

STABILITY OF A PLANAR FRONT IN A CLASS OF REACTION-DIFFUSION SYSTEMS*

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Abstract. We study the planar front solution for a class of reaction diffusion equations in multidimensional space in the case when the essential spectrum of the linearization in the direction of the front touches the imaginary axis. At the linear level, the spectrum is stabilized by using an exponential weight. A priori estimates for the nonlinear terms of the equation governing the evolution of the perturbations of the front are obtained when perturbations belong to the intersection of the exponentially weighted space with the original space without a weight. These estimates are then used to show that in the original norm, initially small perturbations to the front remain bounded, while in the exponentially weighted norm, they algebraically decay in time.

Key words. traveling waves, fronts, planar fronts, stability, nonlinear stability, essential spectrum, convective instability, exponential weights

AMS subject classifications. 35B35, 35K57, 35K55, 35K58, 35B45

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1. Introduction. Planar traveling fronts are solutions to partial differential equations posed on multidimensional infinite domains that move in a preferred direction with constant speed without changing their shape and that are asymptotic to spatially constant steady-state solutions.

Stability theory of the traveling fronts in reaction-diffusion equations is a vast subject that has a long history and is very active today; see, e.g., [H, KP, Sa, VVV] and the literature cited in these books, as well as [BGHL, BKSS, K, KV, LX, LW, R1, R2, R3, R4, TZKS, X] and the bibliographies therein.

The cornerstone of the stability analysis of the fronts (or pulses), in general, is to determine the location of the spectrum of the linearization of the underlying system about the wave. The spectrum may contain isolated eigenvalues of finite algebraic multiplicity and the essential spectrum; the latter may consist of curves and domains filled with spectrum, which is due to the dynamics near the asymptotic rest states of the wave. Presence of unstable discrete eigenvalues points to the absolute instability of the wave when the perturbations to the wave grow exponentially and eventually lead to unrecoverable distortion of the wave. Absence of an unstable spectrum indicates the resilience of the wave to small perturbations, if the nonlinear effects are negligibly small compared to the linear dynamics. In the case when the only unstable spectrum is a subset of the essential spectrum on the imaginary axis, the balance between linear growth and nonlinear effects becomes crucial. We call the essential spectrum marginally unstable if it extends up to the imaginary axis. The marginally unstable

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essential spectrum in reaction-diffusion systems as well as discrete eigenvalues located on the imaginary axis are indicative of an instability. In this paper we are interested in identifying the character of instability of a planar front with a marginally unstable spectrum.

Although a great deal of the literature is devoted to the multidimensional reaction-diffusion equations [BKSS, LMNT, LX, LW, HPSS, T, X, K], the theory in this case is still not as well developed as in the one-dimensional situation. One of the most high impact works is the 1997 paper [K] by Kapitula, who demonstrated, under very general conditions, that the stability of a multidimensional planar front is related to the stability of the associated one-dimensional front profile. More precisely, Kapitula in [K] proved the algebraic decay of perturbations to a planar front in a general reaction-diffusion system in the case when the spectrum of the linearization along the associated one-dimensional front is located in the stable half-plane. The case of a marginally unstable essential spectrum has been open since 1990. In the current work it is settled for a special class of reaction-diffusion systems.

For the problems posed on one-dimensional space that exhibit traveling waves with a marginally unstable essential spectrum, there exists an important technique for stability analysis based on applying exponential weights that are not bounded away from zero. It goes back to the celebrated work [PW, S] and amounts to recalculating the spectrum of the operator obtained by linearizing the equation about the wave in a function space equipped with an exponential weight. Since the exponential weights in some situations may stabilize the system at the linear level by shifting the essential spectrum of the linearization into the stable half-plane, one can then exploit the decay of the related linear semigroup to investigate whether the nonlinear effects in the underlying system are negligible in the introduced exponentially weighted norm. Instability of the essential spectrum in the original norm and stability of the wave in an exponentially weighted norm point to the convective nature of instability [SS] which is the instability characterized by pointwise decay of the perturbations. Also, exponential and polynomial weights have been successfully used to establish stability of traveling waves in viscous conservation laws; see, for example, [JGK], [KM]. Yet another powerful method developed for conservation laws is based on pointwise semigroup techniques (cf., e.g., [ZH]), where certain pointwise estimates are obtained for the integral kernel of the semigroup generated by the linearized equation, which are then used in the variation of constants formula to produce estimates on the solutions of the nonlinear equation.

To the best of our knowledge, the current paper is the first where this technique is used to analyze the stability of multidimensional traveling waves with a marginal or unstable essential spectrum in reaction-diffusion systems. We mention, however, an important paper [BKSS], where Brand et al. successfully used a combination of weights in a reaction-diffusion-convection system to investigate the nonlinear stability of the zero solution. The reaction terms in [BKSS] are assumed to be exponentially localized, unlike the reaction terms considered here.

In a recent series of papers [G, GLSS, GLS, GLS1] the method of exponential weights was used for a traveling front with a marginally unstable essential spectrum in a class of reaction-diffusion systems posed in one space dimension. These equations originate in combustion theory where the combustion front captures the propagation of the highest temperature zone. We refer to [GLSR, BGHL, BaM, SKMS1, VaV] and references therein for the results on existence and spectral stability of the combustion fronts. A recent paper [GLSR] contains an overview of the exponential weights technique and further references. The results were formulated as the orbital stability

of the traveling front in an exponentially weighted norm, against perturbations that belong to the space obtained by intersecting the original space (a Sobolev space or the space of bounded uniformly continuous functions) with the same space but equipped with the exponentially weighted norm. The orbital stability of the wave in the exponentially weighted norm may be interpreted as convective instability.

In the current paper, we investigate nonlinear stability of the planar front of a certain special class of systems of reaction-diffusion equations. The front is assumed to have a special feature: for the associated one-dimensional front, the instability is related not to the presence of unstable eigenvalues but to the essential spectrum touching imaginary axis. More precisely, our objective is to relate the type of stability of the planar front to the convective nature of instability of the associated one-dimensional fronts. The system considered in the current paper has a certain special structure in the reaction term similar to that of the equations studied in [GLSS, GLS, GLS1] for the one-dimensional case. Indeed, our motivation comes from the system

$$(1.1) \quad \begin{cases} u_{1t}(t, x) = \Delta_x u_1(t, x) + u_2(t, x)g(u_1(t, x)), \\ u_{2t}(t, x) = \epsilon \Delta_x u_2(t, x) - \kappa u_2(t, x)g(u_1(t, x)), \end{cases}$$

with frequently considered in combustion problems nonlinearity

$$(1.2) \quad g(u_1) = \begin{cases} e^{-\frac{1}{u_1}} & \text{if } u_1 > 0, \\ 0 & \text{if } u_1 \leq 0, \end{cases}$$

where $u_1, u_2 \in \mathbb{R}$, $t \in \mathbb{R}^+$, $x \in \mathbb{R}^d$ ($d \geq 2$), and the parameters ϵ and κ satisfy $0 \leq \epsilon < 1$ and $\kappa > 0$.

Ultimately, we would like to develop a technique to study nonlinear stability in weighted spaces for a marginally unstable front in the general system of the type

$$(1.3) \quad u_t(t, x) = D \Delta_x u(t, x) + f(u(t, x)), \quad u \in \mathbb{R}^n, \quad x \in \mathbb{R}^d, \quad d \geq 2, \quad t \in \mathbb{R}^+,$$

but here we focus on the case when the diffusion matrix D is the identity matrix. The reaction term in (1.3) satisfies some additional assumptions that are described in detail in section 3. In particular, the nonlinearity $f(u)$ is assumed to have a special “product-triangular” structure,

$$(1.4) \quad f(u) = \begin{pmatrix} A_1 u_1 + \tilde{f}_1(u_1, u_2) u_2 \\ \tilde{f}_2(u_1, u_2) u_2 \end{pmatrix}, \quad \tilde{f}_i : \mathbb{R}^{n_i} \times \mathbb{R}^{n_2} \rightarrow \mathbb{R}^{n_i}, \quad i = 1, 2,$$

where $n_1 + n_2 = n$ for the splitting $u = (u_1, u_2)$ such that $f(u_1, 0) = (A_1 u_1, 0)$ with a constant matrix A_1 , and $\tilde{f}_1$ and $\tilde{f}_2$ are given smooth maps.

Our plan of study can be summarized as follows. For the nonlinear system (1.3), (1.4) with D being the identity matrix we obtain a nonlinear system of evolution equations for perturbations of the front. We apply an exponential weight in the direction of propagation of the front in order to stabilize the marginal essential spectrum of the associated linear one-dimensional operator generated by the front profile. Furthermore, using the “product-triangular” structure of the reaction term as described in (1.4), we are able to control the decay of perturbations of the front with small initial data in the exponentially weighted norm and establish merely boundedness of the component u_1 of the perturbations in the norm without the weight. The decay in the exponentially weighted norm proved in this paper is comparable to the decay obtained in [K, Lemma 3.5]; cf. Proposition 8.6 below. This is a consequence of

the similarity of the spectral pictures for the associated linearization along the one-dimensional front in [K] (in the space with no weight) and in the current paper (in the space with the exponential weight). However, unlike the one-dimensional case (cf. [GLSS, GLS, GLS1]), in the multidimensional situation (in both [K] and the current paper) the decay of perturbations is not exponential but merely algebraic due to the lack of a gap between the spectrum of the multidimensional linearized operator and the imaginary axes. The main new technical difficulty that we had to overcome in this paper is related to the unfavorable influence of the exponential weights at the nonlinear term; this is exactly the point where we use the assumption on the “product-triangular” structure of nonlinearity.

The restriction imposed by choosing the diffusion matrix D to be the identity matrix is not necessarily technical. Distinct diffusion rates are known to be responsible for a variety of dynamical phenomena such as Turing instability or sideband instability. Removing this condition or identifying situations when this condition can be removed is generally speaking an open problem. A stronger assumption sufficient to guarantee asymptotic stability of a planar traveling wave against localized in transverse direction perturbations in the reaction-diffusion systems with nonequal diffusion, in addition to the spectral stability of the associated one-dimensional front, includes quadratic tangency of the dispersion relation near critical Fourier mode. We refer readers to [HS] for the proof and the discussion.

We mention an interesting open question of whether the pointwise semigroup techniques could be applied to the reaction-diffusion systems studied in the current paper. In particular, it is not clear if our assumption on the nonlinearity can play the role of the additional structure that balances out the fact that the system is not a conservation law.

To summarize, for a class of reaction terms in the system (1.3) with the identity diffusion matrix, we developed a technique that effectively combines the approach introduced in [K] with techniques from [GLSS, GLS, GLS1] to prove nonlinear stability in a weighted multidimensional space for a planar front that has an unstable or marginally stable essential spectrum.

2. The plan of the paper and notation. The plan of the paper is as follows. In section 3 we list the assumptions imposed on the system. We study the spectrum of the operator obtained by linearizing the system about the planar front in section 4. Also, in this section we study projections in the multidimensional weighted spaces associated with the spectral projections of the one-dimensional linearization along the front. We obtain estimates of the semigroup generated by the multidimensional linearized operator in section 5. In section 6 we derive a system of partial differential evolution equations for the perturbations of the planar front to be studied and give some preliminary estimates associated with the drift along the front. In section 7 we estimate the nonlinear terms in the system. In this estimate we heavily use the “product-triangular” structure of nonlinearity. In particular, it allows one to control the weighted norm of the nonlinear terms by the product of the weighted and unweighted norms of the perturbation. We complete the proof of stability of the front in section 8. The proof is based on a bootstrap argument where we first control the perturbation in the unweighted norm, then use this and the exponential decay of the linear part in the weighted norm to show decay in the weighted norm, which in turn allows us to establish boundedness of the perturbation in the unweighted norm. The main estimates are given in Proposition 8.6 (for the weighted norm) and in Proposition 8.7 (for the unweighted norm); the main stability result is formulated in Theorem 8.8.

We consistently use the same symbol to denote the space of scalar valued functions and the space of respective vector valued functions, whenever it is clear from the context, e.g., we use the same notation $H^k(\mathbb{R}^d)$ for the Sobolev space of scalar functions and for the Sobolev space of vector functions $(H^k(\mathbb{R}^d))^n$ when $n > 1$.

In the physical space $\mathbb{R}^d$ we associate $z \in \mathbb{R}$ with the direction of the propagation of the planar front. The complement of z we denote $y \in \mathbb{R}^{d-1}$.

For a fixed weight function $\gamma_\alpha(z)$ and $(z, y) \in \mathbb{R}^d$, we denote $H_\alpha^k(\mathbb{R}) = \{v : \gamma_\alpha v \in H^k(\mathbb{R})\}$, and $H_\alpha^k(\mathbb{R}^d) = \{u : (z, y) \mapsto \gamma_\alpha(z)u(z, y) \in H^k(\mathbb{R}^d)\}$. The spaces are equipped with the norm $\|u\|_{H_\alpha^k(\mathbb{R}^d)} = \|\gamma_\alpha u\|_{H^k(\mathbb{R}^d)}$. Also, we use $\mathcal{H}$ to denote the intersection space

$$(2.1) \quad \mathcal{H} := H^k(\mathbb{R}^d) \cap H_\alpha^k(\mathbb{R}^d), \text{ with } \|u\|_{\mathcal{H}} = \max\{\|u\|_{H^k(\mathbb{R}^d)}, \|u\|_{H_\alpha^k(\mathbb{R}^d)}\}.$$

Throughout, $\mathcal{B}(X, Y)$ denotes the space of bounded operators from X to Y , and we abbreviate $\mathcal{B}(X) = \mathcal{B}(X, X)$. We denote by $Sp(\mathcal{T})$ and $Sp_{ess}(\mathcal{T})$ the spectrum and the essential spectrum of the operator $\mathcal{T}$ and by $\text{ran } \mathcal{T}$ and $\ker \mathcal{T}$ its range and nullspace. Throughout the paper, we denote by C a generic positive constant.

3. Assumptions. In this section we introduce the reaction-diffusion system to be studied. In the one-dimensional situation similar assumptions on the system were originally developed in [GLS, GLS1]. We consider the system of reaction-diffusion equations

$$(3.1) \quad u_t(t, x) = \Delta_x u(t, x) + f(u(t, x)),$$

where $u(t, x) \in \mathbb{R}^n$, $x \in \mathbb{R}^d$, $t \in \mathbb{R}^+$, and the function $f(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is smooth.

We assume that this system has a planar wave that moves in the direction of the vector $e = (1, 0, \dots, 0) \in \mathbb{S}^d$ with a certain speed $c > 0$. In the co-moving frame $z = x_1 - ct$, (3.1) reads

$$(3.2) \quad u_t = \Delta u + cu_z + f(u),$$

where $\Delta = \partial_z^2 + \partial_{x_2}^2 + \dots + \partial_{x_n}^2$.

A traveling wave $\phi(z)$ for system (3.1) is a time-independent function of $z \in \mathbb{R}$ such that

$$(3.3) \quad 0 = \frac{d^2 \phi}{dz^2} + c \frac{d\phi}{dz} + f(\phi).$$

We further assume that the wave converges to its rest states $\phi_\pm \in \mathbb{R}^n$ exponentially. The wave is called a front if $\phi_- \neq \phi_+$; otherwise, it is called a pulse. Without loss of generality, we assume that $\phi_- = 0$.

To study the stability of ϕ , we first linearize (3.2) about ϕ . We define the linear variable-coefficient differential expression L by

$$(3.4) \quad L = \Delta + c\partial_z + df(\phi),$$

where $df(\phi)$ is the differential of the function f evaluated at $\phi(\cdot)$. The linear stability of the front is determined by the spectral information of the operator $\mathcal{L}$ associated with L and acting on the Sobolev space $H^k(\mathbb{R}^d)^n$ for $k > 1$. For $k > [\frac{d}{2}]$ the spaces $H^k(\mathbb{R}^d)$ are, in fact, Banach algebras and thus are convenient for the nonlinear stability analysis.

Using the tensor product notation, we write $H^k(\mathbb{R}^d) = H^k(\mathbb{R}) \otimes H^k(\mathbb{R}^{d-1})$ and note that for any $u \in H^k(\mathbb{R})$ and $v \in H^k(\mathbb{R}^{d-1})$ the function $(z, x_2, \dots, x_d) \mapsto u(z)v(x_2, \dots, x_d)$ belongs to $H^k(\mathbb{R}^d)$. From now on, we decompose $x \in \mathbb{R}^d$ as $x = (z, y) \in \mathbb{R} \otimes \mathbb{R}^{d-1}$, where $z = x_1 - ct$ and $y = (x_2, \dots, x_d)$. Thus we can use the decomposition of $\mathcal{L}$ on $H^k(\mathbb{R}^d)$ as follows:

$$\mathcal{L} = \mathcal{L}_1 \otimes I_{H^k(\mathbb{R}^{d-1})^n} + I_{H^k(\mathbb{R})^n} \otimes \Delta_y,$$

where $\mathcal{L}_1$ is associated with the one-dimensional differential expression

$$(3.5) \quad L_1 = \partial_z^2 + c\partial_z + df(\phi)$$

that depends only on z , and

$$(3.6) \quad \Delta_y = \partial_{x_2}^2 + \dots + \partial_{x_d}^2.$$

We next introduce an exponential weight to counteract the marginally unstable essential spectrum. We call $\gamma_\alpha \in C^{k+3}(\mathbb{R})$ the weight function of class $\alpha = (\alpha_-, \alpha_+) \in \mathbb{R}^2$ if $\gamma_\alpha(z) > 0$ for all $z \in \mathbb{R}$, and

$$(3.7) \quad \gamma_\alpha(z) = \begin{cases} e^{\alpha_- z} & \text{for } z \text{ negative, } |z| \text{ large,} \\ e^{\alpha_+ z} & \text{for } z \text{ positive and large.} \end{cases}$$

For a fixed weight function γ_α , let $H_\alpha^k(\mathbb{R}) := \{v : \gamma_\alpha v \in H^k(\mathbb{R})\}$. We then denote

$$H_\alpha^k(\mathbb{R}^d) = H_\alpha^k(\mathbb{R}) \otimes H^k(\mathbb{R}^{d-1}) = \{u : (\gamma_\alpha \otimes I_{H^k(\mathbb{R}^{d-1})})u \in H^k(\mathbb{R}^d)\}$$

with the norm $\|u\|_{H_\alpha^k(\mathbb{R}^d)} = \|\gamma_\alpha u\|_{H^k(\mathbb{R}^d)}$. Here, $(\gamma_\alpha \otimes I_{H^k(\mathbb{R}^{d-1})})u(z, y) := \gamma_\alpha(z)u(z, y)$, $(z, y) \in \mathbb{R}^d$.

DEFINITION 3.1.

1. $\mathcal{L} : H^k(\mathbb{R}^d)^n \rightarrow H^k(\mathbb{R}^d)^n$ is the linear operator given by the formula $u \mapsto Lu$, with L as in (3.4) where $\text{dom } \mathcal{L} = H^{k+2}(\mathbb{R}^d)^n \subset H^k(\mathbb{R}^d)^n$, for $k = 1, 2, \dots$;
2. $\mathcal{L}_1 : H^k(\mathbb{R})^n \rightarrow H^k(\mathbb{R})^n$ is the linear operator given by the formula $u \mapsto L_1 u$ with L_1 as in (3.5), where $\text{dom } \mathcal{L}_1 = H^{k+2}(\mathbb{R})^n \subset H^k(\mathbb{R})^n$;
3. $\Delta_y : H^k(\mathbb{R}^{d-1})^n \rightarrow H^k(\mathbb{R}^{d-1})^n$ is the linear operator given by the formula (3.6), with the domain $H^{k+2}(\mathbb{R}^{d-1})^n$;
4. $\mathcal{L}_\alpha : H_\alpha^k(\mathbb{R}^d)^n \rightarrow H_\alpha^k(\mathbb{R}^d)^n$ is the operator given by the formula $u \mapsto Lu$, with L as in (3.4) and $\text{dom } \mathcal{L}_\alpha = H_\alpha^{k+2}(\mathbb{R}) \otimes H^{k+2}(\mathbb{R}^{d-1})$;
5. $\mathcal{L}_{1,\alpha} : H_\alpha^k(\mathbb{R})^n \rightarrow H_\alpha^k(\mathbb{R})^n$ is the operator given by $u \mapsto L_1 u$, with L_1 as in (3.5) and $\text{dom } \mathcal{L}_{1,\alpha} = H_\alpha^{k+2}(\mathbb{R})^n \subset H_\alpha^k(\mathbb{R})^n$;
6. $\mathcal{L}_\mathcal{H} : \mathcal{H}^n \rightarrow \mathcal{H}^n$ is the linear operator generated by $u \mapsto Lu$ with L as in (3.4), with the domain $H^{k+2}(\mathbb{R}^d) \cap H_\alpha^{k+2}(\mathbb{R}^d)$.

We summarize the assumptions on the system (3.1) considered on $\mathcal{H}$ as follows.

HYPOTHESIS 3.2. The function $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is in $C^{k+3}(\mathbb{R}^n)^n$.

HYPOTHESIS 3.3. The system (3.1) has a C^{k+5} -smooth planar front $\phi(z)$, $z = x_1 - ct$,

$$\lim_{z \rightarrow \pm\infty} \phi(z) = \phi_\pm,$$

for which there exist numbers $K > 0$ and $\omega_- < 0 < \omega_+$ such that

$$\|\phi(z) - \phi_-\|_{\mathbb{R}^n} \leq Ke^{-\omega_- z} \text{ for } z \leq 0 \quad \text{and} \quad \|\phi(z) - \phi_+\|_{\mathbb{R}^n} \leq Ke^{-\omega_+ z} \text{ for } z \geq 0.$$

Without loss of generality, in the rest of the paper we assume that $\phi_- = 0$.

HYPOTHESIS 3.4. *There exists $\alpha = (\alpha_-, \alpha_+) \in \mathbb{R}^2$ such that the following assertions hold:*

- (1) $0 < \alpha_- < -\omega_-$.
- (2) $0 \leq \alpha_+ < \omega_+$.
- (3) *For the linear operator $\mathcal{L}_{1,\alpha} : H_\alpha^k(\mathbb{R})^n \rightarrow H_\alpha^k(\mathbb{R})^n$, there exists $\nu > 0$ such that*

$$\sup\{\operatorname{Re} \lambda : \lambda \in \operatorname{Sp}_{\text{ess}}(\mathcal{L}_{1,\alpha})\} < -\nu,$$

and the only element of $\operatorname{Sp}(\mathcal{L}_{1,\alpha})$ in $\{\lambda : \operatorname{Re} \lambda \geq 0\}$ is a simple eigenvalue 0.

Hypotheses 3.3 and 3.4 imply the following lemma.

LEMMA 3.5. *If Hypotheses 3.3 and 3.4 hold, then*

- (1) $\gamma_\alpha^{-1}\phi$ *is a $C^{k+5}(\mathbb{R})^n$ function that approaches zero exponentially as $z \rightarrow \pm\infty$,*
- (2) $\gamma_\alpha\phi$ *is a $C^{k+5}(\mathbb{R})^n$ function that exponentially approaches infinity as $z \rightarrow \infty$ and zero as $z \rightarrow -\infty$, while $\gamma_\alpha\phi^{(m)}$ approaches zero exponentially as $z \rightarrow \pm\infty$, for any $m = 1, 2, \dots, k+5$.*

In addition, we assume that the nonlinearity in system (3.1) satisfies the following hypothesis.

HYPOTHESIS 3.6. *There exists a splitting $u = (u_1, u_2) \in \mathbb{R}^n$, where $u_1 \in \mathbb{R}^{n_1}$ and $u_2 \in \mathbb{R}^{n_2}$, and $n_2 + n_1 = n$, such that $f(u_1, 0) = (A_1 u_1, 0)$ for some $n_1 \times n_1$ constant matrix A_1 .*

From Hypotheses 3.2 and 3.6 we have the following representation of function f :

$$(3.8) \quad f(u) = \begin{pmatrix} f_1(u_1, u_2) \\ f_2(u_1, u_2) \end{pmatrix} = \begin{pmatrix} A_1 u_1 + \tilde{f}_1(u_1, u_2) u_2 \\ \tilde{f}_2(u_1, u_2) u_2 \end{pmatrix}, \quad f_i : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \rightarrow \mathbb{R}^{n_i}, i = 1, 2,$$

where $\tilde{f}_1$ and $\tilde{f}_2$ are matrix-valued functions of size $n_1 \times n_2$ and $n_2 \times n_2$, respectively.

The system (3.1) in terms of u_1 and u_2 reads

$$\begin{aligned} \partial_t u_1 &= \Delta_x u_1 + f_1(u_1, u_2), \\ \partial_t u_2 &= \Delta_x u_2 + f_2(u_1, u_2), \end{aligned}$$

and the system (3.2) reads

$$\begin{aligned} \partial_t u_1 &= (\partial_{zz} + \Delta_y) u_1 + c \partial_z u_1 + f_1(u_1, u_2), \\ \partial_t u_2 &= (\partial_{zz} + \Delta_y) u_2 + c \partial_z u_2 + f_2(u_1, u_2). \end{aligned}$$

Similarly, we write $\phi(z) = (\phi_1(z), \phi_2(z))$ and $\phi_+ = (\phi_{1,+}, \phi_{2,+})$ and the differential expressions obtained by linearizing (3.3) at 0 and ϕ_+ , respectively, are given by the formulas

$$(3.9) \quad L_1^- = \partial_{zz} + c \partial_z + df(0), \quad L_1^+ = \partial_{zz} + c \partial_z + df(\phi_+).$$

In relation to the linearization about 0, we denote

$$(3.10) \quad L_1^{(1)} = \partial_{zz} + c \partial_z + d_{u_1} f_1(0, 0) = \partial_{zz} + c \partial_z + A_1,$$

$$(3.11) \quad L_1^{(2)} = \partial_{zz} + c \partial_z + d_{u_2} f_2(0, 0),$$

where $d_{u_i}f_i$ is the Jacobian of f_i with respect to u_i , $i = 1, 2$. From (3.8) it follows then that

$$(3.12) \quad L_1^- = \begin{pmatrix} L_1^{(1)} & d_{u_2}f_1(0,0) \\ 0 & L_1^{(2)} \end{pmatrix}.$$

We write L_1 defined in (3.5) as follows:

$$(3.13) \quad L_1 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} L_1^{(1)} & d_{u_2}f_1(0,0) \\ 0 & L_1^{(2)} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + (df(\phi) - df(0)) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

We now define the operators $\mathcal{L}_1^{(1)}$ and $\mathcal{L}_1^{(2)}$ as prescribed in item (2) of Definition 3.1. The next hypothesis implies, in part, the stability of the end state $(0,0)$ located behind the front.

HYPOTHESIS 3.7. *In addition to Hypotheses 3.4 and 3.6, we assume that the following is true:*

- (1) *The analytic semigroup generated by the operator $\mathcal{L}_1^{(1)}$ on $H^k(\mathbb{R})^{n_1}$ induced by (3.10) in $H^k(\mathbb{R})$ is bounded, that is, there exists $K > 0$ such that $\|e^{t\mathcal{L}_1^{(1)}}\|_{\mathcal{B}(H^k(\mathbb{R}))} \leq K$ for all $t \geq 0$.*
- (2) *The spectrum $\text{Sp}(\mathcal{L}_1^{(2)})$ of the operator $\mathcal{L}_1^{(2)}$ on $H^k(\mathbb{R})^{n_2}$ introduced by (3.11) is located strictly to the left of the imaginary axis, that is, $\sup\{\text{Re } \lambda : \lambda \in \text{Sp}(\mathcal{L}_1^{(2)})\} < 0$. Therefore, there exist constants $\rho > 0$ and $K > 0$ such that $\|e^{t\mathcal{L}_1^{(2)}}\|_{\mathcal{B}(H^k(\mathbb{R}))} \leq Ke^{-\rho t}$ for all $t \geq 0$.*

Remark 3.8. Hypothesis 3.7 implies that (a) $\sup\{\text{Re } \lambda : \lambda \in \text{Sp}(\mathcal{L}_1^{(1)})\} \leq 0$; (b) $\sup\{\text{Re } \lambda : \lambda \in \text{Sp}(\mathcal{L}_1^-)\} \leq 0$, which in turn implies that $\sup\{\text{Re } \mu : \mu \in \text{Sp}(d_{u_1}f_1(0,0))\} \leq 0$, while $\sup\{\text{Re } \mu : \mu \in \text{Sp}(d_{u_2}f_2(0,0))\} < 0$. More information on the relation of Hypothesis 3.7 and location of the spectra of the matrices $\text{Sp}(d_{u_i}f_i(0,0))$ can be found in [GLSS, Appendix A].

4. Spectrum and projection operators. In this section we discuss the projection operator on the central direction that corresponds to the isolated zero eigenvalue of the linear operator $\mathcal{L}_{1,\alpha}$ associated with (3.5), on the weighted space $H_\alpha^k(\mathbb{R})^n$, and describe the central projection for the operator $\mathcal{L}_\alpha$ in $H_\alpha^k(\mathbb{R}^d)^n$.

We recall that for a closed densely defined operator $\mathcal{T}$, the resolvent set $\rho(\mathcal{T})$ is the set of $\lambda \in \mathbb{C}$ such that $\mathcal{T} - \lambda I$ has a bounded inverse. The complement of $\rho(\mathcal{T})$ is the spectrum $\text{Sp}(\mathcal{T})$. It includes the discrete spectrum, $\text{Sp}_d(\mathcal{T})$, which is the set of isolated eigenvalues of $\mathcal{T}$ of finite algebraic multiplicity. The rest of the spectrum is called the essential spectrum and denoted by $\text{Sp}_{\text{ess}}(\mathcal{T})$.

The spectrum of the linearization touching the imaginary axis complicates the stability analysis of system (3.1) in multidimensional space. In the one-dimensional case [GLS], the authors have imposed the hypotheses under which the front is spectrally stable in $H_\alpha^1(\mathbb{R})^n$, i.e., the linear operator associated with the one-dimensional differential expression $L_1 = D\partial_z^2 + c\partial_z + df(\phi)$ has only one simple, isolated eigenvalue at 0 while the rest of the spectrum is located to the left of the imaginary axis. More precisely, let L_1^- and L_1^+ be defined as in (3.9). By [GLS, Lemma 3.5], the rightmost boundary of the corresponding $\text{Sp}_{\text{ess}}(\mathcal{L}_{1,\alpha})$ is the rightmost boundary of the set $\text{Sp}(\mathcal{L}_{1,\alpha}^-) \cup \text{Sp}(\mathcal{L}_{1,\alpha}^+)$, where

$$\begin{aligned} \text{Sp}(\mathcal{L}_{1,\alpha}^-) = \{ \lambda \in \mathbb{C} \mid \exists \theta \in \mathbb{R} : \det(-\theta^2 + i\theta(c - 2\alpha_-)I - \lambda I \\ + (\alpha_-^2 - c\alpha_-)I + df(0)) = 0 \}, \end{aligned}$$

$$\begin{aligned} \text{Sp}(\mathcal{L}_{1,\alpha}^+) = \{ \lambda \in \mathbb{C} \mid \exists \theta \in \mathbb{R} : \det(-\theta^2 + i\theta(c - 2\alpha_+)I - \lambda I \\ + (\alpha_+^2 - c\alpha_+)I + df(\phi_+)) = 0 \}. \end{aligned}$$

It is assumed that the rightmost boundary is located strictly to the left of the imaginary axis. Thus, in the one-dimensional case the essential spectrum of the linearization in the exponentially weighted space is located in the open left plane.

For the multidimensional case the situation is by far more complicated. Here, one concern regarding the spectrum of $\mathcal{L}_\alpha$ is that the zero eigenvalue, which is an isolated eigenvalue for a one-dimensional operator considered on $H_\alpha^k(\mathbb{R})$, is no longer an isolated point of the spectrum of $\mathcal{L}_\alpha$ in $H_\alpha^k(\mathbb{R}^d)$.

In Figure 1, we illustrate the influence of the exponential weight on the location of the essential spectrum and the issue arising in the multidimensional system when the same method of passing to the exponentially weighted spaces is applied.

Indeed, the following proposition holds, showing that the essential spectrum of $\mathcal{L}_\alpha$ is no longer bounded away from the imaginary axis on the weighted space $H_\alpha^k(\mathbb{R}^d)^n$ as it was for $d = 1$.

PROPOSITION 4.1. *Let $d > 1$, let the assumptions of Hypothesis 3.4 hold, and let the linear operators $\mathcal{L}_\alpha$ and $\mathcal{L}_{1,\alpha}$ be the operators defined according to Definition 3.1 associated with L and L_1 introduced in (3.4) and (3.5), respectively. Each point $\eta \in \text{Sp}(\mathcal{L}_{1,\alpha})$ generates a horizontal half-line $\{\lambda \in \mathbb{C} : \text{Re } \lambda \leq \text{Re } \eta, \text{Im } \lambda = \text{Im } \eta\}$ that belongs to the essential spectrum $\text{Sp}_{\text{ess}}(\mathcal{L}_\alpha)$. In particular, the half-line $\{\lambda \in \mathbb{R} : \text{Re } \lambda \leq 0\}$ belongs to the essential spectrum of $\mathcal{L}_\alpha$.*

Proof. The result follows from [RS4, Theorem XIII.34, Theorem XIII.35, and Corollary 1]. Indeed, since $\mathcal{L}_{1,\alpha}$ and $I_n \Delta_y$ are the generators of bounded analytic semigroups on Hilbert spaces $H_\alpha^k(\mathbb{R})$ and $H^k(\mathbb{R}^{d-1})$, respectively, we have

$$\text{Sp}(\mathcal{L}_{1,\alpha} \otimes I_{H^k(\mathbb{R}^{d-1})} + I_{H_\alpha^k(\mathbb{R})} \otimes \Delta_y) = \text{Sp}(\mathcal{L}_{1,\alpha}) + \text{Sp}(\Delta_y),$$

which implies the conclusions of the proposition. $\square$

Since by Hypothesis 3.4, 0 is a simple, isolated eigenvalue of $\mathcal{L}_{1,\alpha}$, we can define the Riesz spectral projection P_α of $\mathcal{L}_{1,\alpha}$ on $H_\alpha^k(\mathbb{R})^n$ onto the one-dimensional space $\ker(\mathcal{L}_{1,\alpha})$. The projection P_α commutes with $e^{t\mathcal{L}_{1,\alpha}}$ for all $t \geq 0$. Since the operator

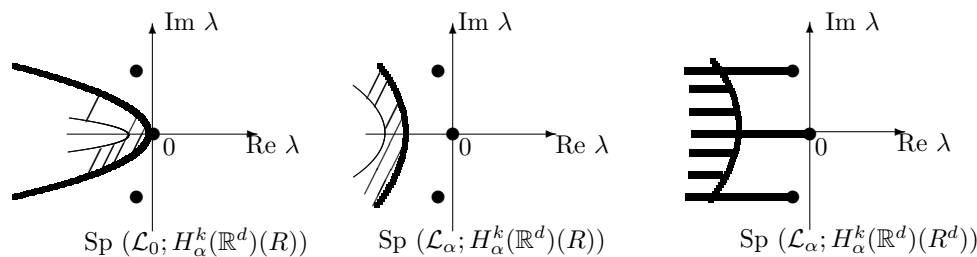


FIG. 1. The first panel: the rightmost boundary of the essential spectrum and the eigenvalue at the origin of the linearization of (1.1)–(1.2) about the front in the space with no exponential weight. The second panel: the rightmost boundary of the essential spectrum and the eigenvalue at the origin of the linearization of (1.1)–(1.2) about the front in the case of a one-dimensional spatial variable in the exponentially weighted space. The essential spectrum in the one-dimensional case is bounded away from the imaginary axis. The third panel: the multidimensional case. The essential spectrum in the weighted space is not bounded away from the imaginary axis.

$\mathcal{L}_{1,\alpha}$ is Fredholm of index zero, standard operator theory (see, e.g., [DL, Lemma 2.13]) yields that $H_\alpha^k(\mathbb{R})^n = \text{ran } \mathcal{L}_{1,\alpha} \oplus \ker \mathcal{L}_{1,\alpha}$ and $\ker P_\alpha = \text{ran } \mathcal{L}_{1,\alpha}$.

Hypothesis 3.4 implies that $\text{ran } P_\alpha = \ker \mathcal{L}_{1,\alpha}$ is spanned by ϕ' . Reasoning as in [K] or as in the proof of Lemma 3.8 in [GLS], that is, by invoking Palmer's theorem [Pa], one can show that there exists a unique H^k -smooth function $\tilde{e} : \mathbb{R} \rightarrow \mathbb{R}^n$ such that the function $\gamma_\alpha^{-1}(\cdot)\tilde{e}(\cdot)$ is exponentially decaying, and $\tilde{e}$ solves the adjoint equation $L_{1,\alpha}^* \tilde{e} = 0$ and satisfies $\int_{\mathbb{R}} (\tilde{e}(s), \phi'(s))_{\mathbb{R}^n} ds = 1$, where $(\cdot, \cdot)_{\mathbb{R}^n}$ is the standard inner product in $\mathbb{R}^n$. Then for $V \in H_\alpha^k(\mathbb{R})^n$, the operator P_α can be written as follows:

$$(P_\alpha V)(z) = \left(\int_{\mathbb{R}} (\tilde{e}(s), V(s))_{\mathbb{R}^n} ds \right) \phi'(z), \quad z \in \mathbb{R}.$$

Let $Q_\alpha = I - P_\alpha$ be the projection in $H_\alpha^k(\mathbb{R})^n$ onto $\text{ran } \mathcal{L}_{1,\alpha}$ with kernel $\ker(\mathcal{L}_{1,\alpha})$. The operator Q_α also commutes with $e^{t\mathcal{L}_{1,\alpha}}$ for all $t \geq 0$. Next, for $U \in H_\alpha^k(\mathbb{R})^n \otimes H^k(\mathbb{R}^{d-1})^n$ we denote

$$(4.1) \quad (\pi_\alpha U)(y) = \int_{\mathbb{R}} (\tilde{e}(s), U(s, y))_{\mathbb{R}^n} ds$$

and introduce an operator on $H_\alpha^k(\mathbb{R}^d)^n = H_\alpha^k(\mathbb{R})^n \otimes H^k(\mathbb{R}^{d-1})^n$ defined by

$$\mathcal{P}U = (P_\alpha \otimes I_{H^k(\mathbb{R}^{d-1})}) U,$$

so that

$$(\mathcal{P}U)(z, y) = \left(\int_{\mathbb{R}} (\tilde{e}(s), U(s, y))_{\mathbb{R}^n} ds \right) \phi'(z) = (\pi_\alpha U)(y) \phi'(z), \quad (z, y) \in \mathbb{R}^d.$$

In what follows we frequently use the following lemma from [RS1, p. 299].

LEMMA 4.2. *Let A and B be bounded operators on Hilbert spaces H_1 and H_2 . Then*

$$\|A \otimes B\|_{\mathcal{B}(H_1 \otimes H_2)} = \|A\|_{\mathcal{B}(H_1)} \|B\|_{\mathcal{B}(H_2)}.$$

We now show that π_α and $\mathcal{P}$ have the following properties.

LEMMA 4.3. *Let $k \geq [\frac{d+1}{2}]$ and $\alpha = (\alpha_-, \alpha_+) \in \mathbb{R}_+^2$ be as in Hypothesis 3.4. Then*

$$\mathcal{P} \in \mathcal{B}(H_\alpha^k(\mathbb{R}^d) \cap H^k(\mathbb{R}^d)) \quad \text{and} \quad \pi_\alpha \in \mathcal{B}(H^k(\mathbb{R}^d) \cap H_\alpha^k(\mathbb{R}^d), H^k(\mathbb{R}^{d-1})).$$

Moreover,

$$\pi_\alpha \in \mathcal{B}(L_\alpha^1(\mathbb{R}) \otimes L^1(\mathbb{R}^{d-1}), L^1(\mathbb{R}^{d-1})).$$

Proof. Since $\|\gamma_\alpha^{-1}(z)\tilde{e}(z)\|_{\mathbb{R}^n} \rightarrow 0$ exponentially fast as $|z| \rightarrow \infty$, there exist $\zeta_- < 0 < \zeta_+$ and $K > 0$ such that $\|\gamma_\alpha^{-1}(z)\tilde{e}(z)\|_{\mathbb{R}^n} \leq Ke^{-\zeta_- z}$ for $z \leq 0$, and $\|\gamma_\alpha^{-1}(z)\tilde{e}(z)\|_{\mathbb{R}^n} \leq Ke^{-\zeta_+ z}$ for $z \geq 0$. We pick $U \in H^k(\mathbb{R}^d)^n \cap H_\alpha^k(\mathbb{R}^d)^n$, and first consider the L^2 -norm, so that

$$\begin{aligned} \|\pi_\alpha U\|_{L^2(\mathbb{R}^{d-1})}^2 &= \int_{\mathbb{R}^{d-1}} \left| \int_{\mathbb{R}} (\gamma_\alpha^{-1}(s)\tilde{e}(s), \gamma_\alpha(s)U(s, y))_{\mathbb{R}^n} ds \right|^2 dy \\ &\leq \int_{\mathbb{R}^{d-1}} \left(\int_{\mathbb{R}} \|\gamma_\alpha^{-1}(s)\tilde{e}(s)\|_{\mathbb{R}^n}^2 ds \right) \left(\int_{\mathbb{R}} \|\gamma_\alpha(s)U(s, y)\|_{\mathbb{R}^n}^2 ds \right) dy \end{aligned}$$

by Hölder's inequality. Since

$$\|\gamma_\alpha^{-1}(s)\tilde{e}(s)\|_{\mathbb{R}^n} \leq \begin{cases} Ke^{-\zeta-s} & \text{for } s \leq 0, \\ Ke^{-\zeta+s} & \text{for } s \geq 0, \end{cases}$$

then

$$(4.2) \quad \int_{\mathbb{R}} \|\gamma_\alpha^{-1}(s)\tilde{e}(s)\|_{\mathbb{R}^n}^2 ds \leq K \left(\int_{-\infty}^0 e^{-2\zeta-s} ds + \int_0^\infty e^{-2\zeta+s} ds \right) \leq C$$

for some constant $C > 0$. Thus,

$$(4.3) \quad \|\pi_\alpha U\|_{L^2(\mathbb{R}^{d-1})}^2 \leq C \|\gamma_\alpha U\|_{L^2(\mathbb{R}^d)}^2 \leq C \max\{\|U\|_{L^2(\mathbb{R}^d)}^2, \|U\|_{L_\alpha^2(\mathbb{R}^d)}^2\} \leq C \|U\|_{\mathcal{H}}^2.$$

For H^k -norms, we use the equivalent Sobolev norm (see, e.g., [NS, p. 316]) given as follows: Let $x = (z, y) \in \mathbb{R}^d$ and $y = (x_2, \dots, x_d) \in \mathbb{R}^{d-1}$; then

$$\|f\|_{H^k(\mathbb{R}^{d-1})} \sim \|f\|_{L^2(\mathbb{R}^{d-1})} + \sum_{a_2+\dots+a_d=k} \left\| \frac{\partial^k}{\partial x_2^{a_2} \dots \partial x_d^{a_d}} f \right\|_{L^2(\mathbb{R}^{d-1})},$$

where the sum extends over all $(d-1)$ -tuples $(a_2, \dots, a_d)$ of nonnegative integers with $\sum_{i=2}^d a_i = k$, and $\frac{\partial^{a_i}}{\partial x_i^{a_i}}$ is the a_i th differentiation of functions with respect to x_i , $i = 2, \dots, d$.

We already have the estimates for $\|\pi_\alpha U\|_{L^2(\mathbb{R}^{d-1})}$ for $U \in H_\alpha^k(\mathbb{R}^d)^n \cap H^k(\mathbb{R}^d)^n$. From Hölder's inequality and (4.2) it follows that

$$\begin{aligned} & \left\| \frac{\partial^k}{\partial x_2^{a_2} \dots \partial x_d^{a_d}} \pi_\alpha U \right\|_{L^2(\mathbb{R}^{d-1})}^2 \\ & \leq \int_{\mathbb{R}^{d-1}} \left(\int_{\mathbb{R}} \|\gamma_\alpha^{-1}(s)\tilde{e}(s)\|_{\mathbb{R}^n} \left\| \gamma_\alpha(s) \frac{\partial^k}{\partial x_2^{a_2} \dots \partial x_d^{a_d}} U(s, y) \right\|_{\mathbb{R}^n} ds \right)^2 dy \\ & \leq \int_{\mathbb{R}^{d-1}} \left(\int_{\mathbb{R}} \|\gamma_\alpha^{-1}(s)\tilde{e}(s)\|_{\mathbb{R}^n}^2 ds \right) \left(\int_{\mathbb{R}} \left\| \gamma_\alpha(s) \frac{\partial^k U(s, y)}{\partial x_2^{a_2} \dots \partial x_d^{a_d}} \right\|_{\mathbb{R}^n}^2 ds \right) dy \\ (4.4) \quad & \leq C \|U\|_\alpha^2 \leq C \|U\|_{\mathcal{H}}^2, \end{aligned}$$

thus implying $\pi_\alpha \in \mathcal{B}(H_\alpha^k(\mathbb{R}^d)^n \cap H^k(\mathbb{R}^d)^n, H^k(\mathbb{R}^{d-1}))$.

For the L^1 -norm of $\pi_\alpha U$, analogously,

$$(4.5) \quad \|\pi_\alpha U\|_{L^1(\mathbb{R}^{d-1})} \leq C \|\gamma_\alpha U\|_{L^1(\mathbb{R}^d)} \leq C \|U\|_{L_\alpha^1(\mathbb{R}) \otimes L^1(\mathbb{R}^{d-1})}.$$

We now consider $\mathcal{P}U$ for $U \in H_\alpha^k(\mathbb{R}^d)^n$ noting that $\mathcal{P} = P_\alpha \otimes I_{H^k(\mathbb{R}^{d-1})}$. As shown in [GLS, section 3.3], the projection P_α is a bounded operator from $H_\alpha^k(\mathbb{R}) \cap H^k(\mathbb{R})$ to $H_\alpha^k(\mathbb{R}) \cap H^k(\mathbb{R})$. Therefore, by Lemma 4.2 we have

$$\|\mathcal{P}\|_{\mathcal{B}(\mathcal{H})} = \|P_\alpha\|_{\mathcal{B}(H_\alpha^k(\mathbb{R}) \cap H^k(\mathbb{R}))} \|I\|_{\mathcal{B}(H^k(\mathbb{R}^{d-1}))} \leq C,$$

which completes the proof. $\square$

LEMMA 4.4. *The operator $\mathcal{P}$ is a bounded operator (i) from $H_\alpha^k(\mathbb{R}^d)$ to $H_\alpha^k(\mathbb{R}^d)$; (ii) from $\mathcal{H}$ to $H_\alpha^k(\mathbb{R}^d)$; (iii) from $H_\alpha^k(\mathbb{R}^d)$ to $H^k(\mathbb{R}^d)$; and (iv) from $\mathcal{H}$ to $H^k(\mathbb{R}^d)$. The complementary projection $\mathcal{Q} = \mathcal{I} - \mathcal{P}$ is a bounded operator (i) from $H_\alpha^k(\mathbb{R}^d)$ to $\mathcal{H}$ and (ii) from $\mathcal{H}$ to $\mathcal{H}$.*

Proof. Indeed, Lemma 4.3 and the definition $\mathcal{P}U(z, y) = (\pi_\alpha U)(y)\phi'(z)$, $z, y \in \mathbb{R}^d$, imply that

$$\begin{aligned}\|\mathcal{P}U\|_{H_\alpha^k(\mathbb{R}^d)} &= \|\pi_\alpha U\|_{H^k(\mathbb{R}^{d-1})} \|\phi'\|_{H_\alpha^k(\mathbb{R})} \leq C \|U\|_{H_\alpha^k(\mathbb{R}^d)} \|\phi'\|_{H_\alpha^k(\mathbb{R})} \leq C \|U\|_{\mathcal{H}} \|\phi'\|_{H_\alpha^k(\mathbb{R})}, \\ \|\mathcal{P}U\|_{H^k(\mathbb{R}^d)} &= \|\pi_\alpha U\|_{H^k(\mathbb{R}^{d-1})} \|\phi'\|_{H^k(\mathbb{R})} \leq C \|U\|_{H_\alpha^k(\mathbb{R}^d)} \|\phi'\|_{H^k(\mathbb{R})} \leq C \|U\|_{\mathcal{H}} \|\phi'\|_{H^k(\mathbb{R})},\end{aligned}$$

and the statement above follows. $\square$

The projection P_α is initially defined as the Riesz projection for the operator $\mathcal{L}_{1,\alpha}$. To verify that $\mathcal{P}\mathcal{L}_\alpha = \mathcal{L}_\alpha\mathcal{P}$, we recall that $P_\alpha\mathcal{L}_{1,\alpha} = \mathcal{L}_{1,\alpha}P_\alpha$, which implies that $\mathcal{L}_\alpha$ and $\mathcal{P}$ commute since $\mathcal{P}\mathcal{L}_\alpha = P_\alpha\mathcal{L}_{1,\alpha} \otimes I_{H^k(\mathbb{R}^{d-1})^n} + P_\alpha \otimes \Delta_y$ and $\mathcal{L}_\alpha\mathcal{P} = \mathcal{L}_{1,\alpha}P_\alpha \otimes I_{H^k(\mathbb{R}^{d-1})^n} + P_\alpha \otimes \Delta_y$.

Remark 4.5. When the diffusion matrix D in (1.3) is not a multiple of an identity matrix, the relation $\mathcal{P}\mathcal{L}_\alpha = \mathcal{L}_\alpha\mathcal{P}$ does not hold in general. Indeed, $\mathcal{L}_\alpha = \mathcal{L}_{1,\alpha} \otimes I_{H^k(\mathbb{R}^{d-1})^n} + DI_{H_\alpha^k(\mathbb{R})^n} \otimes \Delta_y$, and, in general, D doesn't commute with P_α . This is the main obstacle that prevents us from dealing with nonscalar diffusion matrices.

5. The semigroup estimates. In this section we provide estimates for the semigroups generated by the linear operators $\mathcal{L}_\alpha$, $\mathcal{L}_\mathcal{H}$, Δ_y , and $\mathcal{L}^{(i)}$ for $i = 1, 2$; cf. (3.4), (3.10), and (3.11), and see Lemmas 5.1, 5.2, 5.3, and 5.5 below. Hypothesis 3.4 implies the following standard fact about analytic semigroups.

LEMMA 5.1. *If $\nu > 0$ is such that $\sup\{\operatorname{Re} \lambda : \lambda \in \operatorname{Sp}_{\text{ess}}(\mathcal{L}_{1,\alpha})\} < -\nu$, then there exists $K > 0$ such that $\|e^{t\mathcal{L}_{1,\alpha}}Q_\alpha\|_{\mathcal{B}(H_\alpha^k(\mathbb{R}))} \leq Ke^{-\nu t}$ for $t \geq 0$.*

Moreover, the following lemma is true.

LEMMA 5.2. *Assume Hypothesis 3.4. If $\sup\{\operatorname{Re} \lambda : \lambda \in \operatorname{Sp}(\mathcal{L}_{1,\alpha}) \text{ and } \lambda \neq 0\} < -\nu$, for some $\nu > 0$, then there exists $K > 0$ such that $\|e^{t\mathcal{L}_\alpha}\mathcal{Q}\|_{\mathcal{B}(H_\alpha^k(\mathbb{R}^d))} \leq Ke^{-\nu t}$ for all $t \geq 0$.*

Proof. Since $\mathcal{Q} = Q_\alpha \otimes I_{H^k(\mathbb{R}^{d-1})}$ and $\mathcal{L}_\alpha = \mathcal{L}_{1,\alpha} \otimes I_{H^k(\mathbb{R}^{d-1})} + I_{H_\alpha^k(\mathbb{R})} \otimes \Delta_y$, by the proof of [RS4, Theorem XIII.35] we have $e^{t\mathcal{L}_\alpha}\mathcal{Q} = e^{t\mathcal{L}_{1,\alpha}}Q_\alpha \otimes e^{t\Delta_y}I_{H^k(\mathbb{R}^{d-1})}$. The operators $\mathcal{L}_{1,\alpha}$ and Δ_y both generate bounded semigroups on $\operatorname{ran} Q_\alpha = \operatorname{ran} \mathcal{L}_{1,\alpha}$ and $H^k(\mathbb{R}^{d-1})$ (cf. Lemmas 5.1 and 5.5.a); thus by Lemma 4.2 we infer

$$\|e^{t\mathcal{L}_{1,\alpha}}Q_\alpha \otimes e^{t\Delta_y}I_{H^k(\mathbb{R}^{d-1})}\|_{\mathcal{B}(H_\alpha^k(\mathbb{R}^d))} = \|e^{t\mathcal{L}_{1,\alpha}}Q_\alpha\|_{\mathcal{B}(H_\alpha^k(\mathbb{R}))} \|e^{t\Delta_y}\|_{\mathcal{B}(H^k(\mathbb{R}^{d-1}))},$$

which completes the proof. $\square$

We consider the operator $\mathcal{L}^-$ on $H^k(\mathbb{R}^d)$ associated with the differential expression

$$(5.1) \quad L^- = L_1^- \otimes I_{H^k(\mathbb{R}^{d-1})} + I_{H^k(\mathbb{R})} \otimes \Delta_y,$$

where L_1^- is defined in (3.12), and let

$$(5.2) \quad \begin{aligned}L^{(1)} &= \Delta_x + c\partial_z + A_1 = L_1^{(1)} \otimes I_{H^k(\mathbb{R}^{d-1})} + I_{H^k(\mathbb{R})} \otimes \Delta_y, \\ L^{(2)} &= \Delta_x + c\partial_z + d_{u_2}f_2(0) = L_1^{(2)} \otimes I_{H^k(\mathbb{R}^{d-1})} + I_{H^k(\mathbb{R})} \otimes \Delta_y,\end{aligned}$$

where A_1 is introduced in Hypothesis 3.6, and $L_1^{(i)}$, $i = 1, 2$, are as in (3.10) and (3.11). Thus

$$(5.3) \quad L^- = \begin{pmatrix} L^{(1)} & d_{u_2}f_1(0) \\ 0 & L^{(2)} \end{pmatrix},$$

and the linearization (3.4) about the front is given by the formula

$$(5.4) \quad L = L^- + (df(\phi) - df(0)) \otimes I_{H^k(\mathbb{R}^{d-1})}.$$

As in [GLS, Lemma 8.2(1)], the operator $df(\phi) - df(0)$ is a bounded operator from $H_\alpha^k(\mathbb{R})$ into $H^k(\mathbb{R})$. We therefore have

$$(5.5) \quad (df(\phi) - df(0)) \otimes I_{H^k(\mathbb{R}^{d-1})} \in \mathcal{B}(H_\alpha^k(\mathbb{R}^d), H^k(\mathbb{R}^d)).$$

LEMMA 5.3. *Assume Hypotheses 3.7. Let $\mathcal{L}^{(i)}$, $i = 1, 2$, be the operators given by the differential expressions (5.2) on $H^k(\mathbb{R}^d)$. If $\sup\{\operatorname{Re} \lambda : \lambda \in \operatorname{Sp}(\mathcal{L}_1^{(2)}) \text{ and } \lambda \neq 0\} < -\rho$, for some $\rho > 0$, then there exists $K > 0$ such that*

$$(5.6) \quad \|e^{t\mathcal{L}^{(1)}}\|_{\mathcal{B}(H^k(\mathbb{R}^d))} \leq K, \quad \|e^{t\mathcal{L}^{(2)}}\|_{\mathcal{B}(H^k(\mathbb{R}^d))} \leq Ke^{-\rho t},$$

for all $t \geq 0$. Moreover, the operator $\mathcal{L}^-$ given by the differential expression (5.1) generates a bounded semigroup on $H^k(\mathbb{R}^d)$, that is,

$$(5.7) \quad \|e^{t\mathcal{L}^-}\|_{\mathcal{B}(H^k(\mathbb{R}^d))} \leq K \text{ for all } t \geq 0.$$

Proof. We shall use the fact [RS4, Theorem XIII.35] that

$$e^{t\mathcal{L}^{(i)}} = e^{t(\mathcal{L}_1^{(i)} \otimes I_{H^k(\mathbb{R}^{d-1})} + I_{H^k(\mathbb{R})} \otimes \Delta_y)} = e^{t\mathcal{L}_1^{(i)}} \otimes e^{t\Delta_y} \quad \text{for } i = 1, 2.$$

By Hypothesis 3.7(1), the operator $\mathcal{L}_1^{(1)}$ generates a bounded semigroup on $H^k(\mathbb{R})$, thus, by Lemma 4.2, $\|e^{t\mathcal{L}_1^{(1)}} \otimes e^{t\Delta_y}\| = \|e^{t\mathcal{L}_1^{(1)}}\| \|e^{t\Delta_y}\| < K$ for some $K > 0$ and all $t \geq 0$. Similarly, from Hypothesis 3.7(2) and Lemma 4.2, $\|e^{t\mathcal{L}_1^{(2)}} \otimes e^{t\Delta_y}\| = \|e^{t\mathcal{L}_1^{(2)}}\| \|e^{t\Delta_y}\| < Ke^{-\rho t}$ for some $K > 0$ and all $t \geq 0$.

To prove (5.7), we notice that the triangular structure of the operator $\mathcal{L}^-$ yields the triangular structure of the semigroup $e^{t\mathcal{L}^-}$, that is,

$$(5.8) \quad e^{t\mathcal{L}^-} = \begin{pmatrix} e^{t\mathcal{L}^{(1)}} & \int_0^t e^{(t-s)\mathcal{L}^{(1)}} \partial_{u_2} f_1(0) e^{s\mathcal{L}^{(2)}} ds \\ 0 & e^{t\mathcal{L}^{(2)}} \end{pmatrix}.$$

Equation (5.8) and inequalities (5.6) imply (5.7). $\square$

We next use Lemmas 5.2 and 5.3 to show that the semigroup generated by the operator $\mathcal{L}$ on $\mathcal{H}$ is also bounded.

LEMMA 5.4. *Assume Hypotheses 3.7. Let $\mathcal{L}_{\mathcal{H}}$ be the operator given by the differential expressions (3.4) on $\mathcal{H} = H^k(\mathbb{R}^d) \cap H_\alpha^k(\mathbb{R}^d)$. There exists $K > 0$ such that $\|e^{t\mathcal{L}_{\mathcal{H}}}\|_{\mathcal{B}(\mathcal{H})} \leq K$ for all $t \geq 0$.*

Proof. Let the operator $\mathcal{Q}_{\mathcal{H}}$ be given by restricting $\mathcal{Q}$ to $\mathcal{H}$; then by Lemma 5.2

$$(5.9) \quad \|e^{t\mathcal{L}_{\mathcal{H}}}\mathcal{Q}_{\mathcal{H}}\|_{\mathcal{B}(H_\alpha^k(\mathbb{R}^d))} \leq Ke^{-\nu t}.$$

Therefore, it remains to estimate $\|e^{t\mathcal{L}_{\mathcal{H}}}\mathcal{Q}_{\mathcal{H}}\|_{\mathcal{B}(\mathcal{H}, H^k(\mathbb{R}^d))}$ and $\|e^{t\mathcal{L}_{\mathcal{H}}}\mathcal{P}_{\mathcal{H}}\|_{\mathcal{B}(\mathcal{H})}$.

Since $\operatorname{ran} \mathcal{Q}_{\mathcal{H}} = \operatorname{ran} \mathcal{L}_\alpha \cap \mathcal{H}^n$ and $\mathcal{Q}_{\mathcal{H}}$ commutes with $\mathcal{L}_{\mathcal{H}}$ and $e^{t\mathcal{L}_{\mathcal{H}}}$, the variation of constant formula and (5.4) yield

$$e^{t\mathcal{L}_{\mathcal{H}}} = e^{t\mathcal{L}^-} + \int_0^t e^{(t-s)\mathcal{L}^-} ((df(\phi) - df(0)) \otimes I_{H^k(\mathbb{R}^{d-1})}) e^{s\mathcal{L}_{\mathcal{H}}} ds,$$

from where, by (5.5) and Lemma 4.4, as well as (5.7) and (5.9),

$$\begin{aligned} \|e^{t\mathcal{L}_\mathcal{H}} \mathcal{Q}_\mathcal{H}\|_{\mathcal{B}(\mathcal{H}, H^k(\mathbb{R}^d))} &\leq \|e^{t\mathcal{L}^-}\|_{\mathcal{B}(H^k(\mathbb{R}^d))} \|\mathcal{Q}_\mathcal{H}\|_{\mathcal{B}(\mathcal{H}, H^k(\mathbb{R}^d))} \\ &+ \int_0^t \|e^{(t-s)\mathcal{L}^-}\|_{\mathcal{B}(H^k(\mathbb{R}^d))} \|df(\phi) \\ &- df(0)\|_{\mathcal{B}(H_\alpha^k(\mathbb{R}^d), H^k(\mathbb{R}^d))} \|e^{s\mathcal{L}_\mathcal{H}} \mathcal{Q}_\mathcal{H}\|_{\mathcal{B}(H_\alpha^k(\mathbb{R}^d))} \|\mathcal{Q}_\mathcal{H}\|_{\mathcal{B}(\mathcal{H}, H_\alpha^k(\mathbb{R}^d))} ds < K. \end{aligned}$$

Combined with (5.9) this shows that the semigroup $\{e^{t\mathcal{L}_\mathcal{H}} \mathcal{Q}_\mathcal{H}\}_{t \geq 0}$ is bounded in $\text{ran } \mathcal{Q}_\mathcal{H}$.

We note that $\mathcal{H} = \text{ran } \mathcal{P}_\mathcal{H} \oplus \text{ran } \mathcal{Q}_\mathcal{H}$ and $e^{t\mathcal{L}_\mathcal{H}} = e^{t\mathcal{L}_\mathcal{H}} \mathcal{P}_\mathcal{H} \oplus e^{t\mathcal{L}_\mathcal{H}} \mathcal{Q}_\mathcal{H}$. In order to finish the proof of Lemma 5.4, we will need to show that the semigroup $\{e^{t\mathcal{L}_\mathcal{H}} \mathcal{P}_\mathcal{H}\}$ is bounded in $\text{ran } \mathcal{P}_\mathcal{H}$. Since $\mathcal{P}_\mathcal{H}$ projects onto the kernels of $\mathcal{L}$ defined on $H^k(\mathbb{R}^d)$ and $\mathcal{L}_\alpha$ defined on $H_\alpha^k(\mathbb{R}^d)$, then, by Lemma 4.4, $e^{t\mathcal{L}_\alpha} \mathcal{P}_\mathcal{H} = \mathcal{P}_\mathcal{H}$ and $e^{t\mathcal{L}} \mathcal{P}_\mathcal{H} = \mathcal{P}_\mathcal{H}$, where $\mathcal{P}_\mathcal{H} \in \mathcal{B}(\mathcal{H}, H^k(\mathbb{R}^d))$ and $\mathcal{P}_\mathcal{H} \in \mathcal{B}(H_\alpha^k(\mathbb{R}^d))$, and, therefore, for all $t \geq 0$,

$$\begin{aligned} \|e^{t\mathcal{L}} \mathcal{P}_\mathcal{H}\|_{\mathcal{B}(\mathcal{H}, H^k(\mathbb{R}^d))} &= \|\mathcal{P}_\mathcal{H}\|_{\mathcal{B}(\mathcal{H}, H^k(\mathbb{R}^d))} \leq K \text{ and} \\ \|e^{t\mathcal{L}_\alpha} \mathcal{P}_\mathcal{H}\|_{\mathcal{B}(H_\alpha^k(\mathbb{R}^d))} &= \|\mathcal{P}_\mathcal{H}\|_{\mathcal{B}(H_\alpha^k(\mathbb{R}^d))} \leq K. \end{aligned} \quad \square$$

We also recall the following standard estimates; see, e.g., [K, Lemma 3.2].

LEMMA 5.5. *The semigroup $S_{\Delta_y}(t)$ generated by the linear operator Δ_y for all $t > 0$ satisfies the following decay estimates with some $\beta > 0$:*

- (a) $\|S_{\Delta_y}(t)u\|_{H^k(\mathbb{R}^{d-1})} \leq C\|u\|_{H^k(\mathbb{R}^{d-1})},$
- (b) $\|S_{\Delta_y}(t)u\|_{H^k(\mathbb{R}^{d-1})} \leq C(1+t)^{-\frac{d-1}{4}}\|u\|_{L^1(\mathbb{R}^{d-1})} + Ce^{-\beta t}\|u\|_{H^k(\mathbb{R}^{d-1})},$
- (c) $\|\nabla_y S_{\Delta_y}(t)u\|_{H^k(\mathbb{R}^{d-1})} \leq Ct^{-1/2}\|u\|_{H^k(\mathbb{R}^{d-1})},$
- (d) $\|\nabla_y S_{\Delta_y}(t)u\|_{H^k(\mathbb{R}^{d-1})} \leq C(1+t)^{-\frac{d+1}{4}}\|u\|_{L^1(\mathbb{R}^{d-1})} + Ct^{-\frac{1}{2}}e^{-\beta t}\|u\|_{H^k(\mathbb{R}^{d-1})}.$

6. The system of evolution equations. In this section we derive the system of evolution equations (6.15) governing the perturbation of the planar front, by following [K] with modifications needed to accommodate the presence of the weight.

We denote $\text{ran } \mathcal{P} = \{U \in H_\alpha^k(\mathbb{R}^d)^n : U = \mathcal{P}U\}$ and $\text{ran } \mathcal{Q} = \{U \in H_\alpha^k(\mathbb{R}^d)^n : U = \mathcal{Q}U\}$. In fact, if $U \in \text{ran } \mathcal{Q}$, then $\pi_\alpha U = 0$ because $\mathcal{P}U = 0$. Hypothesis 3.4 and Lemma 3.5 imply that $\phi' \in \mathcal{H}^n$; therefore if $v \in \mathcal{H}^n \hookrightarrow H_\alpha^k(\mathbb{R}^d)^n$, then $\mathcal{P}v \in \mathcal{H}^n$, and then $\mathcal{Q}v = (I - \mathcal{P})v \in \mathcal{H}^n$. Hence we may define $\mathcal{P}_\mathcal{H}$ and $\mathcal{Q}_\mathcal{H}$ to be the restrictions of $\mathcal{P}$ and $\mathcal{Q}$ to $\mathcal{H}^n$. Since $\mathcal{H}^n \hookrightarrow H_\alpha^k(\mathbb{R}^d)^n$, the operators $\mathcal{P}_\mathcal{H}$ and $\mathcal{Q}_\mathcal{H}$ are also bounded. It follows from Lemmas 4.3 and 4.4 that $\mathcal{H}^n = \text{ran } \mathcal{P}_\mathcal{H} \oplus \text{ran } \mathcal{Q}_\mathcal{H}$, where $\text{ran } \mathcal{P}_\mathcal{H} = \text{ran } \mathcal{L}_\alpha \cap \mathcal{H}^n$.

The following lemma shows that for any sufficiently small $\tilde{v} \in \mathcal{H}^n$, there exists a unique pair $(v, q) \in \text{ran } \mathcal{Q}_\mathcal{H} \times H^k(\mathbb{R}^{d-1})$ such that $\phi + \tilde{v}$ can be uniquely expressed by means of (v, q) .

LEMMA 6.1. *Assume Hypothesis 3.4 and $k \geq [\frac{d+1}{2}]$. For any $\tilde{v} \in \mathcal{H}^n$ small enough, there exists $(v, q) \in \text{ran } \mathcal{Q}_\mathcal{H} \times H^k(\mathbb{R}^{d-1})$ such that*

$$(6.1) \quad \phi(z) + \tilde{v}(z, y) = \phi(z - q(y)) + v(z, y), \quad (z, y) \in \mathbb{R}^d.$$

Proof. As in the proof of [K, Lemma 2.2], for any $q \in H^k(\mathbb{R}^{d-1})$ we write

$$\phi(z - q(y)) - \phi(z) = -q(y) \int_0^1 \phi'(z - sq(y)) ds.$$

Since $q \in H^k(\mathbb{R}^{d-1}) \hookrightarrow L^\infty(\mathbb{R}^{d-1})$, we have

$$|\phi'(z - sq(y))| \leq Ke^{-\omega_\pm(z - sq(y))} \leq Ce^{-\omega_\pm z},$$

where C is a constant that depends on q . By Hypothesis 3.4 then

$$\int_{\mathbb{R}} |\phi'(z - sq(y))|^2 \gamma_\alpha^2(z) dz \leq C \left(\int_0^\infty e^{-2\omega_+ z} e^{2\alpha_+ z} dz + \int_{-\infty}^0 e^{-2\omega_- z} e^{2\alpha_- z} dz \right),$$

and, thus, $\|\phi'(\cdot - sq(\cdot))q(\cdot)\|_{L_\alpha^2(\mathbb{R}^d)}^2 \leq C\|q\|_{L^2(\mathbb{R}^{d-1})}^2$, so $\phi(\cdot - q) - \phi(\cdot) \in \mathcal{H}^n$ if $q \in H^k(\mathbb{R}^{d-1})$.

We then write (6.1) as

$$(6.2) \quad \tilde{v}(z, y) = v(z, y) - q(y) \int_0^1 \phi'(z - sq(y)) ds$$

and apply π_α (see (4.1)). Since $v \in \text{ran } \mathcal{Q}_\mathcal{H} = \ker \mathcal{P}_\mathcal{H}$,

$$\pi_\alpha(\tilde{v}(z, y)) = -q(y) \left(\int_0^1 \pi_\alpha(\phi'(z - sq(y))) ds \right).$$

We consider the mapping $(q, \tilde{v}) \mapsto \mathcal{G}(q, \tilde{v})$ defined by

$$\mathcal{G}(q(y), \tilde{v}(z, y)) = \pi_\alpha(\tilde{v}(z, y)) + q(y) \left(\int_0^1 \pi_\alpha(\phi'(z - sq)) ds \right)$$

as a mapping from $H^k(\mathbb{R}^{d-1}) \times \mathcal{H}^n$ to $H^k(\mathbb{R}^{d-1})$ such that $\mathcal{G}(0, 0) = 0$ and $\frac{\partial \mathcal{G}}{\partial q}(0, 0) = I$. For any $\tilde{v}$ near $\tilde{v} = 0$, the implicit function theorem yields the existence of a unique q as a function of $\tilde{v}$ so that $\mathcal{G}(q, \tilde{v}) = 0$.

So, given a $\tilde{v}$, we first find q from the equation $\mathcal{G}(q, \tilde{v}) = 0$ and then, to identify v that corresponds to that q , we apply $\mathcal{Q}_\mathcal{H}$ to (6.2) and set $v = \mathcal{Q}_\mathcal{H} \tilde{v}$, thus obtaining the following formula:

$$v = \mathcal{Q}_\mathcal{H} \tilde{v} + \mathcal{Q}_\mathcal{H} \left(q \int_0^1 \phi'(\cdot - sq) ds \right). \quad \square$$

Since the coordinate system $(v, q) \in \text{ran } \mathcal{Q}_\mathcal{H} \otimes H^k(\mathbb{R}^{d-1})$ is well defined by Lemma 6.1, we can decompose solutions of (3.2) that are close to the front ϕ as a sum of a spatial translation component, i.e., the component in the direction of the front $\phi(z - q(y, t))$, and a normal component v , so that $v = v(\cdot, y, t)$ belongs to $\text{ran } \mathcal{Q}_\alpha = \text{ran } \mathcal{L}_{1, \alpha}$, for each $(y, t) \in \mathbb{R}^{d-1} \times \mathbb{R}^+$. In other words, we can write a solution u of (3.2) in $\mathcal{H}^n$ as

$$(6.3) \quad u(z, y, t) = \phi(z - q(y, t)) + v(z, y, t), \quad (z, y) \in \mathbb{R}^d,$$

where $(v, q) \in \text{ran } \mathcal{Q}_\mathcal{H} \otimes H^k(\mathbb{R}^{d-1})$. For convenience, in what follows, we denote $\phi_q(z) = \phi(z - q)$.

We substitute (6.3) into (3.2). Repeating computations from [K, section 2], we see that v solves the equation

$$(6.4) \quad \partial_t v = Lv + (df(\phi_q) - df(\phi))v + N(\phi_q, v)v + (\partial_t q - \Delta_y q)\phi'_q + (\nabla_y q \cdot \nabla_y q)\phi''_q,$$

where L is the differential expression defined in (3.4), $\nabla_y q = (\partial_{x_2} q, \dots, \partial_{x_d} q)$, and

$$(6.5) \quad N(u, v) = \int_0^1 df(u + sv) - df(u)ds$$

is an $n \times n$ matrix-valued function of (u, v) .

We assume that $v(\cdot, \cdot, t) \in \text{ran } \mathcal{Q}_{\mathcal{H}} \cap \mathcal{H}^n$ for every $t \geq 0$, that is, $\mathcal{P}_{\mathcal{H}}v = 0$, and apply the projection $\mathcal{P}_{\mathcal{H}}$ to (6.4), thus obtaining an equation for q ,

$$(6.6) \quad (-\pi_{\alpha}\phi'_q)\partial_t q = (\pi_{\alpha}\phi''_q)(\nabla_y q \cdot \nabla_y q) - (\pi_{\alpha}\phi'_q)\Delta_y q + \pi_{\alpha}((df(\phi_q) - df(\phi))v + N(\phi_q, v)v).$$

The following result is proved in [K, Lemma 2.3]. It shows that $\pi_{\alpha}(\phi'_q)(y)$ is not close to zero.

LEMMA 6.2. *There are positive constants δ_0 and C such that if $\|q\|_{L^{\infty}(\mathbb{R}^{d-1})} \leq \delta_0$, then for all $y \in \mathbb{R}^{d-1}$*

$$1 - C\delta_0 \leq 1 - C\|q\|_{L^{\infty}(\mathbb{R}^{d-1})} \leq |\pi_{\alpha}(\phi'_q)(y)| \leq 1 + C\|q\|_{L^{\infty}(\mathbb{R}^{d-1})} \leq 1 + C\delta_0, \\ C(1 - \delta_0) \leq C(1 - \|q\|_{L^{\infty}(\mathbb{R}^{d-1})}) \leq |\pi_{\alpha}(\phi''_q)(y)| \leq C(1 + \|q\|_{L^{\infty}(\mathbb{R}^{d-1})}) \leq C(1 + \delta_0).$$

For δ_0 as in Lemma 6.2, we assume that $\|q\|_{L^{\infty}(\mathbb{R}^{d-1})} \leq \delta_0$ and denote

$$(6.7) \quad G(v, q) = (df(\phi_q) - df(\phi))v + N(\phi_q, v)v, \quad K_1(q) = -\frac{\pi_{\alpha}\phi''_q}{\pi_{\alpha}\phi'_q}, \quad K_2(q) = -\frac{1}{\pi_{\alpha}\phi'_q}.$$

Lemma 6.2 allows us to divide both sides of (6.6) by $\pi_{\alpha}\phi'_q$ and obtain

$$(6.8) \quad \partial_t q = \Delta_y q + K_1(q)(\nabla_y q) \cdot (\nabla_y q) + K_2(q)\pi_{\alpha}(G(v, q)).$$

The following lemma is proved by minor modifications of the argument leading to [K, equation (2.23)]. It will be used to derive various estimates for nonlinearities in evolution equations studied below.

LEMMA 6.3. *Let the functions $K_1 = K_1(q)(y)$ and $K_2 = K_2(q)(y)$ for $q \in H^k(\mathbb{R}^{d-1})$ be defined as in (6.7). There exist positive constants δ_0 and C such that for $\|q\|_{H^k(\mathbb{R}^{d-1})} \leq \delta_0$ we have*

$$(6.9) \quad \|K_i(q)\|_{L^{\infty}(\mathbb{R}^{d-1})} \leq C(1 + \|q\|_{H^k(\mathbb{R}^{d-1})}), \quad i = 1, 2.$$

Moreover, the formulas for K_i , $i = 1, 2$, define locally Lipschitz mappings $q \mapsto K_i(q)$ from $H^k(\mathbb{R}^{d-1})$ to $L^{\infty}(\mathbb{R}^{d-1})$.

We return to the task of deriving the evolution equation for the perturbation $\phi_q + v$. Applying the projection operator $\mathcal{Q}_{\mathcal{H}}$ to (6.4) yields the equation

$$(6.10) \quad \partial_t v = Lv + \mathcal{Q}_{\mathcal{H}}(G(v, q) + (\partial_t q - \Delta_y q)\phi'_q + (\nabla_y q) \cdot (\nabla_y q)\phi''_q),$$

where $G(v, q)$ is defined in (6.7). Combining (6.10) and (6.8) we have the system

$$(6.11) \quad \begin{aligned} \partial_t v &= Lv + \mathcal{Q}_{\mathcal{H}}(G(v, q) + (\partial_t q - \Delta_y q)\phi'_q + (\nabla_y q) \cdot (\nabla_y q)\phi''_q), \\ \partial_t q &= \Delta_y q + K_1(q)(\nabla_y q) \cdot (\nabla_y q) + K_2(q)\pi_{\alpha}G(v, q). \end{aligned}$$

We further denote

$$w(y) = \nabla_y q(y), \quad y \in \mathbb{R}^{d-1},$$

$$(6.12) \quad F_1(v, q, w) = G(v, q) + (\partial_t q - \Delta_y q) \phi'_q + (w \cdot w) \phi''_q,$$

$$(6.13) \quad F_2(v, q, w) = K_1(q)(w \cdot w) + K_2(q) \pi_\alpha G(v, q).$$

Using (6.8) in (6.13), we obtain a relation between F_1 and F_2 ,

$$(6.14) \quad F_1(v, q, w) = G(v, q) + F_2(v, q, w) \phi'_q + (w \cdot w) \phi''_q.$$

From (6.4), using $v \in \text{ran } \mathcal{Q}_\mathcal{H}$ and $\phi'_q \in \ker P_\mathcal{H}$, we obtain

$$\mathcal{P}_\mathcal{H} (G(v, q) + (\partial_t q - \Delta_y q) \phi'_q + (w \cdot w) \phi''_q) = 0,$$

which implies that $F_1(v, q, w) = \mathcal{Q}_\mathcal{H} F_1(v, q, w) = G(v, q) + (\partial_t q - \Delta_y q) \phi'_q + (w \cdot w) \phi''_q$. Thus applying ∇_y to (6.11) we finally arrive to the system for $(v, q, w) \in \text{ran } \mathcal{Q}_\mathcal{H} \times H^k(\mathbb{R}^{d-1}) \times H^k(\mathbb{R}^{d-1})^{d-1}$ that we shall study,

$$(6.15) \quad \begin{aligned} \partial_t v &= Lv + F_1(v, q, w), \\ \partial_t q &= \Delta_y q + F_2(v, q, w), \\ \partial_t w &= \Delta_y w + \nabla_y \cdot F_2(v, q, w). \end{aligned}$$

7. Estimates for the nonlinear terms. In this section we obtain estimates for the nonlinear terms in (6.15). Below we use the fact [AF, Theorem 4.39] that, for $2k > d$, the Sobolev embedding yields the inequality

$$(7.1) \quad \|uv\|_{H^k(\mathbb{R}^d)} \leq C \|u\|_{H^k(\mathbb{R}^d)} \|v\|_{H^k(\mathbb{R}^d)}.$$

LEMMA 7.1. For $k \geq [\frac{d+1}{2}]$, the following assertions hold:

- (1) If $u, v \in H^k(\mathbb{R}^d)$, then $uv \in H^k(\mathbb{R}^d)$. Moreover, there exists a constant $C > 0$ such that $\|uv\|_{H^k(\mathbb{R}^d)} \leq C \|u\|_{H^k(\mathbb{R}^d)} \|v\|_{H^k(\mathbb{R}^d)}$ for all $u, v \in H^k(\mathbb{R}^d)$.
- (2) If $u, v \in \mathcal{H}$, then $uv \in H^k_\alpha(\mathbb{R}^d)$. Moreover, there exists a constant $C > 0$ such that $\|uv\|_{H^k_\alpha(\mathbb{R}^d)} \leq C \|u\|_{H^k(\mathbb{R}^d)} \|v\|_{H^k_\alpha(\mathbb{R}^d)}$ for all $u, v \in \mathcal{H}$.
- (3) If $u, v \in \mathcal{H}$, then $uv \in \mathcal{H}$. Moreover, there exists a constant $C > 0$ such that $\|uv\|_\mathcal{H} \leq C \|u\|_\mathcal{H} \|v\|_\mathcal{H}$ for all $u, v \in \mathcal{H}$.

Proof. The proof is similar to the one-dimensional estimates in [GLS, Proposition 7.1].

LEMMA 7.2. For $k \geq [\frac{d+1}{2}]$, if $q_1, q_2 \in H^k(\mathbb{R}^{d-1})$ and $\psi \in H^{k+2}(\mathbb{R})$ is such that $\psi'(z) \rightarrow 0$ exponentially as $z \rightarrow \pm\infty$; then the function $\sigma(z, y) = \psi'(z - q_1(y))q_2(y)$, $(z, y) \in \mathbb{R}^d$, satisfies

$$\|\sigma\|_{H^k(\mathbb{R}^d)} \leq C \|q_2\|_{H^k(\mathbb{R}^{d-1})},$$

where $C = C(\|\psi\|_{L^\infty(\mathbb{R})}, \|\psi'\|_{H^{k+1}(\mathbb{R})}, \|q_1\|_{H^k(\mathbb{R}^{d-1})})$ is bounded in each ball $\{q_1 : \|q_1\|_{H^k(\mathbb{R}^{d-1})} \leq K\}$.

Proof. The derivatives of σ are given by

$$(7.2) \quad \begin{aligned} \frac{\partial \sigma}{\partial z} &= \psi''(z - q_1(y))q_2(y), \\ \frac{\partial \sigma}{\partial x_j} &= \psi''(z - q(y))q_2(y) \frac{\partial q_1}{\partial x_j} + \psi'(z - q_1(y)) \frac{\partial q_2}{\partial x_j}, \quad j = 2, \dots, d. \end{aligned}$$

Since ψ' is exponentially decaying to 0, we have

$$(7.3) \quad \|\sigma\|_{L^2(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^{d-1}} \left(\int_{\mathbb{R}} |\psi'(z - q_1(y))|^2 dz \right) |q_2(y)|^2 dy \leq C \|q_2\|_{H^k(\mathbb{R}^{d-1})}.$$

Similarly,

$$(7.4) \quad \left\| \psi''(z - q(y)) q_2(y) \frac{\partial q_1(y)}{\partial x_j} \right\|_{L^2(\mathbb{R}^d)} \leq C \|q_2\|_{L^\infty(\mathbb{R}^{d-1})} \left\| \frac{\partial q_1}{\partial x_j} \right\|_{L^2(\mathbb{R}^{d-1})} \\ \leq C \|q_2\|_{H^k(\mathbb{R}^{d-1})} \|q_1\|_{H^k(\mathbb{R}^{d-1})}.$$

The statement of the lemma then is proved by a calculation similar to the proof of Proposition A.3 in the appendix. Indeed, instead of (A.7) in the proof of Proposition A.3, we may use relations

$$k - \frac{d-1}{2} = k - \frac{d}{2} + \frac{1}{2} > n_i - \frac{d}{p_i} + \frac{1}{p_i} = n_i - \frac{d-1}{p_i},$$

which proves the embedding $H^k(\mathbb{R}^{d-1}) \hookrightarrow W^{n_i, p_i}(\mathbb{R}^{d-1})$ by Lemma A.1 in the appendix. $\square$

Using Lemma 7.2 we now prove the following estimates of the $H^k(\mathbb{R}^d)$ -norm and the weighted norm of the nonlinear term $G(v, q)$ introduced in (6.7).

PROPOSITION 7.3. *Assume Hypotheses 3.2 and 3.3. For $k \geq [\frac{d+1}{2}]$, the following assertions hold:*

- (1) *Formula $(v, q) \mapsto (df(\phi_q) - df(\phi))v$ defines a mapping from $H^k(\mathbb{R}^d)^n \times H^k(\mathbb{R}^{d-1})$ to $H^k(\mathbb{R}^d)^n$ that is locally Lipschitz on any set of the form $\{(v, q) : \|v\|_{H^k(\mathbb{R}^d)} + \|q\|_{H^k(\mathbb{R}^{d-1})} \leq K\}$. On such a set there is a constant C_K depending on K such that*

$$\|(df(\phi_q) - df(\phi))v\|_{H^k(\mathbb{R}^d)} \leq C_K \|q\|_{H^k(\mathbb{R}^{d-1})} \|v\|_{H^k(\mathbb{R}^d)}.$$

- (2) *Formula $(v, q) \mapsto (df(\phi_q) - df(\phi))v$ defines a mapping from $\mathcal{H}^n \times H^k(\mathbb{R}^{d-1})$ to $\mathcal{H}^n$ that is locally Lipschitz on any set of the form $\{(v, q) : \|v\|_{\mathcal{H}} + \|q\|_{H^k(\mathbb{R}^{d-1})} \leq K\}$. On such a set there is a constant C_K depending on K such that*

$$\|(df(\phi_q) - df(\phi))v\|_{H_\alpha^k(\mathbb{R}^d)} \leq C_K \|q\|_{H^k(\mathbb{R}^{d-1})} \|v\|_{H_\alpha^k(\mathbb{R}^d)},$$

and, therefore,

$$\|(df(\phi_q) - df(\phi))v\|_{\mathcal{H}} \leq C_K \|q\|_{H^k(\mathbb{R}^{d-1})} \|v\|_{\mathcal{H}}.$$

Proof. We define $p(q, v) \in H^k(\mathbb{R}^d)$ for $q \in H^k(\mathbb{R}^{d-1})$ and $v \in H^k(\mathbb{R}^d)$ by the formula

$$p(q, v)(z, y) = \left(df(\phi(z - q(y))) - df(\phi(z)) \right) v(x),$$

so that

$$(7.5) \quad p(q, v) = \int_0^1 \frac{d}{ds} df(\phi(\cdot - sq)) v ds = - \int_0^1 d^2 f(\phi(\cdot - sq)) (\phi'(\cdot - sq) q, v) ds.$$

Since $x \mapsto d^2 f(\phi(z - sq(y)))$ is a smooth function with bounded derivatives, using Lemma 7.2 we conclude that

$$p(q, v) \in H^k(\mathbb{R}^d) \text{ and } \|p(q, v)\|_{H^k(\mathbb{R}^d)} \leq C_K \|q\|_{H^k(\mathbb{R}^{d-1})} \|v\|_{H^k(\mathbb{R}^d)}.$$

We then multiply (7.5) by γ_α and infer

$$\|p(q, v)\|_{H_\alpha^k(\mathbb{R}^d)} \leq C_K \|q\|_{H^k(\mathbb{R}^{d-1})} \|v\|_{H_\alpha^k(\mathbb{R}^d)}.$$

To show the local Lipschitz estimates for $p(q, v)$ and $\gamma_\alpha p(q, v)$, we pass to components in the vector equation (7.5). It is enough to show that the map $(q_1, q_2, v) \mapsto \tilde{l}(q_1, q_2, v)$ defined by

$$\tilde{l}(q_1, q_2, v)(z, y) = l(\psi(z - q_1(y)))\psi'(z - q_1(y))q_2(y)v(x), \quad x = (z, y) \in \mathbb{R}^d$$

is a locally Lipschitz map from $H^k(\mathbb{R}^{d-1}) \times H^k(\mathbb{R}^{d-1}) \times H^k(\mathbb{R}^d)$ into $H^k(\mathbb{R}^d)$. Here $\psi : \mathbb{R} \rightarrow \mathbb{R}$ is a function that is exponentially decaying to some constants $\psi_\pm$ as $z \rightarrow \pm\infty$, the derivatives $\psi^{(m)}(z) \rightarrow 0$, $m = 1, 2, \dots, k+2$, as $z \rightarrow \pm\infty$ exponentially, and $l : \mathbb{R} \rightarrow \mathbb{R}$ is a C^{k+3} function with bounded derivatives.

Recall that $k \geq [\frac{d+1}{2}]$ and thus $H^k(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$. The derivatives of l are bounded, so by Lemma 7.2 we have

$$\|\tilde{l}(q_1, q_2, v)\|_{H^k(\mathbb{R}^d)} \leq C \|l\|_{C^{k+3}} \|q_2\|_{H^k(\mathbb{R}^{d-1})} \|v\|_{H^k(\mathbb{R}^d)},$$

and thus the map $\tilde{l}$ is well defined.

We will now proceed with the local Lipschitz estimates for $\tilde{l}$. To show the estimate for the variation in q_1 , we fix q_2, v and write

$$\begin{aligned} \tilde{l}(q_1, q_2, v) - \tilde{l}(\bar{q}_1, q_2, v) &= \left(l(\psi(\cdot - q_1)) - l(\psi(\cdot - \bar{q}_1)) \right) \psi'(\cdot - q_1) q_2 v \\ &\quad + l(\psi(\cdot - \bar{q}_1)) (\psi'(\cdot - q_1) - \psi'(\cdot - \bar{q}_1)) q_2 v, \end{aligned}$$

where

$$\begin{aligned} l(\psi(z - q_1(y))) - l(\psi(z - \bar{q}_1(y))) &= \int_0^1 \frac{d}{ds} l(\psi(z - q_1(y) - (s-1)(q_1(y) - \bar{q}_1(y)))) ds \\ &= - \int_0^1 l'(\psi(\cdot)) \psi'(\cdot) (q_1(y) - \bar{q}_1(y)) ds. \end{aligned}$$

Applying Lemma 7.2 again we get

$$\|l(\psi(\cdot - q_1)) - l(\psi(\cdot - \bar{q}_1))\|_{H^k(\mathbb{R}^d)} \leq C_K \|q_1 - \bar{q}_1\|_{H^k(\mathbb{R}^{d-1})}.$$

On the other hand,

$$\begin{aligned} \psi'(z - q_1(y)) - \psi'(z - \bar{q}_1(y)) &= \int_0^1 \frac{d}{ds} \psi'(z - q_1(y) - (s-1)(q_1(y) - \bar{q}_1(y))) ds \\ &= - \int_0^1 \psi''(\cdot) (q_1(y) - \bar{q}_1(y)) ds. \end{aligned}$$

Another application of Lemma 7.2 yields

$$\|\tilde{l}(q_1, q_2, v) - \tilde{l}(\bar{q}_1, q_2, v)\|_{H^k(\mathbb{R}^d)} \leq C_K \|q_1 - \bar{q}_1\|_{H^k(\mathbb{R}^{d-1})}.$$

The estimate for the variation in q_2 is similar. The estimate for the variation in v follows from Proposition A.3 in the appendix by fixing q_1 and q_2 .

Multiplying $\tilde{l}$ by γ_α and working with $l(\psi(\cdot - q_1))\psi'(\cdot - q_1)q_2\gamma_\alpha v$ gives the local Lipschitz estimate of $p(q, v)$ in the weighted norm. $\square$

The next statement concerns the nonlinearity N defined in (6.5).

PROPOSITION 7.4. Assume Hypotheses 3.2 and 3.3, and let $k \geq [\frac{d+1}{2}]$.

- (1) Formula $(v, q) \mapsto N(\phi_q, v)$ defines a mapping from $H^k(\mathbb{R}^d)^n \times H^k(\mathbb{R}^{d-1})$ to $H^k(\mathbb{R}^d)^{n^2}$ that is locally Lipschitz and $\mathcal{O}(\|v\|_{H^k(\mathbb{R}^d)})$ as $\|v\|_{H^k(\mathbb{R}^d)} \rightarrow 0$ uniformly on any bounded neighborhood of $(0, 0)$ in $H^k(\mathbb{R}^d)^n \times H^k(\mathbb{R}^{d-1})$.
- (2) Formula $(v, q) \mapsto N(\phi_q, v)v$ defines a mapping from $H^k(\mathbb{R}^d)^n \times H^k(\mathbb{R}^{d-1})$ to $H^k(\mathbb{R}^d)^n$ that is locally Lipschitz on any bounded neighborhood of $(0, 0)$ in $H^k(\mathbb{R}^d)^n \times H^k(\mathbb{R}^{d-1})$.

Proof. To prove (1), we note first that

$$N(\phi_q, v) = \int_0^1 \int_0^1 \frac{d}{d\tau} (df(\phi_q + s\tau v)) d\tau ds = \int_0^1 \int_0^1 d^2 f(\phi_q + s\tau v) s v d\tau ds.$$

It is enough to show that the following map $\tilde{l} : H^k(\mathbb{R}^{d-1}) \times H^k(\mathbb{R}^d) \times H^k(\mathbb{R}^d) \rightarrow H^k(\mathbb{R}^d)$ is locally Lipschitz. We define

$$\tilde{l}(q, u, v)(z, y) = l(\psi(z - q(y)), u(x))v(x), \quad x = (z, y) \in \mathbb{R}^d,$$

where $\psi : \mathbb{R} \rightarrow \mathbb{R}$ is a function exponentially decaying to some constants $\psi_{\pm}$ as $z \rightarrow \pm\infty$ and $\psi^{(m)}(z) \rightarrow 0$ exponentially, for any $m = 1, 2, \dots$, and $l : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a C^{k+3} -smooth bounded function with bounded derivatives.

Again, $k \geq [\frac{d+1}{2}]$, so that $H^k(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$, and then $\|\tilde{l}(q, u, v)\|_{L^2(\mathbb{R}^d)} \leq \|l\|_{C^{k+3}} \|v\|_{L^2(\mathbb{R}^d)}$. If l'_j denotes the derivative with respect to the j th variable, then

$$\begin{aligned} \frac{\partial \tilde{l}}{\partial z} &= l'_1(\cdot) \psi'(z - q(y))v(x) + l'_2(\cdot) \frac{\partial u}{\partial z} v(x) + l(\cdot) \frac{\partial v}{\partial z}, \\ \frac{\partial \tilde{l}}{\partial x_j} &= -l'_1(\cdot) \psi'(z - q(y))v(x) \frac{\partial q}{\partial x_j} + l'_2(\cdot) \frac{\partial u}{\partial x_j} v(x) + l(\cdot) \frac{\partial v}{\partial x_j}, \quad j = 2, \dots, d. \end{aligned}$$

Since l'_1 , l'_2 , ψ' , and $\frac{\partial q}{\partial x_j}$ are bounded, $\tilde{l}(q, u, v) \in H^1(\mathbb{R}^d)$. A similar calculation (cf. the proof of Proposition A.3) with $q(z, y)$ in the proof replaced by $\phi'(z - q(y))$ shows that $\tilde{l}(q, u, v) \in H^k(\mathbb{R}^d)$. Thus, the map $\tilde{l}$ is well defined. Next we proceed with the proof of the local Lipschitz property.

Variation in q gives

$$\begin{aligned} \tilde{l}(q, u, v) - \tilde{l}(\bar{q}, u, v) &= \int_0^1 \frac{d}{ds} l\left(\psi(z - q - (s-1)(q - \bar{q})), u\right) v ds \\ &= - \int_0^1 l'_1(\psi(\cdot), u) \psi'(\cdot)(q - \bar{q}) v ds. \end{aligned}$$

Since l'_1 and its derivatives are bounded, the main part of the estimate

$$\|\tilde{l}(q, u, v) - \tilde{l}(\bar{q}, u, v)\|_{H^k(\mathbb{R}^d)} \leq C_K \|q - \bar{q}\|_{H^k(\mathbb{R}^{d-1})}$$

on sets of the form $\{(q, u, v) : \|q\|_{H^k(\mathbb{R}^{d-1})} + \|u\|_{H^k(\mathbb{R}^d)} + \|v\|_{H^k(\mathbb{R}^d)} \leq K\}$ is reduced to Lemma 7.2.

For variation in u , the estimate

$$\begin{aligned} \|\tilde{l}(q, u, v) - \tilde{l}(q, \bar{u}, v)\|_{H^k(\mathbb{R}^d)} &= \|l(\psi(z - q), u) - l(\psi(z - q), \bar{u})\|_{H^k(\mathbb{R}^d)} v \\ &\leq C_K \|u - \bar{u}\|_{H^k(\mathbb{R}^d)} \end{aligned}$$

follows from Proposition A.3 considered for the mapping $u \mapsto l(\psi_q, u)$ from $H^k(\mathbb{R}^d)$ into $H^k(\mathbb{R}^d)$.

The estimate for the variation in v also follows from Proposition A.3 for fixed q and u . This concludes the proof of, the first assertion in part (1) of Proposition 7.4.

Using the Lipschitz property and the property $N(\phi_q, 0) = 0$ we conclude that

$$\|N(\phi_q, v)\|_{H^k(\mathbb{R}^d)} = \|N(\phi_q, v) - N(\phi_q, 0)\|_{H^k(\mathbb{R}^d)} \leq C_K \|v\|_{H^k(\mathbb{R}^d)}$$

on any set of the form $\{(v, q) : \|q\|_{H^k(\mathbb{R}^{d-1})} + \|v\|_{H^k(\mathbb{R}^d)} \leq K\}$ as required.

The proof of part (2) follows from part (1) since $H^k(\mathbb{R}^d)$ is an algebra (see (7.1)); for instance, the estimate of the variation in q is

$$\begin{aligned} \|N(\phi_q, v) - N(\phi_{\bar{q}}, v)\|_{H^k(\mathbb{R}^d)} &= \|(\tilde{l}(q, u, v) - \tilde{l}(\bar{q}, u, v))v\|_{H^k(\mathbb{R}^d)} \\ &\leq \|\tilde{l}(q, u, v) - \tilde{l}(\bar{q}, u, v)\|_{H^k(\mathbb{R}^d)} \|v\|_{H^k(\mathbb{R}^d)} \\ &\leq C_K \|q - \bar{q}\|_{H^k(\mathbb{R}^{d-1})} \|v\|_{H^k(\mathbb{R}^d)}. \end{aligned}$$

The estimate for variation in v follows from Proposition A.3, where we fix q and consider the map $v \rightarrow l(\psi_q, v)v$. $\square$

PROPOSITION 7.5. *Assume Hypotheses 3.2 and 3.3 and let $k \geq [\frac{d+1}{2}]$.*

- (1) *If $v \in \mathcal{H}^n$, then $N(\phi_q, v)v \in H_\alpha^k(\mathbb{R}^d)^n$, and for any ball of radius K centered at $(0, 0)$ in $\mathcal{H}^n \times H^k(\mathbb{R}^{d-1})$ there is a constant $C_K > 0$ depending on K such that for any (v, q) in the ball one has*

$$\|N(\phi_q, v)v\|_{H_\alpha^k(\mathbb{R}^d)} \leq C_K \|v\|_{H^k(\mathbb{R}^d)} \|v\|_{H_\alpha^k(\mathbb{R}^d)}.$$

- (2) *Formula $(v, q) \mapsto N(\phi_q, v)$ defines a mapping from $\mathcal{H}^n \times H^k(\mathbb{R}^{d-1})$ to $\mathcal{H}^{n^2}$ that is locally Lipschitz on any bounded neighborhood of $(0, 0)$ in $\mathcal{H}^n \times H^k(\mathbb{R}^{d-1})$.*
- (3) *Formula $(v, q) \mapsto N(\phi_q, v)v$ defines a mapping from $\mathcal{H}^n \times H^k(\mathbb{R}^{d-1})$ to $\mathcal{H}^n$ that is locally Lipschitz on any bounded neighborhood of $(0, 0)$ in $\mathcal{H}^n \times H^k(\mathbb{R}^{d-1})$.*

Proof. (1) Using (7.1) and Proposition 7.4(1) we infer

$$\begin{aligned} \|N(\phi_q, v)v\|_{H_\alpha^k(\mathbb{R}^d)} &= \|N(\phi_q, v)\gamma_\alpha v\|_{H^k(\mathbb{R}^d)} \\ &\leq C \|N(\phi_q, v)\|_{H^k(\mathbb{R}^d)} \|\gamma_\alpha v\|_{H^k(\mathbb{R}^d)} \leq C_K \|v\|_{H^k(\mathbb{R}^d)} \|v\|_{H_\alpha^k(\mathbb{R}^d)}. \end{aligned}$$

To show the local Lipschitz property in parts (2) and (3) of the proposition, we note that

$$\gamma_\alpha N(\phi_q, v) = \int_0^1 \int_0^1 d^2 f(\phi_q + s\tau v) s \gamma_\alpha v d\tau ds.$$

The Lipschitz assertion then is proved by repeating arguments from Proposition 7.4(1) and (2) for $\tilde{l}(q, u, \gamma_\alpha v)$. $\square$

PROPOSITION 7.6. *Assume Hypotheses 3.2 and 3.3, and let $k \geq [\frac{d+1}{2}]$. The formula $(q, w) \rightarrow (w \cdot w)\phi_q''$ defines a locally Lipschitz mapping from $H^k(\mathbb{R}^{d-1}) \times H^k(\mathbb{R}^{d-1})^{d-1}$ to $\mathcal{H}^n$ on any bounded set of the form $\{(q, w) : \|q\|_{H^k(\mathbb{R}^{d-1})} + \|w\|_{H^k(\mathbb{R}^{d-1})} \leq K\}$, and the mapping satisfies*

$$\|(w \cdot w)\phi_q''\|_{H^k(\mathbb{R}^d)} \leq C_K \|w\|_{H^k(\mathbb{R}^{d-1})}^2 \quad \text{and} \quad \|((w \cdot w)\phi_q'')\|_{H_\alpha^k(\mathbb{R}^d)} \leq C_K \|w\|_{H^k(\mathbb{R}^{d-1})}^2.$$

Proof. Recall that, by Hypothesis 3.4, ϕ'' and its derivatives are exponentially decaying to 0 as $z \rightarrow \pm\infty$, and, by Lemma 3.5, $\gamma_\alpha \phi^{(m)}$, for $m = 1, \dots, k+1$, is exponentially decaying to 0 as $z \rightarrow \pm\infty$.

For a fixed $q \in H^k(\mathbb{R}^{d-1})$, to show the local Lipschitz estimate in w , we use the Sobolev embedding $H^k(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ and inequality (7.1) for $H^k(\mathbb{R}^{d-1})$ with $k \geq [\frac{d+1}{2}] > \frac{(d-1)+1}{2}$ and observe that, using Lemma 7.2 with $q_2 = w \cdot w - \bar{w} \cdot \bar{w}$, we have

$$\begin{aligned} \|(w \cdot w - \bar{w} \cdot \bar{w})\phi_q''\|_0 &\leq C\|(w - \bar{w}) \cdot (w + \bar{w})\|_{H^k(\mathbb{R}^{d-1})} \\ &\leq C\|w + \bar{w}\|_{H^k(\mathbb{R}^{d-1})}\|w - \bar{w}\|_{H^k(\mathbb{R}^{d-1})} \\ &\leq C_K\|w - \bar{w}\|_{H^k(\mathbb{R}^{d-1})}, \\ \|(w \cdot w - \bar{w} \cdot \bar{w})\phi_q''\|_{H_\alpha^k(\mathbb{R}^d)} &\leq C\|(w - \bar{w}) \cdot (w + \bar{w})\|_{H^k(\mathbb{R}^{d-1})} \\ &\leq C\|w + \bar{w}\|_{H^k(\mathbb{R}^{d-1})}\|w - \bar{w}\|_{H^k(\mathbb{R}^{d-1})} \\ &\leq C_K\|w - \bar{w}\|_{H^k(\mathbb{R}^{d-1})} \end{aligned}$$

with some $C_K > 0$ that depends on K . This completes the proof.

The local Lipschitz estimate in q is proved similarly to Proposition 7.4 using Lemma 7.2. $\square$

PROPOSITION 7.7. *Assume Hypotheses 3.2 and 3.3, and let $k \geq [\frac{d+1}{2}]$. Formula (6.7) for $G(v, q)$, formula (6.12) for $F_1(v, q, w)$, and formula (6.13) for $F_2(v, q, w)$ define locally Lipschitz mappings from $\mathcal{H}^n \times H^k(\mathbb{R}^{d-1}) \times H^k(\mathbb{R}^{d-1})^{d-1}$ to $\mathcal{H}^n$, $\mathcal{H}^n$, and $H^k(\mathbb{R}^{d-1})$, respectively, on any set of the form $\{(v, q, w) : \|v\|_{\mathcal{H}} + \|q\|_{H^k(\mathbb{R}^{d-1})} + \|w\|_{H^k(\mathbb{R}^{d-1})} < K\}$. Moreover, if $\|v\|_{\mathcal{H}} + \|q\|_{H^k(\mathbb{R}^{d-1})} + \|w\|_{H^k(\mathbb{R}^{d-1})} < K$, then for some $C_K > 0$ depending on K one has*

$$\begin{aligned} \text{(a)} \quad &\|G(v, q)\|_{H_\alpha^k(\mathbb{R}^d)} \leq C_K(\|v\|_{H^k(\mathbb{R}^d)} + \|q\|_{H^k(\mathbb{R}^{d-1})})\|v\|_{H_\alpha^k(\mathbb{R}^d)}, \\ \text{(b)} \quad &\|F_1(v, q, w)\|_{H_\alpha^k(\mathbb{R}^d)} \leq C_K(\|v\|_{H^k(\mathbb{R}^d)}\|v\|_{H_\alpha^k(\mathbb{R}^d)} + \|q\|_{H^k(\mathbb{R}^{d-1})}\|v\|_{H_\alpha^k(\mathbb{R}^d)} \\ &\quad + \|w\|_{H^k(\mathbb{R}^{d-1})}^2), \\ \text{(b')} \quad &\|F_1(v, q, w)\|_{H^k(\mathbb{R}^d)} \leq C_K(\|v\|_{H^k(\mathbb{R}^d)}\|v\|_{H_\alpha^k(\mathbb{R}^d)} + \|q\|_{H^k(\mathbb{R}^{d-1})}\|v\|_{H_\alpha^k(\mathbb{R}^d)} \\ &\quad + \|w\|_{H^k(\mathbb{R}^{d-1})}^2 + \|v\|_{H^k(\mathbb{R}^d)}\|v_2\|_{H^k(\mathbb{R}^d)}), \text{ where } v = (v_1, v_2)^T \\ &\quad \text{with } v_i \in \mathcal{H}^{n_i}, i = 1, 2, \\ \text{(c)} \quad &\|F_2(v, q, w)\|_{H^k(\mathbb{R}^{d-1})} \leq C_K(\|v\|_{H^k(\mathbb{R}^d)}\|v\|_{H_\alpha^k(\mathbb{R}^d)} + \|q\|_{H^k(\mathbb{R}^{d-1})}\|v\|_{H_\alpha^k(\mathbb{R}^d)} + \\ &\quad \|w\|_{H^k(\mathbb{R}^{d-1})}^2), \\ \text{(d)} \quad &\|F_2(v, q, w)\|_{L^1(\mathbb{R}^{d-1})} \leq C_K(\|v\|_{H^k(\mathbb{R}^d)}\|v\|_{H_\alpha^k(\mathbb{R}^d)} + \|q\|_{H^k(\mathbb{R}^{d-1})}\|v\|_{H_\alpha^k(\mathbb{R}^d)} + \\ &\quad \|w\|_{H^k(\mathbb{R}^{d-1})}^2). \end{aligned}$$

Proof. The local Lipschitz property of $(df(\phi_q) - df(\phi))v$ on $\mathcal{H}^n$ has been proved in Proposition 7.3, and the local Lipschitz property of $N(\phi_q, v)v$ on $\mathcal{H}^n \times H^k(\mathbb{R}^{d-1})$ has been proved in Proposition 7.5(3). The local Lipschitz properties of these terms imply the locally Lipschitz property of $G(v, q)$ on $\mathcal{H}^n \times H^k(\mathbb{R}^{d-1})$. The proof of (a) follows from (6.7) and Propositions 7.3 and 7.5.

In formula (6.13) for $F_2(v, q, w)$, we first consider the term $K_1(q)(w \cdot w)$. The Lipschitz estimate of the variation in q follows from the triangular inequality, inequality (7.1), and Lemma 6.3 because

$$\begin{aligned} &\|K_1(q)(w \cdot w - K_1(\bar{q})(\bar{w} \cdot \bar{w}))\|_{H^k(\mathbb{R}^{d-1})} \\ &\leq \|K_1(q) - K_1(\bar{q})\|_{H^k(\mathbb{R}^{d-1})}\|w \cdot w\|_{H^k(\mathbb{R}^{d-1})} \\ &\quad + \|K_1(\bar{q})\|_{H^k(\mathbb{R}^{d-1})}\|(w \cdot w - \bar{w} \cdot \bar{w})\|_{H^k(\mathbb{R}^{d-1})} \\ &\leq C_K(\|q - \bar{q}\|_{H^k(\mathbb{R}^{d-1})} + \|w - \bar{w}\|_{H^k(\mathbb{R}^{d-1})}). \end{aligned}$$

We then consider the term $K_2(q)\pi_\alpha G(v, q)$. The Lipschitz estimate of the variation in q follows from the triangular inequality, (7.1), Lemma 4.3, the Lipschitz property of $G(v, q)$ on $\mathcal{H}^n \times H^k(\mathbb{R}^{d-1})$, Lemma 6.3, and the estimates in part (a) because

$$\begin{aligned} & \|K_2(q)\pi_\alpha G(v, q) - K_2(\bar{q})\pi_\alpha G(v, \bar{q})\|_{H^k(\mathbb{R}^{d-1})} \\ & \leq \|K_2(q)\pi_\alpha G(v, q) - K_2(q)\pi_\alpha G(v, \bar{q})\|_{H^k(\mathbb{R}^{d-1})} \\ & \quad + \|K_2(q)\pi_\alpha G(v, \bar{q}) - K_2(\bar{q})\pi_\alpha G(v, \bar{q})\|_{H^k(\mathbb{R}^{d-1})} \\ & \leq \|K_2(q)\|_{L^\infty(\mathbb{R}^{d-1})} \|\pi_\alpha\|_{\mathcal{B}(\mathcal{H}, H^k(\mathbb{R}^{d-1}))} \|G(v, q) - G(v, \bar{q})\|_{\mathcal{H}} \\ & \quad + \|K_2(q) - K_2(\bar{q})\|_{L^\infty(\mathbb{R}^{d-1})} \|\pi_\alpha\|_{\mathcal{B}(\mathcal{H}, H^k(\mathbb{R}^{d-1}))} \|G(v, \bar{q})\|_{\mathcal{H}} \leq C_K \|q - \bar{q}\|_{H^k(\mathbb{R}^{d-1})}. \end{aligned}$$

The estimate in part (c) follows from (7.1), (4.3), (4.4), and (6.9). Indeed,

$$\begin{aligned} (7.6) \quad & \|K_2(q)\pi_\alpha G(v, q)\|_{H^k(\mathbb{R}^{d-1})} \leq \|K_2(q)\|_{L^\infty(\mathbb{R}^{d-1})} \|\pi_\alpha G(v, q)\|_{H^k(\mathbb{R}^{d-1})} \\ & \leq C_K \|G(v, q)\|_{H_\alpha^k(\mathbb{R}^d)} \leq C_K (\|v\|_{H^k(\mathbb{R}^d)} \|v\|_{H_\alpha^k(\mathbb{R}^d)} + \|v\|_{H_\alpha^k(\mathbb{R}^d)} \|q\|_{H^k(\mathbb{R}^{d-1})}), \end{aligned}$$

$$(7.7) \quad \|K_1(q)(w \cdot w)\|_{H^k(\mathbb{R}^{d-1})} \leq \|K_1(q)\|_{L^\infty(\mathbb{R}^{d-1})} \|w \cdot w\|_{H^k(\mathbb{R}^{d-1})} \leq C_K \|w\|_{H^k(\mathbb{R}^{d-1})}^2.$$

Combining estimates in part (a), (7.6), and (7.7) we have part (c).

For part (b), in formula (6.14) of $F_1(v, q, w)$, we already have the Lipschitz property of $G(v, q)$ mapping into $\mathcal{H}^n$ by part (a) and the Lipschitz property of the term $(w \cdot w)\phi''(q)$ mapping into $\mathcal{H}^n$ by Proposition 7.6. To prove the Lipschitz estimate for $\phi'_q F_2(v, q, w)$ of the variation in v and w , we apply the Lipschitz property of the map $(v, q) \mapsto F_2(v, q, w)$ for a fixed q . To prove the Lipschitz property of the variation in $q \mapsto \phi'_q F_2(v, q, w)$, we use the fact that ϕ' decays exponentially to 0 and Lemma 7.2 with $q_2 = F_2(v, q, w)$. We have the inequality

$$\|F_1(v, q, w)\|_{H_\alpha^k(\mathbb{R}^d)} \leq \|G(v, q)\|_{H_\alpha^k(\mathbb{R}^d)} + C \|F_2(v, q, w)\|_{H^k(\mathbb{R}^{d-1})} + \|\gamma_\alpha(w \cdot w)\phi_q''\|_{H^k(\mathbb{R}^d)}.$$

We can now use the estimates on $G(v, q)$ in part (a) to deal with the first term, then use the estimates for $F_2(v, q, w)$ in part (c), while the estimates of the last term are given by Proposition 7.6.

Similarly, for part (b'), we refer to Proposition 7.6 and then use Lemma 7.2 to estimate the second term in the formula (6.14) for F_1 ,

$$\|\phi'_q F_2(v, q, w)\|_{H^k(\mathbb{R}^d)} \leq C (\|v\|_{H^k(\mathbb{R}^d)} \|v\|_{H_\alpha^k(\mathbb{R}^d)} + \|q\|_{H^k(\mathbb{R}^{d-1})} \|v\|_{H_\alpha^k(\mathbb{R}^d)} + \|w\|_{H^k(\mathbb{R}^{d-1})}^2).$$

Similarly to the proof of Proposition 7.7(b), the third term in (6.14) is estimated as

$$\|\phi_q''(w \cdot w)\|_{H^k(\mathbb{R}^d)} \leq C \|w\|_{H^k(\mathbb{R}^{d-1})}^2.$$

We obtain an estimate for $G(v, q) = (df(\phi_q) - df(\phi))v + N(\phi_q, v)v$ by using from Lemma 7.9(2)–(3),

$$\begin{aligned} \|G(v, q)\|_{H^k(\mathbb{R}^d)} & \leq \|(df(\phi_q) - df(\phi))v\|_{H^k(\mathbb{R}^d)} + \|N(\phi_q, v)v\|_{H^k(\mathbb{R}^d)} \\ & \leq K (\|q\|_{H^k(\mathbb{R}^{d-1})} \|v\|_{H_\alpha^k(\mathbb{R}^d)} + \|v\|_{H^k(\mathbb{R}^d)} \|v\|_{H_\alpha^k(\mathbb{R}^d)} \\ & \quad + \|v\|_{H^k(\mathbb{R}^d)} \|v_2\|_{H^k(\mathbb{R}^d)}). \end{aligned}$$

Adding the above inequalities for the terms of (6.14) finishes the proof of (b').

To prove part (d), we use the Cauchy–Schwarz inequality, (4.5), Proposition 7.4(1), and Lemma 4.3,

$$\begin{aligned}
 \|\pi_\alpha G(v, q)\|_{L^1(\mathbb{R}^{d-1})} &\leq C\|\gamma_\alpha G(v, q)\|_{L^1(\mathbb{R}^d)} \\
 &\leq C(\|\gamma_\alpha N(\phi_q, v)v\|_{L^1(\mathbb{R}^d)} + \|\gamma_\alpha(df(\phi_q) - df(\phi))v\|_{L^1(\mathbb{R}^d)}) \\
 &\leq C(\|N(\phi_q, v)\|_{L^2(\mathbb{R}^d)}\|\gamma_\alpha v\|_{L^2(\mathbb{R}^d)} \\
 &\quad + \|df(\phi_q) - df(\phi)\|_{L^2(\mathbb{R}^d)}\|\gamma_\alpha v\|_{L^2(\mathbb{R}^d)}) \\
 &\leq C_K(\|v\|_{H^k(\mathbb{R}^d)}\|v\|_{H_\alpha^k(\mathbb{R}^d)} \\
 &\quad + \|q\|_{H^k(\mathbb{R}^{d-1})}\|v\|_{H_\alpha^k(\mathbb{R}^d)}).
 \end{aligned}$$

Similarly, using the Cauchy–Schwarz inequality, we infer

$$\|w \cdot w\|_{L^1(\mathbb{R}^{d-1})} \leq \|w\|_{L^2(\mathbb{R}^{d-1})}\|w\|_{L^2(\mathbb{R}^{d-1})} \leq \|w\|_{H^k(\mathbb{R}^{d-1})}^2,$$

and thus we have

$$\begin{aligned}
 \|F_2(v, q, w)\|_{L^1(\mathbb{R}^{d-1})} &\leq C(\|\pi_\alpha G(v, q)\|_{L^1(\mathbb{R}^{d-1})} + \|w \cdot w\|_{L^1(\mathbb{R}^{d-1})}) \\
 &\leq C_K(\|v\|_{H^k(\mathbb{R}^d)}\|v\|_{H_\alpha^k(\mathbb{R}^d)} + \|q\|_{H^k(\mathbb{R}^{d-1})}\|v\|_{H_\alpha^k(\mathbb{R}^d)} \\
 &\quad + \|w\|_{H^k(\mathbb{R}^{d-1})}^2).
 \end{aligned}$$

This finishes the proof of the required inequalities in part (d). $\square$

We next formulate lemmas whose proofs resemble the proofs in [GLS, Lemmas 8.1, 8.2, and 8.3]. We will use them later to prove boundedness of the components of the solutions in $H^k(\mathbb{R}^d)$ -norm.

LEMMA 7.8. *Assume Hypotheses 3.2 and 3.3, and let $k \geq \frac{[d+1]}{2}$. Then the entries of the matrix-valued function $(df(\phi) - df(0))\gamma_\alpha^{-1}$ belong to $H^k(\mathbb{R})$.*

Proof. This follows from the formula $df(\phi) - df(0) = \phi \int_0^1 d^2 f(s\phi) ds$, where $f(\cdot)$ is a C^{k+3} smooth function by Hypothesis 3.2 and from the fact that $\phi\gamma_\alpha^{-1} \in H^k(\mathbb{R})$ using Lemma 3.5(1). $\square$

We will now use Lemma 7.2 to prove an analogue of Proposition 7.3(1) with $\|v\|_{H^k(\mathbb{R}^d)}$ in the right-hand side replaced by $\|v\|_{H_\alpha^k(\mathbb{R}^d)}$ and Proposition 7.5(1) with $\|\cdot\|_{H_\alpha^k(\mathbb{R}^d)}$ in the left-hand side replaced by $\|\cdot\|_{H^k(\mathbb{R}^d)}$. We recall that $\phi = \phi(z)$ and that the function $(z, y) \mapsto (df(\phi(z - q(y))) - df(0))v(y)$ is in $H^k(\mathbb{R}^d)$.

LEMMA 7.9. *Assume Hypotheses 3.2 and 3.3, and let $k \geq \frac{[d+1]}{2}$. For each $K > 0$, there is a constant $C_K > 0$ such that if $q \in H^k(\mathbb{R}^{d-1})$ and $v \in \mathcal{H}$ satisfy $\|v\|_{\mathcal{H}} + \|q\|_{H^k(\mathbb{R}^{d-1})} \leq K$, then*

- (1) $\|(df(\phi) - df(0))v\|_{H^k(\mathbb{R}^d)} \leq C_K\|v\|_{H_\alpha^k(\mathbb{R}^d)};$
- (2) $\|(df(\phi_q) - df(\phi))v\|_{H^k(\mathbb{R}^d)} \leq C_K\|q\|_{H^k(\mathbb{R}^{d-1})}\|v\|_{H_\alpha^k(\mathbb{R}^d)};$
- (3) *for (v, q) in a bounded neighborhood of $(0, 0)$ in $\mathcal{H}^n \times H^k(\mathbb{R}^{d-1})$, and $v = (v_1, v_2)^T$ with $v_i \in \mathcal{H}^{n_i}$, $i = 1, 2$, one has for $N(\cdot, \cdot)$ defined in (6.5),*

$$\|N(\phi_q, v)v\|_{H^k(\mathbb{R}^d)} \leq C_K\|v\|_{H^k(\mathbb{R}^d)}(\|v\|_{H_\alpha^k(\mathbb{R}^d)} + \|v_2\|_{H^k(\mathbb{R}^d)}).$$

Proof. Lemma 7.8 and (7.1) yield (1) since

$$\|(df(\phi) - df(0))v\|_{H^k(\mathbb{R}^d)} \leq \|(df(\phi) - df(0))\gamma_\alpha^{-1}\|_{H^k(\mathbb{R})}\|\gamma_\alpha v\|_{H^k(\mathbb{R}^d)} \leq C_K\|v\|_{H_\alpha^k(\mathbb{R}^d)}.$$

To prove (2), we write, as in (7.5),

$$(7.8) \quad (df(\phi_q) - df(\phi))v = - \int_0^1 d^2 f(\phi(z - sq(y))) (\gamma_\alpha^{-1} \phi'(z - sq(y))q, \gamma_\alpha v) ds.$$

We next use an argument similar to the one in Lemma 7.2 to prove that

$$(7.9) \quad \|(df(\phi_q) - df(\phi))v\|_{H^k(\mathbb{R}^d)} \leq C_K \|q\|_{H^k(\mathbb{R}^{d-1})} \|v\|_{H_\alpha^k(\mathbb{R}^d)}.$$

Indeed, the main steps in the proof of (7.9) are as follows.

We consider (7.8) componentwise. The proof of (7.9) is then reduced to a proof of the inequality

$$(7.10) \quad \|\sigma\|_{H^k(\mathbb{R}^d)} \leq C \|q\|_{H^k(\mathbb{R}^{d-1})},$$

where $\sigma(z, y) = \gamma_\alpha^{-1}(z) \psi'(z - q(y))q(y)$, $x = (z, y) \in \mathbb{R}^d$, and ψ as required in Lemma 7.2 has exponentially decaying derivatives. Indeed, as soon as (7.10) is proved, the inequality

$$\|\phi'(\cdot - sq(\cdot))q(\cdot)v(\cdot)\|_{H^k(\mathbb{R}^d)} \leq \|\sigma\|_{H^k(\mathbb{R}^d)} \|v\|_{H_\alpha^k(\mathbb{R}^d)}$$

yields (7.9) from (7.8).

To prove (7.10), we denote $m(z, y) = \gamma_\alpha(z - q(y))\gamma_\alpha^{-1}(z)$ so that

$$\sigma(z, y) = m(z, y)(\gamma_\alpha^{-1}\psi')(z - q(y))q(y).$$

We note that $\gamma_\alpha^{-1}(z)\psi'(z)$ exponentially decays at $z \rightarrow \pm\infty$. Using $q \in H^k(\mathbb{R}^{d-1}) \hookrightarrow L^\infty(\mathbb{R}^{d-1})$ and formula (3.7) for $\gamma_\alpha(z)$, we conclude that $m(z, y) = e^{-\alpha - q(y)}$ for $z \leq -r$ and $m(z, y) = e^{-\alpha + q(y)}$ for $z \geq r$ for some large $r > 0$ uniformly in $y \in \mathbb{R}^{d-1}$; moreover, $\gamma_\alpha(-q(\cdot)) \in L^\infty(\mathbb{R}^{d-1})$, with the norm bounded by a constant that depends on K . Similarly to the calculation in (7.3), the $L^2(\mathbb{R}^d)$ -norm of σ can be estimated as

$$(7.11) \quad \|\sigma\|_{L^2(\mathbb{R}^d)} \leq \|m\|_{L^\infty(\mathbb{R}^{d-1})}, \quad \|(\gamma_\alpha^{-1}\psi')(\cdot - q(\cdot))q(\cdot)\|_{L^2(\mathbb{R}^{d-1})} \leq C_K \|q\|_{H^k(\mathbb{R}^{d-1})}.$$

We now show how to estimate the $L^2(\mathbb{R}^d)$ -norm of the derivatives of σ . The z -derivative,

$$\frac{\partial \sigma}{\partial z} = (\gamma_\alpha^{-1})'(z)\psi'(z - q(y))q(y) + \gamma_\alpha^{-1}(z)\psi''(z - q(y))q(y),$$

is the sum of two terms that can be handled similarly to (7.11). Taking derivatives with respect to x_j , $j = 2, \dots, d$, yields, as in (7.2),

$$(7.12) \quad \frac{\partial \sigma}{\partial x_j} = \gamma_\alpha^{-1}\psi''(z - q(y))\frac{\partial q}{\partial x_j}q(y) + \gamma_\alpha^{-1}(z)\psi'(z - q(y))\frac{\partial q}{\partial x_j}.$$

The $L^2(\mathbb{R}^d)$ -norm of the first term can be estimated as in (7.4) and (7.11), that is, by a calculation similar to (7.3), we have

$$\begin{aligned} & \left\| \gamma_\alpha^{-1}(\cdot)\psi''(\cdot - q(\cdot))\frac{\partial q}{\partial x_j}q \right\|_{L^2(\mathbb{R}^d)} \\ & \leq \|\mu\|_{L^\infty(\mathbb{R}^{d-1})} \|q\|_{L^\infty(\mathbb{R}^{d-1})} \left\| (\gamma_\alpha^{-1}\psi'')(\cdot - q(\cdot))\frac{\partial q}{\partial x_j} \right\|_{L^2(\mathbb{R}^d)} \end{aligned}$$

$$\begin{aligned} &\leq \|\gamma_\alpha(-q(\cdot))\|_{L^\infty(\mathbb{R}^{d-1})} \|q\|_{L^\infty(\mathbb{R}^{d-1})} \left\| (\gamma_\alpha^{-1} \phi'')(\cdot - q(\cdot)) \frac{\partial q}{\partial x_j} \right\|_{L^2(\mathbb{R}^d)} \\ &\leq C_K \|q\|_{H^k(\mathbb{R}^{d-1})}, \end{aligned}$$

where C_K is a K -dependent constant different from the constant in (7.11). A similar calculation works for the second term in (7.12). This proves assertion (7.10) for $k = 1$. We conclude the proof of assertion (2) by pointing out that higher order derivatives are handled as described in the proof of Proposition A.3.

To prove (3), we recall the following representation of the nonlinearity $v = (v_1, v_2)^T \mapsto N(\phi_q, v)v$ borrowed from the proof of [GLS, Lemma 8.3]:

$$N(\phi_q, v)v = I_1(v) + I_2(v) + I_3(v) + I_4(v) + I_5(v),$$

where $\phi_q = (\phi_1(z - q), \phi_2(z - q))^T = (\phi_{1,q}, \phi_{2,q})^T$, $v = (v_1, v_2)^T$,

$$\begin{aligned} I_1(v) &= \int_0^1 (\partial_{u_1} r(\phi_q + tv) - \partial_{u_1} r(\phi_q)) v_1 \phi_{2,q} dt, \\ I_2(v) &= \int_0^1 (\partial_{u_1} r(\phi_q + tv) v_1) tv_2 dt, \quad I_3(v) = \int_0^1 (\partial_{u_2} r(\phi_q + tv) - \partial_{u_2} r(\phi_q)) v_2 \phi_{2,q} dt, \\ I_4(v) &= \int_0^1 (\partial_{u_2} r(\phi_q + tv) v_2) tv_2 dt, \quad I_5(v) = \int_0^1 (r(\phi_q + tv) - r(\phi_q)) v_2 dt, \end{aligned}$$

and the $n \times n$ matrix-valued C^k function $r = r(u_1, u_2)$ is given by

$$r(u_1, u_2) = \int_0^1 \partial_{u_2} f(u_1, su_2) ds.$$

The proof of the required estimates for each I_j , $j = 1, 2, \dots, 5$, is similar to the proof of assertion (2) above and uses Lemma 3.5. For instance, for $j = 1$, passing in the integral to the third derivative of f (which is a C^k -bounded function by Hypothesis 3.2), we reduce the problem to obtaining an estimate for $\|v v_1 \phi_{2,q}\|_{H^k(\mathbb{R}^d)}$. If we write $v_1 \phi_{2,q} = (\gamma_\alpha v_1)(\gamma_\alpha^{-1} \phi_{2,q})$ and use that $H^k(\mathbb{R}^d)$ is an algebra, then, in order to prove that

$$(7.13) \quad \|I_1(v)\|_{H^k(\mathbb{R}^d)} \leq C \|v\|_{H^k(\mathbb{R}^d)} \|v_1\|_{H_\alpha^k(\mathbb{R}^d)},$$

it suffices to show that the $H^k(\mathbb{R}^d)$ -norm of $\sigma(z, y) = \gamma_\alpha^{-1}(z) \phi_2(z - q(y)) w(x)$ with $w = \gamma_\alpha v_1$ is bounded by $C \|w\|_{H^k(\mathbb{R}^d)}$. This follows because $q \in H^k(\mathbb{R}^{d-1}) \hookrightarrow L^\infty(\mathbb{R}^{d-1})$ yields the existence of a large $r > 0$ such that, uniformly for $y \in \mathbb{R}^{d-1}$, we have

$$|\gamma_\alpha^{-1}(z) \phi_2(z - q(y))| \leq \begin{cases} K e^{-\alpha - z} e^{-\omega - (z - q(y))}, & z \leq -r, \\ K e^{-\alpha + z} (|\phi_2| + e^{-\omega + (z - q(y))}), & z \leq -r. \end{cases} \quad \square$$

Using $e^q \in L^\infty(\mathbb{R}^{d-1})$ and Hypothesis 3.4 we conclude that $\gamma_\alpha^{-1}(\cdot) \phi_2(\cdot - q(\cdot))$ is bounded. A similar argument, as in the proof of (2) above, applies for the derivatives of σ . This completes the proof of (7.13). For $j = 2, \dots, 5$, the estimates $\|I_j(v)\|_{H^k(\mathbb{R}^d)} \leq C \|v\|_{H^k(\mathbb{R}^d)} \|v_2\|_{H^k(\mathbb{R}^d)}$ are straightforward since each integral has a factor v_2 and both derivatives of f and $\phi_{2,q}$ are k -smooth with bounded derivatives. Combining the estimates for $j = 1, \dots, k$ yields assertion (3).

8. Nonlinear stability.

8.1. Local in time existence and bounds. In this section we analyze the system (6.15). We denote

$$(8.1) \quad \mathcal{X} = \text{ran } \mathcal{Q}_{\mathcal{H}} \times H^k(\mathbb{R}^{d-1}) \times H^k(\mathbb{R}^{d-1})^{d-1}.$$

We also will assume as before that $k \geq [\frac{d+1}{2}]$. Let $S_{\mathcal{L}_{\mathcal{H}}}(t) = e^{t\mathcal{L}_{\mathcal{H}}}$ be the semigroup generated by the operator $\mathcal{L}_{\mathcal{H}}$ (see Definition 3.1(6)). Let (v^0, q^0, w^0) be the initial perturbation to the front. Since $S_{\Delta_y}(t)\nabla F_2 = \nabla_y S_{\Delta_y}(t)F_2$, the variation of constants formula implies that the mild solution to (6.15) on $\mathcal{X}$ satisfy the equations

$$(8.2) \quad \begin{aligned} v(t) &= S_{\mathcal{L}_{\mathcal{H}}}(t)v^0 + \int_0^t S_{\mathcal{L}_{\mathcal{H}}}(t-s)F_1(v(s), q(s), w(s))ds, \\ q(t) &= S_{\Delta_y}(t)q^0 + \int_0^t S_{\Delta_y}(t-s)F_2(v(s), q(s), w(s))ds, \\ w(t) &= S_{\Delta_y}(t)w^0 + \int_0^t \nabla_y S_{\Delta_y}(t-s)F_2(v(s), q(s), w(s))ds. \end{aligned}$$

Next we formulate a statement that shows the existence and uniqueness of the mild solutions of (8.2).

PROPOSITION 8.1. *For any initial data $(v^0, q^0, w^0) \in \mathcal{X}$ system (6.15) has a unique mild solution (that is, a solution of (8.2)) $(v(t), q(t), w(t)) \in \mathcal{X}$ in the maximal interval $0 \leq t < t_{\max}$, where $0 < t_{\max} \leq \infty$.*

The proof can be found in [K, Lemma 3.4]. We just mention that the proof only uses the fact that $\mathcal{L}_{\mathcal{H}}$ generates a strongly continuous semigroup, even though we know that $\mathcal{L}_{\mathcal{H}}$ generates a bounded strongly continuous semigroup. Indeed, since the operator $\mathcal{L}_{\mathcal{H}}$ generates a strongly continuous semigroup and the nonlinearities F_1 and F_2 are locally Lipschitz with Lipschitz constant C_K on the set $\{(v, q, w) : \|v\|_{\mathcal{H}} + \|q\|_{H^k(\mathbb{R}^{d-1})} + \|w\|_{H^k(\mathbb{R}^{d-1})} < K\}$, the estimate from Lemma 5.5(c), which is integrable at $t = 0$, yields the statement of Proposition 8.1.

For (6.15) on $\mathcal{X}$ from (8.1) we combine Proposition 8.1 and [SY, Theorem 64.2] to obtain the next lemma.

LEMMA 8.2. *For each $\delta > 0$, if $0 < \gamma < \delta$, there exists T ($0 < T \leq \infty$) such that the following is true: if $(v^0, q^0, w^0) \in \mathcal{X}$ satisfies*

$$(8.3) \quad \|v^0\|_{\mathcal{H}} + \|q^0\|_{H^k(\mathbb{R}^{d-1})} + \|w^0\|_{H^k(\mathbb{R}^{d-1})} \leq \gamma$$

and $0 \leq t < T$, then the solution $(v(t), q(t), w(t)) \in \mathcal{X}$ of (8.2) with the initial data (v^0, q^0, w^0) is defined and satisfies

$$(8.4) \quad \|v(t)\|_{\mathcal{H}} + \|q(t)\|_{H^k(\mathbb{R}^{d-1})} + \|w(t)\|_{H^k(\mathbb{R}^{d-1})} \leq \delta.$$

DEFINITION 8.3. *Let $T(\delta, \gamma)$ denote the supremum of all T such that (8.4) holds for all $0 \leq t < T$ whenever (8.3) is satisfied.*

Having established the local in time existence of the solution of (6.15), we show next the algebraic decay and boundedness of the solution.

COROLLARY 8.4. *For any $K > 0$, there exists $\delta_0 < K$ such that for any γ and δ satisfying $0 < \gamma < \delta < \delta_0$, the mild solution $V(t) = (v(t), q(t), w(t))$ of (6.15)*

satisfying $\|V(t)\|_{\mathcal{X}} \leq \delta$ on the interval $t \in [0, T(\delta, \gamma))$ is continuous with respect to the initial data $V^0 = (v^0, q^0, w^0)$ satisfying $\|V^0\|_{\mathcal{X}} \leq \gamma$. Moreover,

$$(8.5) \quad \|V(t)\|_{\mathcal{X}} \leq C(K)\|V^0\|_{\mathcal{X}} \text{ for all } t \in [0, \min\{1, T(\delta, \gamma)\}],$$

where $C(K)$ is a constant that depends on K but is independent of δ and γ .

Proof. Since the estimate in Lemma 5.5 is integrable at $t = 0$, the continuity with respect to initial data is a simple modification of the standard argument; see [SY, Theorem 64.2].

By Lemmas 5.4 and 5.5 the semigroup $\{T(t)\}_{t \geq 0} = \{S_{\mathcal{L}_{\mathcal{H}}}(t) \oplus S_{\Delta_y}(t) \oplus S_{\Delta_y}(t)\}_{t \geq 0}$ is bounded. We define $M = \max\{\sup\{\|T(t)\|_{\mathcal{B}(\mathcal{X})} : t \geq 0\}, C\}$, where C is the constant from Lemma 5.5(c). The variation of constant formula (8.2) and Proposition 7.7 and Lemma 8.2 together with assumption $0 \leq t < \min\{1, T(\delta, \gamma)\} \leq 1$, for all $t \in [0, \min\{1, T(\delta, \gamma)\})$, yield

$$\begin{aligned} \|V(t)\|_{\mathcal{X}} &\leq M\|V^0\|_{\mathcal{X}} + MC_K \delta \int_0^t (t-s)^{-1/2} \|V(s)\|_{\mathcal{X}} ds \\ &\leq M\|V^0\|_{\mathcal{X}} + 2MC_K \delta \sup_{T(\delta, \gamma) \geq 0} \|V(t)\|_{\mathcal{X}}. \end{aligned}$$

We then choose $\delta_0 \leq \min\{K, 1/4MC_K\}$ and conclude that for any $0 < \delta < \delta_0$ and $0 < \gamma < \delta$, then $\|V(t)\|_{\mathcal{X}} \leq C(K)\|V^0\|_{\mathcal{X}}$ for some $C(K)$ depending on K for all $t \in [0, \min\{1, T(\delta, \gamma)\})$. $\square$

8.2. The algebraic decay of solutions in weighted norm. In this subsection we show that the weighted norm of the solution $v(t) = (v_1, v_2)$ of (6.15) decays algebraically as $t \rightarrow \infty$, the $H^k(\mathbb{R}^d)$ -norm of $v_2(t)$ also decays algebraically as $t \rightarrow \infty$, while the $H^k(\mathbb{R}^d)$ -norm of $v_1(t)$ is bounded provided the initial value of the solution is sufficiently small. For the initial data $(v^0, q^0, w^0) \in \text{ran } \mathcal{Q}_{\mathcal{H}} \times H^k(\mathbb{R}^{d-1}) \times H^k(\mathbb{R}^{d-1})$, we denote the size of the initial values by

$$(8.6) \quad E_k = \|v^0\|_{\mathcal{H}} + \|q^0\|_{H^{k+1}(\mathbb{R}^{d-1})} + \|w^0\|_{W^{1,1}(\mathbb{R}^{d-1})}.$$

We assume $q^0 \in H^{k+1}(\mathbb{R}^{d-1}) \cap W^{1,1}(\mathbb{R}^{d-1})$ so that when (6.15) has a mild solution, $w(t)$ satisfies $w(t) = \nabla_y q(t)$ and $w(t) \in H^k(\mathbb{R}^{d-1})^{d-1} \cap L^1(\mathbb{R}^{d-1})^{d-1}$, thus (8.6) contains the norm of $\|w^0\|_{L^1(\mathbb{R}^{d-1})}$ in the last term.

The following estimates are proved by direct computation in [X].

LEMMA 8.5. Suppose $a, b, c > 0$; then

- (1) $\int_0^{t/2} (1+t-s)^{-b} (1+s)^{-c} ds \leq C(1+t)^{-a}$, if $a \leq b$, $a \leq b+c-1$, $c \neq 1$, or if $a < b$, $c = 1$;
- (2) $\int_{t/2}^t (1+t-s)^{-b} (1+s)^{-c} ds \leq C(1+t)^{-a}$, if $a \leq c$, $a \leq b+c-1$, $b \neq 1$, or if $a < c$, $b = 1$;
- (3) $\int_0^t e^{-b(t-s)} (1+s)^{-c} ds \leq C(1+t)^{-c}$.

We now show that the weighted norm of $v(t)$ and the $H^k(\mathbb{R}^{d-1})$ -norms of $q(t)$ and $w(t)$, in fact, decay to zero algebraically as long as t grows but the $\mathcal{H}$ -norm of $v(t)$ and the $H^k(\mathbb{R}^{d-1})$ norms of $q(t)$ and $w(t)$ remain small.

PROPOSITION 8.6. Assume Hypotheses 3.2, 3.3, and 3.7, and let $k \geq [\frac{d+1}{2}]$. Choose $\nu > 0$ as in Lemma 5.2. There exist $\delta_1 > 0$ and $C_1 > 0$ such that for every $\delta \in (0, \delta_1)$ and every γ with $0 < \gamma < \delta$, if $E_k < \gamma$, then the solution $(v(t), q(t), w(t))$ of (6.15) with the initial data (v^0, q^0, w^0) for $t \in [0, T(\delta, \gamma))$ satisfies the estimates

$$\begin{aligned}\|v(t)\|_{H_\alpha^k(\mathbb{R}^d)} &\leq C_1(1+t)^{-\frac{d+1}{2}} E_k, \\ \|q(t)\|_{H^k(\mathbb{R}^{d-1})} &\leq C_1(1+t)^{-\frac{d-1}{4}} E_k, \\ \|w(t)\|_{H^k(\mathbb{R}^{d-1})} &\leq C_1(1+t)^{-\frac{d+1}{4}} E_k.\end{aligned}$$

Proof. In Corollary 8.4 we have discussed the solution of (6.15) in a time period $[0, \min\{1, T(\delta, \gamma)\})$; therefore, without loss of generality, we may assume that $T(\delta, \gamma) > 1$.

We recall that $\text{ran } \mathcal{Q}_{\mathcal{H}} = \text{ran } \mathcal{L}_\alpha \cap \mathcal{H}^n$; thus for $v \in \text{ran } \mathcal{Q}_{\mathcal{H}}$ we can replace $\mathcal{L}_{\mathcal{H}}$ by $\mathcal{L}_\alpha$ in (8.2). Applying the semigroup estimates from Lemmas 5.5 and 5.2 to (8.2) yields

$$\begin{aligned}(8.7) \quad \|v(t)\|_{H_\alpha^k(\mathbb{R}^d)} &\leq C \left(e^{-\nu t} \|v^0\|_{H_\alpha^k(\mathbb{R}^d)} + \int_0^t e^{-\nu(t-s)} \|F_1(s)\|_{H_\alpha^k(\mathbb{R}^d)} ds \right), \\ \|q(t)\|_{H^k(\mathbb{R}^{d-1})} &\leq C \left((1+t)^{-\frac{d-1}{4}} \|q^0\|_{L^1(\mathbb{R}^{d-1})} + e^{-\beta t} \|q^0\|_{H^k(\mathbb{R}^{d-1})} \right. \\ &\quad \left. + \int_0^t e^{-\beta(t-s)} \|F_2(s)\|_{H^k(\mathbb{R}^{d-1})} ds + \int_0^t (1+t-s)^{-\frac{d-1}{4}} \|F_2(s)\|_{L^1(\mathbb{R}^{d-1})} ds \right), \\ \|w(t)\|_{H^k(\mathbb{R}^{d-1})} &\leq C \left((1+t)^{-\frac{d+1}{4}} \|q^0\|_{L^1(\mathbb{R}^{d-1})} + t^{-1/2} e^{-\beta t} \|q^0\|_{H^k(\mathbb{R}^{d-1})} \right. \\ &\quad \left. + \int_0^t (t-s)^{-\frac{1}{2}} e^{-\beta(t-s)} \|F_2(s)\|_{H^k(\mathbb{R}^{d-1})} ds + \int_0^t (1+t-s)^{-\frac{d+1}{4}} \|F_2(s)\|_{L^1(\mathbb{R}^{d-1})} ds \right),\end{aligned}$$

where $F_{1,2}(s) = F_{1,2}(v(s), q(s), w(s))$. For $t > 1$, there exist C such that $e^{-\nu t} \|v^0\|_{H_\alpha^k(\mathbb{R}^d)} \leq C e^{-\nu t} E_k$ and

$$\begin{aligned}(1+t)^{-\frac{d-1}{4}} \|q^0\|_{L^1(\mathbb{R}^{d-1})} + e^{-\beta t} \|q^0\|_{H^k(\mathbb{R}^{d-1})} &\leq C(1+t)^{-\frac{d-1}{4}} E_k, \\ (1+t)^{-\frac{d+1}{4}} \|q^0\|_{L^1(\mathbb{R}^{d-1})} + t^{-1/2} e^{-\beta t} \|q^0\|_{H^k(\mathbb{R}^{d-1})} &\leq C(1+t)^{-\frac{d+1}{4}} E_k.\end{aligned}$$

For any δ' and γ such that $0 < \gamma < \delta'$, if $E_k < \gamma$, then, by Lemma 8.2,

$$\|v(s)\|_{\mathcal{H}} + \|q(s)\|_{H^k(\mathbb{R}^{d-1})} + \|w(s)\|_{H^k(\mathbb{R}^{d-1})} < \delta' \text{ for all } s \in [0, T(\delta', \gamma)).$$

Within this bounded set, Proposition 7.7(b) and (c) states that

$$\begin{aligned}\|F_1(v(s), q(s), w(s))\|_{H_\alpha^k(\mathbb{R}^d)} &\leq C_{\delta'} (\|v(s)\|_{H^k(\mathbb{R}^d)} \|v(s)\|_{H_\alpha^k(\mathbb{R}^d)} \\ &\quad + \|q(s)\|_{H^k(\mathbb{R}^{d-1})} \|v(s)\|_{H_\alpha^k(\mathbb{R}^d)} + \|w(s)\|_{H^k(\mathbb{R}^{d-1})}^2), \\ \|F_2(v(s), q(s), w(s))\|_{H^k(\mathbb{R}^{d-1})} &\leq C_{\delta'} (\|v(s)\|_{H^k(\mathbb{R}^d)} \|v(s)\|_{H_\alpha^k(\mathbb{R}^d)} \\ &\quad + \|q(s)\|_{H^k(\mathbb{R}^{d-1})} \|v(s)\|_{H_\alpha^k(\mathbb{R}^d)} + \|w(s)\|_{H^k(\mathbb{R}^{d-1})}^2).\end{aligned}$$

Using Proposition 7.7(d) the inequalities (8.7) can be rewritten as follows:

$$\begin{aligned}\|v(t)\|_{H_\alpha^k(\mathbb{R}^d)} &\leq C e^{-\nu t} E_k \\ &\quad + C C_{\delta'} \int_0^t e^{-\nu(t-s)} (\|v(s)\|_{H^k(\mathbb{R}^d)} \|v(s)\|_{H_\alpha^k(\mathbb{R}^d)} + \|q(s)\|_{H^k(\mathbb{R}^{d-1})} \|v(s)\|_{H_\alpha^k(\mathbb{R}^d)} \\ &\quad + \|w(s)\|_{H^k(\mathbb{R}^{d-1})}^2) ds,\end{aligned}$$

$$\begin{aligned}
\|q(t)\|_{H^k(\mathbb{R}^{d-1})} &\leq C(1+t)^{-\frac{d-1}{4}} E_k \\
&+ CC_{\delta'} \int_0^t e^{-\beta(t-s)} (\|v(s)\|_{H^k(\mathbb{R}^d)} \|v(s)\|_{H_\alpha^k(\mathbb{R}^d)} + \|q(s)\|_{H^k(\mathbb{R}^{d-1})} \|v(s)\|_{H_\alpha^k(\mathbb{R}^d)} \\
&+ \|w(s)\|_{H^k(\mathbb{R}^{d-1})}^2) ds \\
&+ CC_{\delta'} \int_0^t (1+t-s)^{-\frac{d-1}{4}} ((\|v(s)\|_{H^k(\mathbb{R}^d)} + \|q(s)\|_{H^k(\mathbb{R}^{d-1})}) \|v(s)\|_{H_\alpha^k(\mathbb{R}^d)} \\
&+ \|w(s)\|_{H^k(\mathbb{R}^{d-1})}^2) ds
\end{aligned}$$

and

$$\begin{aligned}
\|w(t)\|_{H^k(\mathbb{R}^{d-1})} &\leq C(1+t)^{-\frac{d+1}{4}} E_k \\
&+ CC_{\delta'} \int_0^t (t-s)^{-\frac{1}{2}} e^{-\beta(t-s)} ((\|v(s)\|_{H^k(\mathbb{R}^d)} + \|q(s)\|_{H^k(\mathbb{R}^{d-1})}) \|v(s)\|_{H_\alpha^k(\mathbb{R}^d)} \\
&+ \|w(s)\|_{H^k(\mathbb{R}^{d-1})}^2) ds \\
&+ CC_{\delta'} \int_0^t (1+t-s)^{-\frac{d+1}{4}} ((\|v(s)\|_{H^k(\mathbb{R}^d)} + \|q(s)\|_{H^k(\mathbb{R}^{d-1})}) \|v(s)\|_{H_\alpha^k(\mathbb{R}^d)} \\
&+ \|w(s)\|_{H^k(\mathbb{R}^{d-1})}^2) ds.
\end{aligned}$$

We denote

$$\begin{aligned}
M_v(t) &= \sup_{0 < s \leq t} (1+s)^{\frac{d+1}{2}} \|v(s)\|_{H_\alpha^k(\mathbb{R}^d)}, \\
M_q(t) &= \sup_{0 < s \leq t} (1+s)^{\frac{d-1}{4}} \|q(s)\|_{H^k(\mathbb{R}^{d-1})}, \\
M_w(t) &= \sup_{0 < s \leq t} (1+s)^{\frac{d+1}{4}} \|w(s)\|_{H^k(\mathbb{R}^{d-1})}
\end{aligned}$$

and note that for each $\delta < \delta'$, and $0 < \gamma < \delta$, if $E_k < \gamma$, by Lemma 8.2, for all $s \in (0, T(\delta, \gamma))$, we have $\|v(s)\|_{H^k(\mathbb{R}^d)} \leq \|v(s)\|_{\mathcal{H}} < \delta$, and therefore

$$\begin{aligned}
\|v(t)\|_{H_\alpha^k(\mathbb{R}^d)} &\leq Ce^{-\nu t} E_k + CC_{\delta'} \left(\delta M_v(t) \int_0^t e^{-\nu(t-s)} (1+s)^{-\frac{d+1}{2}} ds \right. \\
&+ M_v(t) M_q(t) \int_0^t e^{-\nu(t-s)} (1+s)^{-\frac{3d+1}{4}} ds + M_w^2(t) \int_0^t e^{-\nu(t-s)} (1+s)^{-\frac{d+1}{2}} ds \Big), \\
\|q(t)\|_{H^k(\mathbb{R}^{d-1})} &\leq C(1+t)^{-\frac{d-1}{4}} E_k + CC_{\delta'} \left(\delta M_v(t) \int_0^t e^{-\beta(t-s)} (1+s)^{-\frac{d+1}{2}} ds \right. \\
&+ M_v(t) M_q(t) \int_0^t e^{-\beta(t-s)} (1+s)^{-\frac{3d+1}{4}} ds + M_w^2(t) \int_0^t e^{-\beta(t-s)} (1+s)^{-\frac{d+1}{2}} ds \\
&+ \delta M_v(t) \int_0^t (1+t-s)^{-\frac{d-1}{4}} (1+s)^{-\frac{d+1}{2}} ds \\
&+ M_v(t) M_q(t) \int_0^t (1+t-s)^{-\frac{d-1}{4}} (1+s)^{-\frac{3d+1}{4}} ds \\
&+ M_w^2(t) \int_0^t (1+t-s)^{-\frac{d-1}{4}} (1+s)^{-\frac{d+1}{2}} ds \Big),
\end{aligned}$$

$$\begin{aligned}
\|w(t)\|_{H^k(\mathbb{R}^{d-1})} &\leq C(1+t)^{-\frac{d+1}{4}} E_k + CC_{\delta'} \left(\delta M_v(t) \int_0^t (t-s)^{-\frac{1}{2}} e^{-\beta(t-s)} (1+s)^{-\frac{d+1}{2}} ds \right. \\
&\quad + M_v(t) M_q(t) \int_0^t (t-s)^{-\frac{1}{2}} e^{-\beta(t-s)} (1+s)^{-\frac{3d+1}{4}} ds \\
&\quad + M_w^2(t) \int_0^t (t-s)^{-\frac{1}{2}} e^{-\beta(t-s)} (1+s)^{-\frac{d+1}{2}} ds \\
&\quad + \delta M_v(t) \int_0^t (1+t-s)^{-\frac{d+1}{4}} (1+s)^{-\frac{d+1}{2}} ds \\
&\quad + M_v(t) M_q(t) \int_0^t (1+t-s)^{-\frac{d+1}{4}} (1+s)^{-\frac{3d+1}{4}} ds \\
&\quad \left. + M_w^2(t) \int_0^t (1+t-s)^{-\frac{d+1}{4}} (1+s)^{-\frac{d+1}{2}} ds \right).
\end{aligned}$$

By Lemma 8.5 then

$$\begin{aligned}
\|v(t)\|_{H_\alpha^k(\mathbb{R}^d)} &\leq C e^{-\nu t} E_k \\
&\quad + CC_{\delta'} \left(\delta M_v(t) (1+t)^{-\frac{d+1}{2}} + M_v(t) M_q(t) (1+t)^{-\frac{3d+1}{4}} \right. \\
&\quad \left. + M_w^2(t) (1+t)^{-\frac{d+1}{2}} \right), \\
\|q(t)\|_{H^k(\mathbb{R}^{d-1})} &\leq C(1+t)^{-\frac{d-1}{4}} E_k + CC_{\delta'} \left(\delta M_v(t) + M_v(t) M_q(t) + M_w^2(t) \right) (1+t)^{-\frac{d-1}{4}}, \\
\|w(t)\|_{H^k(\mathbb{R}^{d-1})} &\leq C(1+t)^{-\frac{d+1}{4}} E_k + CC_{\delta'} \left(\delta M_v(t) + M_v(t) M_q(t) + M_w^2(t) \right) (1+t)^{-\frac{d+1}{4}}.
\end{aligned}$$

One then has for some $C > 0$,

$$\begin{aligned}
(1+t)^{\frac{d+1}{2}} \|v(t)\|_{H_\alpha^k(\mathbb{R}^d)} &\leq C \left((1+t)^{\frac{d+1}{2}} e^{-\nu t} E_k + \delta M_v(t) \right. \\
&\quad \left. + M_v(t) M_q(t) (1+t)^{-\frac{d-1}{4}} + M_w^2(t) \right), \\
(1+t)^{\frac{d-1}{4}} \|q(t)\|_{H^k(\mathbb{R}^{d-1})} &\leq C \left(E_k + \delta M_v(t) + M_v(t) M_q(t) + M_w^2(t) \right), \\
(1+t)^{\frac{d+1}{4}} \|w(t)\|_{H^k(\mathbb{R}^{d-1})} &\leq C \left(E_k + \delta M_v(t) + M_v(t) M_q(t) + M_w^2(t) \right).
\end{aligned}$$

Since $M_v(t)$, $M_q(t)$, $M_w(t)$ are increasing functions, it can be concluded that for $t \in [1, T(\gamma, \delta))$,

$$\begin{aligned}
(8.8) \quad M_v(t) &\leq C E_k + C(\delta M_v(t) + M_v(t) M_q(t) + M_w^2(t)), \\
M_q(t) &\leq C E_k + C(\delta M_v(t) + M_v(t) M_q(t) + M_w^2(t)), \\
M_w(t) &\leq C E_k + C(\delta M_v(t) + M_v(t) M_q(t) + M_w^2(t)).
\end{aligned}$$

If we set $M(t) = M_v(t) + M_q(t) + M_w(t)$; then by (8.8), for all $t \in [1, T(\gamma, \delta))$ and for some $C > 0$ that depends on δ' ,

$$M(t) \leq C E_k + C \delta M(t) + C M^2(t).$$

Note that by Corollary 8.4 $M(t) \leq C(\delta') E_k$ for $0 \leq t \leq 1$ and some constant $C(\delta') > 0$. Choose $\delta_1 \leq \min\{1/2C, \delta'\}$ and $0 < \gamma < \delta < \delta_1$; then absorbing the term $\frac{1}{2} M(t)$ into the left-hand side, we have $M(t) \leq 2C E_k + 2C M^2(t)$ for all $t \in [0, T(\delta, \gamma))$. Since this inequality holds for all $t \in [0, T(\delta, \gamma))$, by continuity of $M(\cdot)$, the expression $M(t)$ cannot “jump” over the first root of the respective quadratic equation. This root,

in turn, can be controlled by $K_1 E_k$ as long as E_k is sufficiently small. Indeed, let $M_1 = 1 - \sqrt{1 - 16C^2 E_k}/4C$ be the first root of the equation $2CM^2 - M + 2CE_k = 0$. If $E_k < 1/16C^2$, then

$$M_1 = \frac{16C^2 E_k}{4C(1 + \sqrt{1 - 16C^2 E_k})} < 4CE_k.$$

Since $M(t)$ is continuous and $M(0) = E_k$, it follows that if $\delta_1 \leq \min\{\delta', 1/2C, 1/16C^2\}$, then for all $\delta \in (0, \delta_1)$ and $0 < E_k \leq \gamma < \delta$ (see Lemma 8.2) we have $M(t) \leq M_1 \leq K_1 E_k$ for some $C_1 > 0$ and all $t \in [0, T(\delta, \gamma))$. $\square$

8.3. The boundedness of solutions in $H^k(\mathbb{R}^d)$ -norm. In this subsection, we show that the $H^k(\mathbb{R}^d)$ -norm of the solution $v(t)$ remains bounded for all t . Together with the decay of the weighted norm for large t this implies smallness of $\mathcal{H}$ -norm of the solution when the initial conditions are small, which is the key step in the bootstrap argument used in Theorem 8.8 proved below.

We point out that the algebraic decay of $\|v(t, \cdot, \cdot)\|_{H^k_\alpha(\mathbb{R}^d)}$ implies convergence of v to 0 for z close to ∞ , but it does not provide any information about the properties of the solution at $z = -\infty$. Indeed, since the weight function $\gamma_\alpha(z)$ with $\alpha_+ \geq 0$ either is 1 or grows exponentially as $z \rightarrow \infty$, the algebraic decay of the solution in the weighted norm may be achieved only if the solution decays to 0 for large positive z faster than the weight grows. On the other hand, since $\gamma_\alpha(z)$ exponentially converges to 0 as $z \rightarrow -\infty$, it is possible that v grows at $-\infty$ but that growth is compensated by the decay of the weight. It is the “product-triangular” structure of the nonlinearity that allows us to show the boundedness of the perturbations in the norm without a weight.

The following is the analogue of Proposition 8.6 for $H^k(\mathbb{R}^d)$ -norm.

PROPOSITION 8.7. *Assume Hypotheses 3.2, 3.3, and 3.7 and let $k \geq [\frac{d+1}{2}]$. Choose $\rho > 0$ to satisfy*

$$\sup\{\operatorname{Re} \lambda : \lambda \in \operatorname{Sp}(\mathcal{L}_1^{(2)})\} < -\rho$$

and δ_1 as indicated in Proposition 8.6. There exist $\delta_2 \in (0, \delta_1)$ and $C_2 > 0$ such that for every $\delta \in (0, \delta_2)$ and every γ with $0 < \gamma < \delta$, the following is true: if $E_k \leq \gamma$, then the solution to (6.15) for $t \in [0, T(\delta, \gamma))$ satisfies the estimates

$$(8.9) \quad \|v_1(t)\|_{H^k(\mathbb{R}^d)} \leq C_2 E_k;$$

$$(8.10) \quad \|v_2(t)\|_{H^k(\mathbb{R}^d)} \leq C_2 (1+t)^{-\frac{d+1}{2}} E_k.$$

Proof. Using (3.13), we write the first equation in (6.15) as follows:

$$(8.11) \quad \partial_t v_1 = L^{(1)} v_1 + d_{u_2} f(0, 0) v_2 + H_1(q, w, v_1, v_2),$$

$$(8.12) \quad \partial_t v_2 = L^{(2)} v_2 + H_2(q, w, v_1, v_2),$$

where

$$\begin{pmatrix} H_1(v_1, v_2, q, w) \\ H_2(v_1, v_2, q, w) \end{pmatrix} = F_1(v, q, w) + (df(\phi) - df(0))v.$$

Since $(v_1, v_2, q, w)(t)$ is a fixed solution of (6.15) in $\mathcal{H}^n \times H^k(\mathbb{R}^{d-1}) \times H^k(\mathbb{R}^{d-1})$, we may regard (8.11)–(8.12) as a nonautonomous linear system on $H^k(\mathbb{R}^d)^n$. The mild solutions of (8.11) and (8.12) satisfy the system of integral equations

$$(8.13) \quad v_1(t) = e^{t\mathcal{L}^{(1)}} v_1^0 + \int_0^t e^{(t-s)\mathcal{L}^{(1)}} (d_{u_2} f(0, 0) v_2(s) + H_1(v(s), q(s), w(s))) ds,$$

$$(8.14) \quad v_2(t) = e^{t\mathcal{L}^{(2)}} v_2^0 + \int_0^t e^{(t-s)\mathcal{L}^{(2)}} H_2(v(s), q(s), w(s)) \, ds.$$

As in the proof of Proposition 8.6, we may assume that $t \in [1, T(\delta, \gamma))$.

From Lemma 5.3 we know that $\|e^{t\mathcal{L}^{(2)}}\|_{\mathcal{B}(H^k(\mathbb{R}^d))} \leq K e^{-\rho t}$. By the definition of $T(\delta, \gamma)$, for $0 < \delta < \delta_1$, if $0 < \gamma < \delta$ and $E_k < \gamma$, then for all $s \in [1, T(\gamma, \delta))$

$$\|v(s)\|_{\mathcal{H}} + \|q(s)\|_{H^k(\mathbb{R}^{d-1})} + \|w(s)\|_{H^k(\mathbb{R}^{d-1})} < \delta < \delta_1.$$

It follows from Lemmas 7.9 and part (b') of Proposition 7.7 that there exists a constant $C_{\delta_1} > 0$ such that

$$(8.15) \quad \begin{aligned} \|H_i(v_1(s), v_2(s), q(s), w(s))\|_{H^k(\mathbb{R}^d)} &\leq C_{\delta_1} (\|v(s)\|_{H_\alpha^k(\mathbb{R}^d)} + \|v(s)\|_{H^k(\mathbb{R}^d)} \|v_2(s)\|_{H^k(\mathbb{R}^d)} \\ &\quad + \|v(s)\|_{H^k(\mathbb{R}^d)} \|v(s)\|_{H_\alpha^k(\mathbb{R}^d)} + \|q(s)\|_{H^k(\mathbb{R}^{d-1})} \|v(s)\|_{H_\alpha^k(\mathbb{R}^d)} \\ &\quad + \|w(s)\|_{H^k(\mathbb{R}^{d-1})}^2) \end{aligned}$$

for $i = 1, 2$. Thus by Proposition 8.6, Lemma 8.5, and (8.15), and also because $\|v(t)\|_{H^k(\mathbb{R}^d)} < \delta$, formula (8.14) yields, for some $K > 0$ and all $t \in [1, T(\gamma, \delta))$,

$$\begin{aligned} \|v_2(t)\|_{H^k(\mathbb{R}^d)} &\leq K e^{-\rho t} E_k + K \int_0^t e^{-\rho(t-s)} C_{\delta_1} (\|v(s)\|_{H_\alpha^k(\mathbb{R}^d)} \\ &\quad + \|v(s)\|_{H^k(\mathbb{R}^d)} \|v_2(s)\|_{H^k(\mathbb{R}^d)} \\ &\quad + \|v(s)\|_{H^k(\mathbb{R}^d)} \|v(s)\|_{H_\alpha^k(\mathbb{R}^d)} + \|q(s)\|_{H^k(\mathbb{R}^{d-1})} \|v(s)\|_{H_\alpha^k(\mathbb{R}^d)} \\ &\quad + \|w(s)\|_{H^k(\mathbb{R}^{d-1})}^2) ds. \end{aligned}$$

We define $M_{v_2}(t) = \sup_{0 < s \leq t} (1+s)^{\frac{d+1}{2}} \|v_2(s)\|_{H^k(\mathbb{R}^d)}$ and use Lemma 8.5 to obtain

$$\begin{aligned} \|v_2(t)\|_{H^k(\mathbb{R}^d)} &\leq K(1+t)^{-\frac{d+1}{2}} E_k + K \delta M_{v_2}(t) \int_0^t e^{-\rho(t-s)} (1+s)^{-\frac{d+1}{2}} ds \\ &\leq K(E_k + \delta M_{v_2}(t))(1+t)^{-\frac{d+1}{2}}, \end{aligned}$$

and thus, $(1+t)^{\frac{d+1}{2}} \|v_2(t)\|_{H^k(\mathbb{R}^d)} \leq K E_k + K \delta M_{v_2}(t)$. Because $M_{v_2}(t)$ is increasing, we conclude that

$$M_{v_2}(t) \leq K E_k + K \delta M_{v_2}(t).$$

Choosing $\delta_2 < \min\{\delta_1, 1/2K\}$, we obtain that if $0 < \delta < \delta_2$ and $0 < \gamma < \delta$, then

$$\|v_2(t)\|_{H^k(\mathbb{R}^d)} \leq \tilde{C}_2 (1+t)^{-\frac{d+1}{2}} E_k$$

for some $\tilde{C}_2 > 0$ on the time interval $t \in [1, T(\gamma, \delta))$, thus finishing the proof of (8.10).

To prove (8.9), we first use Lemma 5.3 in (8.13) to infer $\|e^{t\mathcal{L}^{(1)}}\|_{\mathcal{B}(H^k(\mathbb{R}^d))} \leq K$. We then find an estimate for the solution to (8.13) based on (8.15),

$$\begin{aligned} \|v_1(t)\|_{H^k(\mathbb{R}^d)} &\leq K E_k + K C_{\delta_1} \int_0^t [C \|v_2(s)\|_{H^k(\mathbb{R}^d)} + \|v(s)\|_{H_\alpha^k(\mathbb{R}^d)} \\ &\quad + \|v(s)\|_{H^k(\mathbb{R}^d)} \|v_2(s)\|_{H^k(\mathbb{R}^d)} \\ &\quad + \|v(s)\|_{H^k(\mathbb{R}^d)} \|v(s)\|_{H_\alpha^k(\mathbb{R}^d)} + \|q(s)\|_{H^k(\mathbb{R}^{d-1})} \|v(s)\|_{H_\alpha^k(\mathbb{R}^d)} \\ &\quad + \|w(s)\|_{H^k(\mathbb{R}^{d-1})}^2] ds, \end{aligned}$$

for some $C > 0$. Since $0 < E_k < \gamma < \delta < \delta_2$, for $t \in [1, T(\gamma, \delta))$ we have $\|v(t)\|_{H^k(\mathbb{R}^d)} < \delta$ by Lemma 8.2 and Definition 8.3 of $T(\delta, \gamma)$. Therefore,

$$\begin{aligned} \|v_1(t)\|_{H^k(\mathbb{R}^d)} &\leq KE_k + KC_{\delta_1} \int_0^t (C\|v_2(s)\|_{H^k(\mathbb{R}^d)} + \|v(s)\|_{H_\alpha^k(\mathbb{R}^d)} + \delta\|v_2(s)\|_{H^k(\mathbb{R}^d)} \\ &\quad + \delta\|v(s)\|_{H_\alpha^k(\mathbb{R}^d)} + \|q\|_{H^k(\mathbb{R}^{d-1})}\|v(s)\|_{H_\alpha^k(\mathbb{R}^d)} + \|w(s)\|_{H^k(\mathbb{R}^{d-1})}^2) ds \\ &\leq KE_k + K(C + \delta) \int_0^t \|v_2(s)\|_{H^k(\mathbb{R}^d)} ds + K(1 + \delta) \int_0^t \|v(s)\|_{H_\alpha^k(\mathbb{R}^d)} ds \\ &\quad + K \int_0^t \|q\|_{H^k(\mathbb{R}^{d-1})}\|v(s)\|_{H_\alpha^k(\mathbb{R}^d)} ds + K \int_0^t \|w(s)\|_{H^k(\mathbb{R}^{d-1})}^2 ds. \end{aligned}$$

Finally, we apply Proposition 8.6, (8.10), and Lemma 8.5 to obtain that

$$\begin{aligned} \|v_1(t)\|_{H^k(\mathbb{R}^d)} &\leq K \left(E_k + C_{\delta_1}(C + \delta)E_k \int_0^t (1+s)^{-\frac{d+1}{2}} ds \right. \\ &\quad + C_1(1 + \delta)E_k \int_0^t (1+s)^{-\frac{d+1}{2}} ds \\ &\quad \left. + C_1^2 E_k \int_0^t (1+s)^{-\frac{3d+1}{4}} ds + C_1^2 E_k \int_0^t (1+s)^{-\frac{d+1}{2}} ds \right) \leq \tilde{C}_2 E_k. \end{aligned}$$

The proposition then holds with $K_2 = \max\{\tilde{C}_2, \tilde{C}_2\}$. $\square$

8.4. Global existence and bounds. We now present the main result of this paper. It relies on a bootstrap argument based on Propositions 8.6 and 8.7. The constant δ_0 in the next theorem can be taken to be $\delta_0 = \delta_2$, where δ_2 is chosen as in Proposition 8.7.

THEOREM 8.8. *Assume Hypotheses 3.2, 3.3, 3.4, and 3.7 and let $k \geq [\frac{d+1}{2}]$. There exist positive constants δ_0 and C such that for each $0 < \delta < \delta_0$ there exists $0 < \eta < \delta$ such that the following is true. Let $(v^0, q^0, w^0) \in \mathcal{H}^n \times H^k(\mathbb{R}^{d-1}) \times H^k(\mathbb{R}^{d-1})^{d-1}$ be the initial condition satisfying*

$$E_k = \|v^0\|_{\mathcal{H}} + \|q^0\|_{H^k(\mathbb{R}^{d-1})} + \|q^0\|_{W^{1,1}(\mathbb{R}^{d-1})} \leq \eta$$

and let $(v(t), q(t), w(t)) \in \mathcal{H}^n \times H^k(\mathbb{R}^{d-1}) \times H^k(\mathbb{R}^{d-1})^{d-1}$ be the solution of the evolution equation (6.15) with the initial condition (v^0, q^0, w^0) . Then for all $t > 0$,

- (1) $(v(t), q(t), w(t))$ is defined in $\mathcal{H}^n \times H^k(\mathbb{R}^{d-1}) \times H^k(\mathbb{R}^{d-1})^{d-1}$;
- (2) $\|v(t)\|_{\mathcal{H}} + \|q(t)\|_{H^k(\mathbb{R}^{d-1})} + \|w(t)\|_{H^k(\mathbb{R}^{d-1})} \leq \delta$;
- (3) $\|v(t)\|_{H_\alpha^k(\mathbb{R}^d)} \leq C(1+t)^{-\frac{d+1}{2}} E_k$;
- (4) $\|q(t)\|_{H^k(\mathbb{R}^{d-1})} \leq C(1+t)^{-\frac{d-1}{4}} E_k$;
- (5) $\|w(t)\|_{H^k(\mathbb{R}^{d-1})} \leq C(1+t)^{-\frac{d+1}{4}} E_k$;
- (6) $\|v_1(t)\|_{H^k(\mathbb{R}^d)} \leq CE_k$;
- (7) $\|v_2(t)\|_{H^k(\mathbb{R}^d)} \leq C(1+t)^{-\frac{d+1}{2}} E_k$.

Proof. We choose $\delta_0 = \delta_2$, with δ_2 from Proposition 8.7, and then we fix $C > \max\{1, C_1, C_2\}$ with C_1 and C_2 from Propositions 8.6 and 8.7, respectively. We take γ such that $0 < \gamma < \delta < \delta_0$ and set $\eta = C^{-1}\gamma/3$. Let $(v^0, q^0, w^0) \in \text{ran } \mathcal{Q}_{\mathcal{H}} \times H^k(\mathbb{R}^{d-1}) \times H^k(\mathbb{R}^{d-1})^{d-1}$ be the initial value of the solution $(v(t), q(t), w(t)) \in \text{ran } \mathcal{Q}_{\mathcal{H}} \times H^k(\mathbb{R}^{d-1}) \times H^k(\mathbb{R}^{d-1})^{d-1}$ of (3.2) such that $E_k \leq \eta$. Since $\eta < \gamma < \delta$,

we can apply Propositions 8.6 and 8.7 with γ replaced by η and conclude that for all $t \in [0, T(\gamma, \eta))$ assertions (1)–(7) of the theorem hold.

We claim that $T(\delta, \eta) = \infty$; thus the theorem holds as soon as the claim is proved. To prove the claim, we fix any $T \in (0, T(\delta, \eta))$. At that T , we note that by assertions (3), (6), and (7) and the definition of the $\mathcal{H}$ norm, $\|v(T, v^0, q^0)\|_{\mathcal{H}} \leq \sqrt{2}CE_k$ and therefore

$$\begin{aligned} & \|v(T, v^0, q^0, w^0)\|_{\mathcal{H}} + \|q(T, v^0, q^0, w^0)\|_{H^k(\mathbb{R}^{d-1})} + \|w(T, v^0, q^0, w^0)\|_{H^k(\mathbb{R}^{d-1})} \\ & \leq (2 + \sqrt{2})CE_k \leq (2 + \sqrt{2})C\eta = \gamma. \end{aligned}$$

We now apply Lemma 8.2 to the solution with the initial data $(v(T), q(T), w(T))$. This lemma says that for all $t \in [0, T(\delta, \gamma))$ we have the inequality

$$\|v(t+T)\|_{\mathcal{H}} + \|q(t+T)\|_{H^k(\mathbb{R}^{d-1})} + \|w(t+T)\|_{H^k(\mathbb{R}^{d-1})} \leq \delta,$$

so, if $E_k \leq \eta$, then $\|v(t)\|_{\mathcal{H}} + \|q(t)\|_{H^k(\mathbb{R}^{d-1})} + \|w(t)\|_{H^k(\mathbb{R}^{d-1})} \leq \delta$ for all $t \in [0, T + T(\delta, \gamma))$. By Definition 8.3 that means that $T(\delta, \eta) \geq T + T(\delta, \gamma)$ for each $T \in (0, T(\delta, \gamma))$ and therefore $T(\delta, \gamma) = \infty$, which completes the proof. $\square$

Appendix A. Lipschitz properties of the Nemytskij operator. In this appendix we prove the Lipschitz properties of the Nemytskij operator (A.1) induced by the nonlinear term in system (3.2) that we consider. In order to do so, we need the following lemma from [RS] and a generalized Hölder's inequality (see, e.g., [WZ]).

LEMMA A.1. *For Sobolev spaces $W^{k,p}(\mathbb{R}^d)$ and $W^{k_0,p_0}(\mathbb{R}^d)$, if $k > k_0$ and $k - \frac{d}{p} > k_0 - \frac{d}{p_0}$, then the Sobolev embedding $W^{k,p}(\mathbb{R}^d) \hookrightarrow W^{k_0,p_0}(\mathbb{R}^d)$ holds.*

LEMMA A.2. *Assume that $r \in (0, \infty)$ and $p_1, \dots, p_n \in (0, \infty]$ are such that $\sum_{k=1}^n \frac{1}{p_k} = \frac{1}{r}$. Then for all μ -measurable real or complex-valued functions $f_1, \dots, f_n$,*

$$\left\| \prod_{k=1}^n f_k \right\|_{L^r(\mu)} \leq \prod_{k=1}^n \|f_k\|_{L^{p_k}(\mu)}.$$

In particular, $f_k \in L^{p_k}(\mu)$ for all $k \in \{1, \dots, n\}$ implies that $\prod_{k=1}^n f_k \in L^r(\mu)$.

Next we formulate an analogue of [GLS, Proposition 7.2].

PROPOSITION A.3. *Assume that $m : (q, u) \mapsto m(q, u) \in \mathbb{R}$ is a function from $C^{k+1}(\mathbb{R}^2)$ with $k \geq [\frac{d+1}{2}]$. Consider the formula*

$$(A.1) \quad (q(x), u(x), v(x)) \mapsto m(q(x), u(x))v(x),$$

where $q(\cdot), u(\cdot), v(\cdot) : \mathbb{R}^d \mapsto \mathbb{R}$, and the variable $x = (x_1, \dots, x_d) \in \mathbb{R}^d$.

- (1) *Formula (A.1) defines a mapping from $H^k(\mathbb{R}^d) \times H^k(\mathbb{R}^d)^2$ to $H^k(\mathbb{R}^d)$ that is locally Lipschitz on any set of the form $\{(q, u, v) : \|q\|_{H^k(\mathbb{R}^d)} + \|u\|_{H^k(\mathbb{R}^d)} + \|v\|_{H^k(\mathbb{R}^d)} \leq K\}$.*
- (2) *Formula (A.1) defines a mapping from $H^k(\mathbb{R}^d) \times \mathcal{H}^2$ to $\mathcal{H}$ that is locally Lipschitz on any set of the form $\{(q, u, v) : \|q\|_{H^k(\mathbb{R}^d)} + \|u\|_{\mathcal{H}} + \|v\|_{\mathcal{H}} \leq K\}$.*

Proof. In the following proof, we will abbreviate the norm of q, u, v , for instance, $H^k(\mathbb{R}^d)$ to H^k since q, u, v are all $\mathbb{R}^d \rightarrow \mathbb{R}$. We shall use the equivalent Sobolev norm (see, e.g., [NS]):

$$\|f\|_{H^k} \sim \|f\|_{L^2} + \sum_{a_1+\dots+a_d=k} \left\| \frac{\partial^k}{\partial x_1^{a_1} \dots \partial x_d^{a_d}} f \right\|_{L^2},$$

where the sum is over all d -tuples $(a_1, \dots, a_d)$ of nonnegative integers such that $\sum_{l=1}^d a_l = k$, and $\frac{\partial^{a_l}}{\partial x_l^{a_l}}$ is the a_l th differentiation of functions with respect to x_l , $l = 1, \dots, d$.

For variation in q , we write $m(q + \bar{q}, u) - m(q, u) = \bar{q} \left(\int_0^1 m_q(q + t\bar{q}, u) dt \right)$, and, using embedding of $H^k(\mathbb{R}^d)$ in $L^\infty(\mathbb{R}^d)$, we obtain

$$\|m(q + \bar{q}, u)v - m(q, u)v\|_{L^2} \leq \|m\|_{C^1(\mathbb{R}^2)} \|\bar{q}\|_{L^\infty} \|v\|_{L^2} \leq \|m\|_{C^1(\mathbb{R}^2)} \|\bar{q}\|_{H^k} \|v\|_{L^2}.$$

The estimate of $m(q + \bar{q}, u)v - m(q, u)v$ in $L_\alpha^2(\mathbb{R}) \otimes L^2(\mathbb{R}^{d-1})$ follows from

$$\begin{aligned} \|\gamma_\alpha(m(q + \bar{q}, u)v - m(q, u)v)\|_{L^2} &\leq \|m\|_{C^1(\mathbb{R}^2)} \|\bar{q}\|_{L^\infty} \|\gamma_\alpha v\|_{L^2} \\ &\leq \|m\|_{C^1(\mathbb{R}^2)} \|\bar{q}\|_{H^k} \|\gamma_\alpha v\|_{L^2}. \end{aligned}$$

We denote $m_1(q, \bar{q}, u) = \int_0^1 m_q(q + t\bar{q}, u) dt$. To estimate the derivatives of $m_1(q, \bar{q}, u)\bar{q}v$ in $L^2(\mathbb{R}^d)$, we need the general Leibniz rule [O]: if $f_1, \dots, f_m$ are all n -times differentiable functions, then their product $f_1 \dots f_m$ is also n -times differentiable and its n th derivative is given by

$$(f_1 f_2 \dots f_m)^{(n)} = \sum_{k_1+k_2+\dots+k_m=n} \binom{n}{k_1, k_2, \dots, k_m} \prod_{1 \leq t \leq m} f_t^{(k_t)},$$

where the sum extends over all m -tuples $(k_1, \dots, k_m)$ of nonnegative integers such that $\sum_{l=1}^m k_l = n$ and $\binom{n}{k_1, k_2, \dots, k_m} = \frac{n!}{k_1! k_2! \dots k_m!}$ are the multinomial coefficients. We then have

$$\begin{aligned} &\frac{\partial^k}{\partial x_1^{a_1} \dots \partial x_d^{a_d}} (m_1(q, \bar{q}, u)\bar{q}v) \\ &= \frac{\partial^{a_1}}{\partial x_1^{a_1}} \dots \frac{\partial^{a_{d-1}}}{\partial x_{d-1}^{a_{d-1}}} \sum_{b_d+c_d+e_d=a_d} \binom{a_d}{b_d, c_d, e_d} \frac{\partial^{b_d} m_1}{\partial x_d^{b_d}} \cdot \frac{\partial^{c_d} \bar{q}}{\partial x_d^{c_d}} \cdot \frac{\partial^{e_d} v}{\partial x_d^{e_d}} \\ &= \frac{\partial^{a_1+\dots+a_{d-2}}}{\partial x_1^{a_1} \dots \partial x_{d-2}^{a_{d-2}}} \sum_{b_d+c_d+e_d=a_d} \binom{a_d}{b_d, c_d, e_d} \\ &\quad \sum_{b_{d-1}+c_{d-1}+e_{d-1}=a_{d-1}} \binom{a_{d-1}}{b_{d-1}, c_{d-1}, e_{d-1}} \frac{\partial^{b_{d-1}+b_d} m_1}{\partial x_{d-1}^{b_{d-1}} \partial x_d^{b_d}} \cdot \frac{\partial^{c_{d-1}+c_d} \bar{q}}{\partial x_{d-1}^{c_{d-1}} \partial x_d^{c_d}} \cdot \frac{\partial^{e_{d-1}+e_d} v}{\partial x_{d-1}^{e_{d-1}} \partial x_d^{e_d}} = \dots \\ &= \sum_{b_d+c_d+e_d=a_d} \binom{a_d}{b_d, c_d, e_d} \dots \sum_{b_1+c_1+e_1=a_1} \binom{a_1}{b_1, c_1, e_1} \frac{\partial^{b_1+\dots+b_d} m_1}{\partial x_1^{b_1} \dots \partial x_d^{b_d}} \\ &\quad \cdot \frac{\partial^{c_1+\dots+c_d} \bar{q}}{\partial x_1^{c_1} \dots \partial x_d^{c_d}} \cdot \frac{\partial^{e_1+\dots+e_d} v}{\partial x_1^{e_1} \dots \partial x_d^{e_d}}, \end{aligned}$$

where $a_1 + \dots + a_d = k$.

We now refer to the higher chain formula (see, e.g., [Ts, Lemma 1]). We consider a mapping

$$M : x \in X \subset \mathbb{R}^d \xrightarrow{g} (q(x), \bar{q}(x), u(x)) \in G \subset \mathbb{R}^3 \xrightarrow{h} m_1 \in \mathbb{R},$$

where X, G are open subsets of $\mathbb{R}^d$ and $\mathbb{R}^3$, respectively, and g, h are sufficiently smooth functions. We denote $(g_1(x), g_2(x), g_3(x)) = (q(x), \bar{q}(x), u(x))$. For each i in

the set J_s of integers $1, 2, \dots, s$, where $s = b_1 + \dots + b_d$, let t_i denote one of the independent variables $x_1, \dots, x_d$. A partition of J_s is a family of pairwise disjoint nonempty subsets of J_s whose union is J_s . Sets in a partition are called blocks. A block's function is to assign a label to each block of a partition. The set of all block functions from a partition P of J_s into J_3 is denoted by P_3 . The set of all partitions of J_s is denoted by P_s . We then have

$$(A.2) \quad \frac{\partial^s m_1(g(x))}{\partial t_1 \cdots \partial t_s} = \sum_{P \in P_s} \sum_{\lambda \in P_3} \left\{ \left(\prod_{B \in P} \frac{\partial}{\partial g_{\lambda(B)}} \right) m_1 \right\} \left\{ \prod_{B \in P} \left[\left(\prod_{b \in B} \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right] \right\}.$$

$B \in P$ means that B runs through the list of all of the blocks of the partition P . The number of blocks in the partition P is denoted by $|P|$. The partition P then can be written as $P = \{B_1, \dots, B_{|P|}\}$. Let $|B_i|$ be the size of the block B_i , $i = 1, 2, \dots, |P|$. For fixed multinomial coefficients $\binom{a_i}{b_i, c_i, e_i}$ ($i = 1, \dots, d$), fixed $P \in P_s$ and $\lambda \in P_3$ we need to estimate the following term in both $L^2(\mathbb{R}^d)$ and $L^2_\alpha(\mathbb{R}) \otimes L^2(\mathbb{R})^{d-1}$:

$$\left\{ \left(\prod_{B \in P} \frac{\partial}{\partial g_{\lambda(B)}} \right) m_1 \right\} \left\{ \prod_{B \in P} \left[\left(\prod_{b \in B} \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right] \right\} \frac{\partial^{c_1 + \dots + c_d} \bar{q}}{\partial x_1^{c_1} \cdots \partial x_d^{c_d}} \frac{\partial^{e_1 + \dots + e_d} v}{\partial x_1^{e_1} \cdots \partial x_d^{e_d}},$$

where $g_{\lambda(B_i)}$ is one of $(g_1, g_2, g_3) = (q, \bar{q}, u)$. To obtain the estimates we distinguish several cases.

Case 1.1. If $b_1 + \dots + b_d \neq 0$, $c_1 + \dots + c_d \neq 0$ and $e_1 + \dots + e_d \neq 0$, we use Lemma A.2 with $\frac{1}{2} = \sum_{i=1}^{|P|+2} \frac{1}{p_i}$, where p_i will be chosen below. If we denote $P = \{B_1, B_2, \dots, B_{|P|}\}$ and $l = |P| + 2$ (note that $3 \leq l \leq k$), and introduce the l -tuple

$$(n_1, n_2, \dots, n_l) = (|B_1|, |B_2|, \dots, |B_{l-2}|, c_1 + \dots + c_d, e_1 + \dots + e_d),$$

then

$$(A.3) \quad \begin{aligned} & \left\| \left\{ \left(\prod_{B \in P} \frac{\partial}{\partial g_{\lambda(B)}} \right) m_1 \right\} \left\{ \prod_{B \in P} \left[\left(\prod_{b \in B} \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right] \right\} \frac{\partial^{c_1 + \dots + c_d} \bar{q}}{\partial x_1^{c_1} \cdots \partial x_d^{c_d}} \frac{\partial^{e_1 + \dots + e_d} v}{\partial x_1^{e_1} \cdots \partial x_d^{e_d}} \right\|_{L^2} \\ & \leq \|m_1^{(|P|)}\|_{L_\infty(\mathbb{R}^3)} \left\| \frac{\partial^{c_1 + \dots + c_d} \bar{q}}{\partial x_1^{c_1} \cdots \partial x_d^{c_d}} \right\|_{L^{p_{l-1}}} \left\| \frac{\partial^{e_1 + \dots + e_d} v}{\partial x_1^{e_1} \cdots \partial x_d^{e_d}} \right\|_{L^{p_l}} \prod_{i=1}^{l-2} \left\| \left(\prod_{b \in B_i} \frac{\partial}{\partial t_b} \right) g_{\lambda(B_i)} \right\|_{L^{p_i}} \\ & \leq \|m_1^{(|P|)}\|_{L_\infty(\mathbb{R}^3)} \|g_{\lambda(B_1)}\|_{W^{n_1, p_1}} \cdots \|g_{\lambda(B_{l-2})}\|_{W^{n_{l-2}, p_{l-2}}} \|\bar{q}\|_{W^{n_{l-1}, p_{l-1}}} \|v\|_{W^{n_l, p_l}}, \end{aligned}$$

where $W^{k,p}$ are the Sobolev spaces of k times differentiable functions from L^p .

In (A.3), $\sum_{i=1}^{|P|} |B_i| = \sum_{j=1}^d b_j$ because P is a partition of the $b_1 + \dots + b_d$ indices of

$$\underbrace{(x_1, \dots, x_1)}_{b_1 \text{ times}}, \dots, \underbrace{(x_d, \dots, x_d)}_{b_d \text{ times}}$$

and $\{B_1, \dots, B_{|P|}\}$ are all blocks in the partition P . Since all $b_1 + \dots + b_d$, $c_1 + \dots + c_d$, and $e_1 + \dots + e_d$ are nonzero, it is obvious that $k > n_i$. By Lemma A.1 then, in order to prove that $W^{k,2}(\mathbb{R}^d) = H^k(\mathbb{R}^d) \hookrightarrow W^{n_i, p_i}(\mathbb{R}^d)$, we must show that $k - \frac{d}{2} > n_i - \frac{d}{p_i}$.

If we choose $\frac{1}{p_i} = \left(\frac{1}{2} - \frac{k}{d}\right)\frac{1}{l} + \frac{n_i}{d}$, then $\sum_{i=1}^l \frac{1}{p_i} = \frac{1}{2}$, and

$$n_i - \frac{d}{p_i} = n_i - d \left(\left(\frac{1}{2} - \frac{k}{d} \right) \frac{1}{l} + \frac{n_i}{d} \right) = \frac{k}{l} - \frac{d}{2l}.$$

Since $k \geq [\frac{d+1}{2}]$ and $l > 2$, we can conclude that

$$k - \frac{d}{2} > \frac{1}{l} \left(k - \frac{d}{2} \right) = n_i - \frac{d}{p_i}, i = 1, \dots, l.$$

Therefore $H^k(\mathbb{R}^d)$ can be embedded into $W^{n_i, p_i}(\mathbb{R}^d)$, $i = 1, \dots, l$. Following (A.3), we then have

$$\begin{aligned} (A.4) \quad & \left\| m_1^{(|P|)} \cdot \prod_{B \in P} \left[\left(\prod_{b \in B} \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right] \cdot \frac{\partial^{c_1 + \dots + c_d} \bar{q}}{\partial x_1^{c_1} \dots \partial x_d^{c_d}} \cdot \frac{\partial^{e_1 + \dots + e_d} v}{\partial x_1^{e_1} \dots \partial x_d^{e_d}} \right\|_{L^2} \\ & \leq \|m_1^{(|P|)}\|_{L^\infty} \|\bar{q}\|_{W^{n_{l-1}, p_{l-1}}} \|v\|_{W^{n_l, p_l}} \prod_{i=1}^{|P|} \|g_{\lambda(B_i)}\|_{W^{n_i, p_i}} \\ & \leq C \|m_1^{(|P|)}\|_{L^\infty} \|\bar{q}\|_{H^k} \|v\|_{H^k} \prod_{i=1}^{|P|} \|g_{\lambda(B_i)}\|_{H^k}. \end{aligned}$$

Case 1.2. If $c_1 + \dots + c_d = e_1 + \dots + e_d = 0$ and $|P| \neq 0$, we use the Sobolev embedding $H^k(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ and Lemma 7.1(1), so that, similarly to Case 1.1,

$$\begin{aligned} & \left\| m_1^{(|P|)} \cdot \prod_{B \in P} \left[\left(\prod_{b \in B} \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right] \cdot \frac{\partial^{c_1 + \dots + c_d} \bar{q}}{\partial x_1^{c_1} \dots \partial x_d^{c_d}} \cdot \frac{\partial^{e_1 + \dots + e_d} v}{\partial x_1^{e_1} \dots \partial x_d^{e_d}} \right\|_{L^2} \\ & \leq \|m_1^{(|P|)}\|_{L^\infty(\mathbb{R}^3)} \left\| \prod_{B \in P} \left[\left(\prod_{b \in B} \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right] \right\|_{L^2} \|\bar{q}v\|_{L^\infty} \\ & \leq C \|m_1^{(|P|)}\|_{L^\infty(\mathbb{R}^3)} \left\| \prod_{B \in P} \left[\left(\prod_{b \in B} \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right] \right\|_{L^2} \|\bar{q}\|_{H^k} \|v\|_{H^k}. \end{aligned}$$

We denote $l = |P|$. When $|P| = 1$, we use the inequality

$$\left\| \frac{\partial^k}{\partial x_1^{a_1} \dots \partial x_d^{a_d}} g_i \right\|_{L^2} \leq \|u\|_{H^k},$$

where g_i is one of $(g_1, g_2, g_3) = (q, \bar{q}, u)$, and thus obtain

$$\begin{aligned} \left\| m_1^{(|P|)} \cdot \frac{\partial^k g_i}{\partial x_1^{a_1} \dots \partial x_d^{a_d}} \cdot \bar{q}v \right\|_{L^2} & \leq \|m_1^{(1)}\|_{L^\infty(\mathbb{R}^3)} \|g_i\|_{H^k} \|\bar{q}v\|_{H^k} \\ & \leq C \|m\|_{C^2(\mathbb{R}^2)} \|g_i\|_{H^k} \|\bar{q}\|_{H^k} \|v\|_{H^k}. \end{aligned}$$

When $l \geq 2$, let $P = \{B_1, B_2, \dots, B_l\}$ and the l -tuple $(n_1, \dots, n_l) = (|B_1|, \dots, |B_l|)$. We then use Lemmas A.1 and A.2 with $\frac{1}{p_i} = (\frac{1}{2} - \frac{k}{d})\frac{1}{l} + \frac{n_i}{d}$, $i = 1, \dots, l$, to obtain

$$\begin{aligned} \left\| \prod_{B \in P} \left[\left(\prod_{b \in B} \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right] \right\|_{L^2} & \leq \prod_{i=1}^l \left\| \left(\prod_{b \in B_i} \frac{\partial}{\partial t_b} \right) g_{\lambda(B_i)} \right\|_{L^{p_i}} \\ & \leq \prod_{i=1}^l \|g_{\lambda(B_i)}\|_{W^{n_i, p_i}} \leq \prod_{i=1}^l \|g_{\lambda(B_i)}\|_{H^k}, \end{aligned}$$

from which we conclude that

$$\begin{aligned} & \left\| m_1^{(l)}(q, \bar{q}, u) \prod_{B \in P} \left[\left(\prod_{b \in B} \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right] \bar{q}v \right\|_{L^2} \\ & \leq \|m_1^{(l)}\|_{L^\infty(\mathbb{R}^3)} \|\bar{q}\|_{H^k} \|v\|_{H^k} \prod_{i=1}^l \|g_{\lambda(B_i)}\|_{H^k}. \end{aligned}$$

Case 1.3. If $b_1 + \cdots + b_d = c_1 + \cdots + c_d = 0$ and $e_1 + \cdots + e_d \neq 0$, we are evaluating the term $m_1(q, \bar{q}, u) \bar{q} \frac{\partial^k v}{\partial x_1^{a_1} \cdots \partial x_d^{a_d}}$ on $L^2(\mathbb{R}^d)$, based on the embedding $H^k \hookrightarrow L^\infty$:

$$\begin{aligned} \left\| m_1(q, \bar{q}, u) \bar{q} \frac{\partial^k v}{\partial x_1^{a_1} \cdots \partial x_d^{a_d}} \right\|_{L^2} &\leq \|m_1\|_{L^\infty(\mathbb{R}^3)} \|\bar{q}\|_{L^\infty} \left\| \frac{\partial^k v}{\partial x_1^{a_1} \cdots \partial x_d^{a_d}} \right\|_{L^2} \\ &\leq C \|m\|_{C^1(\mathbb{R}^2)} \|\bar{q}\|_{H^k} \|v\|_{H^k}. \end{aligned}$$

Similarly, if $b_1 + \cdots + b_d = e_1 + \cdots + e_d = 0$ and $c_1 + \cdots + c_d \neq 0$, we have

$$\left\| m_1(q, \bar{q}, u) v \frac{\partial^k \bar{q}}{\partial x_1^{a_1} \cdots \partial x_d^{a_d}} \right\|_{L^2} \leq C \|m\|_{C^1(\mathbb{R}^2)} \|v\|_{H^k} \|\bar{q}\|_{H^k}.$$

Case 1.4. If $b_1 + \cdots + b_d = 0$, but $c_1 + \cdots + c_d \neq 0$, and $e_1 + \cdots + e_d \neq 0$, then we set

$$(n_1, n_2) = (b_1 + \cdots + b_d, e_1 + \cdots + e_d)$$

and apply Lemmas A.1 and A.2 with $\frac{1}{p_i} = (\frac{1}{2} - \frac{k}{d})\frac{1}{l} + \frac{n_i}{d}$, $i = 1, 2$,

$$\begin{aligned} &\left\| m_1(q, \bar{q}, u) \cdot \frac{\partial^{c_1+\cdots+c_d} \bar{q}}{\partial x_1^{c_1} \cdots \partial x_d^{c_d}} \cdot \frac{\partial^{e_1+\cdots+e_d} v}{\partial x_1^{e_1} \cdots \partial x_d^{e_d}} \right\|_{L^2} \\ &\leq \|m_1\|_{L^\infty(\mathbb{R}^3)} \left\| \frac{\partial^{c_1+\cdots+c_d} \bar{q}}{\partial x_1^{c_1} \cdots \partial x_d^{c_d}} \right\|_{L^{p_1}} \left\| \frac{\partial^{e_1+\cdots+e_d} v}{\partial x_1^{e_1} \cdots \partial x_d^{e_d}} \right\|_{L^{p_2}} \\ &\leq C \|m\|_{C^1(\mathbb{R}^2)} \|\bar{q}\|_{W^{n_1, p_1}} \|v\|_{W^{n_2, p_2}} \leq C \|m\|_{C^1(\mathbb{R}^2)} \|\bar{q}\|_{H^k} \|v\|_{H^k}. \end{aligned}$$

Case 1.5. If $c_1 + \cdots + c_d = 0$, $|P| \neq 0$ and $e_1 + \cdots + e_d \neq 0$, we, similarly, using Lemmas A.1 and A.2 with $\frac{1}{p_i} = (\frac{1}{2} - \frac{k}{d})\frac{1}{|P|+1} + \frac{n_i}{d}$, $i = 1, \dots, |P| + 1$, obtain

$$\begin{aligned} &\|m_1^{(|P|)}(q, \bar{q}, u) \cdot \prod_{B \in P} \left[\left(\prod_{b \in B} \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right] \cdot \bar{q} \cdot \frac{\partial^{e_1+\cdots+e_d} v}{\partial x_1^{e_1} \cdots \partial x_d^{e_d}}\|_{L^2} \\ &\leq C \|m\|_{C^{|P|+1}(\mathbb{R}^2)} \|\bar{q}\|_{H^k} \|v\|_{H^k} \prod_{i=1}^{|P|} \|g_{\lambda(B_i)}\|_{H^k}, \end{aligned}$$

and, if $e_1 + \cdots + e_d = 0$, $|P| \neq 0$ and $c_1 + \cdots + c_d \neq 0$, we obtain

$$\begin{aligned} (A.5) \quad &\|m_1^{(|P|)}(q, \bar{q}, u) \cdot \prod_{B \in P} \left[\left(\prod_{b \in B} \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right] \cdot \frac{\partial^{c_1+\cdots+c_d} \bar{q}}{\partial x_1^{c_1} \cdots \partial x_d^{c_d}} \cdot v\|_{L^2} \\ &\leq C \|m\|_{C^{|P|+1}(\mathbb{R}^2)} \|v\|_{H^k} \|\bar{q}\|_{H^k} \prod_{i=1}^{|P|} \|g_{\lambda(B_i)}\|_{H^k}. \end{aligned}$$

Since $|P| \leq k$, the inequalities (A.4)–(A.5) imply

$$\begin{aligned} &\|m_1^{(|P|)}(q, \bar{q}, u) \cdot \prod_{B \in P} \left[\left(\prod_{b \in B} \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right] \cdot \frac{\partial^{c_1+\cdots+c_d} \bar{q}}{\partial x_1^{c_1} \cdots \partial x_d^{c_d}} \cdot \frac{\partial^{e_1+\cdots+e_d} v}{\partial x_1^{e_1} \cdots \partial x_d^{e_d}}\|_{L^2} \\ &\leq \|m\|_{C^{k+1}(\mathbb{R}^2)} \|\bar{q}\|_{H^k} \|v\|_{H^k} \prod_{i=1}^{|P|} \|g_{\lambda(B_i)}\|_{H^k}. \end{aligned}$$

Similarly,

$$\begin{aligned}
& \left\| \gamma_\alpha \left(m_1^{(|P|)}(q, \bar{q}, u) \cdot \prod_{B \in P} \left[\left(\prod_{b \in B} \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right] \cdot \frac{\partial^{c_1+\dots+c_d} \bar{q}}{\partial x_1^{c_1} \dots \partial x_d^{c_d}} \frac{\partial^{e_1+\dots+e_d} v}{\partial x_1^{e_1} \dots \partial x_d^{e_d}} \right) \right\|_{L^2} \\
& \leq \|m_1^{(|P|)}\|_{L^\infty(\mathbb{R}^3)} \prod_{i=1}^{|P|} \left\| \left(\prod_{b \in B_i} \frac{\partial}{\partial t_b} \right) g_{\lambda(B_i)} \right\|_{L^{p_i}} \\
& \quad \times \left\| \frac{\partial^{c_1+\dots+c_d} \bar{q}}{\partial x_1^{c_1} \dots \partial x_d^{c_d}} \right\|_{L^{p|P|+1}} \left\| \gamma_\alpha \frac{\partial^{e_1+\dots+e_d} v}{\partial x_1^{e_1} \dots \partial x_d^{e_d}} \right\|_{L^{p|P|+2}} \\
& \leq \|m_1\|_{C^{|P|}(\mathbb{R}^3)} \|\gamma_\alpha v\|_{W^{n|P|+2, p|P|+2}} \prod_{i=1}^{|P|} \|g_{\lambda(B_i)}\|_{W^{n_i, p_i}} \\
& \leq \|m\|_{C^{|P|+1}(\mathbb{R}^2)} \|\bar{q}\|_{H^k} \|\gamma_\alpha v\|_{H^k} \prod_{i=1}^{|P|} \|g_{\lambda(B_i)}\|_{H^k}.
\end{aligned}$$

The case $|P| = 0$, $c_1 + \dots + c_d = 0$ or $e_1 + \dots + e_d = 0$ can be treated analogously.

For variations in u , the representation $m(q, u + \bar{u}) - m(q, u) = \bar{u} \int_0^1 m_u(q, u + t\bar{u}) dt$ yields

$$\|m(q, u + \bar{u})v - m(q, u)v\|_{L^2} \leq \|m\|_{C^1(\mathbb{R}^2)} \|\bar{u}\|_{L^\infty} \|v\|_{L^2},$$

which, by the Sobolev embedding $H^k(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$, implies

$$\|m(q, u + \bar{u})v - m(q, u)v\|_{L^2} \leq \|m\|_{C^1(\mathbb{R}^2)} \|\bar{u}\|_{H^k} \|v\|_{L^2},$$

and

$$\|\gamma_\alpha (m(q, u + \bar{u})v - m(q, u)v)\|_{L^2} \leq \|m\|_{C^1(\mathbb{R}^2)} \|\bar{u}\|_{H^k} \|\gamma_\alpha v\|_{L^2}.$$

Let $m_2(q, u, \bar{u}) = \int_0^1 m_u(q, u + t\bar{u}) dt$, $g_1(x) = q(x)$, $g_2(x) = u(x)$, and $g_3(x) = \bar{u}(x)$.

For $\sum_{t=1}^d a_t = k$, we then have

$$\begin{aligned}
& \frac{\partial^k}{\partial x_1^{a_1} \dots \partial x_d^{a_d}} m_2(g_1, g_2, g_3) \bar{u} v \\
& = \frac{\partial^{a_1+\dots+a_{d-1}}}{\partial x_1^{a_1} \dots \partial x_{d-1}^{a_{d-1}}} \sum_{b_d+c_d+e_d=a_d} \binom{a_d}{b_d, c_d, e_d} \frac{\partial^{b_d} m_2}{\partial x_d^{b_d}} \frac{\partial^{c_d} \bar{u}}{\partial x_d^{c_d}} \frac{\partial^{e_d} v}{\partial x_d^{e_d}} \\
& = \dots \\
& = \sum_{b_d+c_d+e_d=a_d} \binom{a_d}{b_d, c_d, e_d} \dots \\
& \quad \sum_{b_1+c_1+e_1=a_1} \binom{a_1}{b_1, c_1, e_1} \frac{\partial^{b_1+\dots+b_d} m_2}{\partial x_1^{b_1} \dots \partial x_d^{b_d}} \frac{\partial^{c_1+\dots+c_d} \bar{u}}{\partial x_1^{c_1} \dots \partial x_d^{c_d}} \frac{\partial^{e_1+\dots+e_d} v}{\partial x_1^{e_1} \dots \partial x_d^{e_d}}.
\end{aligned}$$

The same argument as the one that lead to (A.2) implies

$$\frac{\partial^s m_2(g(x))}{\partial t_1 \dots \partial t_s} = \sum_{P \in P_s} \sum_{\lambda \in P_3} \left\{ \left(\prod_{B \in P} \frac{\partial}{\partial g_{\lambda(B)}} \right) m_2 \right\} \left\{ \prod_{B \in P} \left[\left(\prod_{b \in B} \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right] \right\}.$$

For fixed multinomial coefficients $\binom{a_i}{b_i, c_i, e_i}$ ($i = 1, \dots, d$), $P \in P_s$, and $\lambda \in P_3$, we shall find a bound on the term

$$\left\{ \left(\prod_{B \in P} \frac{\partial}{\partial g_{\lambda(B)}} \right) m_2 \right\} \left\{ \prod_{B \in P} \left[\left(\prod_{b \in B} \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right] \right\} \frac{\partial^{c_1+\dots+c_d} \bar{u}}{\partial x_1^{c_1} \dots \partial x_d^{c_d}} \frac{\partial^{e_1+\dots+e_d} v}{\partial x_1^{e_1} \dots \partial x_d^{e_d}}$$

in both $L^2(\mathbb{R}^d)$ and $L_\alpha^2(\mathbb{R}) \otimes L^2(\mathbb{R}^{d-1})$. In order to do that we consider the following cases.

Case 2.1. If $|P|, c_1 + \dots + c_d, e_1 + \dots + e_d > 0$, we denote $l = |P| + 2$ ($2 < l \leq k$) and $P = \{B_1, \dots, B_{l-2}\}$, and use Lemmas A.1 and A.2 with

$$(n_1, \dots, n_l) = \{|B_1|, \dots, |B_{l-2}|, c_1 + \dots + c_d, e_1 + \dots + e_d\}$$

and $\frac{1}{p_i} = (\frac{1}{2} - \frac{k}{d})\frac{1}{l} + \frac{n_i}{d}$ for $i = 1, 2, \dots, l$, to obtain the inequality

(A.6)

$$\begin{aligned} & \left\| \left\{ \left(\prod_{B \in P} \frac{\partial}{\partial g_{\lambda(B)}} \right) m_2 \right\} \left\{ \prod_{B \in P} \left[\left(\prod_{b \in B} \frac{\partial}{\partial t_b} \right) y_{\lambda(B)} \right] \right\} \frac{\partial^{c_1 + \dots + c_d} \bar{u}}{\partial x_1^{c_1} \dots \partial x_d^{c_d}} \frac{\partial^{e_1 + \dots + e_d} v}{\partial x_1^{e_1} \dots \partial x_d^{e_d}} \right\|_{L^2} \\ & \leq \|m_2\|_{C^{l-2}(\mathbb{R}^3)} \left\| \frac{\partial^{c_1 + \dots + c_d} \bar{u}}{\partial x_1^{c_1} \dots \partial x_d^{c_d}} \right\|_{L^{p_{l-1}}} \left\| \frac{\partial^{e_1 + \dots + e_d} v}{\partial x_1^{e_1} \dots \partial x_d^{e_d}} \right\|_{L^{p_l}} \prod_{i=1}^{l-2} \left\| \left(\prod_{b \in B_i} \frac{\partial}{\partial t_b} \right) g_{\lambda(B_i)} \right\|_{L^{p_i}} \\ & \leq \|m\|_{C^{l-1}(\mathbb{R}^2)} \|g_{\lambda(B_1)}\|_{W^{n_1, p_1}} \dots \|g_{\lambda(B_{l-2})}\|_{W^{n_{l-2}, p_{l-2}}} \|\bar{u}\|_{W^{n_{l-1}, p_{l-1}}} \|v\|_{W^{n_l, p_l}} \\ & \leq C \|m\|_{C^{l-1}(\mathbb{R}^2)} \|g_{\lambda(B_1)}\|_{H^k} \dots \|g_{\lambda(B_{l-2})}\|_{H^k} \|\bar{u}\|_{H^k} \|v\|_{H^k}. \end{aligned}$$

When not all $|P|, c_1 + \dots + c_d, e_1 + \dots + e_d$ are positive, we have the following cases.

Case 2.2. If $|P| = 0$ and $c_1 + \dots + c_d, e_1 + \dots + e_d > 0$, we apply Lemmas A.1 and A.2 with $l = 2$, $(n_1, n_2) = (c_1 + \dots + c_d, e_1 + \dots + e_d)$ and $\frac{1}{p_i} = (\frac{1}{2} - \frac{k}{d})\frac{1}{l} + \frac{n_i}{d}$ so that $H^k(\mathbb{R}^d) \hookrightarrow W^{n_i, p_i}(\mathbb{R}^d)$ for $i = 1, 2$, and obtain

$$\begin{aligned} & \left\| m_2 \frac{\partial^{c_1 + \dots + c_d} \bar{u}}{\partial x_1^{c_1} \dots \partial x_d^{c_d}} \frac{\partial^{e_1 + \dots + e_d} v}{\partial x_1^{e_1} \dots \partial x_d^{e_d}} \right\|_{L^2} \\ & \leq \|m\|_{C^1(\mathbb{R}^2)} \left\| \frac{\partial^{c_1 + \dots + c_d} \bar{u}}{\partial x_1^{c_1} \dots \partial x_d^{c_d}} \right\|_{L^{p_1}} \left\| \frac{\partial^{e_1 + \dots + e_d} v}{\partial x_1^{e_1} \dots \partial x_d^{e_d}} \right\|_{L^{p_2}} \\ & \leq \|m\|_{C^1(\mathbb{R}^2)} \|\bar{u}\|_{W^{n_1, p_1}} \|v\|_{W^{n_2, p_2}} \leq \|m\|_{C^1(\mathbb{R}^2)} \|\bar{u}\|_{H^k} \|v\|_{H^k}. \end{aligned}$$

Case 2.3. If $e_1 + \dots + e_d = 0$, and $|P|, c_1 + \dots + c_d > 0$, let $l = |P| + 1$ ($2 \leq l \leq k$) and $P = \{B_1, \dots, B_{l-1}\}$; then from Lemmas A.1 and A.2 with $(n_1, \dots, n_l) = (|B_1|, \dots, |B_{l-1}|, c_1 + \dots + c_d)$ and $\frac{1}{p_i} = (\frac{1}{2} - \frac{k}{d})\frac{1}{l} + \frac{n_i}{d}$, $i = 1, \dots, l$, and the Sobolev embedding $H^k(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$, we obtain

$$\begin{aligned} & \left\| \left\{ \left(\prod_{B \in P} \frac{\partial}{\partial g_{\lambda(B)}} \right) m_2 \right\} \left\{ \prod_{B \in P} \left[\left(\prod_{b \in B} \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right] \right\} \left(\frac{\partial^{c_1 + \dots + c_d} \bar{u}}{\partial x_1^{c_1} \dots \partial x_d^{c_d}} \right) v \right\|_{L^2} \\ & \leq \|m_2\|_{C^{l-1}(\mathbb{R}^3)} \|v\|_{L^\infty} \left\| \left\{ \prod_{B \in P} \left[\left(\prod_{b \in B} \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right] \right\} \frac{\partial^{c_1 + \dots + c_d} \bar{u}}{\partial x_1^{c_1} \dots \partial x_d^{c_d}} \right\|_{L^2} \\ & \leq C \|m_2\|_{C^{l-1}(\mathbb{R}^3)} \|v\|_{H^k} \prod_{i=1}^{|P|} \left\| \left(\prod_{b \in B_i} \frac{\partial}{\partial t_b} \right) g_{\lambda(B_i)} \right\|_{L^{p_i}} \left\| \frac{\partial^{c_1 + \dots + c_d} \bar{u}}{\partial x_1^{c_1} \dots \partial x_d^{c_d}} \right\|_{L^{p_l}} \\ & \leq C \|m_2\|_{C^{l-1}(\mathbb{R}^3)} \|\bar{u}\|_{W^{n_l, p_l}} \|v\|_{H^k} \prod_{i=1}^{|P|} \|g_{\lambda(B_i)}\|_{W^{n_i, p_i}} \\ & \leq C \|m\|_{C^l(\mathbb{R}^2)} \|v\|_{H^k} \prod_{i=1}^{|P|} \|g_{\lambda(B_i)}\|_{H^k} \|\bar{u}\|_{H^k}. \end{aligned}$$

Similarly, if $c_1 + \dots + c_d = 0$ and $|P|, e_1 + \dots + e_d > 0$, we have

$$\begin{aligned}
& \left\| \left\{ \left(\prod_{B \in P} \frac{\partial}{\partial g_{\lambda(B)}} \right) m_2 \right\} \left\{ \prod_{B \in P} \left[\left(\prod_{b \in B} \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right] \right\} \left(\frac{\partial^{e_1 + \dots + e_d}}{\partial x_1^{e_1} \dots \partial x_d^{e_d}} v \right) \bar{u} \right\|_{L^2} \\
& \leq \|m_2\|_{C^{l-1}(\mathbb{R}^3)} \|\bar{u}\|_{L^\infty} \left\| \left\{ \prod_{B \in P} \left[\left(\prod_{b \in B} \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right] \right\} \frac{\partial^{e_1 + \dots + e_d}}{\partial x_1^{e_1} \dots \partial x_d^{e_d}} v \right\|_{L^2} \\
& \leq C \|m_2\|_{C^{l-1}(\mathbb{R}^3)} \|\bar{u}\|_{H^k} \left\| \frac{\partial^{e_1 + \dots + e_d} v}{\partial x_1^{e_1} \dots \partial x_d^{e_d}} \right\|_{L^{p_l}} \prod_{i=1}^{l-1} \left\| \left(\prod_{b \in B_i} \frac{\partial}{\partial t_b} \right) g_{\lambda(B_i)} \right\|_{L^{p_i}} \\
& \leq C \|m_2\|_{C^{l-1}(\mathbb{R}^3)} \|\bar{u}\|_{H^k} \|v\|_{W^{n_l, p_l}} \prod_{i=1}^{l-1} \|g_{\lambda(B_i)}\|_{W^{n_i, p_i}} \\
& \leq C \|m\|_{C^l(\mathbb{R}^2)} \|\bar{u}\|_{H^k} \|v\|_{H^k} \prod_{i=1}^{l-1} \|g_{\lambda(B_i)}\|_{H^k}.
\end{aligned}$$

Case 2.4. If $|P| = e_1 + \dots + e_d = 0$ and $c_1 + \dots + c_d \neq 0$, we use the Sobolev embedding $H^k(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$, and we obtain

$$\begin{aligned}
& \left\| m_2(g_1, g_2, g_3) \left(\frac{\partial^{a_1 + \dots + a_d} \bar{u}}{\partial x_1^{a_1} \dots \partial x_d^{a_d}} v \right) \right\|_{L^2} \leq \|m_2\|_{L^\infty(\mathbb{R}^3)} \left\| \frac{\partial^{a_1 + \dots + a_d} \bar{u}}{\partial x_1^{a_1} \dots \partial x_d^{a_d}} \right\|_{L^2} \|v\|_{L^\infty} \\
& \leq C \|m\|_{C^1(\mathbb{R}^2)} \|\bar{u}\|_{H^k} \|v\|_{H^k},
\end{aligned}$$

or, for $|P| = c_1 + \dots + c_d = 0$ and $e_1 + \dots + e_d \neq 0$,

$$\begin{aligned}
& \left\| m_2(g_1, g_2, g_3) \bar{u} \frac{\partial^{a_1 + \dots + a_d} v}{\partial x_1^{a_1} \dots \partial x_d^{a_d}} \right\|_{L^2} \leq \|m_2\|_{L^\infty(\mathbb{R}^3)} \|\bar{u}\|_{L^\infty} \left\| \frac{\partial^{a_1 + \dots + a_d} v}{\partial x_1^{a_1} \dots \partial x_d^{a_d}} \right\|_{L^2} \\
& \leq C \|m\|_{C^1(\mathbb{R}^2)} \|\bar{u}\|_{H^k} \|v\|_{H^k}.
\end{aligned}$$

Case 2.5. If $c_1 + \dots + c_d = e_1 + \dots + e_d = 0$ and $|P| \neq 0$, we denote $l = |P|$ ($1 \leq l \leq k$) and $P = \{B_1, \dots, B_l\}$: when $l = 1$, $g_{\lambda(B)} = g_i$, $i = 1, 2, 3$. We then use the inequality

$$\left\| \frac{\partial^k g_i}{\partial x_1^{a_1} \dots \partial x_d^{a_d}} \right\|_{L^2} \leq \|g_i\|_{H^k},$$

Sobolev embedding $H^k(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$, and Lemma 7.1(1) to obtain for $i = 1, 2, 3$,

$$\begin{aligned}
& \left\| m_2^{(1)} \left(\frac{\partial^k}{\partial x_1^{a_1} \dots \partial x_d^{a_d}} g_i \right) \bar{u} v \right\|_{L^2} \\
& \leq \|m\|_{C^1(\mathbb{R}^2)} \|g_i\|_{H^k} \|\bar{u} v\|_{L^\infty} \leq \|m\|_{C^1(\mathbb{R}^2)} \|g_i\|_{H^k} \|\bar{u} v\|_{H^k} \\
& \leq \|m\|_{C^1(\mathbb{R}^2)} \|g_i\|_{H^k} \|\bar{u}\|_{H^k} \|v\|_{H^k}.
\end{aligned}$$

When $l \geq 2$, we use the Sobolev embedding $H^k(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, Lemma A.1, and Lemma A.2 with $(n_1, \dots, n_l) = (|B_1|, \dots, |B_l|)$ and $\frac{1}{p_i} = (\frac{1}{2} - \frac{k}{d}) \frac{1}{l} + \frac{n_i}{d}$ for $i = 1, \dots, l$ and Lemma 7.1(1) to obtain

$$\begin{aligned}
(A.7) \quad & \left\| \left\{ \left(\prod_{B \in P} \frac{\partial}{\partial g_{\lambda(B)}} \right) m_2 \right\} \left\{ \prod_{B \in P} \left[\left(\prod_{b \in B} \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right] \right\} \bar{u} v \right\|_{L^2} \\
& \leq \|m_2\|_{C^l(\mathbb{R}^3)} \left\| \left\{ \prod_{B \in P} \left[\left(\prod_{b \in B} \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right] \right\} \right\|_{L^2} \|\bar{u} v\|_{L^\infty}
\end{aligned}$$

$$\begin{aligned}
&\leq C \|m_2\|_{C^l(\mathbb{R}^3)} \|\bar{u}v\|_{H^k} \prod_{i=1}^l \left\| \left(\prod_{b \in B_i} \frac{\partial}{\partial t_b} \right) g_{\lambda(B_i)} \right\|_{L^{p_i}} \\
&\leq C \|m_2\|_{C^l(\mathbb{R}^3)} \|\bar{u}\|_{H^k} \|v\|_{H^k} \prod_{i=1}^l \|g_{\lambda(B_i)}\|_{W^{n_i, p_i}} \\
&\leq C \|m\|_{C^{l+1}(\mathbb{R}^2)} \|\bar{u}\|_{H^k} \|v\|_{H^k} \prod_{i=1}^l \|g_{\lambda(B_i)}\|_{H^k}.
\end{aligned}$$

Combining above inequalities (A.6)–(A.7), we can conclude that

$$\begin{aligned}
&\left\| \left\{ \left(\prod_{B \in P} \frac{\partial}{\partial g_{\lambda(B)}} \right) m_2 \right\} \left\{ \prod_{B \in P} \left[\left(\prod_{b \in B} \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right] \right\} \frac{\partial^{c_1+\dots+c_d} \bar{u}}{\partial x_1^{c_1} \dots \partial x_d^{c_d}} \frac{\partial^{e_1+\dots+e_d} v}{\partial x_1^{e_1} \dots \partial x_d^{e_d}} \right\|_{L^2} \\
&\leq \|m\|_{C^{k+1}(\mathbb{R}^2)} \|\bar{u}\|_{H^k} \|v\|_{H^k} \prod_{i=1}^{|P|} \|g_{\lambda(B_i)}\|_{H^k}.
\end{aligned}$$

Similarly, we let $l = |P| + 2$ ($2 < l \leq k$), $P = \{B_1, \dots, B_{l-2}\}$ and $\gamma_\alpha = e^{\alpha x_1}$, and use Lemmas A.1 and A.2 with $(n_1, \dots, n_l) = \{|B_1|, \dots, |B_{l-2}|, c_1 + \dots + c_d, e_1 + \dots + e_d\}$ and $\frac{1}{p_i} = (\frac{1}{2} - \frac{k}{d})\frac{1}{l} + \frac{n_i}{d}$ for $i = 1, 2, \dots, l$, to obtain

$$\begin{aligned}
&\left\| \gamma_\alpha m_2^{(|P|)} \left\{ \prod_{B \in P} \left[\left(\prod_{b \in B} \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right] \right\} \frac{\partial^{c_1+\dots+c_d} \bar{u}}{\partial x_1^{c_1} \dots \partial x_d^{c_d}} \frac{\partial^{e_1+\dots+e_d} v}{\partial x_1^{e_1} \dots \partial x_d^{e_d}} \right\|_{L^2} \\
&\leq C \|m\|_{C^{k+1}(\mathbb{R}^2)} \|g_{\lambda(B_1)}\|_{H^k} \dots \|g_{\lambda(B_{l-2})}\|_{H^k} \|\bar{u}\|_{H^k} \|\gamma_\alpha v\|_{H^k}.
\end{aligned}$$

The cases when $|P|, c_1 + \dots + c_d = 0$ or $e_1 + \dots + e_d = 0$ can be treated analogously.

Finally, for variations in v , we write $m(q, u)(v + \bar{v}) - m(q, u)v = m(q, u)\bar{v}$, and, therefore,

$$\begin{aligned}
&\|m(q, u)\bar{v}\|_{L^2} \leq \|m\|_{L^\infty} \|\bar{v}\|_{L^2}, \\
&\|\gamma_\alpha m(q, u)\bar{v}\|_{L^2} \leq \|m\|_{L^\infty} \|\gamma_\alpha \bar{v}\|_{L^2}.
\end{aligned}$$

By the general Leibniz rule,

$$\frac{\partial^k}{\partial x_1^{a_1} \dots \partial x_d^{a_d}} (m(q, u)\bar{v}) = \sum_{b_d=0}^{a_d} \binom{a_d}{b_d} \dots \sum_{b_1=0}^{a_1} \binom{a_1}{b_1} \frac{\partial^{b_1+\dots+b_d} m}{\partial x_1^{b_1} \dots \partial x_d^{b_d}} \frac{\partial^{a_1-b_1+\dots+a_d-b_d} \bar{v}}{\partial x_1^{a_1-b_1} \dots \partial x_d^{a_d-b_d}},$$

where $\binom{a_j}{b_j} = \frac{a_j!}{b_j!(a_j-b_j)!}$. Let $s = b_1 + \dots + b_d$, $(g_1(x), g_2(x)) = (q(x), u(x))$. We use the higher chain formula, for each i in the set J_s of integers $1, 2, \dots, s$. Let again t_i denote one of the independent variables $x_1, \dots, x_d$. We consider a partition of J_s . The set of all block functions from a partition P of J_s into J_2 is P_2 and P_s is the set of all partitions of J_s . Then

$$\frac{\partial^s m(q, u)}{\partial t_1 \dots \partial t_s} = \sum_{P \in P_s} \sum_{\lambda \in P_2} \left\{ \left(\prod_{B \in P} \frac{\partial}{\partial g_{\lambda(B)}} \right) m \right\} \left\{ \prod_{B \in P} \left[\left(\prod_{b \in B} \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right] \right\}.$$

Thus, for fixed binomial coefficients $\binom{a_j}{b_j}$, $j = 1, \dots, d$, for fixed partition $P \in P_s$ and block function $\lambda \in P_2$ we need to estimate

$$\left\{ \left(\prod_{B \in P} \frac{\partial}{\partial g_{\lambda(B)}} \right) m \right\} \left\{ \prod_{B \in P} \left[\left(\prod_{b \in B} \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right] \right\} \frac{\partial^{a_1-b_1+\dots+a_d-b_d} \bar{v}}{\partial x_1^{a_1-b_1} \dots \partial x_d^{a_d-b_d}}$$

in $L^2(\mathbb{R}^d)$ and $L_\alpha^2(\mathbb{R}) \otimes L^2(\mathbb{R}^{d-1})$ -norms. To do so we consider the following cases.

Case 3.1. If $b_1 + \dots + b_d = 0$, then

$$\begin{aligned} & \left\| \left\{ \left(\prod_{B \in P} \frac{\partial}{\partial g_{\lambda(B)}} \right) m \right\} \left\{ \prod_{B \in P} \left[\left(\prod_{b \in B} \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right] \right\} \frac{\partial^{a_1-b_1+\dots+a_d-b_d}}{\partial x_1^{a_1-b_1} \dots \partial x_d^{a_d-b_d}} \bar{v} \right\|_{L^2} \\ &= \left\| m(q, u) \frac{\partial^{a_1+\dots+a_d}}{\partial x_1^{a_1} \dots \partial x_d^{a_d}} \bar{v} \right\|_{L^2} \leq \|m\|_{L^\infty(\mathbb{R}^2)} \left\| \frac{\partial^{a_1+\dots+a_d}}{\partial x_1^{a_1} \dots \partial x_d^{a_d}} \bar{v} \right\|_{L^2} \leq \|m\|_{L^\infty(\mathbb{R}^2)} \|\bar{v}\|_{H^k}. \end{aligned}$$

Case 3.2. If $0 < b_1 + \dots + b_d < a_1 + \dots + a_d$, let $P = \{B_1, B_2, \dots, B_{l-1}\}$, and

$$(n_1, \dots, n_l) = (|B_1|, \dots, |B_{l-1}|, a_1 - b_1 + \dots + a_d - b_d),$$

then we use Lemmas A.1 and A.2 with $\frac{1}{p_i} = (\frac{1}{2} - \frac{k}{d})\frac{1}{l} + \frac{n_i}{d}$, $i = 1, \dots, l$, and obtain

$$\begin{aligned} & \left\| \left\{ \left(\prod_{B \in P} \frac{\partial}{\partial g_{\lambda(B)}} \right) m \right\} \left\{ \prod_{B \in P} \left[\left(\prod_{b \in B} \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right] \right\} \frac{\partial^{a_1-b_1+\dots+a_d-b_d}}{\partial x_1^{a_1-b_1} \dots \partial x_d^{a_d-b_d}} \bar{v} \right\|_{L^2} \\ & \leq \|m^{(l-1)}\|_{L^\infty(\mathbb{R}^2)} \left\| \left\{ \prod_{B \in P} \left[\left(\prod_{b \in B} \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right] \right\} \frac{\partial^{a_1-b_1+\dots+a_d-b_d}}{\partial x_1^{a_1-b_1} \dots \partial x_d^{a_d-b_d}} \bar{v} \right\|_{L^2} \\ & \leq \|m\|_{C^{l-1}(\mathbb{R}^2)} \left\| \frac{\partial^{a_1-b_1+\dots+a_d-b_d} \bar{v}}{\partial x_1^{a_1-b_1} \dots \partial x_d^{a_d-b_d}} \right\|_{L^{p_l}} \prod_{i=1}^{l-1} \left\| \left(\prod_{b \in B_i} \frac{\partial}{\partial t_b} \right) g_{\lambda(B_i)} \right\|_{L^{p_i}} \\ & \leq \|m\|_{C^{l-1}(\mathbb{R}^2)} \|\bar{v}\|_{W^{n_l, p_l}} \prod_{i=1}^{l-1} \|g_{\lambda(B_i)}\|_{W^{n_i, p_i}} \\ & \leq \|m\|_{C^{l-1}(\mathbb{R}^2)} \prod_{i=1}^{l-1} \|g_{\lambda(B_i)}\|_{H^k} \|\bar{v}\|_{H^k}. \end{aligned}$$

Case 3.3. If $b_1 + \dots + b_d = a_1 + \dots + a_d$, when $|P| = 1$, $g_{\lambda(B)} = g_i$, $i = 1, 2$, or 3, we use Sobolev embedding $H^k(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ and obtain

$$\begin{aligned} & \left\| \left\{ \left(\prod_{B \in P} \frac{\partial}{\partial g_{\lambda(B)}} \right) m \right\} \left\{ \prod_{B \in P} \left[\left(\prod_{b \in B} \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right] \right\} \frac{\partial^{a_1-b_1+\dots+a_d-b_d} \bar{v}}{\partial x_1^{a_1-b_1} \dots \partial x_d^{a_d-b_d}} \right\|_{L^2} \\ & \leq \|m^{(1)}\|_{L^\infty(\mathbb{R}^2)} \|\bar{v}\|_{L^\infty} \left\| \frac{\partial^{a_1+\dots+a_d} g_i}{\partial x_1^{a_1} \dots \partial x_d^{a_d}} \right\|_{L^2} \leq \|m\|_{C^1(\mathbb{R}^2)} \|\bar{v}\|_{H^k} \|g_i\|_{H^k}. \end{aligned}$$

When $|P| = l > 1$, let $P = \{B_1, B_2, \dots, B_l\}$ and $(n_1, \dots, n_l) = (|B_1|, \dots, |B_l|)$. Lemmas A.1 and A.2 with $\frac{1}{p_i} = (\frac{1}{2} - \frac{k}{d})\frac{1}{l} + \frac{n_i}{d}$, $i = 1, \dots, l$, and the Sobolev embedding $H^k(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ imply

$$\begin{aligned} & \left\| \left\{ \left(\prod_{B \in P} \frac{\partial}{\partial g_{\lambda(B)}} \right) m \right\} \left\{ \prod_{B \in P} \left[\left(\prod_{b \in B} \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right] \right\} \frac{\partial^{a_1-b_1+\dots+a_d-b_d}}{\partial x_1^{a_1-b_1} \dots \partial x_d^{a_d-b_d}} \bar{v} \right\|_{L^2} \\ & \leq \|m^{(l)}\|_{L^\infty(\mathbb{R}^2)} \|\bar{v}\|_{L^\infty} \left\| \left\{ \prod_{B \in P} \left[\left(\prod_{b \in B} \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right] \right\} \right\|_{L^2} \\ & \leq \|m\|_{C^l(\mathbb{R}^2)} \|\bar{v}\|_{H^k} \prod_{i=1}^l \left\| \left(\prod_{b \in B_i} \frac{\partial}{\partial t_b} \right) g_{\lambda(B_i)} \right\|_{L^{p_i}} \leq \|m\|_{C^l(\mathbb{R}^2)} \|\bar{v}\|_{H^k} \prod_{i=1}^l \|g_{\lambda(B_i)}\|_{W^{n_i, p_i}} \\ & \leq \|m\|_{C^l(\mathbb{R}^2)} \|\bar{v}\|_{H^k} \prod_{i=1}^l \|g_{\lambda(B_i)}\|_{H^k}. \end{aligned}$$

Similarly, let $l = |P| + 1$, $P = \{B_1, \dots, B_{l-1}\}$, $(n_1, \dots, n_l) = (|B_1|, \dots, |B_{l-1}|, a_1 - b_1 + \dots + a_d - b_d)$ and $\gamma_\alpha = e^{\alpha x_1}$, then

$$\begin{aligned} & \left\| \gamma_\alpha \left\{ \left(\prod_{B \in P} \frac{\partial}{\partial g_{\lambda(B)}} \right) m \right\} \left\{ \prod_{B \in P} \left[\left(\prod_{b \in B} \frac{\partial}{\partial t_b} \right) g_{\lambda(B)} \right] \right\} \frac{\partial^{a_1 - b_1 + \dots + a_d - b_d} \bar{v}}{\partial x_1^{a_1 - b_1} \dots \partial x_d^{a_d - b_d}} \right\|_{L^2} \\ & \leq \|m^{(|P|)}\|_{L^\infty(\mathbb{R}^2)} \left\| \gamma_\alpha \frac{\partial^{a_1 - b_1 + \dots + a_d - b_d} \bar{v}}{\partial x_1^{a_1 - b_1} \dots \partial x_d^{a_d - b_d}} \right\|_{L^{p_l}} \prod_{i=1}^{l-1} \left\| \left(\prod_{b \in B_i} \frac{\partial}{\partial t_b} \right) g_{\lambda(B_i)} \right\|_{L^{p_i}} \\ & \leq \|m\|_{C^{l-1}(\mathbb{R}^2)} \|\gamma_\alpha \bar{v}\|_{W^{n_l, p_l}} \prod_{i=1}^{l-1} \|g_{\lambda(B_i)}\|_{W^{n_i, p_i}} \leq \|m\|_{C^{l-1}(\mathbb{R}^2)} \|\gamma_\alpha \bar{v}\|_{H^k} \prod_{i=1}^{l-1} \|g_{\lambda(B_i)}\|_{H^k}. \end{aligned}$$

The case of weighted norm when $b_1 + \dots + b_d = 0$, or $b_1 + \dots + b_d = a_1 + \dots + a_d$ can be considered similarly.

Using Lipschitz estimates for variations in q , u , and v , one can easily show that the mappings are locally Lipschitz on the given sets in $H^k(\mathbb{R}^d)$ and in $H_\alpha^k(\mathbb{R}^d)$; therefore the mappings are also locally Lipschitz on the given sets $\mathcal{H} = H^k(\mathbb{R}^d) \cap H_\alpha^k(\mathbb{R}^d)$. $\square$

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