}{n}} - |L|^{\frac{n-k}{n}} < c_{n,k}^k \min_{H \in G_{n,n-k}} \big(|K \cap H| - |L \cap H|\big).$$

*Proof.* Since $K$ is an infinitely smooth convex body, $||\cdot||_K^{-k}$ is infinitely smooth. Thus $(||\cdot||_K^{-k})^\wedge$ is a continuous function on $\mathbb{R}^n \setminus \{0\}$. Since $K$ is not a $k$-intersection body, by Proposition 2.4, there exists $\xi \in S^{n-1}$ such that $(||\cdot||_K^{-k})^\wedge(\xi) < 0$. By continuity of $(||\cdot||_K^{-k})^\wedge$, there exists a small neighborhood $\Omega$ of $\xi$ where $(||\cdot||_K^{-k})^\wedge$ is still negative. We choose $\Omega$ smaller (if necessary) so that there exists a $(n-k)$-dimensional subspace $H_0$ such that $S^{n-1} \cap H_0^\perp \subset S^{n-1} \setminus \Omega$.

Let $\phi \in C^\infty(S^{n-1})$ be an even non-negative function on $S^{n-1}$ with support in $\Omega \cup -\Omega$. Extend $\phi$ to an even homogeneous function $\phi \cdot r^{-k}$ of degree $-k$ on $\mathbb{R}^n$. The Fourier transform of the distribution $\phi \cdot r^{-k}$ is a homogeneous of degree $-n+k$ function $\psi \cdot r^{-n+k}$, where $\psi$ is an infinitely smooth function on the sphere $S^{n-1}$.

Let $\varepsilon$ be a positive number such that $|B_2^{n-k}|\cdot||\theta||_K^{-n+k} > \varepsilon > 0$ for all $\theta \in S^{n-1}$. Define a star body $L$ by

$$||\theta||_L^{-n+k} = ||\theta||_K^{-n+k} - \delta\psi(\theta) - \frac{\varepsilon}{|B_2^{n-k}|}, \quad \forall\theta \in S^{n-1}. \tag{3.1}$$

Select $\delta$ small enough so that for every $\theta$

$$|\delta\psi(\theta)| < \min\left\{||\theta||_K^{-n+k} - \frac{\varepsilon}{|B_2^{n-k}|},\ \frac{\varepsilon}{|B_2^{n-k}|}\right\}.$$

This condition implies that $L$ contains the origin and that $L \subset K$. Since $L$ has strictly positive curvature, by the argument from [9, p.96], we can select $\varepsilon, \delta$ even smaller (if necessary) to ensure that the body $L$ is convex.

Now we extend the functions in both sides of (3.1) to even homogeneous functions of degree $-n+k$ on $\mathbb{R}^n$ and apply the Fourier transform. We have that for every $\xi \in S^{n-1}$

$$(||\cdot||_L^{-n+k})^\wedge(\xi) = (||\cdot||_K^{-n+k})^\wedge(\xi) - (2\pi)^n\delta\phi(\xi) - \frac{\varepsilon}{|B_2^{n-k}|}(||\cdot||_2^{-n+k})^\wedge(\xi).$$

It follows from (2.5) that for every $\xi \in S^{n-1}$

$$(||\cdot||_L^{-n+k})^\wedge(\xi) = (||\cdot||_K^{-n+k})^\wedge(\xi) - (2\pi)^n\delta\phi(\xi) - \frac{(2\pi)^k(n-k)}{k\omega_k}\varepsilon. \tag{3.2}$$

For every $H \in G_{n,n-k}$ we integrate (3.2) over $S^{n-1}\cap H^\perp$ and get

$$\begin{aligned}\int_{S^{n-1}\cap H^\perp}(||\cdot||_L^{-n+k})^\wedge(\xi)d\xi = &\int_{S^{n-1}\cap H^\perp}(||\cdot||_K^{-n+k})^\wedge(\xi)d\xi - (2\pi)^n\delta\int_{S^{n-1}\cap H^\perp}\phi(\xi)d\xi \\ &- (2\pi)^k(n-k)\varepsilon.\end{aligned} \tag{3.3}$$

Since $S^{n-1}\cap H_0^\perp \subset S^{n-1}\setminus\Omega$, we have $\phi(\xi) = 0$ for all $\xi \in H_0^\perp$. By (2.4), we get

$$\varepsilon = \min_{H\in G_{n,n-k}}(|K\cap H| - |L\cap H|). \tag{3.4}$$

Multiplying both sides of (3.2) by $(||\cdot||_K^{-k})^\wedge$, integrating over the sphere $S^{n-1}$ and using Parseval's formula on the sphere, we get

$$\begin{aligned}(2\pi)^n\int_{S^{n-1}}||\theta||_K^{-k}||\theta||_L^{-n+k}d\theta =&(2\pi)^n n|K| - (2\pi)^n\delta\int_{S^{n-1}}\phi(\theta)(||\cdot||_K^{-k})^\wedge(\theta)d\theta \\ &- \frac{(2\pi)^k(n-k)}{k\omega_k}\varepsilon\int_{S^{n-1}}(||\cdot||_K^{-k})^\wedge(\theta)d\theta.\end{aligned}$$

Since $\phi$ is a non-negative function supported in $\Omega\cup-\Omega$, where $(||\cdot||_K^{-k})^\wedge$ is negative, the above inequality implies that

$$(2\pi)^n\int_{S^{n-1}}||\theta||_K^{-k}||\theta||_L^{-n+k}d\theta > (2\pi)^n n|K| - \frac{(2\pi)^k(n-k)}{k\omega_k}\varepsilon\int_{S^{n-1}}(||\cdot||_K^{-k})^\wedge(\theta)d\theta. \tag{3.5}$$

By Hölder's inequality,

$$
\begin{aligned}
(2\pi)^n \int_{S^{n-1}} ||\theta||_K^{-k} ||\theta||_L^{-n+k} d\theta \leq & (2\pi)^n \left( \int_{S^{n-1}} ||\theta||_K^{-n} d\theta \right)^{\frac{k}{n}} \left( \int_{S^{n-1}} ||\theta||_L^{-n} d\theta \right)^{\frac{n-k}{n}} \\
= & (2\pi)^n n |K|^{\frac{k}{n}} |L|^{\frac{n-k}{n}}. \qquad (3.6)
\end{aligned}
$$

By (2.5), Parseval's formula on the sphere and Hölder's inequality,

$$
\begin{aligned}
\frac{(2\pi)^k (n-k)}{k\omega_k} \int_{S^{n-1}} (|| \cdot ||_K^{-k})^\wedge(\theta) d\theta = & \frac{1}{|B_2^{n-k}|} \int_{S^{n-1}} (|| \cdot ||_K^{-k})^\wedge(\theta) (|| \cdot ||_2^{-n+k})^\wedge(\xi) d\theta \\
= & \frac{(2\pi)^n}{|B_2^{n-k}|} \int_{S^{n-1}} ||\theta||_K^{-k} d\theta \\
\leq & \frac{(2\pi)^n}{|B_2^{n-k}|} |S^{n-1}|^{\frac{n-k}{n}} \left( \int_{S^{n-1}} ||\theta||_K^{-n} d\sigma(\theta) \right)^{\frac{k}{n}} \\
= & (2\pi)^n n c_{n,k}^k |K|^{\frac{k}{n}}. \qquad (3.7)
\end{aligned}
$$

Combining (3.5), (3.6) and (3.7), we have

$$
(2\pi)^n n |K| < (2\pi)^n n |K|^{\frac{k}{n}} |L|^{\frac{n-k}{n}} + (2\pi)^n n c_{n,k}^k |K|^{\frac{k}{n}} \varepsilon.
$$

This, together with (3.4), implies the result. □

The following result extends (1.4) to sections of lower dimensions.

**Theorem 3.2.** *Let $1 \leq k < n$, and let $L$ be an infinitely smooth origin-symmetric convex body in $\mathbb{R}^n$, with strictly positive curvature, that is not a $k$-intersection body. Then there exists an origin-symmetric convex body $K$ in $\mathbb{R}^n$ such that $L \subset K$ and*

$$
|K|^{\frac{n-k}{n}} - |L|^{\frac{n-k}{n}} > \frac{C^k}{\left(\sqrt{n} M(\bar{L})\right)^k} \max_{H \in G_{n,n-k}} \big( |K \cap H| - |L \cap H| \big),
$$

*where $C > 0$ is an absolute constant.*

*Proof.* Since $L$ is an infinitely smooth convex body, $|| \cdot ||_L^{-k}$ is infinitely smooth. Thus $(|| \cdot ||_L^{-k})^\wedge$ is a continuous function on $\mathbb{R}^n \setminus \{0\}$. Since $L$ is not a $k$-intersection body, by Proposition 2.4, there exists $\xi \in S^{n-1}$ such that $(|| \cdot ||_L^{-k})^\wedge(\xi) < 0$. By continuity of $(|| \cdot ||_L^{-k})^\wedge$ there is a small neighborhood $\Omega$ of $\xi$ where $(|| \cdot ||_L^{-k})^\wedge$ is negative. We choose $\Omega$ smaller (if necessary) so that there exists a $(n-k)$-dimensional subspace $H_0$ such that $S^{n-1} \cap H_0^\perp \subset S^{n-1} \setminus \Omega$.

Let $\phi \in C^\infty(S^{n-1})$ be an even non-negative function on $S^{n-1}$ with support in $\Omega \cup -\Omega$. Extend $\phi$ to an even homogeneous function $\phi \cdot r^{-1}$ of degree $-k$ on $\mathbb{R}^n$. The Fourier transform of $\phi \cdot r^{-k}$ in the sense of distribution is $\psi \cdot r^{-n+k}$ where $\psi$ is an infinitely smooth function on the sphere $S^{n-1}$.

Let $\varepsilon$ be a positive number with $\varepsilon > 0$. Define a star body $K$ by

$$||\theta||_K^{-n+k} = ||\theta||_L^{-n+k} - \delta\psi(\theta) + \frac{\varepsilon}{|B_2^{n-k}|}. \tag{3.8}$$

Choose $\delta$ small enough such that for every $\theta$

$$|\delta\psi(\theta)| < \min\left\{||\theta||_L^{-n+k} + \frac{\varepsilon}{|B_2^{n-k}|}, \frac{\varepsilon}{|B_2^{n-k}|}\right\}.$$

This condition implies that $K$ contains the origin and that $L \subset K$. Since $L$ has strictly positive curvature, we also make $\varepsilon, \delta$ smaller (if necessary) to ensure that the body $K$ is convex.

Now we extend the functions in (3.8) to even homogeneous functions of degree $-n+k$ on $\mathbb{R}^n$. Applying the Fourier transform we get for every $\xi \in S^{n-1}$

$$(||\cdot||_K^{-n+k})^\wedge(\xi) = (||\cdot||_L^{-n+k})^\wedge(\xi) - (2\pi)^n\delta\phi(\xi) + \frac{\varepsilon}{|B_2^{n-k}|}(||\cdot||_2^{-n+k})^\wedge(\xi).$$

Hence, it follows from (2.5) that for every $\xi \in S^{n-1}$

$$(||\cdot||_K^{-n+k})^\wedge(\xi) = (||\cdot||_L^{-n+k})^\wedge(\xi) - (2\pi)^n\delta\phi(\xi) + \frac{(2\pi)^k(n-k)}{k\omega_k}\varepsilon. \tag{3.9}$$

For every $H \in G_{n,n-k}$ we integrate (3.9) over $S^{n-1} \cap H^\perp$ and get

$$\begin{aligned}\int_{S^{n-1}\cap H^\perp}(||\cdot||_K^{-n+k})^\wedge(\xi)d\xi = &\int_{S^{n-1}\cap H^\perp}(||\cdot||_L^{-n+k})^\wedge(\xi)d\xi - (2\pi)^n\delta\int_{S^{n-1}\cap H^\perp}\phi(\xi)d\xi \\ &+ (2\pi)^k(n-k)\varepsilon.\end{aligned} \tag{3.10}$$

Since $S^{n-1} \cap H_0^\perp \subset S^{n-1} \setminus \Omega$, we have $\phi(\xi) = 0$ for $\xi \in H_0^\perp$. By (2.4), we have that

$$\varepsilon = \max_{H\in G_{n,n-k}}(|K \cap H| - |L \cap H|). \tag{3.11}$$

Multiplying both sides of (3.9) by $(||\cdot||_L^{-k})^\wedge$, integrating over the sphere $S^{n-1}$ and using Parseval's formula on the sphere, we get

$$\begin{aligned}(2\pi)^n\int_{S^{n-1}}||\theta||_L^{-k}||\theta||_K^{-n+k}d\theta = &(2\pi)^n n|L| - (2\pi)^n\delta\int_{S^{n-1}}\phi(\theta)(||\cdot||_L^{-k})^\wedge(\theta)d\theta \\ &+ \frac{(2\pi)^k(n-k)}{k\omega_k}\varepsilon\int_{S^{n-1}}(||\cdot||_L^{-k})^\wedge(\theta)d\theta.\end{aligned}$$

Since $\phi$ is a non-negative function with support in $\Omega \cup -\Omega$, where $(||\cdot||_L^{-k})^\wedge$ is negative, the above inequality implies that

$$(2\pi)^n\int_{S^{n-1}}||\theta||_L^{-k}||\theta||_K^{-n+k}d\theta > (2\pi)^n n|L| + \frac{(2\pi)^k(n-k)}{k\omega_k}\varepsilon\int_{S^{n-1}}(||\cdot||_L^{-k})^\wedge(\theta)d\theta. \tag{3.12}$$

By Hölder's inequality,

$$(2\pi)^n \int_{S^{n-1}} ||\theta||_L^{-k} ||\theta||_K^{-n+k} d\theta \le (2\pi)^n n |L|^{\frac{k}{n}} |K|^{\frac{n-k}{n}}. \tag{3.13}$$

By (2.5), Parseval's formula on the sphere and Jensen's inequality,

$$\begin{aligned}
\frac{(2\pi)^k (n-k)}{k\omega_k} \int_{S^{n-1}} (||\cdot||_L^{-k})^\wedge(\theta) d\theta &= \frac{1}{|B_2^{n-k}|} \int_{S^{n-1}} (||\cdot||_L^{-k})^\wedge(\theta) (||\cdot||_2^{-n+k})^\wedge(\xi) d\theta \\
&= \frac{(2\pi)^n |S^{n-1}|}{|B_2^{n-k}|} \int_{S^{n-1}} ||\theta||_L^{-k} d\sigma(\theta) \\
&\ge \frac{(2\pi)^n |S^{n-1}|}{|B_2^{n-k}|} \left( \frac{1}{\int_{S^{n-1}} ||\theta||_L d\sigma(\theta)} \right)^k \\
&= \frac{(2\pi)^n |S^{n-1}|}{|B_2^{n-k}|} \frac{|L|^{\frac{k}{n}}}{M(\bar{L})^k} \\
&\ge (2\pi)^n n C^k \frac{|L|^{\frac{k}{n}}}{\left(\sqrt{n} M(\bar{L})\right)^k}.
\end{aligned} \tag{3.14}$$

From standard estimates for the $\Gamma$-function and (3.12), (3.13), (3.14), we get

$$(2\pi)^n n |L|^{\frac{k}{n}} |K|^{\frac{n-k}{n}} > (2\pi)^n n |L| + (2\pi)^n n C^k \frac{|L|^{\frac{k}{n}}}{\left(\sqrt{n} M(\bar{L})\right)^k} \varepsilon.$$

This, together with (3.11), implies the required result. □

**Acknowledgements.** The first named author was supported in part by the US National Science Foundation grant DMS-1700036. The second named author was partially supported by Fundamental Research Funds for the Central Universities (No. XDJK2016D026) and China Scholarship Council.

# References

[1] S. Artstein, A. Giannopoulos and V. Milman, *Asymptotic geometric analysis*, Mathematical Surveys and Monographs 202, Amer. Math. Soc., Providence, RI, 2015.

[2] I. M. Gelfand, G. E. Shilov, *Generalized functions, vol. 1. Properties and operations*, Academic Press, New York, 1964.

[3] A. Giannopoulos and A. Koldobsky, *Variants of the Busemann-Petty problem and of the Shephard problem*, Int. Math. Res. Not. IMRN 2017, no.3, 921–943.

[4] A. Giannopoulos, A. Koldobsky, *Volume difference inequalities*, to appear in Trans. Amer. Math. Soc.; arXiv: 1608.03435v1.

[5] P. Goodey and W. Weil, *Intersection bodies and ellipsoids*, Mathematika **42** (1995), 295–304.

[6] E. Grinberg and Gaoyong Zhang, *Convolutions, transforms and convex bodies*, Proc. London Math. Soc. (3) **78** (1999), 77–115.

[7] A. Koldobsky, *An application of the Fourier transform to sections of star bodies*, Israel J. Math. **106** (1998), 157–164.

[8] A. Koldobsky, *A functional analytic approach to intersection bodies*, Geom. Funct. Anal. **10** (2000), 1507–1526.

[9] A. Koldobsky, *Fourier analysis in convex geometry*, Mathematical Surveys and Monographs 116, Amer. Math. Soc., Providence, RI, 2005.

[10] A. Koldobsky, *Stability in the Busemann-Petty and Shephard problems*, Adv. Math. **228** (2011), 2145–2161.

[11] A. Koldobsky, *Slicing inequalities for measures of convex bodies*, Adv. Math. **283** (2015), 473–488.

[12] A. Koldobsky and M. Lifshits, *Average volume of sections of star bodies*, Geometric Aspects of Functional Analysis, V. Milman and G. Schechtman, eds., Lecture Notes in Math. **1745** (2000), 119–146.

[13] A. Koldobsky and Dan Ma, *Stability and slicing inequalities for intersection bodies*, Geom. Dedicata **162**, (2013), 325–335.

[14] E. Lutwak, *Intersection bodies and dual mixed volumes*, Adv. Math. **71**, 2 (1988), 232–261.

[15] Gaoyong Zhang, *Sections of convex bodies*, Amer. J. Math. **118** (1996), 319–340.

Alexander Koldobsky:
Department of Mathematics, University of Missouri, Columbia, MO 65211.
E-mail: koldobskiya@missouri.edu
Denghui Wu:
School of Mathematics and Statistics, Southwest University, Chongqing, 400715, China,
Department of Mathematics, University of Missouri, Columbia, MO 65211.
E-mail: wudenghui66@163.com