On the maximal perimeter of sections of the cube

Hermann König (Kiel)*, Alexander Koldobsky **(Columbia)

Abstract

We prove that the $(n-2)$-dimensional surface area (perimeter) of central hyperplane sections of the n-dimensional unit cube is maximal for the hyperplane perpendicular to the vector $(1, 1, 0, \dots, 0)$. This gives a positive answer to a question of Pełczyński who solved the three dimensional case. We study both the real and the complex versions of this problem. We also use our result to show that the answer to an analogue of the Busemann-Petty problem for the surface area is negative in dimensions 14 and higher.

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1 Introduction, volume formulas and results

A remarkable result of Ball [B1] states that the hyperplane section of the n-cube B_∞^n perpendicular to $a_{max} := \frac{1}{\sqrt{2}}(1, 1, 0, \dots, 0)$ has the maximal $(n-1)$-dimensional volume among all hyperplane sections, i.e. for any $a \in S^{n-1} \subset \mathbb{R}^n$

$$vol_{n-1}(B_\infty^n \cap a^\perp) \le vol_{n-1}(B_\infty^n \cap a_{max}^\perp) = \sqrt{2},$$

where $a^\perp$ is the central hyperplane orthogonal to a. Oleszkiewicz and Pełczyński [OP] proved the complex analogue of this result, with the same hyperplane $a_{max}^\perp$.

Pełczyński [P] asked whether the same hyperplane section is also maximal for intersections with the *boundary* of the n-cube, i.e. whether for all $a \in S^{n-1} \subset \mathbb{R}^n$

$$vol_{n-2}(\partial B_\infty^n \cap a^\perp) \le vol_{n-2}(\partial B_\infty^n \cap a_{max}^\perp) = 2((n-2)\sqrt{2} + 1).$$

He proved it for $n = 3$ when $vol_1(\partial B_\infty^3 \cap a^\perp)$ is the *perimeter* of the quadrangle or hexagon of intersection. In this paper, we answer Pełczyński's question affirmatively for all $n \ge 3$. We also solve the complex version of this problem. For simplicity, we continue to call the quantity $vol_{n-2}(\partial B_\infty^n \cap a^\perp)$ the perimeter of the cubic section.

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Ball [B2] used his result to prove that the answer to the Busemann-Petty problem is negative in dimensions 10 and higher. The Busemann-Petty problem asks the following question. Suppose that origin-symmetric convex bodies K, L in $\mathbb{R}^n$ satisfy

$$vol_{n-1}(K \cap a^\perp) \leq vol_{n-1}(L \cap a^\perp)$$

for all $a \in S^{n-1}$. Does it follow that the n-dimensional volume of K is smaller than that of L, i.e. $vol_n K \leq vol_n L$? The problem was solved as the result of work of many mathematicians, and the answer is affirmative for $n \leq 4$, and it is negative for $n \geq 5$; see [K] for details. Ball's result was one of the steps of the solution. He showed that the answer is negative when $n \geq 10$, K is the unit cube and L is the Euclidean ball of certain size in $\mathbb{R}^n$.

We consider the following analogue of the Busemann-Petty problem for the surface area. Suppose that $n \geq 3$, and origin-symmetric convex bodies K, L in $\mathbb{R}^n$ satisfy

$$vol_{n-2}(\partial K \cap a^\perp) \leq vol_{n-2}(\partial L \cap a^\perp)$$

for all $a \in S^{n-1}$, i.e. the surface area (perimeter) of every central hyperplane section of K is smaller than the same for L. Does it follow that the surface area of K is smaller than that of L, i.e. $vol_{n-1}(\partial K) \leq vol_{n-1}(\partial L)$? We prove in Section 4 that the answer is negative for $n \geq 14$ and higher, when K is the unit cube and L is the Euclidean ball of appropriate size in $\mathbb{R}^n$.

To formulate our results precisely, let us introduce the following notations. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $\alpha = \frac{1}{2}$ for $\mathbb{K} = \mathbb{R}$ and $\alpha = \frac{1}{\sqrt{\pi}}$ for $\mathbb{K} = \mathbb{C}$. Let $|| \cdot ||_\infty$ and $| \cdot |$ denote the maximum and the Euclidean norm on $\mathbb{K}^n$, respectively. Then

$$B_\infty^n := \{x \in \mathbb{K}^n \mid ||x||_\infty \leq \alpha\}$$

is the n-cube of volume 1 in $\mathbb{K}^n$. For $\mathbb{K} = \mathbb{C}$, we follow the usual definition of volume by identifying $\mathbb{K}^n$ with $\mathbb{R}^{2n}$, i.e. we consider vol_{2n} and vol_{2n-2} for the polydisc B_∞^n and its complex hyperplane sections, respectively. For $a \in \mathbb{K}^n$ with $|a| = 1$ and $t \in \mathbb{K}$, the *parallel section function* A is defined by

$$A_{l(n-1)}(a, t) := vol_{l(n-1)}(B_\infty^n \cap (a^\perp + \alpha t a)),$$

where $l = 1$ if $\mathbb{K} = \mathbb{R}$ and $l = 2$ if $\mathbb{K} = \mathbb{C}$ and $a^\perp := \{x \in \mathbb{K}^n \mid \langle x, a \rangle = 0\}$. This gives the volume of the hyperplane section of the n-cube perpendicular to a and at distance αt to the origin. We put $A_{l(n-1)}(a) = A_{l(n-1)}(a, 0)$. Then Ball's result and Oleszkiewicz and Pełczyński's complex analogue state that for all $a \in \mathbb{K}^n$ with $|a| = 1$ we have

$$A_{l(n-1)}(a) \leq A_{l(n-1)}(a_{max}) = (\sqrt{2})^l.$$

The lower bound $1 = A_{l(n-1)}(a_{min}) \leq A_{l(n-1)}(a)$, $a_{min} = (1, 0, \cdots, 0)$, was shown earlier by Hensley [H].

For $a \in \mathbb{K}^n$ with $|a| = 1$ we define the *perimeter* of the cubic section by $a^\perp$ as

$$P_{l(n-2)}(a) := vol_{l(n-2)}(\partial B_\infty^n \cap a^\perp),$$

with l as before. The main result of this paper answers Pełczyński's problem affirmatively:

Theorem 1. *Let $n \geq 3$ and $a_{max} := \frac{1}{\sqrt{2}}(1,1,0,\cdots,0) \in \mathbb{K}^n$. Then for any $a \in \mathbb{K}^n$ with $|a| = 1$ we have*

$$P_{l(n-2)}(a) \leq P_{l(n-2)}(a_{max}), \tag{1}$$

where $l = 1$ if $\mathbb{K} = \mathbb{R}$ and $l = 2$ if $\mathbb{K} = \mathbb{C}$. We have

$$P_{n-2}(a_{max}) = 2((n-2)\sqrt{2}+1) \quad , \quad \mathbb{K} = \mathbb{R}$$

and

$$P_{2(n-2)}(a_{max}) = 2\pi((n-2)2+1) \quad , \quad \mathbb{K} = \mathbb{C}.$$

For $a \in \mathbb{K}^n$ with $|a| = 1$ let $a^\star$ denote the non-increasing rearrangement of the sequence $(|a_k|)_{k=1}^n$. Since the volume is invariant under coordinate permutations and sign changes, which in the complex case means rotations of coordinate discs, we have

$$A_{l(n-1)}(a,t) = A_{l(n-1)}(a^\star, |t|)$$

and

$$P_{l(n-2)}(a) = P_{l(n-2)}(a^\star).$$

Therefore, we will *generally* assume in this paper that $a = (a_k)_{k=1}^n$ satisfies $a_1 \geq \cdots \geq a_n \geq 0$ and $|a| = 1$ as well as $t \geq 0$. For the parallel section function, the following formulas hold

$$A_{n-1}(a,t) = \frac{2}{\pi} \int_0^\infty \prod_{k=1}^n \frac{\sin(a_k s)}{a_k s} \cos(ts)\ ds \quad , \quad \mathbb{K} = \mathbb{R}, \tag{2}$$

$$A_{2(n-1)}(a,t) = \frac{1}{2} \int_0^\infty \prod_{k=1}^n j_1(a_k s)\ J_0(ts)\ s\ ds \quad , \quad \mathbb{K} = \mathbb{C}, \tag{3}$$

where $j_1(t) = 2\frac{J_1(t)}{t}$ and J_ν denote the Bessel functions of index ν. If $a_k = 0$, $\frac{\sin(a_k s)}{a_k s}$ and $j_1(a_k s)$ have to be read as 1 in formulas (2) and (3). Formula (2) was shown already by Pólya [Po] in 1913, and was used by Ball [B1] in the proof of his result. Both formulas can be proven by taking the Fourier transform of $A_{l(n-1)}(a,\cdot)$, using Fubini's theorem and taking the inverse Fourier transform, cf. e.g. Koldobsky, Theorem 3.1 [K] or König, Koldobsky [KK1] and [KK2]. The $\frac{\sin t}{t}$ and $j_1(t)$ functions occur as Fourier transforms of the interval in $\mathbb{R}$ and the disc in $\mathbb{C} = \mathbb{R}^2$, respectively. For the complex case cf. also Oleszkiewicz, Pełczyński [OP]. To prove Theorem 1, we use the following formulas for the perimeter.

Proposition 2. *For any $a = (a_k)_{k=1}^n \in S^{n-1} \subset \mathbb{R}^n$*

$$P_{n-2}(a) = 2 \sum_{k=1}^n \sqrt{1-a_k^2}\ \frac{2}{\pi} \int_0^\infty \prod_{j=1, j\neq k}^n \frac{\sin(a_j s)}{a_j s}\ \cos(a_k s)\ ds \quad , \quad \mathbb{K} = \mathbb{R}, \tag{4}$$

$$P_{2(n-2)}(a) = 2\pi \sum_{k=1}^n (1-a_k^2)\ \frac{1}{2} \int_0^\infty \prod_{j=1, j\neq k}^n j_1(a_j s)\ J_0(a_k s)\ s\ ds \quad , \quad \mathbb{K} = \mathbb{C}. \tag{5}$$

In Ball's result, the integral in (2) for $t = 0$ is estimated by using Hölder's inequality if $a_1 \le \frac{1}{\sqrt{2}}$, which is natural since in the extremal case ($a_1 = a_2 = \frac{1}{\sqrt{2}}, a_j = 0, j > 3$) the integrand is non-negative. In (4) and (5) we have weighted sums of integrals where the integrands are non-positive in the extremal case. Therefore, estimating $P_{l(n-2)(a)}$ requires further methods in addition to Ball's techniques and inequalities or those of Oleszkiewicz and Pełczyński. One idea is to consider the perimeter estimate as a constrained optimization problem, in view of the following two results. We continue to denote $l = 1$ if $\mathbb{K} = \mathbb{R}$ and $l = 2$ if $\mathbb{K} = \mathbb{C}$.

Proposition 3. *For any $a = (a_k)_{k=1}^n \in S^{n-1} \subset \mathbb{R}^n$ and $k \in \{1, \cdots, n\}$, define*

$$D_k(a) := \left\{ \begin{array}{lcl} \frac{2}{\pi} \int_0^\infty \prod_{j=1, j\neq k}^n \frac{\sin(a_j s)}{a_j s} \ \cos(a_k s) \ ds & , & \mathbb{K} = \mathbb{R} \\ \frac{1}{2} \int_0^\infty \prod_{j=1, j\neq k}^n j_1(a_j s) \ J_0(a_k s) \ s \ ds & , & \mathbb{K} = \mathbb{C} \end{array} \right\}. \quad (6)$$

Then

$$\sum_{k=1}^n D_k(a) = (n-1) \ A_{l(n-1)}(a). \quad (7)$$

Proposition 4. *For any $a = (a_k)_{k=1}^n \in S^{n-1} \subset \mathbb{R}^n$ and $k \in \{1, \cdots, n\}$,*

$$D_k(a) \le A_{l(n-1)}(a). \quad (8)$$

The proof of Proposition 4 also yields the following estimate for the parallel section function

Corollary 5. *For any $a \in \mathbb{K}^n$ with $|a| = 1$ and $t > 0$ we have*

$$A_{l(n-1)}(a,t) \le \left(\frac{2}{1+t^2} \right)^{l/2}.$$

Ball's proof relies on the non-trivial estimate $f(p) \le f(2) = 1$ for the function

$$f(p) := \sqrt{\frac{p}{2}} \ \frac{2}{\pi} \ \int_0^\infty \left| \frac{\sin(t)}{t} \right|^p \ dt \ ,$$

since then in the real case for all $0 < a_n \le \cdots \le a_1 \le \frac{1}{\sqrt{2}}$ with $\sum_{k=1}^n a_k^2 = 1$ we find by using Hölder's inequality with $p_k := a_k^{-2} \ge 2$

$$\begin{aligned} A_{n-1}(a) &\le \prod_{k=1}^n \left(\frac{2}{\pi} \int_0^\infty \left| \frac{\sin(a_k s)}{a_k s} \right|^{a_k^{-2}} \ ds \right)^{a_k^2} \\ &= \left(\prod_{k=1}^n f(a_k^{-2}) \right)^{a_k^2} \sqrt{2} \le \sqrt{2}. \end{aligned} \quad (9)$$

The constrained approximation approach suffices to prove Theorem 1, except when, in the real case, a_1 is in a small interval around $\frac{1}{\sqrt{2}}$. To prove Theorem 1 also in this case, we need additional information on the function f:

Proposition 6. *Define* $f : (1, \infty) \to \mathbb{R}_+$ *by*

$$f(p) := \sqrt{\frac{p}{2}}\ \frac{2}{\pi} \int_0^\infty \left| \frac{\sin(t)}{t} \right|^p \, dt.$$

Then
(a) $\lim_{p\to\infty} f(p) = \sqrt{\frac{3}{\pi}}$ *and* $f(\frac{9}{4}) < \sqrt{\frac{3}{\pi}}$.
(b) $f(\sqrt{2} + \frac{1}{2}) < \frac{51}{50}$.
(c) $f|_{[\sqrt{2}+\frac{1}{2}, \frac{9}{4}]}$ *is decreasing and convex.*

Proposition 7. *For all* $p \geq \frac{9}{4}$, $f(p) \leq \sqrt{\frac{3}{\pi}}$.

Using the convexity of f and the estimates for $f(p)$ for $p = \sqrt{2}+1/2$ and $p = \frac{9}{4}$, we may improve the general estimate (9) for certain sequences a with a_1 close to $\frac{1}{\sqrt{2}}$, which will essentially suffice to prove Theorem 1 in these cases. This works since

$$\sqrt{\frac{3}{\pi}} = \lim_{p\to\infty} f(p) < f(2) = 1,$$

i.e. f has strictly smaller values near ∞ than at 2. In the complex case, the function f is replaced by

$$\tilde{f}(p) := \frac{p}{2}\frac{1}{2} \int_0^\infty |j_1(s)|^p s \, ds \ ,$$

where also $\tilde{f}(p) \leq \tilde{f}(2) = 1$ for all $p \geq 2$, cf. Oleszkiewicz, Pełczyński [OP]. However, in this case $\lim_{p\to\infty} \tilde{f}(p) = \tilde{f}(2) = 1$. Therefore, no analogue of Proposition 6 (a), (b) and Proposition 7 is possible in the complex case. Fortunately, in the complex case, the perimeter formula given by (5) is easier to estimate since it does not contain a square root in the weights of the integrals, and the constraint technique works for all sequences a.

We quickly outline the strategy of the proof of the main result. In section 2, we prove the perimeter formulas of Proposition 2 and the constraints of Propositions 3 and 4 for the integrals occurring in Proposition 2. The constraint optimization problem is then solved to prove Theorem 1, except for the case where $\mathbb{K} = \mathbb{R}$ and a_1 is very close to $\frac{1}{\sqrt{2}}$. In the latter situation, the optimization problem yields an estimate which is weaker than the one claimed in Theorem 1. The problem is that the extremals for the perimeter estimate ($a_1 = a_2 = \frac{1}{\sqrt{2}}$, $a_k = 0, k > 3$) deviate too much from the extremals for the constraints. We also give a direct proof of Theorem 1 in the first non-trivial case $n = 3$, $\mathbb{K} = \mathbb{R}$.

In section 3 we consider the real case when a_1 is close to $\frac{1}{\sqrt{2}}$. In the constraint optimization problem, we estimate the perimeter by $\phi(a_1, a_2)A_{n-1}(a)$ for a specific function ϕ. Ball's estimate $A_{n-1}(a) \leq \sqrt{2}$ is not sufficient to prove Theorem 1 if a_1 is too close to $\frac{1}{\sqrt{2}}$ for this specific function ϕ. However, in this case of a_1, the general estimate $A_{n-1}(a) \leq \sqrt{2}$ may be improved to $A_{n-1}(a) \leq \sqrt{2} - \delta(a_1, a_2)$ for a certain function $\delta(a_1, a_2) > 0$ by improving Ball's estimate $f(p) \leq f(2) = 1$ for $p \geq 2$ for his function f defined above. We show that e.g. $f(p) \leq \sqrt{\frac{3}{\pi}} < 1$

for all $p \geq \frac{9}{4}$. This improves the estimate $A_{n-1}(a) \leq \sqrt{2}$ when $a_1 \leq \frac{2}{3}$, in which case $f(p) \leq \sqrt{\frac{3}{\pi}}$ is applied to $p = a_1^{-2} \geq \frac{9}{4}$. Other cases of $a_2 \leq a_1 < \frac{1}{\sqrt{2}}$ and $a_2 < \frac{1}{\sqrt{2}} < a_1$ are treated by some form of interpolation, using the convexity of f near $p = 2$ which is shown in the proof of Theorem 6, see chapter 3.

The application to the Busemann-Petty type problem is considered in section 4. The Appendix gives some of the technical proofs concerning Ball's function f.

2 Geometric relations and optimization

We start by proving the formulas for the perimeter.

Proof of Proposition 2.
Let $a = (a_k)_{k=1}^n \in S^{n-1}$, $a_1 \geq \cdots a_n \geq 0$ and $x \in \mathbb{K}^n$. We write $a = (a_1, \tilde{a})$, $x = (x_1, \tilde{x})$ with $\tilde{a} = (a_k)_{k=2}^n, \tilde{x} = (x_k)_{k=2}^n \in \mathbb{K}^{n-1}$. This notation is also used in the following proofs. In the real case $\mathbb{K} = \mathbb{R}$, the $(n-1)$-dimensional hyperplane $a^\perp$ intersects the boundary ∂B_∞^n in $2n$ $(n-2)$-dimensional (typically non-central) sections of an $(n-1)$-cube, namely for $x_j = \pm\frac{1}{2}$, $j = 1, \cdots, n$. For $x_1 = -\frac{1}{2}$ we need to calculate

$$vol_{n-2}\{\tilde{x} \in \mathbb{R}^{n-1} \mid \langle \tilde{x}, \tilde{a} \rangle = \frac{1}{2} a_1\}.$$

Let $a_j' := \frac{a_j}{\sqrt{1-a_1^2}}$, $j = 1, \cdots, n$ and $\tilde{a}' := (a_j')_{j=2}^n$. Then $|\tilde{a}'|^2 = \sum_{j=2}^n a_j'^2 = 1$. Using (2), we find

$$\begin{aligned} vol_{n-2}\{\ \tilde{x} \in \mathbb{R}^{n-1} \mid \langle \tilde{x}, \tilde{a} \rangle = \frac{1}{2} a_1\ \} &= A_{n-2}(\tilde{a}', a_1') \\ = \frac{2}{\pi} \int_0^\infty \prod_{j=2}^n \frac{\sin(a_j' r)}{a_j' r} \cos(a_1' r) dr &= \sqrt{1 - a_1^2}\ \frac{2}{\pi} \int_0^\infty \prod_{j=2}^n \frac{\sin(a_j s)}{a_j s}\ \cos(a_1 s) ds\ . \end{aligned}$$

The same holds for $x_1 = +\frac{1}{2}$ and similarly for $x_j = \pm\frac{1}{2}$, so that

$$P_{n-2}(a) = 2\ \sum_{k=1}^n \sqrt{1 - a_k^2}\ \frac{2}{\pi}\ \int_0^\infty \prod_{j=1, j\neq k}^n \frac{\sin(a_j s)}{a_j s}\ \cos(a_k s)\ ds\ ,$$

which proves (4).
In the complex case $\mathbb{K} = \mathbb{C}$, we have to consider the intersection of $a^\perp$ with $x_j = \frac{1}{\sqrt{\pi}} \exp(i\theta)$ for all $\theta \in [0, \pi)$, and use (3) instead of (2). Then

$$\begin{aligned} P_{2(n-2)}(a) &= 2\pi\ \sum_{k=1}^n \frac{1}{2}\ \int_0^\infty \prod_{j=1, j\neq k}^n j_1(a_j' r)\ J_0(a_k' r)\ r\ dr \\ &= 2\pi\ \sum_{k=1}^n (1 - a_k^2)\ \frac{1}{2}\ \int_0^\infty \prod_{j=1, j\neq k}^n j_1(a_j s)\ J_0(a_k s)\ s\ ds\ , \end{aligned}$$

which yields formula (5). □

Pełczyński [P] proved Theorem 1 for $n = 3$ in the real case by considering three affine independent points on the boundary of the cube and their antipodals, calculating the perimeter of the (possibly non-planar) hexagon defined that way. This perimeter then turned out to be maximal in the case that the hexagon degenerates into a rectangle perpendicular to e.g. $(1, 1, 0)$, which is planar. We give the easy direct proof of Theorem 1 for $n = 3$, $\mathbb{K} = \mathbb{R}$ by using Proposition 2.

Proof of Theorem 1 for $n = 3, \mathbb{K} = \mathbb{R}$:
For $\alpha > \beta > 0$ and $\gamma > 0$ we have

$$\frac{2}{\pi}\int_0^\infty \frac{\sin(\alpha s)}{\alpha s}\frac{\sin(\beta s)}{\beta s}\cos(\gamma s)ds = \begin{cases} \frac{1}{\alpha} & , \quad \gamma \le \alpha - \beta \\ \frac{\alpha+\beta-\gamma}{2\alpha\beta} & , \quad \alpha - \beta < \gamma \le \alpha + \beta \\ 0 & , \quad \alpha + \beta < \gamma \end{cases} . \tag{10}$$

Let $a_1 \ge a_2 \ge a_3 \ge 0$, $a_1^2 + a_2^2 + a_3^2 = 1$. Using (10) to calculate the integrals in (4) and distinguishing the cases $a_1 \ge a_2 + a_3$ and $a_1 < a_2 + a_3$, we find that

$$\frac{1}{2}P_1(a) = \left\{ \begin{array}{ll} \frac{1}{a_1}(\sqrt{1-a_2^2} + \sqrt{1-a_3^2}) & , \quad a_1 \ge a_2 + a_3 \\ \sqrt{1-a_1^2}\frac{a_2+a_3-a_1}{2a_2a_3} + \sqrt{1-a_2^2}\frac{a_1+a_3-a_2}{2a_1a_3} + \sqrt{1-a_3^2}\frac{a_1+a_2-a_3}{2a_1a_2} & , \quad a_1 < a_2 + a_3 \end{array} \right\} .$$

In the first case, the hyperplane intersects the cube in a rectangle, in the second case in a hexagon.

i) Assume first that $a_1 \ge a_2 + a_3$. Then $(a_1 - a_2)^2 \ge a_3^2 = 1 - a_1^2 - a_2^2$, $1 - a_2^2 \le a_1^2 + (a_1 - a_2)^2$ and

$$\frac{1}{2}P_1(a) \le \frac{1}{a_1}(\sqrt{a_1^2 + (a_1-a_2)^2} + \sqrt{a_1^2 + a_2^2}) = \sqrt{1-(1-x)^2} + \sqrt{1-x^2},$$

where $0 \le x := \frac{a_2}{a_1} \le 1$. The right side is maximal for $x = 0$ or $x = 1$ with $\frac{1}{2}P_1(a) \le \sqrt{2} + 1$.

ii) If $a_1 < a_2 + a_3$, assume first that $a_1 = a_2 \ge a_3 \ge 0$. Then $\frac{1}{\sqrt{3}} \le a_2 = a_1 \le \frac{1}{\sqrt{2}}$, $1 - a_3^2 = 2a_1^2$ and, as easily seen by the above formula,

$$\frac{1}{2}P_1(a) = \sqrt{2} + \frac{1}{a_1}(\sqrt{1-a_1^2} - \sqrt{\frac{1}{2} - a_1^2}) \le \sqrt{2} + 1.$$

If $a_1 < a_2 + a_3$, but $a_1 > a_2$, define x by $a_3 = x(a_1 - a_2)$ so that $x \ge 1$. Then

$$\frac{1}{2}P_1(a) = \frac{1}{2}(\sqrt{1-a_1^2}\,\frac{x-1}{xa_2} + \sqrt{1-a_2^2}\,\frac{x+1}{xa_1} + \sqrt{1-x^2(a_1-a_2)^2}\,\frac{a_1+a_2-x(a_1-a_2)}{a_1a_2}) =: \frac{1}{2}\psi(x).$$

We have

$$\psi'(x) = (\frac{\sqrt{1-a_1^2}}{a_2} - \frac{\sqrt{1-a_2^2}}{a_1})\frac{1}{x^2} - \sqrt{1-a_3^2}\,\frac{a_1-a_2}{a_1a_2} - \frac{x(a_1-a_2)^2}{\sqrt{1-a_3^2}}\frac{a_1+a_2-x(a_1-a_2)}{a_1a_2}.$$

If the factor of $\frac{1}{x^2}$ is negative, all summands are negative and $\psi'(x) \le 0$. If the factor is positive,

$$\begin{aligned}\psi'(x) &\le (\frac{\sqrt{1-a_1^2}}{a_2} - \frac{\sqrt{1-a_2^2}}{a_1}) - \sqrt{a_1^2+a_2^2}\,\frac{a_1-a_2}{a_1a_2} \\ &= \frac{1}{a_1a_2}(a_1\sqrt{1-a_1^2} - a_2\sqrt{1-a_2^2} - (a_1-a_2)\sqrt{a_1^2+a_2^2}).\end{aligned}$$

This is negative as well: $\phi(y) := y\sqrt{1-y^2}$ satisfies $\phi'(y) = \frac{1-2y^2}{\sqrt{1-y^2}}$, so that $a_1\sqrt{1-a_1^2} - a_2\sqrt{1-a_2^2} = (a_1-a_2)\frac{1-2y^2}{\sqrt{1-y^2}}$ for some $a_2 < y < a_1$. But

$$\frac{1-2y^2}{\sqrt{1-y^2}} \le \frac{1-2a_2^2}{\sqrt{1-a_2^2}} \le \sqrt{1-a_2^2} = \sqrt{a_1^2+a_3^2} \le \sqrt{a_1^2+a_2^2}\ .$$

Hence $\psi'(x) \le 0$, so that $\psi(x) \le \psi(1)$ since $x \ge 1$. Therefore

$$\frac{1}{2}P_1(a) \le \frac{1}{a_1}(\sqrt{1-a_2^2} + \sqrt{1-a_3^2}\ ),$$

and, since $x = 1$, $a_1^2 + a_2^2 + (a_1-a_2)^2 = 1$, $a_2 = \frac{1}{2}(a_1 + \sqrt{2-3a_1^2})$, yielding

$$1-a_2^2 = \frac{1}{2}(1+a_1^2) - \frac{1}{2}a_1\sqrt{2-3a_1^2} =: \phi_-(a_1)\ ,\ 1-a_3^2 = a_1^2+a_2^2 = \frac{1}{2}(1+a_1^2) + \frac{1}{2}a_1\sqrt{2-3a_1^2} =: \phi_+(a_1),$$

with $\frac{1}{\sqrt{2}} \le a_1 \le \sqrt{\frac{2}{3}}$ so that

$$\frac{1}{2}P_1(a) \le \frac{1}{a_1}(\sqrt{\phi_-(a_1)} + \sqrt{\phi_+(a_1)}) \le \sqrt{2}+1,$$

the maximal value being attained for $a_1 = \frac{1}{\sqrt{2}}$. □

For $\mathbb{K} = \mathbb{R}$ and $n = 4$, formula (4) can be integrated exactly by using formula (2.1) of König, Koldobsky [KK1] - instead of (10) in the case $n = 3$ - and distinguishing three cases. The result is

$$\frac{1}{2}P_2(a) := \begin{cases} \sqrt{1-a_1^2}\,\frac{a_2+a_3-a_1}{2a_2a_3} + \sqrt{1-a_2^2}\,\frac{a_1+a_3-a_2}{2a_1a_3} + \sqrt{1-a_3^2}\,\frac{a_1+a_2-a_3}{2a_1a_2} \\ \quad +\sqrt{1-a_4^2}\,(\frac{1}{a_1} - \frac{a_4^2(a_2+a_3-a_1)^2}{4a_1a_2a_3}) \quad ,\ a_1 < a_2+a_3-a_4 \\ \\ -\frac{(a_2+a_3+a_4-a_1)^2}{8a_1a_2a_3a_4}\,(-a_1\sqrt{1-a_1^2} + a_2\sqrt{1-a_2^2} + a_3\sqrt{1-a_3^2} + a_4\sqrt{1-a_4^2}) \\ \quad +\frac{1}{a_1}(\sqrt{1-a_2^2} + \sqrt{1-a_3^2} + \sqrt{1-a_4^2})\ ,\ a_2+a_3-a_4 \le a_1 \le a_2+a_3+a_4 \\ \\ \frac{1}{a_1}(\sqrt{1-a_2^2} + \sqrt{1-a_3^2} + \sqrt{1-a_4^2}) \quad ,\ a_2+a_3+a_4 < a_1 \end{cases} \quad .(11)$$

Proof of Proposition 3.
We first give a geometric proof in the real case.

a) Let $a_1 \geq \cdots \geq a_n \geq 0$, $\sum_{k=1}^n a_k^2 = 1$, $a_k' := \frac{a_k}{\sqrt{1-a_1^2}}$, $\tilde{a}' := (a_k')_{k=2}^n$ and $D_j(a)$ be given as in (6). By transformation of variables

$$D_1(a) = \frac{1}{\sqrt{1-a_1^2}} \, \frac{2}{\pi} \int_0^\infty \prod_{j=2}^n \frac{\sin(a_j' r)}{a_j' r} \, \cos(a_1' r) \, dr = \frac{1}{\sqrt{1-a_1^2}} \, A_{n-2}(\tilde{a}', a_1'),$$

in terms of the $(n-2)$-dimensional volume of the section of B_∞^{n-1} perpendicular to $\tilde{a}'$ and at distance $\frac{1}{2}a_1'$ to the origin of B_∞^{n-1}. Since $\frac{1}{2}\frac{1}{\sqrt{1-a_1^2}}$ is the height of the $(n-1)$-dimensional pyramid $P(1)$ with vertex in 0 and base being the above $(n-2)$-dimensional section,

$$vol_{n-1}(P(1)) = \frac{1}{n-1} \, A_{n-2}(\tilde{a}', a_1') \, \frac{1}{2}\frac{1}{\sqrt{1-a_1^2}} \, .$$

Summing up the volumes of these pyramids, also for opposite sections, yields

$$A_{n-1}(a) = 2\sum_{k=1}^n vol_{n-1}(P(k)) = \frac{1}{n-1}\sum_{k=1}^n D_k(a),$$

which is (7).

b) We now give a second, analytic proof of (7), based on integration by parts, using

$$\frac{d}{ds}\left(\frac{\sin(a_j s)}{a_j s}\right) = \frac{1}{s}\left(\cos(a_j s) - \frac{\sin(a_j s)}{a_j s}\right),$$

if all a_j are > 0. Then

$$\begin{aligned} D_1(a) &= \left[\frac{2}{\pi}\prod_{j=2}^n \frac{\sin(a_j s)}{a_j s}\frac{\sin(a_1 s)}{a_1}\right]_{s=0}^\infty - \frac{2}{\pi}\int_0^\infty \sum_{j=2}^n \prod_{k=2, k\neq j}^n \frac{\sin(a_k s)}{a_k s}\left(\frac{\cos(a_j s)}{s} - \frac{\sin(a_j s)}{a_j s^2}\right)\frac{\sin(a_1 s)}{a_1} ds \\ &= (n-1)A_{n-1}(a) - \sum_{j=2}^n D_j(a) \ , \end{aligned}$$

so that $\sum_{j=1}^n D_j(a) = (n-1)A_{n-1}(a)$. If some a_k are zero, the corresponding $D_k(a)$ equals $A_{n-1}(a)$, and (7) follows by integration by parts only for those k where $a_k \neq 0$.

c) The integration by parts technique also works in the complex case $\mathbb{K} = \mathbb{C}$, using

$$\frac{d}{ds}j_1(s) = 2\frac{d}{ds}\left(\frac{J_1(s)}{s}\right) = -2\frac{J_2(s)}{s} = 2\frac{J_0(s)}{s} - 4\frac{J_1(s)}{s^2}$$

and $\frac{d}{ds}(sJ_1(s)) = sJ_0(s)$. For these formulas on Bessel functions, cf. Watson [W]. □

Proof of Proposition 4.
We first consider the real case. To show $D_k(a) \leq A_{n-1}(a)$, we may assume without loss of generality that $k = 1$ and $a_1 > 0$. We will not use any inequality between the coordinates of

a in this proof, but assume that $a_k \geq 0$ for all k. Again, let $a'_j = \frac{a_j}{\sqrt{1-a_1^2}}$ for $j = 1, \cdots, n$, $\tilde{a}' := (a'_j)_{j=2}^n \in \mathbb{R}^{n-1}$. Then $\sum_{j=2}^n a_j'^2 = 1$ so that by transformation of variables and (6) in dimension $n-1$

$$\begin{aligned} D_1(a) &= \frac{1}{\sqrt{1-a_1^2}} \frac{2}{\pi} \int_0^\infty \prod_{j=2}^n \frac{\sin(a'_j r)}{a'_j r} \cos(a'_1 r)\, dr \\ &= \frac{1}{\sqrt{1-a_1^2}}\, vol_{n-2}\{\ \tilde{x} \in B_\infty^{n-1} \mid \langle \tilde{x}, \tilde{a}' \rangle = \frac{1}{2} a'_1\ \}. \end{aligned}$$

By Brunn-Minkowski, we have for any $t \in \mathbb{R}$ with $|t| \leq a'_1$

$$vol_{n-2}\{\tilde{x} \in B_\infty^{n-1} \mid \langle \tilde{x}, \tilde{a}' \rangle = \frac{1}{2} a'_1\} \leq vol_{n-2}\{\tilde{x} \in B_\infty^{n-1} \mid \langle \tilde{x}, \tilde{a}' \rangle = \frac{1}{2} t\}.$$

Therefore

$$D_1(a) \leq \frac{1}{a'_1} \frac{1}{\sqrt{1-a_1^2}}\, vol_{n-1}\{\tilde{x} \in B_\infty^{n-1} \mid |\langle \tilde{x}, \tilde{a}' \rangle| \leq \frac{1}{2} a'_1\}.$$

Define $T : \mathbb{R}^{n-1} \to \mathbb{R}^n$ by $T(x) := (-\frac{\langle \tilde{x}, \tilde{a}' \rangle}{a'_1}, \tilde{x})$. Then T maps the slab $\{\tilde{x} \in B_\infty^{n-1} \mid |\langle \tilde{x}, \tilde{a}' \rangle| \leq \frac{1}{2} a'_1\}$ in dimension $n-1$ into a central section of B_∞^n,

$$\begin{aligned} T\{\tilde{x} \in B_\infty^{n-1} \mid |\langle \tilde{x}, \tilde{a}' \rangle| \leq \frac{1}{2} a'_1\} &= \{y = (x_1, \tilde{x}) \in B_\infty^n \mid x_1 a_1 + \langle \tilde{x}, \tilde{a}' \rangle = 0\} \\ &= \{y \in B_\infty^n \mid \langle y, a \rangle = 0\}. \end{aligned}$$

Recall here that we normalized B_∞^{n-1} and B_∞^n to have volume 1. Since

$$T^* T = \mathrm{Id} + \frac{1}{a_1'^2}\ \langle \cdot, \tilde{a}' \rangle\ \tilde{a}',$$

T^*T has an $(n-2)$-fold eigenvalue 1 and one eigenvalue $1 + \frac{1}{a_1'^2}$ (with eigenvector $\tilde{a}'$ of norm 1) so that

$$\sqrt{\det(T^*T)} = \frac{1}{a'_1} \sqrt{a_1'^2 + 1} = \frac{1}{a'_1} \frac{1}{\sqrt{1-a_1^2}}.$$

Therefore

$$\begin{aligned} D_1(a) &\leq \sqrt{\det(T^*T)}\ vol_{n-1}\{\tilde{x} \in B_\infty^{n-1} \mid |\langle \tilde{x}, \tilde{a}' \rangle| \leq \frac{1}{2} a'_1\} \\ &= vol_{n-1}\{y \in B_\infty^n \mid \langle y, a \rangle = 0\} = A_{n-1}(a). \end{aligned}$$

The complex case requires only minor modifications. In that case

$$\begin{aligned} D_1(a) &= \frac{1}{1-a_1^2}\ vol_{2(n-1)}\{\tilde{x} \in B_\infty^{n-1} \mid \langle \tilde{x}, \tilde{a}' \rangle = \frac{1}{\sqrt{\pi}} a'_1\} \\ &\leq \frac{1}{a_1'^2} \frac{1}{1-a_1^2}\ vol_{2(n-1)}\{\tilde{x} \in B_\infty^{n-1} \mid |\langle \tilde{x}, \tilde{a}' \rangle| \leq \frac{1}{\sqrt{\pi}} a'_1\}. \end{aligned}$$

Define $T : \mathbb{C}^{n-1} \to \mathbb{C}^n$ also by $T(x) := (-\frac{\langle \tilde{x}, \tilde{a}' \rangle}{a'_1}, \tilde{x})$, mapping the slab in $\mathbb{C}^{n-1}$ into the central section in $\mathbb{C}^n$ defined by $a^\perp$. In the complex case $\sqrt{\det(T^*T)} = \frac{1}{a_1'^2}(1 + a_1'^2) = \frac{1}{a_1'^2} \frac{1}{1-a_1^2}$, so that

$$D_1(a) \leq vol_{2(n-1)}\{y \in B_\infty^n \mid \langle y, a \rangle = 0\} = A_{2(n-1)}(a).$$

□

Proof of Corollary 5.
Let $a \in \mathbb{K}^n$, $|a| = 1$ and $t > 0$. Put $a'_j = \frac{a_j}{\sqrt{1+t^2}}$, $t' := \frac{t}{\sqrt{1+t^2}}$. Then $(a', t') \in \mathbb{K}^{n+1}$, $|(a', t')| = 1$. We get from (2) and (3) by transformation of variables $A_{l(n-1)}(a, t) = \frac{1}{\sqrt{1+t^2}}\ D((a', t'))$, where e.g. in the real case

$$D((a', t')) = \frac{2}{\pi} \int_0^\infty \prod_{j=1}^n \frac{\sin(a'_j s)}{a'_j s}\ \cos(t' s)\ ds.$$

By Proposition 4, applied to $(a', t') \in S^n \subset \mathbb{R}^{n+1}$, $D((a', t')) \leq A_n((a', t')) \leq \sqrt{2}$. Similarly, for $\mathbb{K} = \mathbb{C}$, $D((a', t')) \leq A_{2n}((a', t')) \leq 2$. Therefore

$$A_{l(n-1)} \leq (\frac{2}{1+t^2})^{l/2}.$$

□

In the case that the largest coordinate a_1 of $a \in S^{n-1}$ satisfies $a_1 > \frac{1}{\sqrt{2}}$, Ball [B1] showed by projecting $a^\perp$ onto $a_{min}^\perp = (1, 0, \cdots, 0)^\perp$ that

$$A_{n-1}(a) \leq \frac{1}{a_1}\ A_{n-1}(a_{min}) = \frac{1}{a_1}. \tag{12}$$

The complex analogue of this is, again if $a_1 > \frac{1}{\sqrt{2}}$,

$$A_{2(n-1)}(a) \leq \frac{1}{a_1^2}\ A_{2(n-1)}(a_{min}) = \frac{1}{a_1^2}, \tag{13}$$

cf. Oleszkiewicz, Pełczyński [OP].

We now prove Theorem 1, except in the real case when $a_1 \in (\sqrt{\sqrt{2}-1}, \frac{1}{\sqrt{\sqrt{2}+\frac{1}{2}}}) \simeq (0.643, 0.723)$, i.e. when a_1 is close to $\frac{1}{\sqrt{2}}$. This is done using the constraints given by Propositions 3 and 4.

Proof of Theorem 1.
(a) We first verify the result in the complex case $\mathbb{K} = \mathbb{C}$ which is easier to prove. We have for a_{max}

$$D_1(a_{max}) = D_2(a_{max}) = \frac{1}{2} \int_0^\infty j_1(\frac{s}{\sqrt{2}})\ J_0(\frac{s}{\sqrt{2}})\ s\ ds = 2 \int_0^\infty J_1(t)\ J_0(t)\ dt = 1,$$

$$D_j(a_{max}) = \frac{1}{2} \int_0^\infty j_1(\frac{s}{\sqrt{2}})^2\ s\ ds = 4 \int_0^\infty J_1(t)^2 \frac{dt}{t} = 2\ ,\ j > 2.$$

For these integrals, cf. Gradstein, Ryshik [GR] or Watson [W]. Hence by (5)

$$P_{2(n-1)}(a_{max}) = 2\pi(1 + (n-2)2) = 2\pi(2n-3). \tag{14}$$

Now consider $a = (a_k)_{k=1}^n \in S^{n-1}$ with $a_1 \geq \cdots \geq a_n \geq 0$. By Proposition 2

$$\frac{1}{2\pi} P_{2(n-2)}(a) = \sum_{k=1}^{n} (1 - a_k^2)\ D_k(a),$$

and using Propositions 3 and 4, we have

$$\frac{1}{2\pi} P_{2(n-2)}(a) \leq \sup\{\sum_{k=1}^{n}(1-a_k^2)\ C_k \mid 0 \leq C_k \leq A_{2(n-1)}(a),\ \sum_{k=1}^{n} C_k = (n-1)\ A_{2(n-1)}(a)\}.$$

Since $(1-a_k^2)_{k=1}^n$ is increasing in k, the sum $\sum_{k=1}^n (1-a_k^2)\ C_k$ will be maximal under the given restrictions, if the sequence $(C_k)_{k=1}^n$ is increasing as well which, in fact, means that $C_1 = 0$, $C_2 = \cdots = C_n = A_{2(n-1)}(a)$. Therefore

$$\begin{aligned}\frac{1}{2\pi} P_{2(n-2)}(a) &\leq \sum_{k=2}^{n}(1-a_k^2)\ A_{2(n-1)}(a) \\ &= [(n-1) - \sum_{k=2}^{n} a_k^2]\ A_{2(n-1)}(a) = (n-2+a_1^2)\ A_{2(n-1)}(a).\end{aligned}$$

If $a_1 \leq \frac{1}{\sqrt{2}}$, we use that $A_{2(n-1)}(a) \leq A_{2(n-1)}(a_{max}) = 2$ by [OP], so that with (14)

$$\frac{1}{2\pi} P_{2(n-2)}(a) \leq (n - \frac{3}{2})\ 2 = \frac{1}{2\pi} P_{2(n-2)}(a_{max}).$$

If $a_1 > \frac{1}{\sqrt{2}}$, we use that by (13) $A_{2(n-1)}(a) \leq \frac{1}{a_1^2}$, so that

$$\frac{1}{2\pi} P_{2(n-2)}(a) \leq (n-2+a_1^2)\frac{1}{a_1^2} = \frac{n-2}{a_1^2} + 1 \leq (n-2)\ 2 + 1 = \frac{1}{2\pi} P_{2(n-2)}(a_{max}).$$

This proves Theorem 1 in the case of complex scalars.

(b) In the real case $\mathbb{K} = \mathbb{R}$, we have for a_{max}

$$D_1(a_{max}) = D_2(a_{max}) = \frac{2}{\pi}\int_0^\infty \frac{\sin(\frac{s}{\sqrt{2}})}{\frac{s}{\sqrt{2}}}\ \cos(\frac{s}{\sqrt{2}})\ ds = \frac{1}{\sqrt{2}}\frac{2}{\pi}\int_0^\infty \frac{\sin(t)}{t}\ dt = \frac{1}{\sqrt{2}},$$

$$D_j(a_{max}) = \frac{2}{\pi}\int_0^\infty \left(\frac{\sin(\frac{s}{\sqrt{2}})}{\frac{s}{\sqrt{2}}}\right)^2\ ds = \sqrt{2}\frac{2}{\pi}\int_0^\infty \left(\frac{\sin(t)}{t}\right)^2\ dt = \sqrt{2}\ ,\ j > 2.$$

Hence by (4)

$$P_{n-2}(a_{max}) = 2((n-2)\sqrt{2} + 1). \tag{15}$$

Now let $a = (a_k)_{k=1}^n \in S^{n-1}$ be arbitrary with $a_1 \geq \cdots \geq a_n \geq 0$. By Propositions 2, 3 and 4 we get, similarly as in part (a),

$$\frac{1}{2} P_{n-2}(a) \leq \sup\{\sum_{k=1}^n \sqrt{1-a_k^2}\; C_k \mid 0 \leq C_k \leq A_{n-1}(a),\; \sum_{k=1}^n C_k = (n-1)\; A_{n-1}(a)\}.$$

Since also $(\sqrt{1-a_k^2}\;)_{k=1}^n$ is increasing in k, the supremum is attained for increasing C_k as well and, in fact, for $C_1 = 0$, $C_2 = \cdots = C_k = A_{n-1}(a)$ so that

$$\frac{1}{2} P_{n-2}(a) \leq \sum_{k=2}^n \sqrt{1-a_k^2}\; A_{n-1}(a). \tag{16}$$

Since $\phi(x) = \sqrt{1-x}$ is concave,

$$\frac{1}{n-1} \sum_{k=2}^n \phi(a_k^2) \leq \phi(\frac{1}{n-1} \sum_{k=2}^n a_k^2) = \phi(\frac{1}{n-1}(1-a_1^2)).$$

Hence

$$\frac{1}{2} P_{n-2}(a) \leq (n-1)\sqrt{1 - \frac{1-a_1^2}{n-1}}\; A_{n-1}(a) \leq (n-1-\frac{1-a_1^2}{2})\; A_{n-1}(a). \tag{17}$$

If $a_1 \leq \frac{1}{\sqrt{2}}$, we use that $A_{n-1}(a) \leq \sqrt{2}$ by [B1] to get

$$\frac{1}{2} P_{n-2}(a) \leq (n-2)\sqrt{2} + \frac{3}{4}\sqrt{2}.$$

If $a_1 > \frac{1}{\sqrt{2}}$, we use that $A_{n-1}(a) \leq \frac{1}{a_1}$ by (12) and also find

$$\frac{1}{2} P_{n-2}(a) \leq (n - \frac{3}{2} + a_1^2)\frac{1}{a_1} \leq (n-2)\sqrt{2} + \frac{3}{4}\sqrt{2}.$$

However, $\frac{3}{4}\sqrt{2} \simeq 1.0607 > 1$, so that this does not prove $P_{n-2}(a) \leq P_{n-2}(a_{max})$ for all $a \in S^{n-1}$. However, if a_1 satisfies $a_1 \leq \sqrt{\sqrt{2}-1} \simeq 0.643$, (17) yields

$$\frac{1}{2} P_{n-2}(a) \leq \left(n - \frac{3}{2} + \frac{\sqrt{2}-1}{2}\right)\sqrt{2} = (n-2)\sqrt{2} + 1 = \frac{1}{2} P_{n-2}(a_{max}).$$

For $a_1 > \frac{1}{\sqrt{2}}$, the requirement that $(n-1)\sqrt{1 - \frac{1-a_1^2}{n-1}}\frac{1}{a_1} \leq (n-2)\sqrt{2} + 1$ is strongest for $n = 3$, in which case it means $a_1 \geq \frac{1}{\sqrt{\sqrt{2}+\frac{1}{2}}} \simeq 0.723$. Therefore for any $a \in S^{n-1}$ with $a_1 \notin (\sqrt{\sqrt{2}-1}, \frac{1}{\sqrt{\sqrt{2}+\frac{1}{2}}})$, we have shown $P_{n-2}(a) \leq P_{n-2}(a_{max})$.
Hence Theorem 1 is proved also for real scalars, except in the situation that

$$a_1 \in \left(\sqrt{\sqrt{2}-1}, \frac{1}{\sqrt{\sqrt{2}+\frac{1}{2}}}\right), \tag{18}$$

where the estimate is off by at most $2(\frac{3}{4}\sqrt{2}-1) \simeq 0.121$. This discrepancy occurs since in (16) the extremals for the sum of weights and for the section function A occur for different sequences a. This could possibly be avoided, if one could show how the monotonicity of the sequence $(a_k)_{k=1}^n$ affects the size of the integrals $D_k(a)$, but we have not been to find a result of this type. Instead, we will address the case of (18) by a different method in the next section.

3 Interpolating Ball's function

To prove Theorem 1 also for hyperplane sections perpendicular to a with $a_1 \in (\sqrt{\sqrt{2}-1}, \frac{1}{\sqrt{\sqrt{2}+\frac{1}{2}}})$, $\mathbb{K}=\mathbb{R}$, we will improve the general estimate for $A_{n-1}(a)$ in (17), by using the improved estimates for Ball's integral function f stated in Propositions 6 and 7. The convexity of f allows estimates by interpolation for certain values of a_1 and a_2 near $\frac{1}{\sqrt{2}}$. The technical proof of Proposition 6 is given in the Appendix. The proof of Proposition 7 is a slight modification of Nazarov, Podkorytov's [NP] proof of Ball's inequality $f(p) \leq f(2) = 1$ for $p \geq 2$. Recall that Proposition 7 states that $f(p) \leq \sqrt{\frac{3}{\pi}} < 1$ for all $p \geq \frac{9}{4}$.

Proof of Proposition 7.
Let $f : (1,\infty) \to \mathbb{R}_+$ denote Ball's function, $f(p) := \sqrt{\frac{p}{2}\frac{2}{\pi}} \int_0^\infty |\frac{\sin(t)}{t}|^p \, dt$. Define $g, h : [0,\infty) \to \mathbb{R}_+$ by $g(x) := |\frac{\sin(x)}{x}|$ and $h(x) := \exp(-\frac{x^2}{6})$, with $g(0)=1$, and let $G, H : (0,1] \to \mathbb{R}_+$ denote the distribution functions of g and h, respectively. We claim that there is $y_0 \in (0,1)$ such that

$$H(y) \leq G(y) \text{ for all } 0 < y < y_0 \quad \text{ and } \quad H(y) \geq G(y) \text{ for all } y_0 < y < 1. \tag{19}$$

The distribution function lemma in [NP] then implies that the function $\phi : (1,\infty) \to \mathbb{R}_+$,

$$\phi(p) := \frac{1}{p y_0^p} \int_0^\infty (h(x)^p - g(x)^p) \, dx$$

is increasing in p. Since by Proposition 6 for $p_0 := \frac{9}{4}$

$$\int_0^\infty g(x)^{p_0} = \int_0^\infty \left|\frac{\sin(x)}{x}\right|^{p_0} dx < \sqrt{\frac{3}{\pi}}\frac{\pi}{2}\sqrt{\frac{2}{p_0}} = \sqrt{\frac{2\pi}{3}} = \int_0^\infty \exp(-\frac{3}{8}x^2) dx = \int_0^\infty g(x)^{p_0} dx,$$

we conclude that for all $p \geq \frac{9}{4}$

$$\int_0^\infty \left|\frac{\sin(x)}{x}\right|^p \, dx = \int_0^\infty g(x)^p \, dx < \int_0^\infty h(x)^p \, dx = \sqrt{\frac{3\pi}{2p}},$$

which is equivalent to $f(p) < \sqrt{\frac{3}{\pi}}$. For $p = 2$,

$$\int_0^\infty g(x)^2 \, dx = \frac{\pi}{2} > \sqrt{\frac{3}{\pi}}\frac{\pi}{2} = \int_0^\infty h(x)^2 \, dx.$$

Therefore there is $q \in (2, \frac{9}{4})$ such that

$$0 = \int_0^\infty (h(x)^q - g(x)^q) \, dx = q \int_0^1 y^{q-1}(H(y) - G(y)) \, dy.$$

Hence $H-G$ has at least one zero $y_0 \in (0,1)$. To prove (19), we will show that $H-G$ has *only* one zero. For $m \in \mathbb{N}$, let $y_m := \max\{ g(x) \mid x \in [m\pi,(m+1)\pi]\}$. Since

$$\frac{\sin(x)}{x} = \prod_{n\in\mathbb{N}} (1-\frac{x^2}{(n\pi)^2}),$$

we have for all $0<x<\pi$

$$\ln(\frac{\sin(x)}{x}) = \sum_{n\in\mathbb{N}} \ln(1-\frac{x^2}{(n\pi)^2}) \le -\sum_{n\in\mathbb{N}} \frac{x^2}{(n\pi)^2} = -\frac{x^2}{6},$$

i.e. $g(x)=\frac{\sin(x)}{x} \le \exp(-\frac{x^2}{6}) = h(x)$. Therefore $H-G$ is positive in $(y_1,1)$. To show that $H-G$ has only one zero, it suffices to prove that $(H-G)'>0$ in $(0,y_1)=\cup_{m\in\mathbb{N}}[y_{m+1},y_m)$. Since $H'<0, G'<0$, this means that $|\frac{G'}{H'}|>1$ has to be shown there. We have, similarly as in [NP], $H(y)=h^{-1}(y)=\sqrt{6\ln(\frac{1}{y})}$, $H'(y)=\sqrt{\frac{3}{2}}\frac{1}{y\sqrt{\ln(\frac{1}{y})}}$ and

$$|G'(y)| = \sum_{x>0,\ g(x)=y} \frac{1}{|g'(x)|}.$$

For $y\in(y_{m+1},y_m)$, $g(x)=y$ has one root x_0 in $(0,\pi)$ and two roots $x_{j,1},x_{j,2}$ in $(j\pi,(j+1)\pi)$ for $j=1,\cdots,m$. Easy estimates show $|g'(x_0)|<\frac{1}{2}$, $|g'(x_{j,1})|,|g'(x_{j,2})| \le \frac{1}{\pi j}$, $j\in\mathbb{N}$ so that for all $y\in[y_{m+1},y_m)$ with $y>y_{m+1}>\frac{1}{\pi(m+\frac{3}{2})}$

$$|G'(y)| > 2(1+\sum_{j=1}^{m} \pi j) = 2+\pi m(m+1),$$

$$\begin{aligned}\left|\frac{G'(y)}{H'(y)}\right| &> \sqrt{\frac{2}{3}}(2+\pi m(m+1))y\sqrt{\ln(\frac{1}{y})} \\ &\ge \sqrt{\frac{2}{3}}\ \frac{2+\pi m(m+1)}{\pi(m+\frac{3}{2})}\sqrt{\ln(\pi(m+\frac{3}{2}))} \ge \sqrt{\frac{2}{3}\ln(\frac{5\pi}{2})} > 1.\end{aligned}$$

This means that (19) holds and Proposition 7 is proven. □

Theorem 1 has been shown for $\mathbb{K}=\mathbb{R}$, $n=3$ and for $n\ge 4$ if $a_1 \notin (\sqrt{\sqrt{2}-1}, \frac{1}{\sqrt{\sqrt{2}+\frac{1}{2}}})$. We now consider the remaining cases and assume first that $a_2 \ge \frac{2}{3}$.

Lemma 8. *Assume that $a\in S^{n-1}$, $a_1 \in (\sqrt{\sqrt{2}-1}, \frac{1}{\sqrt{\sqrt{2}+\frac{1}{2}}})$ and $a_2 \ge \frac{2}{3}$. Then*

$$P_{n-2}(a) \le P_{n-2}(a_{max}) = 2((n-2)\sqrt{2}+1).$$

Proof. Since $a_n^2 + \cdots + a_3^2 = 1 - a_1^2 - a_2^2 \le \frac{1}{9}$, we know that $a_n \le \cdots \le a_3 \le \frac{1}{3} < \frac{2}{3} \le a_2 \le a_1 \le \frac{1}{\sqrt{\sqrt{2}+\frac{1}{2}}}$. By concavity of $\sqrt{1-x}$, we find similarly as in (17)

$$\begin{aligned}
\frac{1}{2} P_{n-2}(a) &\le \sum_{k=2}^{n} \sqrt{1-a_k^2}\ A_{n-1}(a) \\
&\le \left[\ (n-2)\sqrt{1 - \frac{\sum_{k=3}^n a_k^2}{n-2}} + \sqrt{1-a_2^2}\ \right]\ A_{n-1}(a) \\
&= \left[\ (n-2)\sqrt{1 - \frac{1-a_1^2-a_2^2}{n-2}} + \sqrt{1-a_2^2}\ \right]\ A_{n-1}(a) \\
&\le \left[\ (n-2) - \frac{1-a_1^2-a_2^2}{2} + \sqrt{1-a_2^2}\ \right]\ A_{n-1}(a). \qquad (20)
\end{aligned}$$

i) Suppose first that $a_1 \le \frac{1}{\sqrt{2}}$. Then $\frac{2}{3} \le a_2 \le a_1 \le \frac{1}{\sqrt{2}}$, $2 \le a_1^{-2} \le a_2^{-2} \le \frac{9}{4}$, and by Hölder's inquality with $p_k := a_k^{-2} \ge \frac{9}{4}$ for $k \ge 3$ and Proposition 7

$$\begin{aligned}
A_{n-1}(a) &= \frac{2}{\pi}\int_0^\infty \prod_{k=1}^n \frac{\sin(a_k s)}{a_k s}\ ds \le \prod_{k=1}^n \Big(\frac{2}{\pi}\ a_k^{-1} \int_0^\infty \left|\frac{\sin(t)}{t}\right|^{a_k^{-2}}\ dt\Big)^{a_k^2} \\
&= \left(\prod_{k=1}^n f(a_k^{-2})^{a_k^2}\right)\sqrt{2} \le \left(\sum_{k=1}^n a_k^2 f(a_k^{-2})\right)\sqrt{2} \\
&\le [\ (1-a_1^2-a_2^2)\sqrt{\frac{3}{\pi}} + a_2^2 f(a_2^{-2}) + a_1^2 f(a_1^{-2})\ ]\sqrt{2},
\end{aligned}$$

where the second inequality follows from the general arithmetic-geometric mean inequality. For $k = 1, 2$ write $a_k^{-2} = \lambda_k 2 + (1-\lambda_k)\frac{9}{4}$, $\lambda_k = 9 - 4a_k^{-2}$. Using the convexity of f, cf. Proposition 6 (c), we find

$$a_k^2 f(a_k^{-2}) \le a_k^2(\lambda_k f(2) + (1-\lambda_k) f(\frac{9}{4})) \le (9a_k^2 - 4) + (4 - 8a_k^2)\sqrt{\frac{3}{\pi}} =: \phi_2(a_k),$$

so that

$$\begin{aligned}
A_{n-1}(a) &\le \left((1-a_1^2-a_2^2)\sqrt{\frac{3}{\pi}} + \phi_2(a_2) + \phi_2(a_1)\right)\sqrt{2} \\
&= \left(9(a_1^2+a_2^2) - 8 + 9(1-a_1^2-a_2^2)\sqrt{\frac{3}{\pi}}\right)\sqrt{2} =: \psi_1(a_1,a_2)\sqrt{2}
\end{aligned}$$

and with (20)

$$\frac{1}{2} P_{n-2}(a) \le \left(n - 2 - \frac{1-a_1^2-a_2^2}{2} + \sqrt{1-a_2^2}\ \right)\ \psi_1(a_1,a_2)\sqrt{2} =: \gamma(a_1,a_2)\sqrt{2}.$$

Calculation shows that $\frac{\partial\gamma}{\partial a_1} \geq 0$, $\frac{\partial\gamma}{\partial a_2} \geq 0$ for all a_1, a_2 in the range considered. Therefore γ is increasing in a_1 and a_2 for all $\frac{2}{3} \leq a_2 \leq a_1 \leq \frac{1}{\sqrt{2}}$ and $\frac{1}{2}P_{n-2}(a)$ is bounded by $\gamma(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\sqrt{2} = (n-2)\sqrt{2} + 1 = \frac{1}{2}P_{n-2}(a_{max})$.

ii) Suppose next that $\frac{1}{\sqrt{2}} < a_1 \leq \frac{1}{\sqrt{\sqrt{2}+\frac{1}{2}}}$. Write $a_1^{-2} = \lambda\ 2 + (1-\lambda)(\sqrt{2}+\frac{1}{2})$. Then by Proposition 6

$$\begin{aligned} a_1^2 f(a_1^{-2}) &\leq a_1^2(\lambda f(2) + (1-\lambda) f(\sqrt{2}+\frac{1}{2})) \\ &\leq \frac{1-(\sqrt{2}+\frac{1}{2})a_1^2 + (2a_1^2-1)\frac{51}{50}}{\frac{3}{2}-\sqrt{2}} =: \phi_1(a_1). \end{aligned}$$

Using this, we find similarly as in part i)

$$\begin{aligned} A_{n-1}(a) &\leq \left(\sum_{k=1}^{n} a_k^2 f(a_k^{-2})\right)\sqrt{2} \\ &\leq \left((1-a_1^2-a_2^2)\sqrt{\frac{3}{\pi}} + \phi_2(a_2) + \phi_1(a_1)\right)\sqrt{2} =: \psi_2(a_1,a_2), \end{aligned}$$

and with (20)

$$\begin{aligned} \frac{1}{2}P_{n-2}(a) &\leq \left((n-2) - \frac{1-a_1^2-a_2^2}{2} + \sqrt{1-a_2^2}\right)\ \min(\psi_2(a_1,a_2)\sqrt{2}, \frac{1}{a_1}) \\ &=: \min(\gamma_1(a_1,a_2), \gamma_2(a_1,a_2)), \end{aligned} \tag{21}$$

where we also used that $A_{n-1}(a) \leq \frac{1}{a_1}$ since $a_1 > \frac{1}{\sqrt{2}}$. It is easy to see that γ_2 is decreasing in a_1 and in a_2 since

$$\frac{\partial\gamma_2}{\partial a_1} = -[\ (n-\frac{5}{2}) + \frac{1}{2}(a_1^2+a_2^2) + \sqrt{1-a_2^2}\ ]\frac{1}{a_1^2} + 1 < 0,$$

$$\frac{\partial\gamma_2}{\partial a_2} = \frac{a_2}{a_1}(1 - \frac{1}{\sqrt{1-a_2^2}}) < 0.$$

A slightly longer calculation and easy estimates show, conversely, that γ_1 is increasing in a_1 and a_2. Consider the line $\frac{1}{\sqrt{2}} - a_2 = 8(a_1 - \frac{1}{\sqrt{2}})$ for $\frac{2}{3} \leq a_2 \leq \frac{1}{\sqrt{2}} \leq a_1 \leq \frac{1}{\sqrt{\sqrt{2}+\frac{1}{2}}}$ (which originates as an approximation of the curve defined by $\gamma_1(a_1,a_2) = \gamma_2(a_1,a_2)$). By (21)

$$\frac{1}{2}P_{n-2}(a) \leq \max\{\gamma_1(a_1,a_2), \gamma_2(a_1,a_2) \mid \frac{1}{\sqrt{2}} - a_2 = 8(a_1 - \frac{1}{\sqrt{2}}), \frac{2}{3} \leq a_2 \leq \frac{1}{\sqrt{2}} \leq a_1 \leq \frac{1}{\sqrt{\sqrt{2}+\frac{1}{2}}}\}.$$

One checks that for all $n \geq 4$

$$\frac{\partial\gamma_1}{\partial t}(t, \frac{1}{\sqrt{2}} - 8t) = \frac{\partial\gamma_1}{\partial a_1} - 8\frac{\partial\gamma_1}{\partial a_2} < 0,$$

$$\frac{\partial \gamma_2}{\partial t}(t, \frac{1}{\sqrt{2}} - 8t) = \frac{\partial \gamma_2}{\partial a_1} - 8\frac{\partial \gamma_2}{\partial a_2} < 0.$$

Therefore

$$\gamma_1(a_1, \frac{9}{\sqrt{2}} - 8a_1) \leq \gamma_1(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = (n-2)\sqrt{2} + 1$$

and

$$\gamma_2(a_1, \frac{9}{\sqrt{2}} - 8a_1) \leq \gamma_1(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = (n-2)\sqrt{2} + 1.$$

Hence $P_{n-2}(a) \leq 2((n-2)\sqrt{2}+1) = P_{n-2}(a_{max})$ for all $\frac{2}{3} \leq a_2 \leq \frac{1}{\sqrt{2}} \leq a_1 \leq \frac{1}{\sqrt{\sqrt{2}+\frac{1}{2}}}$. □

Next, we consider a similar interpolation scheme if $a_2 < \frac{2}{3}$.

Lemma 9. *Assume that $a \in S^{n-1}$, $a_1 \in (\sqrt{\sqrt{2}-1}, \frac{1}{\sqrt{2}})$ and $a_2 < \frac{2}{3}$. Then for all $n \geq 3$*

$$P_{n-2}(a) \leq P_{n-2}(a_{max}) = 2((n-2)\sqrt{2}+1).$$

If $a_1 \in (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{\sqrt{2}+\frac{1}{2}}})$ and $a_2 < \frac{2}{3}$, the same holds for all $n \geq 3$, except for possibly $n = 5$ or $n = 6$.

Proof. i) Suppose first that $a_1 \leq \frac{1}{\sqrt{2}}$. Since $a_2 < \frac{2}{3}$, $f(a_2^{-2}) \leq \sqrt{\frac{3}{\pi}}$ by Proposition 7. If $\sqrt{\sqrt{2}-1} \leq a_1 \leq \frac{2}{3}$, also $f(a_1^{-2}) \leq \sqrt{\frac{3}{\pi}}$. If $\frac{2}{3} < a_1 \leq \frac{1}{\sqrt{2}}$, we again use the convexity of f to get the slightly weaker estimate $a_1^2 f(a_1^{-2}) \leq \phi_2(a_1)$, where ϕ_2 is as in i) of the proof of Lemma 8. This yields

$$\begin{aligned} A_{n-1}(a) &\leq \left(\sum_{k=1}^{n} a_k^2 f(a_k^{-2})\right)\sqrt{2} \leq \left(a_1^2 f(a_1^{-2}) + (1-a_1^2)\sqrt{\frac{3}{\pi}}\right)\sqrt{2} \\ &\leq \left((9a_1^2 - 4) + (5 - 9a_1^2)\sqrt{\frac{3}{\pi}}\right)\sqrt{2} =: \psi_1(a_1)\sqrt{2}, \end{aligned}$$

$$\frac{1}{2}P_{n-2}(a) \leq \left((n-1)\sqrt{1 - \frac{1-a_1^2}{n-1}}\right)\ \psi_1(a_1)\sqrt{2} =: \gamma(a_1).$$

As easily seen, γ is increasing for $a_1 \leq \frac{1}{\sqrt{2}}$, so that

$$\begin{aligned} \frac{1}{2}P_{n-2}(a) &\leq \gamma(\frac{1}{\sqrt{2}}) = (n-1)\sqrt{1 - \frac{1}{2(n-1)}}\ \frac{1}{\sqrt{2}}\ (1 + \sqrt{\frac{3}{\pi}}) \\ &< (n-2)\sqrt{2} + 1 = \frac{1}{2}P_{n-2}(a_{max}) \end{aligned} \tag{22}$$

for all $n \geq 5$.

ii) Assume now that $\frac{1}{\sqrt{2}} < a_1 \leq \frac{1}{\sqrt{\sqrt{2}+\frac{1}{2}}}$ and $a_2 < \frac{2}{3}$. Then again $f(a_2^{-2}) \leq \sqrt{\frac{3}{\pi}}$ and for $f(a_1^{-2})$ we get by interpolation $a_1^2 f(a_1^{-2}) \leq \phi_1(a_1)$, where ϕ_1 is as in part ii) of the proof of Lemma 8. Therefore

$$A_{n-1}(a) \leq \left(\sum_{k=1}^{n} a_k^2 f(a_k^{-2}) \right) \sqrt{2} \leq \left(\phi_1(a_1) + (1-a_1^2)\sqrt{\frac{3}{\pi}} \right) \sqrt{2} =: \psi_2(a_1)\sqrt{2},$$

$$\frac{1}{2} P_{n-2}(a) \leq ((n-1)\sqrt{1 - \frac{1-a_1^2}{n-1}}) \ \min(\psi_2(a_1)\sqrt{2}, \frac{1}{a_1}) =: \min(\gamma_1(a_1), \gamma_2(a_1)), \tag{23}$$

where we also used that $A_{n-1}(a) \leq \frac{1}{a_1}$ holds. Differentiating γ_1 and γ_2, one finds that $\gamma_1' > 0 > \gamma_2'$ in the range of a_1 considered. Therefore, γ_1 is increasing and γ_2 is decreasing. We have, independently of $n \in \mathbb{N}, n \geq 3$, that $\gamma_1(\bar{a_1}) = \gamma_2(\bar{a_1})$ for $\bar{a_1} \simeq 0.71254$ and $\gamma_1(\bar{a_1}) = \gamma_2(\bar{a_1}) \leq (n-2)\sqrt{2} + 1$ for all $n \geq 7$. Then $P_{n-2}(a) \leq P_{n-2}(a_{max})$.

For $n = 6$, this estimate is violated by < 0.006, for $n = 5$ by < 0.015. It is correct for $n = 5$, $a_1 \notin (0.7095, 0.7149)$ and for $n = 6$, $a_1 \notin (0.7115, 0.7133)$.

For $n = 3$ we already proved Theorem 1. For $n = 4$, the above estimates in i) and ii) yield $P_2(a) \leq P_2(a_{max})$ if $a_1 \notin (0.7069, 0.7177)$. However, the explicit formulas given in equation (11) for $P_2(a)$ yield $P_2(a) \leq P_2(a_{max})$ also for these a_1. Only the first or the second case in (11) can occur, since $a_1 < a_2 + a_3 + a_4$ in our situation. The maximum of the second formula occurs for $a_3 = a_4$, with $P_2(a) < 3.6 < 2(\sqrt{2}+1)$. The first expression in (11) yields an even smaller maximal value. We do not give the details. In principle, the explicit formulas in (11) could be used to prove $P_2(a) \leq P_2(a_{max})$ for all $a \in S^3$, as in the case $n = 3$, though this would be more complicated.

Replacing (23) by the slightly stronger estimate

$$\frac{1}{2} P_{n-2}(a) \leq \left((n-2)\sqrt{1 - \frac{1-a_1^2-a_2^2}{n-2}} + \sqrt{1-a_2^2} \right) \ \min(\psi_2(a_1)\sqrt{2}, \frac{1}{a_1}),$$

we get $P_{n-2}(a) \leq P_{n-2}(a_{max})$ for all $a_1 \in (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{\sqrt{2}+\frac{1}{2}}})$ and $a_2 \geq 0.5803$ if $n = 5$ and $a_2 \geq 0.4952$ if $n = 6$. Therefore the only cases left open to prove Theorem 1 are

$$\left\{ \begin{array}{ccccc} n = 5 & , & a_1 \in (0.7095, 0.7149) & , & a_2 \leq 0.5803 \\ n = 6 & , & a_1 \in (0.7115, 0.7133) & , & a_2 \leq 0.4952 \end{array} \right\}, \tag{24}$$

i.e. when $a_1 \simeq \frac{1}{\sqrt{2}}$ and $a_2 < a_1 - \frac{1}{8}$. This case will be treated by using the following Lemma. □

Lemma 10. *For $a_1 \in (0.7095, 0.7149)$, $\frac{1}{\sqrt{10}} \leq a_2 \leq 0.5803$ we have*

$$\frac{2}{\pi} \int_0^\infty \left| \frac{\sin(a_1 s)}{a_1 s} \frac{\sin(a_2 s)}{a_2 s} \right|^{\frac{1}{a_1^2+a_2^2}} ds \leq 0.985\sqrt{2}. \tag{25}$$

We will prove Lemma 10 in the Appendix. Using (25), we finish the proof of Theorem 1 in the remaining cases (24):

By Hölder's inequality and (25)

$$A_{n-1}(a) \leq \Big(\frac{2}{\pi}\int_0^\infty \left|\frac{\sin(a_1 s)}{a_1 s}\frac{\sin(a_2 s)}{a_2 s}\right|^{\frac{1}{a_1^2+a_2^2}} \, ds\Big)^{a_1^2+a_2^2} \prod_{j=3}^n \Big(\frac{2}{\pi}\int_0^\infty \left|\frac{\sin(a_j s)}{a_j s}\right|^{a_j^{-2}} \, ds\Big)^{a_j^2}$$
$$\leq \left((a_1^2+a_2^2)\; 0.985 + (1-a_1^2-a_2^2)\sqrt{\frac{3}{\pi}} \right)\sqrt{2} \leq 0.985\sqrt{2}.$$

Therefore

$$\frac{1}{2}P_{n-2}(a) \leq (n-1)\sqrt{1-\frac{1-a_1^2}{n-1}}\; 0.985\sqrt{2}.$$

For $n=5$ and $n=6$ and $a_1 \leq 0.7149$ this is $< (n-2)\sqrt{2}+1$, so that $P_{n-2}(a) \leq P_{n-2}(a_{max})$ also in the cases (24). This ends the proof of Theorem 1. $\square$

As for the lower estimate of $P_{n-2}(a)$, the natural conjecture would be $P_{l(n-2)}(a) \geq P_{l(n-2)}(a_{min}) = 2\pi^{l-1}(n-1)$, $a_{min} = (1,0,\cdots,0)$, with $l=1$ if $\mathbb{K}=\mathbb{R}$ and $l=2$ if $\mathbb{K}=\mathbb{C}$. We can only prove a slightly weaker estimate.

Proposition 11. *For any $a \in \mathbb{K}^n$ with $|a|=1$*

$$P_{n-2}(a) \geq 2(n-2) \quad , \quad \mathbb{K}=\mathbb{R},$$
$$P_{2(n-2)}(a) \geq 2\pi(n-2) \quad , \quad \mathbb{K}=\mathbb{C}.$$

Proof of Proposition 11.
i) We may assume $a \in S^{n-1}$, $a_1 \geq \cdots \geq a_n \geq 0$. In the complex case $\mathbb{K}=\mathbb{C}$, by Propositions 2, 3 and 4

$$\frac{1}{2\pi}P_{2(n-2)}(a) \geq \min\left\{ \sum_{k=1}^n (1-a_k^2)C_k \mid 0 \leq C_k \leq A_{2(n-1)}(a), \sum_{k=1}^n C_k = (n-1)A_{2(n-1)}(a) \right\}.$$

Since $(1-a_k^2)_{k=1}^n$ is increasing in k, the sum $\sum_{k=1}^n (1-a_k^2)C_k$ is minimized, if the C_k are decreasing, i.e. for $C_1 = \cdots = C_{n-1} = A_{2(n-1)}(a)$ and $C_1 = 0$ so that by Hensley [H] and Oleszkiewicz, Pełczyński [OP]

$$\frac{1}{2\pi}P_{2(n-2)}(a) \geq \sum_{k=1}^{n-1}(1-a_k^2)A_{2(n-1)}(a) = (n-1-(1-a_1^2))A_{2(n-1)}(a)$$
$$\geq (n-2)A_{2(n-1)}(a) \geq (n-2).$$

ii) Similarly, we find in the real case $\mathbb{K}=\mathbb{R}$, using Hensley's lower estimate [H] for the parallel section function A_{n-1}

$$\frac{1}{2}P_{n-2}(a) \geq \sum_{k=1}^{n-1}\sqrt{1-a_k^2}\; A_{n-1}(a) \geq \sum_{k=1}^{n-1}\sqrt{1-a_k^2}.$$

Now $\phi(x) = \sqrt{1-x}$ is concave and decreasing on $[0,1]$. Therefore for any $x_2 < y_2 < y_1 < x_1$ with $x_2^2 + x_1^2 = y_2^2 + y_1^2$ we have that $\phi(y_2^2) + \phi(y_1^2) \geq \phi(x_2^2) + \phi(x_1^2)$, i.e. the sum gets smaller by moving all coordinates towards 0 and 1. Hence

$$\frac{1}{2} P_{n-2}(a) \geq (n-2) + \sqrt{1 - \sum_{j=1}^{n-1} a_j^2} \geq (n-2).$$

$\square$

Remarks.
(a) One possibility to improve the lower estimate in Proposition 11 would be to understand how the monotonicity properties of the sequence $a = (a_k)_{k=1}^n$ affect the size of the integrals $D_k(a)$, since e.g. in the real case

$$P_{n-1}(a) = \sum_{k=1}^{n} \sqrt{1-a_k^2}\; D_k(a).$$

(b) Numerical estimates of Ball's integral function f, $f(p) := \sqrt{\frac{p}{2}} \frac{2}{\pi} \int_0^\infty |\frac{\sin(t)}{t}|^p \, dt$ indicate that $f(p) = \sqrt{\frac{3}{\pi}}$ for some p_1 with $2.165 < p_1 < 2.166$, that f attains its minimum in p_2 with $3.36 < p_2 < 3.37$ and that f is convex for $1 < p < p_0$ with $4.46 < p_0 < 4.47$. The behavior of f near ∞ is well-understood: by a result of Kerman, Ol'hava and Spektor [KOS]

$$f(p) = \sqrt{\frac{3}{\pi}} \left(1 - \frac{3}{20} \frac{1}{p} - \frac{13}{1120} \frac{1}{p^2} + O(\frac{1}{p^3}) \right).$$

4 An application of the Busemann-Petty type

In this section we apply the result of Theorem 1 to the surface area version of the Busemann-Petty problem described in the Introduction.

Theorem 12. *For each $n \geq 14$, there exist origin-symmetric convex bodies K, L in $\mathbb{R}^n$ such that for all $a \in S^{n-1}$*

$$vol_{n-2}(\partial K \cap a^\perp) \leq vol_{n-2}(\partial L \cap a^\perp)$$

but

$$vol_{n-1}(\partial K) > vol_{n-1}(\partial L).$$

Proof. Let $K = B_\infty^n$ be the unit cube in $\mathbb{R}^n$. Let L be the Euclidean ball of radius r in $\mathbb{R}^n$ so that the perimeters of hyperplane sections of L are all equal to the maximal perimeter of sections of K. Namely, for any $a \in S^{n-1}$

$$vol_{n-2}(\partial K \cap a^\perp) \leq vol_{n-2}(\partial K \cap a_{max}^\perp) = 2((n-2)\sqrt{2}+1) = vol_{n-2}(rS^{n-2}) = r^{n-2} \; \frac{2\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2})} \; ,$$

i.e.

$$r = \frac{[((n-2)\sqrt{2}+1)\Gamma(\frac{n-1}{2})]^{\frac{1}{n-2}}}{\pi^{(n-1)/(2(n-2))}}.$$

The desired inequality for the surface areas of K and L happens when

$$vol_{n-1}(\partial B_\infty^n) = 2n > vol_{n-1}(rS^{n-1}) = r^{n-1}\ \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}\ .$$

The latter is equivalent to

$$1 > \frac{\pi^{n/2}}{n\Gamma(\frac{n}{2})} r^{n-1} = \frac{\pi^{n/2}}{n\Gamma(\frac{n}{2})} \frac{[((n-2)\sqrt{2}+1)\Gamma(\frac{n-1}{2})]^{\frac{n-1}{n-2}}}{\pi^{(n-1)^2/(2(n-2))}}$$

$$= \frac{1}{n\Gamma(\frac{n}{2})} \frac{[((n-2)\sqrt{2}+1)\Gamma(\frac{n-1}{2})]^{\frac{n-1}{n-2}}}{\pi^{1/(2(n-2))}} =: BP(n)\ .$$

Then BP is decreasing in n, with $BP(x_0) = 1$ for $x_0 \simeq 13.70$, so $BP(n) < 1$ for all $n \geq 14$. $\square$

A similar argument can be made in the complex case when there are similar counterexamples for all $n \geq 11$.

5 Appendix

In the Appendix, we present the technical proofs of Proposition 6 and of Lemma 10.

Proof of Proposition 6 (a).
The fact that $\lim_{p\to\infty} f(p) = \sqrt{\frac{3}{\pi}}$ is well-known [KOS], following from $\frac{\sin(x)}{x} \leq \exp(-\frac{x^2}{6})$ for $0 \leq x \leq \pi$ and $\sqrt{\frac{p}{2}} \frac{2}{\pi} \int_0^\infty \exp(-\frac{x^2 p}{6})\ dx = \sqrt{\frac{3}{\pi}}$. Now let $p_0 := \frac{9}{4}$. Since

$$\frac{\sin(x)}{x} = \prod_{n\in\mathbb{N}} \left(1 - \frac{x^2}{(n\pi)^2}\right),$$

and $\ln(1-y) \leq -y - \frac{1}{2}y^2$ for $0 \leq y < 1$, we find for $0 < x < \pi$

$$\ln\left(\frac{\sin(x)}{x}\right) = \sum_{n\in\mathbb{N}} \ln(1 - \frac{x^2}{(n\pi)^2}) \leq -\sum_{n\in\mathbb{N}} (\frac{x}{n\pi})^2 - \frac{1}{2}\sum_{n\in\mathbb{N}} (\frac{x}{n\pi})^4 = -\frac{x^2}{6} - \frac{x^4}{180},$$

$$\frac{\sin(x)}{x} \leq \exp(-\frac{x^2}{6} - \frac{x^4}{180}) \quad , \quad x \in (0,\pi).$$

This implies

$$\left(\frac{\sin(x)}{x}\right)^{p_0} \leq \left(\frac{\sin(x)}{x}\right)^2 \exp(-\frac{x^2}{24} - \frac{x^4}{720}) \leq \left(\frac{\sin(x)}{x}\right)^2 (1 - \frac{x^2}{24})\ ,\ x \in (0,\pi),$$

$$I_0 := \sqrt{\frac{p_0}{2}\frac{2}{\pi}\int_0^\pi \left|\frac{\sin(x)}{x}\right|^{p_0}\ dx} \leq \sqrt{\frac{p_0}{2}\frac{2}{\pi}\int_0^\pi \left|\frac{\sin(x)}{x}\right|^2\ (1-\frac{x^2}{24})\ dx}$$
$$= \sqrt{\frac{p_0}{2}\frac{2}{\pi}(Si(2\pi)-\frac{\pi}{48})} \leq 0.91340.$$

Here Si denotes the sine integral function. For $x \in (\pi, 2\pi)$, $|\frac{\sin(x)}{x}|^{1/4} \leq |\frac{\sin(x_0)}{x_0}|^{1/4} \leq 0.683$ where $x_0 \simeq 4.493$. Hence

$$I_1 := \sqrt{\frac{p_0}{2}\frac{2}{\pi}\int_\pi^{2\pi} \left|\frac{\sin(x)}{x}\right|^{p_0}\ dx} \leq 0.683\sqrt{\frac{p_0}{2}\frac{2}{\pi}\int_\pi^{2\pi} \left|\frac{\sin(x)}{x}\right|^2\ dx}$$
$$= 0.683\sqrt{\frac{p_0}{2}\frac{2}{\pi}(Si(4\pi)-Si(2\pi))} \leq 0.03414.$$

For $x \in (k\pi, (k+1)\pi)$, $|\frac{\sin(x)}{x}|^{p_0} \leq (\frac{1}{k\pi})^{1/4}\ (\frac{\sin(x)}{x})^2$ and

$$I_k := \sqrt{\frac{p_0}{2}\frac{2}{\pi}\int_{k\pi}^{(k+1)\pi} \left|\frac{\sin(x)}{x}\right|^{p_0}\ dx} \leq \sqrt{\frac{p_0}{2}\frac{2}{\pi}(\frac{1}{k\pi})^{1/4}\ (Si((2k+1)\pi)-Si(2k\pi))}.$$

Also,

$$J_k := \sqrt{\frac{p_0}{2}\frac{2}{\pi}\int_{k\pi}^{\infty} \left|\frac{\sin(x)}{x}\right|^{p_0}\ dx} \leq \sqrt{\frac{p_0}{2}\frac{2}{\pi}(\frac{1}{k\pi})^{1/4}\int_{k\pi}^{\infty} \left|\frac{\sin(x)}{x}\right|^2\ dx}$$
$$= \sqrt{\frac{p_0}{2}\frac{2}{\pi}(\frac{1}{k\pi})^{1/4}(\frac{\pi}{2}-Si(2k\pi))}.$$

Calculation then shows that $f(p_0) = \sum_{k=0}^5 I_k + J_6 \leq 0.977 < \sqrt{\frac{3}{\pi}}$. □

Proof of Proposition 6 (b).
Now let $p_0 := \sqrt{2}+\frac{1}{2} \simeq 1.9142 < 2$. The claim is that $f(p_0) < \frac{51}{50}$. We note that for $p < 2$, $f(p) > 1$. For $0 < x < \frac{\pi}{2}$ and $p > 1$, we have, similarly as above,

$$\left(\frac{\sin(x)}{x}\right)^p \leq \exp(-\frac{x^2}{6})\ (1-\frac{p}{180}x^4),$$

yielding

$$I_{0,1} := \sqrt{\frac{p_0}{2}\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \left|\frac{\sin(x)}{x}\right|^{p_0}\ dx} \leq \sqrt{\frac{p_0}{2}\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \exp(-\frac{x^2}{6})\ (1-\frac{p_0}{180}x^4)\ dx}$$
$$= \sqrt{\frac{p_0}{2}\frac{2}{\pi}\left[\frac{\pi}{480p_0}(36+\pi^2p_0)\exp(-\frac{\pi^2p_0}{24}) + \frac{\sqrt{6\pi}}{p_0^{3/2}}(\frac{p_0}{2}-\frac{3}{40})\ \mathrm{erf}(\pi\sqrt{\frac{p_0}{24}})\right]} \leq 0.76509,$$

where erf denotes the standard error function, $\mathrm{erf}(x) := \frac{2}{\sqrt{\pi}}\int_0^\infty \exp(-t^2)\ dt$. For $x \in (\frac{\pi}{2}, \pi)$, Taylor expansion at $x = \pi$ yields an approximation

$$\left(\frac{\sin(x)}{x}\right)^2 \leq (1-\frac{x}{\pi})^2\sum_{j=0}^6 c_j(1-\frac{x}{\pi})^j,$$

where $c_0 = 1$, $c_1 = 2$, $c_2 = -(\frac{\pi^2}{3} - 3)$ etc. Taking the $\frac{p_0}{2}$-th power of this gives an estimate for $\left(\frac{\sin(x)}{x}\right)^{p_0}$, $x \in (\frac{\pi}{2}, \pi)$ of the form

$$(\frac{\sin(x)}{x})^{p_0} \leq (1 - \frac{x}{\pi})^{p_0} \sum_{j=0}^{6} d_j (1 - \frac{x}{\pi})^j \ ,$$

$d_0 = 1$, $d_1 = 1.9142$ etc., where the right hand side may be integrated exactly, giving

$$I_{0,2} := \sqrt{\frac{p_0}{2}} \frac{2}{\pi} \int_{\frac{\pi}{2}}^{\pi} \left| \frac{\sin(x)}{x} \right|^{p_0} \ dx \leq 0.13531.$$

In (π, ∞), we use approximations of $(\sin(x))^2$ and $|\sin(x)|^{p_0}$ instead of those for $|\frac{\sin(x)}{x}|^{p_0}$. For $x \in (0, 2\pi)$, we have the alternating series estimate

$$(\sin(x))^2 \leq \sum_{j=0}^{4} (-1)^j c_j (x - \frac{\pi}{2})^{2j} \ , \ c_0 = c_1 = 1, c_2 = \frac{1}{3}, c_3 = \frac{2}{45}, c_4 = \frac{1}{315}.$$

Taking the $\frac{p_0}{2}$-th power of this gives an estimate for

$$(\sin(x))^{p_0} \leq \sum_{j=0}^{4} (-1)^j d_j (x - \frac{\pi}{2})^{2j} =: l(x) \ , \ d_0 = 1, d_1 = \frac{p_0}{2}, d_2 = \frac{p_0^2}{8} - \frac{p_0}{12} \text{ etc.}$$

Using this, we find for $l \in \mathbb{N}$

$$\begin{aligned} \sqrt{\frac{p_0}{2}} \frac{2}{\pi} \int_{\pi}^{(l+1)\pi} \left| \frac{\sin(x)}{x} \right|^{p_0} \ dx &= \sqrt{\frac{p_0}{2}} \frac{2}{\pi} \sum_{k=1}^{l} \int_0^{\pi} \frac{\sin(x)^{p_0}}{(x + k\pi)^{p_0}} \ dx \\ &=: \sum_{k=1}^{l} I_k \leq \sqrt{\frac{p_0}{2}} \frac{2}{\pi} \sum_{k=1}^{l} \int_0^{\pi} \frac{l(x)}{(x + k\pi)^{p_0}} \ dx. \end{aligned}$$

The integrals $\int_0^{\pi} \frac{l(x)}{(x+k\pi)^{p_0}} \ dx$ can be integrated exactly. One finds, choosing $l = 5$,

$$(I_{0,1} + I_{0,2}) + (\sum_{k=1}^{5} I_k) \leq 0.90040 + 0.09383 = 0.99423.$$

The remaining integral over $(6\pi, \infty)$ is estimated by

$$\begin{aligned} \sqrt{\frac{p_0}{2}} \frac{2}{\pi} \int_{6\pi}^{\infty} \left| \frac{\sin(x)}{x} \right|^{p_0} \ dx &\leq \sqrt{\frac{p_0}{2}} \frac{2}{\pi} \sum_{k=6}^{\infty} \frac{1}{(k\pi)^{p_0}} \int_0^{\pi} l(x) dx \\ &\leq \sqrt{\frac{p_0}{2}} \frac{2}{\pi} \frac{1}{\pi^{p_0}} \ (\zeta(p_0) - \sum_{k=1}^{5} \frac{1}{k^{p_0}}) \ 1.6 \leq 0.02567. \end{aligned}$$

Hence $f(p_0) \leq 0.99423 + 0.02567 = 1.0199 < \frac{51}{50}$. □

Proof of Proposition 6 (c).
(a) We claim that $f|_{[\sqrt{2}+\frac{1}{2},\frac{9}{4}]}$ is convex. Put $g(p,x) := \sqrt{p}\ |\frac{\sin(x)}{x}|^p$ and $h(p,x) := 4p^2(\ln|\frac{\sin(x)}{x}|)^2 + 4p(\ln|\frac{\sin(x)}{x}|) - 1$. Then

$$\frac{\partial^2 g}{\partial p^2}(p,x) = h(p,x)\ \frac{1}{4p^{\frac{3}{2}}}\ \left|\frac{\sin(x)}{x}\right|^p, x > 0.$$

We will show that for all $p \in [\sqrt{2}+\frac{1}{2}, \frac{9}{4}]$

$$\int_0^\infty h(p,x)\ \left|\frac{\sin(x)}{x}\right|^p\ dx \geq \frac{1}{5} > 0. \tag{26}$$

This yields that $f''(p) > 0$, so that f is convex in this interval. We remark that f is negative in $0 < x < x_p$ and non-negative for $x > x_p$, where $x_p \simeq 1.8$ for p in the given range: More precisely, note that

$$4p^2(\ln y)^2 + 4p(\ln y) - 1 = 0 \ \Leftrightarrow\ \ln y = -\frac{1 \pm \sqrt{2}}{2p} \ \Leftrightarrow\ y = \exp(-\frac{1 \pm \sqrt{2}}{2p})\ .$$

For $y = |\frac{\sin(x)}{x}| < 1$, we need the plus sign and $y_p = \exp(-\frac{1+\sqrt{2}}{2p}) = \frac{\sin(x_p)}{x_p}$ yields a unique solution x_p, decreasing with p,

$$x_p = \begin{cases} 1.8205 & p = \sqrt{2}+1/2 \\ 1.7863 & p = 2 \\ 1.6965 & p = 9/4 \end{cases},$$

$$y_p = \begin{cases} 0.5323 & p = \sqrt{2}+1/2 \\ 0.5469 & p = 2 \\ 0.5848 & p = 9/4 \end{cases}.$$

We estimate $\frac{\partial^2 g}{\partial p^2}(p,x)$ from below on the intervals $(0,x_p)$, (x_p,π) and $(\pi,2\pi)$ and integrate that.

(b) For all $0 < x < x_p$, $h(p,x) < 0$. For these we have to estimate $(\frac{\sin(x)}{x})^p$ from above to estimate the integrand in (26) from below. We again use

$$\left(\frac{\sin(x)}{x}\right)^p \leq \exp(-\frac{x^2 p}{6})\ (1 - \frac{p}{180}x^4)$$

as in the previous proof. Replacing $\ln(\frac{\sin(x)}{x})$ in $h(p,x)$ by $-\frac{x^2}{6} - \frac{x^4}{180}$ increases $h(p,x)$ by at most 10^{-3}, and only for $x \leq 1$, so that

$$\begin{aligned} &\int_0^{x_p} h(p,x)\ \left(\frac{\sin(x)}{x}\right)^p\ dx \\ &\quad \geq \int_0^{x_p} \exp(-\frac{x^2 p}{6})\ (1 - \frac{p}{180}x^4)\ (4p^2(\frac{x^2}{6} + \frac{x^4}{180})^2 + 4p(-\frac{x^2}{6} - \frac{x^4}{180}) - 1)\ dx - 10^{-3} \\ &\quad \geq \int_0^{x_p} \exp(-\frac{x^2 p}{6})\ (-1 - \frac{2}{3}px^2 + (\frac{p^2}{9} - \frac{p}{60})x^4 + \frac{p^2}{900}x^6)\ dx - 10^{-3} \\ &\quad = -\frac{1}{3}x_p\ (3 + px_p^2 + \frac{p}{100}x_p^4)\ \exp(-\frac{p}{6}x_p^2) - 10^{-3} =: \gamma_1(p)\ . \end{aligned}$$

The second inequality follows by expanding the product of both polynomials and easy lower estimates. Actually, the leading term in x^6 is $\frac{p^2}{90}x^6$, but the lesser value $\frac{p^2}{900}x^6$ was chosen to allow for an exact integration without error functions. We note that px_p^2 and $\gamma_1(p)$ are both increasing in p, γ_1 with negative values,

$$\gamma_1(p) = \begin{cases} -2.015 & p = \sqrt{2} + 1/2 \\ -1.971 & p = 2 \\ -1.858 & p = 9/4 \end{cases} .$$

(c) For $x \in (x_p, \pi)$, $h(p, x) > 0$. For $y = \frac{\sin(x)}{x}$, $\frac{dy}{dx} = \frac{\cos(x)}{x} - \frac{\sin(x)}{x^2}$. We have $-0.4362 \le \frac{dy}{dx} \le \frac{1}{\pi} \simeq 0.3183$ for all $x \in (1.6965, \pi)$. Therefore

$$\int_{x_p}^{\pi} h(p,x) \left|\frac{\sin(x)}{x}\right|^p dx \ge \int_{x_p}^{\pi} h(p,x) \left|\frac{\sin(x)}{x}\right|^p \left|\frac{d(\frac{\sin(x)}{x})}{dx}\right| \frac{1}{0.4362}\, dx$$
$$\ge 2.2926 \int_0^{y_p} (4p^2(\ln y)^2 + 4p(\ln y) - 1)\, y^p\, dy =: I \ ,$$

substituting $y = \frac{\sin(x)}{x}$, which maps (x_p, π) bijectively onto $(0, y_p)$. The last integral can be calculated explicitly,

$$I = 2.2926 \left(\frac{4p^2}{1+p} (\ln y_p)^2 - \frac{4p(p-1)}{(1+p)^2} (\ln y_p) + \frac{3p^2 - 6p - 1}{(1+p)^3} \right) y_p^{p+1} =: \gamma_2(p) \ .$$

The function γ_2 is increasing in p, too, with positive values. One has e.g.

$$\gamma_2(p) = \begin{cases} 0.898 & p = \sqrt{2} + 1/2 \\ 0.916 & p = 2 \\ 0.956 & p = 9/4 \end{cases} .$$

(d) For $x \in (\pi, 2\pi)$, $h(p,x)$ attains large values: if $p \le 2$, $h(p,x) \ge 21$ and if $p > 2$, even $h(p,x) > 24$. We find for $p \le 2$

$$\int_{\pi}^{2\pi} h(p,x) \left|\frac{\sin(x)}{x}\right|^p dx \ge 21 \int_{\pi}^{2\pi} \left|\frac{\sin(x)}{x}\right|^p dx \ge 21 \int_{\pi}^{2\pi} \left|\frac{\sin(x)}{x}\right|^2 dx$$
$$= 21(Si(4\pi) - Si(2\pi)) \simeq 1.554 =: \gamma_3(p),$$

and for $p > 2$, using Hölder's inequality,

$$\int_{\pi}^{2\pi} h(p,x) \left|\frac{\sin(x)}{x}\right|^p dx \ge 24 \int_{\pi}^{2\pi} \left|\frac{\sin(x)}{x}\right|^p dx$$
$$\ge 24 \frac{1}{\pi^{p/2-1}} \left(\int_{\pi}^{2\pi} \left|\frac{\sin(x)}{x}\right|^2 dx \right)^{p/2} = 24\pi \left(\frac{1}{\pi}[Si(4\pi) - Si(2\pi)]\right)^{p/2} =: \gamma_3(p) \ .$$

The function γ_3 is decreasing in $[2, \frac{9}{4}]$, $\gamma_3(2) \simeq 1.776$, $\gamma_3(\frac{9}{4}) \simeq 1.112$.

We conclude from the estimates in (b), (c) and (d) that

$$\int_0^\infty h(p,x)\ \left|\frac{\sin(x)}{x}\right|^p\ dx \geq \int_0^{2\pi} h(p,x)\ \left|\frac{\sin(x)}{x}\right|^p\ dx \geq \gamma_1(p)+\gamma_2(p)+\gamma_3(p) =: \gamma(p)\ .$$

This way, we find that $\gamma(p) \geq 0.43$ for $\sqrt{2}+\frac{1}{2} \leq p \leq 2$ and $\gamma(p) \geq 0.21$ for $2 < p \leq \frac{9}{4}$. Hence $f''|_{[\sqrt{2}+\frac{1}{2},\frac{9}{4}]} > 0$, and f is convex there.
Since $f(\sqrt{2}+\frac{1}{2}) > 1 > f(\frac{9}{4})$ and f is convex, f is decreasing in $[\sqrt{2}+\frac{1}{2},\frac{9}{4}]$. □

It remains to prove Lemma 10 which is only used for $\mathbb{K}=\mathbb{R}$, $n=5$ or $n=6$ and a_1 in a very small interval near $\frac{1}{\sqrt{2}}$. Therefore we only outline the essential parts of the proof. Basically, the integral in (25) is strictly less than $\sqrt{2}$ since a_1 and a_2 deviate by at least $\frac{1}{8}$, and so there is some cancelation in the product $\sin(a_1 s)\sin(a_2 s)$.

Proof of Lemma 10. We have to estimate with $d := \frac{a_2}{a_1}$

$$\frac{2}{\pi}\int_0^\infty \left|\frac{\sin(a_1 r)}{a_1 r}\frac{\sin(a_2 r)}{a_2 r}\right|^{\frac{1}{a_1^2+a_2^2}}\ dr = \sqrt{2}\frac{1}{\sqrt{2}a_1}\frac{2}{\pi}\int_0^\infty \left|\frac{\sin(s)}{s}\frac{\sin(ds)}{ds}\right|^{\frac{1}{a_1^2(1+d^2)}}\ ds =: \sqrt{2}J\ .$$

Note that, by assumption, $\sqrt{2}a_1 \simeq 1$, $a_1^{-2} \simeq 2$. We have to show $J \leq 0.985$.

a) For $0 < s < \pi$, $\ln(\frac{\sin(x)}{x}) \leq -\frac{s^2}{6} - \frac{s^4}{180} - O(s^6)$, with only negative terms in the series. This yields

$$\left|\frac{\sin(s)}{s}\frac{\sin(ds)}{ds}\right|^{\frac{1}{a_1^2(1+d^2)}} \leq \exp(-\frac{s^2}{6a_1^2})\ (1-\frac{s^4}{180\ a_1^2}\frac{1+d^4}{1+d^2})\ ,$$

Integration of the right side over $[0,\pi]$ yields

$$\begin{aligned} I_0 &:= \frac{1}{\sqrt{2}a_1}\frac{2}{\pi}\int_0^\pi \left|\frac{\sin(s)}{s}\frac{\sin(ds)}{ds}\right|^{\frac{1}{a_1^2(1+d^2)}}\ ds \\ &\leq \sqrt{\frac{3}{\pi}}\mathrm{erf}(\frac{\pi}{\sqrt{6}a_1})(1-\frac{3}{20}a_1^2\frac{1+d^4}{1+d^2}) + \frac{\sqrt{2}}{60}\exp(-\frac{\pi^2}{6a_1^2})\frac{\pi^2+9a_1^2}{a_1}\frac{1+d^4}{1+d^2} = \gamma_1(a_1,d)\ . \end{aligned}$$

b) For $\pi \leq s \leq 2\pi$, we use Hölder's inequality with exponent $p = 2a_1^2(1+d^2)$,

$$\begin{aligned} I_1 &:= \frac{1}{\sqrt{2}a_1}\frac{2}{\pi}\int_\pi^{2\pi} \left|\frac{\sin(s)}{s}\frac{\sin(ds)}{ds}\right|^{\frac{1}{a_1^2(1+d^2)}}\ ds \\ &\leq \frac{1}{\sqrt{2}a_1}\left(\frac{2}{\pi}\int_\pi^{2\pi}\left|\frac{\sin(s)}{s}\frac{\sin(ds)}{ds}\right|^2\ ds\right)^{\frac{1}{2a_1^2(1+d^2)}}\ (\frac{2}{\pi}\pi)^{1-\frac{1}{2a_1^2(1+d^2)}} \\ &= \frac{\sqrt{2}}{a_1}[\ (Si(2\pi)-Si(4\pi))\frac{1}{3\pi d^2} + (Si(2\pi d)-Si(4\pi d))\frac{d}{3\pi} \\ &\quad - \sum_\pm (Si(2\pi(1\pm d)) - Si(4\pi(1\pm d)))\frac{(1\pm d)^3}{6\pi d^2} - \sin(\pi d)^2\ \frac{\cos(2\pi d)}{3\pi^2 d^2}\ ]^{\frac{1}{2a_1^2(1+d^2)}} =: \gamma_2(a_1,d)\ . \end{aligned}$$

The function γ_2 is increasing in a_1, since $\frac{A^{1/a_1^2}}{a_1}$ is increasing in a_1 if $A < \exp(-\frac{a_1^2}{2}) \simeq \exp(-\frac{1}{4})$ which is satisfied if A is the appropriate power of the integral.

c) For $2\pi \le s < \infty$, we again use Hölder's inequality with exponent $p = 2a_1^2(1+d^2)$, but differently

$$\begin{aligned} I_2 &:= \frac{1}{\sqrt{2}a_1}\frac{2}{\pi}\int_{2\pi}^{\infty}\left|\frac{\sin(s)}{s}\frac{\sin(ds)}{ds}\right|^{\frac{1}{a_1^2(1+d^2)}} ds \\ &\le \frac{1}{\sqrt{2}a_1}\frac{2}{\pi}\left(\int_{2\pi}^{\infty}\left|\frac{\sin(s)}{s}\right|^2 ds\right)^{\frac{1}{2a_1^2(1+d^2)}} \left(\int_{2\pi}^{\infty}\left|\frac{1}{ds}\right|^{\frac{2}{a_1^2(1+d^2)-1}} ds\right)^{1-\frac{1}{2a_1^2(1+d^2)}} . \end{aligned}$$

Here we estimated $|\sin(ds)|$ by 1. This yields

$$I_2 \le \frac{\sqrt{2}}{\pi a_1}\left(\frac{\pi}{2} - Si(4\pi)\right)^{\frac{1}{2a_1^2(1+d^2)}} \left[\left(\frac{1}{2\pi d}\right)^{\frac{2}{2a_1^2(1+d^2)-1}} \frac{2\pi(2a_1^2(1+d^2)-1)}{3-2a_1^2(1+d^2)} \right]^{1-\frac{1}{2a_1^2(1+d^2)}} =: \gamma_3(a_1,d) .$$

In conclusion, $J = I_0+I_1+I_2 \le (\gamma_1+\gamma_2+\gamma_3)(a_1,d) =: \gamma(a_1,d)$, which, for the range of a_1, a_2 and $d = \frac{a_2}{a_1}$ considered, is bounded by $J \le 0.985$. The function γ essentially does not depend on a_1 since it is considered only in a tiny interval. Nevertheless, its maximum (for the relevant values of d) is attained for the maximal choice $a_1 = 0.7149$. As a function of d, the maximum is attained for the maximal value of d, which is $d = \frac{0.5803}{0.7095} \le 0.818$, i.e. $J \le \gamma(0.7149, 0.818) \le 0.985$. This is not surprising, since the cancelation effect in $\sin(s)\sin(ds)$ is minimal if d is maximal. Obviously, the main contribution to J comes from I_0, which is slightly larger than 0.92 whereas I_1 and I_2 each contribute about 0.03. □

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Mathematisches Seminar
Universität Kiel
24098 Kiel, Germany
hkoenig@math.uni-kiel.de

Department of Mathematics
University of Missouri
Columbia, MO 65211, USA
koldobskiya@missouri.edu