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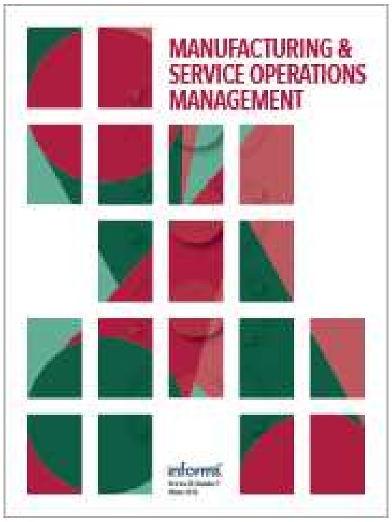

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Dynamic Pricing of Wireless Internet Based on Usage and Stochastically Changing Capacity

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Abstract. *Problem definition*: Inspired by new developments in dynamic spectrum access, we study the dynamic pricing of wireless Internet access when demand and capacity (bandwidth) are stochastic. *Academic/practical relevance*: The demand for wireless Internet access has increased enormously. However, the spectrum available to wireless service providers is limited. The industry has, thus, altered conventional license-based spectrum access policies through unlicensed spectrum operations. The additional spectrum obtained through these operations has stochastic capacity. Thus, the pricing of this service by the service provider has novel challenges. The problem considered in this paper is, therefore, of high practical relevance and new to the academic literature. *Methodology*: We study this pricing problem using a Markov decision process model in which customers are posted dynamic prices based on their bandwidth requirement and the available capacity. *Results*: We characterize the structure of the optimal pricing policy as a function of the system state and of the input parameters. Because it is impossible to solve this problem for practically large state spaces, we propose a heuristic dynamic pricing policy that performs very well, particularly when the ratio of capacity to demand rate is low. *Managerial implications*: We demonstrate the value of using a dynamic heuristic pricing policy compared with the myopic and optimal static policies. The previous literature has studied similar systems with fixed capacity and has characterized conditions under which myopic policies perform well. In contrast, our setting has dynamic (stochastic) capacity, and we find that identifying good state-dependent heuristic pricing policies is of greater importance. Our heuristic policy is computationally more tractable and easier to implement than the optimal dynamic and static pricing policies. It also provides a significant performance improvement relative to the myopic and optimal static policies when capacity is scarce, a condition that holds for the practical setting that motivated this research.

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Keywords: dynamic pricing • wireless Internet • Markov decision process • stochastic capacity

1. Introduction

Wireless networks have revolutionized public life, especially following the invention of smartphones in 2007. It is estimated that by 2020, mobile data traffic will grow by a factor of 10 with a thousandfold increase in demand in certain urban areas (Beltran et al. 2016), and the number of connected mobile devices is expected to reach 11.6 billion (Cisco Visual Networking Index 2016). These increased demands place a significant burden on mobile service providers (SPs) because of the scarcity of available wireless spectrum.

Today's wireless networks are characterized by a *fixed spectrum assignment policy*, with which the resource, that is, the spectrum, is allocated to users, such as government and broadcasters, in a *license-based manner* according to national policies, technical characteristics, and international agreements. Although SPs try to utilize their share, a large portion of the assigned spectrum is used sporadically, and geographical variations in the utilization of assigned spectrum range from 15% to 85% (Federal Communications Commission (FCC) 2002). This leads to the *spectrum crisis*, in which access policies to spectrum are questioned; see, for example, Chen (2012).

The wireless industry has been altering conventional license-based spectrum access policies through unlicensed spectrum operational approaches. The most prominent solution has been WiFi, which exclusively operates on unlicensed spectrum (e.g., 2.4 GHz, 5 GHz). More recently, the spectrum crisis has forced government agencies to adopt unlicensed spectrum operational approaches for other purposes. This leads to *dynamic spectrum access* (DSA), in which resources change dynamically with respect to the usage of the shared, unlicensed spectrum by several users. Examples of unlicensed spectrum operation include the following.

1.1. TV White Space

TV channels have traditionally been allocated to broadcast companies throughout the nation. Because of population density dynamics, not all TV channels are used in all locations; for example, although broadcast companies use all TV channels in locations such as Manhattan, few channels are used in less populated cities or rural areas. This leaves unused TV channels in certain areas, referred to as *TV white spaces*, which can be repurposed for other wireless communication. In 2008, the FCC allowed unlicensed devices, that is, television band devices (TVBDs), to operate in TV white spaces in the United States (FCC 2010). TVBDs operate on available TV bands based on information received from TV spectrum databases or local spectrum sensing operations. Sites such as Google Spectrum and Microsoft Whitespaces have established nationwide spectrum databases that dynamically allocate spectrum for unlicensed usage.

1.2. TV Black Space

Although suitable for rural areas, the FCC's current approach leaves populated urban areas, in which the spectrum crisis is most significant, with few TV white spaces (Harrison et al. 2010). For example, four out of the five largest cities by population have less than four TV white space channels. Thus, recent research studies have focused on finer-grade DSA schemes. Accordingly, an unlicensed user can access a TV channel if none of the TV receivers in its immediate vicinity is watching that channel. Hence, the spectrum can be accessed based on dynamic behaviors of licensed users, that is, TV viewers (Zhao et al. 2014, Zhang and Knightly 2015). Zhao et al. (2014) has shown that spectrum access based on TV viewership leads to up to a fivefold increase in available capacity in New York.

1.3. 3.5-GHz Citizens Broadband Radio Service (CBRS)

In 2015, the FCC adopted a spectrum-sharing policy to make 150 MHz of spectrum available for mobile usage in the 3.5-GHz band previously allocated to the U.S. Department of Defense (FCC 2015), using a three-tiered sharing mechanism: At the top tier, incumbent users, for example, federal operations and satellite services, have exclusive rights to the spectrum in their vicinity. The priority access tier allows use of part of the spectrum in a census track for a limited amount of time. The general authorized access tier may use the remaining spectrum at no cost.

1.4. Long-Term Evolution Unlicensed (LTE-U)

Initially spearheaded by the LTE-U Forum, SPs have begun standardizing 4G network operation on the unlicensed 5-GHz band to provide cellular networks access to additional unlicensed spectrum by sharing the spectrum with WiFi networks and other LTE operators. LTE-U would allow 4G networks to utilize the 5-GHz WiFi band when network demand exceeds available licensed resources (Zhang et al. 2015). LTE-U tries to utilize unused 5-GHz channels based on WiFi and LTE measurements. This results in three types of channels: (i) if a channel is unused by WiFi, the full capacity of that channel may be available to the SP for a certain duration; (ii) if a channel is unused by WiFi, the capacity may be shared by two or more SPs; (iii) if a channel is used by WiFi, LTE-U shares the channel with the WiFi network (Bhattarai et al. 2016).

These emerging DSA solutions result in the following challenges to network operators:

- *Nonexclusive use:* DSA approaches are characterized by the operator's use of unlicensed channels, which, by definition, are not exclusively reserved. As a result, unlicensed use of the spectrum provides capacity to the operator in a nondeterministic fashion.
- *Dynamic channel availability:* The availability of an unlicensed channel depends on factors that cannot be controlled by the network operator. For example, in TV white space, TV black space, and CBRS, the number of available channels changes with time depending on the behaviors of licensed users (i.e., TV viewers or federal operations).
- *Varying channel capacity:* In DSA solutions, two available channels may not provide the same capacity. For example, in LTE-U, if an unused 5-GHz channel is found, the operator can utilize the maximum capacity provided by that channel. However, in some cases, the channel capacity may need to be shared with a WiFi network, leading to a lower and dynamically changing available capacity. As a result, different types of *channel classes* exist because of their available capacity as well as the amount of time they are available to the network operator.

These three factors imply that the capacity of the wireless network varies dynamically. In addition, it is highly likely that the capacity will not be sufficient to provide a good quality Internet service to users at the peak hours of demand, especially in urban areas. To be able to manage the peak-hour demand, we propose to use a dynamic pricing scheme in which the fees charged for an Internet connection change in real time.

To study this problem, we consider a network in which an SP monitors the availability of unlicensed channels and posts prices to the incoming customers based on the available bandwidth. If an arriving customer is willing to accept the posted price, the customer starts the Internet connection; otherwise, the customer departs from the system. We consider a system with multiple classes of customers, in which classes have different bandwidth requirements based on the type of activity to be performed, for example, video streaming versus web surfing versus checking email. The connection fee paid by an arriving customer depends on the posted price at the time of arrival and the connection duration. A key feature of this setting is that the available bandwidth is a stochastic process because unlicensed channels become available for Internet use if they are not actively used by licensed users, and they become unavailable when licensed users start using the channel. This implies that the available bandwidth may decrease at random, and thus, customers that are in service may need to be dropped by the SP. A key challenge in managing the network is to maximize the revenue earned from admitted customers while limiting the cost of dropped customers.

For this setting, we develop an optimal dynamic pricing strategy to maximize the revenue earned by an SP using DSA policies on unlicensed bands while considering the limited capacity as well as the potential losses associated with dropped customers. We model the system as a Markov decision process (MDP) in which customers arrive as a stochastic process with stochastic service times, and unlicensed channels become available (providing bandwidth) as a stochastic process with stochastic times until the channel becomes unavailable. Thus, the state of the system includes both the number of customers of each class being served and the number of channels of each class available for use by the system. Because of the high-dimension state, the optimal policy is difficult to compute and impractical to implement. Therefore, we develop several heuristic policies in which the customer classes and/or channel classes are aggregated. We use a comprehensive set of numerical experiments to demonstrate that the proposed heuristic performs well with respect to the optimal policy. We also compare the performance of the proposed heuristic, which is dynamic (i.e., state-dependent), with non-state-dependent (i.e., static) pricing policies. We find that the proposed heuristic performs best, particularly when the ratio of capacity to demand rate is low, a condition that holds for the practical setting that motivated this research.

2. Literature Review

The research in this paper has some similarity to the previous literature in two areas: (i) dynamic pricing in MDP models and (ii) admission control in systems with multiple classes of demand and stochastic service rates. A key distinguishing feature of our model, compared with this previous work, is that, in addition to having stochastic interarrival and service times, our model also has stochastic capacity availability.

The most closely related previous work is that of Gans and Savin (2007), henceforth GS, who focus on dynamic pricing of rental assets (e.g., cars) when there is stochastic demand from two classes of customers, that is, contract customers (who may be admitted at a prenegotiated price or rejected) and walk-in customers (who are quoted dynamic prices based on the number of customers already in the system). GS focus on characterizing the conditions under which myopic pricing policies are optimal for certain customer classes. Although some of our analysis builds on GS, there are a number of key differences:

- In GS, all customers have the same capacity requirement, which is normalized to one unit (e.g., every customer wants to rent a single car), and the total available capacity (e.g., rental car fleet size) is fixed and known. In contrast, in our model, the capacity requirement varies by customer class but is exogenously specified, and the total available capacity evolves as a stochastic process as channels become available or unavailable.
- In our model, capacity may decrease, potentially requiring the SP to drop customers from service at a cost. Thus, we must determine a *drop-off policy*, which specifies the set of customers to be dropped in a given state, in addition to a dynamic pricing policy.
- GS make admission decisions for contract customers and pricing decisions for walk-in customers, and the latter is based on the class of the arriving customer and the current number of customers of each class in the system. In our model, there are no contract customers, and our dynamic pricing decision is based on the class of the arriving customer, the number of customers of each class in the system, and the current available capacity.
- GS focus on characterizing conditions under which preferred customer classes exist and for which myopic policies perform optimally or close to optimally. In contrast, we focus on developing heuristic policies that are not myopic, that is, which are state dependent.

Other research considers dynamic pricing and/or admission control for network services in which an SP has available capacity that may be used to satisfy the demands of customers who arrive stochastically and have class-dependent capacity requirements. This previous work takes the SP's total available capacity as fixed and known. For example, Paschalidis and Tsitsiklis (2000) consider dynamic pricing policies in such settings. They study the case in which the available capacity is large and there are many users with very small capacity requirements. They demonstrate that a static pricing policy is optimal in the limit. Mutlu et al. (2009) consider a setting with fixed capacity and two customer classes (primary and secondary users), in which each class has the same service rate and each customer uses one unit of capacity. The SP makes pricing decisions for arriving secondary users, and the price depends on the number of customers in the system. Other literature on network services with stochastic arrivals considers admission control with exogenously specified prices (Zhao et al. 2010, Turhan et al. 2012, Yahav et al. 2013).

Another class of relevant research considers queuing models of admission control problems in loss systems with multiple classes of demand. In much of this work, the resource or capacity requirement (e.g., number of servers) is the same (and generally normalized to one) for all customer classes; see, for example, Ulukus et al. (2011), Turhan et al. (2012), and Carrizosa et al. (1998). Because our model may require the dropping of customers who have been admitted to the system, the most relevant papers are those that consider preemption or termination of a customer, including Turhan et al. (2012) and Ulukus et al. (2011). However, in contrast to these papers, our model allows the bandwidth requirement per customer to vary by customer class. Thus, it is most similar to the literature on stochastic knapsacks, in which there are a fixed number of servers with the number of servers required per customer varying by customer class; see, for example, Altman et al. (2001) and Ross and Tsang (1989). Our model also has similarities to the literature on loss systems with batch arrivals, including Örmeci and Burnetas (2004). However, all of these referenced works assume that the total available capacity (number of servers) is fixed and known. Further, although these papers consider the admission control problem, we manage admission and maximize expected discounted profit by setting the optimal price for each arriving customer.

There is also recent literature that considers service systems that use self-scheduling servers, for example, drivers in ride-sharing companies or agents in work-from-home call centers. These self-scheduling servers are not employees in the traditional sense, and their schedules are not dictated by the company, that is, the servers determine their own work schedules. In such business environments, as in the problem setting that we consider, both customer demand and service capacity change stochastically. Hence, to better match supply and demand, dynamic pricing may be used to manage demand (i.e., the customers). Alternatively, dynamic wages can be used to manage supply (i.e., the servers). In particular, surge pricing is often used as a tool to motivate servers to show up to work at required times. Cachon et al. (2017), Gurvich et al. (2017), and Ibrahim (2017) study the role of dynamic wages in such systems. Our paper is different from this line of work in three ways: (i) These works try to match capacity, that is, the number of self-scheduling servers, to demand, and we try to match demand to capacity using admission control through pricing. (ii) In these works, each customer is served by a single server (i.e., single unit of capacity), and in our model, the customers' bandwidth (i.e., capacity) requirements vary by customer class. (iii) In these works, a server cannot terminate a customer's service before service is completed. Thus, in contrast to our setting, customer drop-off is not a concern. There is also recent work on time-dependent pricing schemes in wireless networks (Zhang et al. 2014, Chang et al. 2015, Huang et al. 2016, Tsai et al. 2017). In these schemes, data pricing for the next day is determined based on information from the previous day(s). Prices are allowed to vary from one time slot to another (e.g., hourly) to encourage users to shift demand from peak to off-peak hours. However, prices are not adjusted in real time based on the system state, and the network capacity is assumed to be fixed and known.

3. Problem Formulation

We next introduce our model for the SP's problem. The notation and model formulation used in this paper are similar to GS. However, our model and analysis are more complex because capacity is a stochastic process rather than constant.

3.1. Model Assumptions and Notation

In this section, we first outline our assumptions and notation for the customer and channel classes. We then describe the overall system state and how it evolves over time.

3.1.1. Customer Classes. There are M customer classes, indexed by $i = 1, 2, \ldots, M$. Customers in class i arrive according to a Poisson process with rate λ_i. The service time for class i customers is exponentially distributed with rate μ_i. The bandwidth (capacity) requirement for a class i customer is r_i, where $0 < r_i < 1$. For convenience, we define $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_M)$, $\mathbf{r} = (r_1, \ldots, r_M)$, and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_M)$. In general, we use bold to denote vectors. The customer classes are ordered according to the following assumption.

Assumption 1. $r_M \leq r_{M-1} \leq \cdots \leq r_2 \leq r_1 < 1$.

We standardize the total bandwidth provided by a single channel (of any class) to one. Thus, because $r_i < 1$, several customers can be simultaneously served by a single channel.

The number of class i customers in the system at time t is denoted by $N_i(t)$. We let $\mathbf{N}(t) = (N_1(t), \ldots, N_M(t))$, and we define $N(t) = \sum_{i=1}^{M} N_i(t)$. We drop the argument t when it is not required for clarity.

The SP must select the price to post to each arriving customer. The price set for Internet usage per unit of time is $\mathcal{U} = \{u_1, \ldots, u_L\}$, where the prices are ordered such that $u_1 < u_2 < \ldots < u_L$. Because each customer class has a different bandwidth requirement, r_i, we scale the price set $\mathcal{U}$ by r_i to find the price set for customer class i. For example, the set of prices that can be posted to an arriving class i customer, denoted by $\mathcal{U}_i = \{u_{i1}, \ldots, u_{iL}\}$, may be obtained by setting $u_{i\ell} = r_i u_\ell$, for all $\ell = 1, \ldots, L$. These prices are fees to be paid by the customers of class i per unit of time usage.

A customer from class i accepts the posted price $u_{i\ell}$ with probability $p_{i\ell}$, for $l = 1, 2, \ldots, L$, where these probabilities are assumed to follow Assumption 2.

Assumption 2. (i) $p_{i1} \geq p_{i2} \geq \cdots \geq p_{iL}$ *is a monotone sequence for* $i = 1, \ldots, M$; (ii) u_{iL} *is defined such that* $p_{iL} = 0$ *for* $i = 1, \ldots, M$.

Part (i) implies that a customer's willingness to accept an offered price decreases as the price increases. Part (ii) implies that the maximum price, u_{iL}, is sufficiently large so that it is always rejected. The SP can reject an arriving customer by posting this *reject price*.

The total connection fee of a customer depends on the price the customer accepts and the length of the customer's connection time. We let γ denote the continuous-time discount rate, where $\gamma > 0$. The expected discounted revenue from a class i customer that accepts the offered price $u_{i\ell}$ is $w_{i\ell} = u_{i\ell}/(\mu_i + \gamma)$, where $w_{i1} < \cdots < w_{iL}$. When a class i customer accepts the posted price $u_{i\ell}$, the present value of the customer's expected total connection fee at the time of the customer's connection is $w_{i\ell}$. In the remainder of the paper, we refer to $w_{i\ell}$ as the *discounted price*.

There may be times when it is necessary to drop a customer who is currently receiving service. In that case, the realized service time no longer follows an exponential distribution, which significantly complicates the analysis. To ensure tractability, we adopt the following convention: The service times for customer class i follow an exponential (μ_i) distribution. When a customer of class i arrives and the SP posts the price, the actual service time is unobserved. However, after deciding to accept the price, and before entering into service, the customer realizes the service time, reports this to the SP, and makes the appropriate payment. Thus, we assume that the customer pays upon admission. If the customer is later dropped because a channel becomes unavailable, the SP may refund part of this payment or offer a fixed penalty fee (discussed further as follows) to the customer.

3.1.2. Channel Classes.

There are D channel classes, and the classes differ in their degree of availability to the SP. The number of channels of class d available to be used by the SP at time t is denoted by $R_d(t)$, where $R_d(t) \leq R_{\max,d}$, with $\mathbf{R}(t) = (R_1(t), \ldots, R_D(t))$ and $R(t) = \sum_{d=1}^{D} R_d(t)$. We refer to $\mathbf{R}(t)$ or $R(t)$ as the *capacity*. We drop the argument t when it is not required for clarity. We use $R_{\max,d}$ to denote the maximum number of channels of class d that can ever be available to the SP, and we define $R_{\max} = \sum_{d=1}^{D} R_{\max,d}$. We refer to $R_{\max}$ as the *maximum capacity*. As described in Section 1, $R_d(t)$ is a stochastic process that increases (decreases) in increments of one unit as channels become available (unavailable). Once a channel from class d becomes available, it remains available for a period of time that follows an exponential distribution with rate ω_d, and then it becomes unavailable for a period of time that is exponentially distributed with rate ϕ_d. We define $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_D)$ and $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_D)$.

Because the number of available channels can decrease at any time, it is possible that the total required capacity, that is, $\mathbf{N}(t)'\mathbf{r}$, where $\mathbf{N}(t)'$ denotes the transpose of $\mathbf{N}(t)$, may at some time exceed the capacity, $R(t)$. In that case, some customers currently in service need to be dropped, that is, have their service immediately terminated. In this case, we assume the SP incurs a fixed drop-off fee, denoted by K, which satisfies Assumption 3.

Assumption 3. *The drop-off fee* K *satisfies* $w_{i1} < \cdots < w_{i,L-1} < K$ *for* $i = 1, \ldots, M$.

The fixed fee, K, incurred by the SP may include a penalty fee or partial refund paid to the customer and/or a goodwill cost incurred by the SP because of customer dissatisfaction.

3.1.3. System State.

The state of the system at time t, denoted by $\mathbf{S}(t)$, is given by the number of customers of each class currently in the system as well as the number of available channels from each class, that is, $\mathbf{S}(t) = (\mathbf{N}(t), \mathbf{R}(t))$. An incoming class i customer can be accepted only if $\mathbf{N}(t)'\mathbf{r} + r_i \leq R(t)$. Otherwise, that customer must be rejected by the SP by posting the reject price, u_{iL}. Hence, the state space is $\mathcal{S} = \{(\mathbf{N}, \mathbf{R}) : \mathbf{N}'\mathbf{r} \leq R, R_d \leq R_{\max,d}, d = 1, \ldots, D\}$.

3.2. Model Formulation

We study a continuous-time model with exponential intertransition times. Thus, the embedded stochastic process is a continuous-time Markov chain. The state of the system changes at each arrival and departure (of customers and channels). The SP must choose the price to post whenever an arrival occurs and which customer(s) to drop when capacity decreases, that is, a channel becomes unavailable and the remaining capacity is insufficient to handle the customers currently in the system. At these decision epochs, the decisions are based on the state of the system, $\mathbf{S}(t)$. This decision process is an infinite horizon, continuous-time MDP. See Puterman (1994) for details.

By using uniformization, we build the discrete-time equivalent of the system (Lippman 1975). The maximum possible rate out of any state is $\Gamma = \sum_{i=1}^{M} \lambda_i + \sum_{i=1}^{M} \mu_i \lfloor R_{\max}/r_i \rfloor + \sum_{d=1}^{D} (\omega_d + \phi_d) R_{\max,d} + \gamma$. Without loss of generality, we assume that Γ is one.

The value function, $v(\mathbf{N}, \mathbf{R})$, is the total expected discounted stream of future rewards and penalties given the system is in state $(\mathbf{N}, \mathbf{R})$ and is operated optimally. The optimality equation is as follows.

$$v(\mathbf{N},\mathbf{R}) = \sum_{i=1}^{M} \lambda_i H_i(v(\mathbf{N},\mathbf{R})) + \sum_{i=1}^{M} \mu_i N_i v(\mathbf{N}-\mathbf{e}_i^M, \mathbf{R}) \tag{1}$$

$$+ \sum_{d=1}^{D} \phi_d (R_{\max,d} - R_d) v(\mathbf{N}, \mathbf{R} + \mathbf{e}_d^D)$$
$$+ \sum_{d=1}^{D} \omega_d R_d G_d(v(\mathbf{N},\mathbf{R})) \tag{2}$$

$$+ \left[\sum_{i=1}^{M} \mu_i \left(\left\lfloor \frac{R_{\max}}{r_i} \right\rfloor - N_i \right) \right.$$
$$\left. + \sum_{d=1}^{D} (\phi_d R_d + \omega_d (R_{\max,d} - R_d)) \right] v(\mathbf{N},\mathbf{R}), \tag{3}$$

where $\mathbf{e}_i^M$ $(\mathbf{e}_d^D)$ is an M-dimensional (D-dimensional) unit vector with a one in the ith spot (dth spot) and zero in the remaining spots.

The function $H_i(v(\mathbf{N},\mathbf{R}))$ represents the SP's pricing decision when a class i customer arrives while the system is in state $(\mathbf{N},\mathbf{R})$. We write H_i as follows.

$$H_i(v(\mathbf{N},\mathbf{R})) = \begin{cases} \max_{\ell} [p_{i\ell}(v(\mathbf{N}+\mathbf{e}_i^M, \mathbf{R}) + w_{i\ell}) + (1-p_{i\ell}) v(\mathbf{N},\mathbf{R})] & \text{if } \mathbf{N}'\mathbf{r} + r_i \le R, \\ v(\mathbf{N},\mathbf{R}) & \text{if } \mathbf{N}'\mathbf{r} + r_i > R. \end{cases} \tag{4}$$

The pricing decisions are made only at arrival events with rate λ_i. Hence, rather than tracking revenues in continuous time, we track expected discounted revenues upon arrival.

The second term in the definition of $v(\mathbf{N},\mathbf{R})$ represents transitions resulting from customer departures, and class i customers depart at rate $\mu_i N_i$ when there are N_i customers of class i currently being served. The third term in the definition of $v(\mathbf{N},\mathbf{R})$ represents transitions resulting from channels becoming available, and class d channels become available at rate $\phi_d(R_{\max,d} - R_d)$ when there are R_d class d channels currently in use.

In the fourth term in the definition of $v(\mathbf{N},\mathbf{R})$, class d channels become unavailable at rate $\omega_d R_d$. The function $G_d(v(\mathbf{N},\mathbf{R}))$ represents the SP's drop-off decision when a class d channel becomes unavailable while the system is in state $(\mathbf{N},\mathbf{R})$. We write G_d as follows.

$$G_d(v(\mathbf{N},\mathbf{R})) = \begin{cases} \max_{\hat{\mathbf{N}}} \left[v(\hat{\mathbf{N}}, \mathbf{R}-\mathbf{e}_d^D) - K \sum_{i=1}^{M} (N_i - \hat{N}_i) \right] & \text{if } \mathbf{N}'\mathbf{r} > R-1, \\ v(\mathbf{N}, \mathbf{R}-\mathbf{e}_d^D) & \text{if } \mathbf{N}'\mathbf{r} \le R-1, \end{cases} \tag{5}$$

where $\hat{\mathbf{N}} = (\hat{N}_1, \ldots, \hat{N}_M)$, $\hat{\mathbf{N}}'\mathbf{r} \le R-1$, and $\hat{N}_i \le N_i$, $i = 1, 2, \ldots, M$.

Because the drop-off fee, K, used in the G_d function is fixed per customer, we might intuitively expect that customers requiring the highest bandwidth would be dropped first. Hence, in this paper, we also study a *bandwidth-based drop-off function*, denoted by G_d^B, which drops customers according to their bandwidth requirements; that is, class 1 customers are dropped first, followed by class 2 customers, etc., until the capacity is sufficient to satisfy the bandwidth requirements of the remaining customers. In the following section, we compare the optimal and bandwidth-based drop-off policies in detail. We find that, in most cases, it is optimal to drop the customers with the highest bandwidth requirements first. However, in some cases, it is optimal to drop a customer requiring a lower bandwidth if that customer's expected service time is relatively long (see Theorem 3).

The final term in $v(\mathbf{N},\mathbf{R})$ represents fictitious transitions to the current state $(\mathbf{N},\mathbf{R})$. The value iteration operator, T, is defined as follows, where f is any arbitrary function:

$$Tf(\mathbf{N},\mathbf{R}) = \sum_{i=1}^{M} \lambda_i H_i(f(\mathbf{N},\mathbf{R})) + \sum_{i=1}^{M} \mu_i N_i f(\mathbf{N}-\mathbf{e}_i^M, \mathbf{R})$$
$$+ \sum_{d=1}^{D} \phi_d (R_{\max,d} - R_d) f(\mathbf{N}, \mathbf{R}+\mathbf{e}_d^D)$$
$$+ \sum_{d=1}^{D} \omega_d R_d G_d(f(\mathbf{N},\mathbf{R}))$$
$$+ \left[\sum_{i=1}^{M} \mu_i \left(\left\lfloor \frac{R_{\max}}{r_i} \right\rfloor - N_i \right) \right.$$
$$\left. + \sum_{d=1}^{D} (\phi_d R_d + \omega_d (R_{\max,d} - R_d)) \right] f(\mathbf{N},\mathbf{R}). \tag{6}$$

Because T is a contraction mapping, it can be repeatedly applied to find the value function through successive approximation. If v_0 represents an initial estimate of v, then one pass of the value iteration algorithm (VIA) produces $v_1 = Tv_0$, and q applications of T produce $v_q = Tv_{q-1} = T^q v_0$. Because the state space, $\mathcal{S}$, and the one-period payments are finite, there exists a unique v such that $Tv = v$ and $\lim_{q\to\infty} v_q = v$. See theorem 6.2.5 in Puterman (1994). Hence, VIA can be used to find the v values. See section 6.3 in Puterman (1994). The rewards and transition probabilities are stationary. Because the state space, $\mathcal{S}$; the action space, $\mathcal{U}_i$ for $i = 1, \ldots, M$; and the one-period payments are finite, there exists an optimal deterministic stationary policy. See theorem 6.2.10 in Puterman (1994).

3.3. Dynamic Pricing Policy

The optimal dynamic pricing policy for the SP is found by solving the optimization in the $H_i(v(\mathbf{N},\mathbf{R}))$ function, which determines the optimal price to offer a class i

customer given the state of the system. To characterize this pricing policy, define $A_i(\mathbf{N}, \mathbf{R})$ to be the opportunity cost associated with the acceptance of an arriving class i customer when the system is in state $(\mathbf{N}, \mathbf{R})$, where $A_i(\mathbf{N},\mathbf{R}) = v(\mathbf{N},\mathbf{R}) - v(\mathbf{N} + \mathbf{e}_i^M, \mathbf{R}) \geq 0$. As in Gans and Savin (2007), we let $g_i\,(w_{i\ell}|A_i(\mathbf{N},\mathbf{R})) = p_{i\ell}(-A_i(\mathbf{N},\mathbf{R}) + w_{i\ell})$ denote the expected net gain from offering the discounted price $w_{i\ell}$ to a class i customer given the state $(\mathbf{N}, \mathbf{R})$ and let $g_i^*(\mathbf{N}, \mathbf{R}) \equiv g_i^*(A_i(\mathbf{N}, \mathbf{R}))$ denote the expected net gain from offering the *optimal* discounted price to a class i customer. In other words, $g_i^*(A_i(\mathbf{N}, \mathbf{R})) = \max_{w_{i\ell}} g_i\,(w_{i\ell}|A_i(\mathbf{N}, \mathbf{R})) = \max_\ell [p_{i\ell}\,(-A_i\,(\mathbf{N}, \mathbf{R}) + w_{i\ell})]$. Note that finding $g_i^*(A_i(\mathbf{N}, \mathbf{R}))$ is equivalent to the optimization in $H_i(v(\mathbf{N}, \mathbf{R}))$. Finally, let $w_i^*(\mathbf{N}, \mathbf{R}) \equiv w_{i\ell_i^*(A_i(\mathbf{N},\mathbf{R}))}$, where $\ell_i^*(A_i(\mathbf{N}, \mathbf{R})) = \arg\max_\ell[p_{i\ell}(-A_i(\mathbf{N}, \mathbf{R}) + w_{i\ell})]$, denote the optimal discounted price to offer a class i customer who arrives while the system is in state $(\mathbf{N}, \mathbf{R})$. The optimal discounted prices are dependent on the state, $(\mathbf{N},\mathbf{R})$, only through the opportunity cost, A_i. Therefore, for notational simplicity, we suppress the $(\mathbf{N}, \mathbf{R})$ notation when it is not required for clarity; that is, we write $g_i^*(A_i)$ and $\ell_i^*(A_i)$. We can now state the following result.

Theorem 1. *The expected net gain from offering the optimal price to an arriving class i customer, $g_i^*(A_i)$, is nonincreasing in A_i, and the optimal discounted price, $w_{i\ell_i^*(A_i)}$, is nondecreasing in A_i.*

Thus, to prove structural properties for the optimal dynamic pricing policy, it is sufficient to understand the behavior of A_i, the opportunity cost associated with the acceptance of an arriving class i customer. The proof of Theorem 1 is given in Online Appendix A.

When $A_i = 0$, let $m_i = \arg\max_\ell(p_{i\ell} w_{i\ell})$. The price w_{im_i} is the *myopic price*, that is, the price that maximizes the immediate expected revenue from a class i customer. Because the optimal discounted price, $w_{i\ell_i^*(A_i)}$, is nondecreasing in A_i, the myopic price is a lower bound on the optimal price in any state.

Corollary 1. *The optimal discounted prices satisfy $w_i^*(\mathbf{N}, \mathbf{R}) \geq w_{im_i}$ for all $(\mathbf{N}, \mathbf{R})$.*

Theorem 1 and Corollary 1 together imply that if the condition

$$p_{im_i}(-A_i + w_{im_i}) \geq p_{i\ell}(-A_i + w_{i\ell}) \text{ for all } \ell = m_i + 1, \dots, L \tag{7}$$

is satisfied, then w_{im_i} is the optimal discounted price to offer to a class i customer. Equation (7) implies that if

$$A_i \leq (p_{im_i} w_{im_i} - p_{il} w_{i\ell})/(p_{im_i} - p_{i\ell}) \tag{8}$$

for all $\ell = m_i + 1, \dots, L$, then w_{im_i} is the optimal discounted price to offer to a class i customer. Hence, when the opportunity cost associated with the acceptance of an arriving customer is sufficiently small, the myopic price is the optimal price. However, if the opportunity cost is larger than a threshold, the optimal price is greater than the myopic price. Similar results are proven in the literature for various revenue management problems. See, for example, Bitran and Caldentey (2003).

4. Structural Properties of the MDP

Next, for several different problem settings, we analyze the MDP formulation to derive insights into the structure of the SP's optimal dynamic pricing policy.

4.1. *M* Customer Classes and *D* Channel Classes

For the general setting, the total expected discounted stream of future rewards and penalties decreases as we have more customers in service and increases as we have more channel availability. In other words, more congestion leads to lower expected profit. Also, for a given set of parameter values, if we only increase the arrival rate to the system, this larger demand leads to higher expected profit. Specifically, we have the following theorem:

Theorem 2. *The optimal value function $v(\mathbf{N}, \mathbf{R})$ has the following properties:*

1. *$v(\mathbf{N}, \mathbf{R})$ is nonincreasing in N_j; that is, $v(\mathbf{N} + \mathbf{e}_j^M, \mathbf{R}) \leq v(\mathbf{N}, \mathbf{R})$, for $j = 1, \dots, M$.*
2. *$v(\mathbf{N}, \mathbf{R})$ is nondecreasing in R_d; that is, $v(\mathbf{N}, \mathbf{R} + \mathbf{e}_d^D) \geq v(\mathbf{N}, \mathbf{R})$, for $d = 1, \dots, D$.*
3. *$v(\mathbf{N}, \mathbf{R})$ is a nondecreasing function of the arrival rates, λ_j, $j = 1, \dots,$ M.*

The proof of Theorem 2 is given in Online Appendix B. We next show how the different service rates and remaining capacity in a given state impact the value function.

Theorem 3. *Assume that $\mu_j \leq \mu_k$ for all j, k = 1, …, M, and $j < k$; then*

$$v(\mathbf{N} + \mathbf{e}_j^M, \mathbf{R}) \leq v(\mathbf{N} + \mathbf{e}_k^M, \mathbf{R}). \tag{9}$$

The proof of Theorem 3 is given in Online Appendix C. When $j < k$, Assumption 1 implies $r_j \geq r_k$. Hence, the remaining (i.e., unused) capacity is smaller in state $(\mathbf{N} + \mathbf{e}_j^M, \mathbf{R})$ than in state $(\mathbf{N} + \mathbf{e}_k^M, \mathbf{R})$. According to Theorem 3, when $\mu_j \leq \mu_k$, admitting a class k customer rather than a class j customer takes the system to a state with relatively larger value because of the larger remaining capacity in state $(\mathbf{N} + \mathbf{e}_k^M, \mathbf{R})$. However, when $\mu_j > \mu_k$, the opposite might hold; that is, it might be preferable to admit a class j customer with higher bandwidth requirement rather than a class k customer with smaller bandwidth requirement. This is because, when $\mu_j > \mu_k$, the expected service time of a class j customer is smaller than that of a class k customer; that is, $1/\mu_j < 1/\mu_k$.

This theorem has implications for the optimal drop-off policy, represented by the G_d function in Equation (5). In most cases, when there is a shortage of resources, that is, when $R > \mathbf{N}'\mathbf{r}$, it is optimal to drop the customer(s) with the highest bandwidth requirement, which we refer to as the bandwidth-based drop-off policy. However, when customers requiring lower bandwidth have longer expected service times than customers requiring higher bandwidth, the bandwidth-based drop-off policy may not be optimal. Thus, when the turnover rate of low bandwidth customers is small, that is, they stay in the system for a long time, it may be optimal to drop those low-bandwidth customers first.

In the light of this theorem, we also considered an alternative heuristic drop-off policy, which drops customers based on the magnitude of r_i/μ_i; that is, customers with the largest value of r_i/μ_i are dropped first. Our experimental results indicate that this alternative drop-off policy does not perform considerably better than the proposed bandwidth-based drop-off policy.

The following theorem, whose proof is in Online Appendix D, demonstrates that the bandwidth-based drop-off policy and the optimal drop-off policy always drop the same number of customers. However, the classes of the dropped customers may not be the same.

Theorem 4. *The total number of customers that will be dropped according to the optimal drop-off function G_d in Equation (5) and the bandwidth-based drop-off function G_d^B are exactly the same.*

In the general case of M customer classes and D channel classes, structural properties for the value function, such as concavity and submodularity, are not provable. Even in the special case of $M = 2$ customer classes and D channel classes, the concavity of the optimal value function in N_1 and N_2 is not provable. A similar observation can be made about Gans and Savin (2007) and Ulukus et al. (2011); that is, these authors are also not able to prove the concavity of the value function when there are two customer classes with different service rates. Because we are unable to prove concavity, we cannot prove that the optimal price, $w_i^*(N_1, N_2, \mathbf{R})$, is nondecreasing in N_i, $i = 1, 2$. In fact, experimental results presented in Section 6 indicate that, in some states, the optimal prices for class i, $w_i^*(N_1, N_2, \mathbf{R})$, are decreasing in N_i, a result that may seem counterintuitive.

Similarly, when $M = 2$, submodularity of the optimal value function in N_1 and N_2 is not provable, and thus, we cannot prove that the optimal discounted price for an arriving class i customer, $w_i^*(N_1, N_2, \mathbf{R})$, is nondecreasing in N_j, $i, j = 1, 2$, $i \neq j$. However, if we use the bandwidth-based drop-off function, G_d^B, we can show that the optimal value function is submodular in N_1 and N_2, and thus, $w_i^*(N_1, N_2, \mathbf{R})$ is nondecreasing in N_j, $i, j = 1, 2$, $i \neq j$. In other words, the price offered to an arriving class i customer will generally be increasing in the number of class j customers already in the system. The fact that submodularity of the value function is dependent on the use of the bandwidth-based drop-off policy can be explained by Theorem 3. Under the optimal drop-off policy, it is sometimes optimal to drop the low-bandwidth customer(s) of class 2 first even when there are high-bandwidth class 1 customers in the system. However, under the bandwidth-based drop-off policy, class 1 customers are always dropped first.

4.2. Similar Customer Classes

We next consider the case in which the customer classes are *similar*, that is, have a common service rate and bandwidth requirement, $\mu_1 = \ldots = \mu_M = \mu$ and $r_1 = \ldots = r_M = r$, but their arrival rates and price sensitivities may differ. In this case, we do not need to distinguish between the customer classes once they are admitted, which reduces the size of the state space. Because the bandwidth requirements are the same for all classes, the price sets will be the same; that is, $\mathcal{U}_1 = \ldots = \mathcal{U}_M = \{ru_1, \ldots, ru_L\}$. Because the service rates are identical, we have $w_{1\ell} = \ldots = w_{M\ell} = w_\ell$, $\ell = 1, \ldots, L$.

4.2.1. *M* Similar Customer Classes and *D* Channel Classes.

In the setting with a general number of customer classes and channel classes, the state is $(N, \mathbf{R})$, where N is the total number of customers that are currently in the system. From Theorems 3 and 4, we can conclude that the bandwidth-based drop-off policy is optimal in this case. The formulation of the bandwidth-based drop-off function in this case is as follows.

$$G_d^B(v(N,\mathbf{R})) = \begin{cases} v\left(\left\lfloor \frac{R-1}{r} \right\rfloor, \mathbf{R} - \mathbf{e}_d^D\right) - K\left(N - \left\lfloor \frac{R-1}{r} \right\rfloor\right) & \\ & \text{if } Nr > R-1, \\ v(N, \mathbf{R} - \mathbf{e}_d^D) & \text{if } Nr \le R-1, \end{cases} \tag{10}$$

for $d = 1, \ldots, D$. Because it is easier to prove results using the bandwidth-based drop-off function formulation, G_d^B is used in all of the proofs in this section.

Given this problem formulation, we can derive some important properties for the value function, and they are useful for characterizing the optimal dynamic pricing policy. The properties proven in Theorem 2 apply to the value function $v(N, \mathbf{R})$ as well.

Theorem 5. *The optimal value function $v(N,\mathbf{R})$ has the following properties:*

1. $v(N,\mathbf{R})$ *is concave in N; that is,* $v(N+2,\mathbf{R})-v(N+1,\mathbf{R}) \le v(N+1,\mathbf{R})-v(N,\mathbf{R})$.
2. $v(N,\mathbf{R})$ *is supermodular on* $\mathcal{S}$*; that is,* $v(N+1,\mathbf{R}+\mathbf{e}_d^D)-v(N+1,\mathbf{R}) \ge v(N,\mathbf{R}+\mathbf{e}_d^D)-v(N,\mathbf{R})$.

The proof of Theorem 5 is analogous to the proof of Theorem 2. Notice that, when the customer classes are similar, we are able to prove the concavity of the value function. In the literature, it is not uncommon for the value function to be concave in the number of customers in the system when the service characteristics of all customer classes are the same (in our case, the service rates and bandwidth requirements are the same). See, for example, Miller (1969) and Gans and Savin (2007).

The supermodularity result in Theorem 5 implies that the value of having an additional channel of any class is increasing in the number of customers in the system. This property is different from monotonicity of the value function with respect to the number of servers, a result that is typically proven in previous literature, for example, Aktaran-Kalayci and Ayhan (2009). Although monotonicity implies that the value increases as the number of servers increases, supermodularity considers how that rate of increase in the value function depends on the number of customers in the system. Further, when the previous literature considers how a change in the number of servers affects the value function, the change in the number of servers is permanent (i.e., the literature compares a setting in which the system has s servers to a setting in which the system has $s+1$ servers). In contrast, in our setting, we are considering how the value function changes with a change in the system state (i.e., the available channels $\mathbf{R}$), which is a temporary change (i.e., the number of channels fluctuates over time as channels become available and unavailable).

We next discuss the optimal dynamic pricing policy. We use $A(N,\mathbf{R})$ to represent the opportunity cost associated with the acceptance of an arriving customer when the system is in state $(N,\mathbf{R})$, where $A(N,\mathbf{R}) = v(N,\mathbf{R}) - v(N+1,\mathbf{R}) \ge 0$. We drop the subscript on A_i because we do not need to differentiate by customer class. From Theorem 5, we know that v is concave in N and supermodular in N and $\mathbf{R}$. Hence, $A(N,\mathbf{R})$ is nondecreasing in N and nonincreasing in R_d. Therefore, using Theorem 1, we can state the following result:

Corollary 2. *The expected net gain from offering the optimal price to an arriving class i customer,* $g_i^*(N,\mathbf{R})$*, is nonincreasing in N and nondecreasing in* R_d*, and the optimal discounted prices are nondecreasing in N and nonincreasing in* R_d*; that is,*

$$w_i^*(N,\mathbf{R}) \le w_i^*(N+1,\mathbf{R}) \quad \textit{and} \quad w_i^*(N,\mathbf{R}) \ge w_i^*(N,\mathbf{R}+\mathbf{e}_d^D).$$

Similar relationships hold for $u_i^*(N,\mathbf{R})$.

Thus, the optimal prices are increasing in the number of customers in the system, N, and decreasing in the number of available channels, $R=\sum_{d=1}^{D} R_d$. Previous literature, for example, Aktaran-Kalayci and Ayhan (2009), has also found that the optimal prices are decreasing in the number of servers and in the capacity of the system. However, in that literature, the number of servers and capacity are exogenously specified parameters. In contrast, in our setting, the number of available channels is part of the system state and varies stochastically.

Overall, we have that the prices are decreasing in the *remaining capacity*, which can be measured by $R-N$. Using Corollary 1 and Equation (8), we can also observe that myopic pricing is optimal when the remaining capacity is sufficiently large.

We next present some sensitivity analysis results for the optimal pricing policy.

Theorem 6. *The optimal pricing policy has the following properties:*

1. $w_i^*(N,\mathbf{R})$ *and* $u_i^*(N,\mathbf{R})$ *are nondecreasing functions of the arrival rates,* λ_j, $j=1,\ldots,M$.
2. $w_i^*(N,\mathbf{R})$ *is a nonincreasing function of the service rate,* μ.
3. $w_i^*(N,\mathbf{R})$ *and* $u_i^*(N,\mathbf{R})$ *are nondecreasing functions of the drop-off fee, K.*

The proof of Theorem 6 is given in Online Appendix E. Theorem 6.1 indicates that an increase in the arrival rate for any given customer class causes an increase in optimal prices of *all* classes. Intuitively, an increase in the arrival rate for any class increases the congestion in the system, requiring higher prices. Theorem 6.2 indicates that when customers can be cleared from the system more quickly, optimal prices can be reduced. Notice that the relationship presented in Theorem 6.2 holds only for $w_i^*(N,\mathbf{R})$ and is not valid for $u_i^*(N,\mathbf{R})$. Recall that $u_i^* = w_i^*(\mu+\gamma)$. When μ increases and w_i^* decreases, the behavior of u_i^* is unclear. Theorem 6.3 indicates that an increase in the drop-off fee, K, increases the optimal prices for all customer classes when the customer classes are similar.

Similar results are common in the literature on pricing of queueing systems. For example, Aktaran-Kalayci and Ayhan (2009) provide similar monotonicity results for the optimal prices with respect to the arrival and service rates for an $M/M/s/K$ system with a single customer class. In addition, for a large class of single-dimension queueing systems, Cil et al. (2009)

discuss structural properties of the value function (and, specifically, for the opportunity cost, which is analogous to A_i as defined previously) that are sufficient to ensure that the optimal prices have similar monotonicity properties with respect to the arrival and service rates. However, it is important to note that none of the systems studied in these previous works considers capacity that evolves as a stochastic process with the potential for customer drop-offs. Thus, these previous works also do not present analogous results regarding the monotonicity of the prices with respect to the drop-off fee.

We next consider the impact of customer's price sensitivity on the optimal prices.

Theorem 7. *For classes j and k, if $p_{j\ell}/p_{k\ell}$ is an increasing function of ℓ for $\ell = 1, \ldots, L-1$, then $w_j^*(N, \mathbf{R}) \geq w_k^*(N, \mathbf{R})$ for all $(N, \mathbf{R})$. A similar result holds for $u_j^*(N, \mathbf{R})$ and $u_k^*(N, \mathbf{R})$.*

The proof of Theorem 7 is given in Online Appendix F. The theorem implies that customers who are more price sensitive will be offered lower prices. Gans and Savin (2007) prove a similar result for walk-in customers. The condition in Theorem 7, that is, p_{jl}/p_{kl} is an increasing function of l, implies that class k customers are more price sensitive than class j customers. Recall that p_{il} denotes the probability that a class i customer will accept the posted price ru_l, where the prices are ordered such that $u_1 < u_2 < \cdots < u_L$. Assumption 2 implies that $p_{i1} \geq p_{i2} \geq \cdots \geq p_{iL}$ for all i. Thus, for all customer classes, the probability of accepting a posted price decreases as the price increases. Hence, in the ratio p_{jl}/p_{kl}, both p_{jl} and p_{kl} are decreasing in l. The condition that p_{jl}/p_{kl} is an increasing function of l implies that p_{kl} decreases more rapidly in l than does p_{jl}, which implies that class k customers are more price sensitive than class j customers.

4.2.2. *M* Similar Customer Classes and Two Channel Classes.

In this section, we consider a setting with M similar customer classes and just $D = 2$ channel classes to consider how the relative magnitudes of the channel availability (ω_d) and unavailability (ϕ_d) rates affect the optimal prices. We present two theorems to demonstrate the results.

Theorem 8. *Suppose the two channel classes have the same availability rate and maximum capacity; that is, $\omega_1 = \omega_2 = \omega$ and $R_{max,1} = R_{max,2} = 0.5R_{max}$. However, the unavailability rates may differ; that is, $\phi_1 \geq \phi_2$. Then the optimal pricing policy satisfies $w_i^*(N, R_1+1, R_2) \geq w_i^*(N, R_1, R_2+1)$. A similar relationship holds for $u_i^*(N, R_1, R_2)$.*

The proof of Theorem 8 uses Theorem 1 and is analogous to the proof Theorem 5. Under the conditions of Theorem 8, the probability of a channel becoming *unavailable* in states (N, $R_1 + 1$, R_2) and (N, R_1, $R_2 + 1$) are the same and equal to $(R_1 + R_2 + 1)\omega$. However, the probability of a channel becoming *available* in state (N, $R_1 + 1$, R_2) is $(0.5R_{\max} - R_1 - 1)\phi_1 + (0.5R_{\max} - R_2)\phi_2$. This is less than or equal to the probability of a channel becoming *available* in state (N, R_1, $R_2 + 1$), which is $(0.5R_{\max} - R_1)\phi_1 + (0.5R_{\max} - R_2 - 1)\phi_2$. Hence, there is relatively less capacity in state (N, $R_1 + 1$, R_2) than in state (N, R_1, $R_2 + 1$), which leads to higher optimal discounted prices in state (N, $R_1 + 1$, R_2).

Theorem 9. *Suppose the two channel classes have the same unavailability rate and maximum capacity; that is, $\phi_1 = \phi_2 = \phi$ and $R_{max,1} = R_{max,2} = 0.5R_{max}$. However, the availability rates may differ; that is, $\omega_1 \geq \omega_2$. Then the optimal pricing policy satisfies $w_i^*(N, R_1+1, R_2) \geq w_i^*(N, R_1, R_2+1)$. A similar relationship holds for $u_i^*(N, R_1, R_2)$.*

The proof of Theorem 9 uses Theorem 1 and is analogous to the proof Theorem 5. Under the conditions given in Theorem 9, the probability of a channel becoming *available* in states (N, $R_1 + 1$, R_2) and (N, R_1, $R_2 + 1$) are the same and equal to $(0.5R_{\max} - R_1 - R_2 - 1)\phi$. However, the probability of a channel becoming *unavailable* in state (N, $R_1 + 1$, R_2) is $(R_1 + 1)\omega_1 + R_2\omega_2$. This is larger than or equal to the probability of a channel becoming *unavailable* in state (N, R_1, $R_2 + 1$), which is $R_1\omega_1 + (R_2 + 1)\omega_2$. Hence, there is relatively less capacity in state (N, $R_1 + 1$, R_2) than in state (N, R_1, $R_2 + 1$), which leads to higher optimal discounted prices in state (N, $R_1 + 1$, R_2).

5. Heuristic Policies

The optimal dynamic pricing policy can be determined through the VIA when the state space is reasonably small. However, for realistic problem sizes, solving the VIA is infeasible. Therefore, we propose three heuristics that approximate the optimal dynamic pricing policy by aggregating either the customer classes, the channel classes, or both. In these heuristics, we utilize the VIA in the aggregate state space. Thus, the pricing policy obtained from the heuristic is suboptimal for the original (nonaggregate) state space. In Section 6, we numerically evaluate the performance of these proposed dynamic pricing heuristics and compare their performance with both the myopic and optimal static pricing policies.

5.1. *N*-Heuristic

In the N-heuristic, which is motivated by the optimal policy for the model in Section 4.2.1 with M similar customer classes and D channel classes, we aggregate the customer classes into a single average class. Let $N(t)$ be the number of customers in the system at time t, where $N(t) = \sum_{i=1}^{M} N_i(t)$. The aggregate state is $\mathbf{S}(t) = (N(t), R_1(t), \ldots, R_D(t))$. Thus, once a customer is admitted to the system, the heuristic does not keep track of the customer's class. Therefore, we model the system using the average bandwidth requirement, $\bar{r}$, and average service rate, $\bar{\mu}$, across the customer classes. However, the heuristic does consider the customer class at the time of a customer arrival to the system; that is, the price posted to an arriving customer depends on that customer's price sensitivity (through the $p_{i\ell}$ probabilities). Further, the customer's bandwidth requirement, r_i, is considered when determining whether the customer can be admitted to the system. Thus, an incoming class i customer can be accepted if $N(t)\bar{r} + r_i \le R(t)$, where $R(t) = \sum_{d=1}^{D} R_d(t)$, and the state space is $\mathcal{S} = \{(N, \mathbf{R}) : N\,\bar{r} \le R, R_d \le R_{\max,d}, d = 1, \ldots, D\}$.

The VIA for the N-heuristic can be used to determine a pricing policy. However, because the heuristic aggregates customers once they are admitted to the system, it cannot be used to determine a drop-off policy that considers the classes of the customers that are currently in the system. When the heuristic is implemented, however, it is necessary to have a drop-off policy to determine how to drop customers when the bandwidth requirement exceeds the available capacity. Thus, for the implementation of the N-heuristic, we propose to use the bandwidth-based drop-off policy presented in Section 3.2. As discussed previously, because of Theorems 3 and 4, this policy is known to be optimal under certain conditions.

5.2. *R*-Heuristic

In the R-heuristic, in contrast to the N-heuristic, we aggregate the channel classes into a single average class. We model the system using the average unavailability rate, $\bar{\phi}$, and average availability rate, $\bar{\omega}$, across the classes. The aggregate state is $\mathbf{S}(t) = (N_1(t), \ldots, N_M(t), R(t))$. An incoming class i customer can be accepted if $\mathbf{N}(t)'\mathbf{r} + r_i \le R(t)$. Thus, the state space is $\mathcal{S} = \{(\mathbf{N}, R) : \mathbf{N}'\mathbf{r} \le R, R \le R_{\max}\}$. Unlike the N-heuristic, the R-heuristic provides both a pricing policy and a drop-off policy because the customers' classes can be distinguished while they are in the system. An interesting question, which is considered in Section 6, is whether we lose more, relative to the optimal policy, by aggregating customer classes (the N-heuristic) or channel classes (the R-heuristic).

5.3. (*N*, *R*)-Heuristic

In the (N, R)-heuristic, we aggregate both the customer classes and the channel classes. We model the system using the average bandwidth requirement, $\bar{r}$; average service rate, $\bar{\mu}$; and the average channel rates, $\bar{\phi}$ and $\bar{\omega}$. The aggregate state is $\mathbf{S}(t) = (N(t), R(t))$. As in the N-heuristic, the (N, R)-heuristic does consider the customer class at the time of a customer arrival to the system. However, once admitted, the customer's class is no longer tracked. An incoming class i customer can be accepted if $N(t)\bar{r} + r_i \le R(t)$. Thus, the state space is $\mathcal{S} = \{(N, R) : N\bar{r} \le R, R \le R_{\max}\}$. Similar to the N-heuristic, the (N, R)-heuristic provides a pricing policy but not a drop-off policy. Hence, in implementation of the (N, R)-heuristic, we assume that the bandwidth-based drop-off policy will be used.

The algorithm for the (N, R)-heuristic is

1. Set $v_0(N, R) = 0$. Specify $\varepsilon > 0$, and set $q = 0$. Compute the class i price set $\mathcal{U}_i = \{u_{i1}, u_{i2}, \ldots, u_{iL}\}$ from $\mathcal{U} = \{u_1, u_2, \ldots, u_L\}$ by $u_{i\ell} = r_i u_\ell$ for $\ell = 1, \ldots, L$.

2. For each $(N, R) \in \mathcal{S}$ $(N\bar{r} \le R$ and $R \le R_{\max})$, compute $v_{q+1}(N, R)$ by

$$\begin{aligned} v_{q+1}(N, R) = \Bigg\{ & \sum_{i=1}^{M} \lambda_i H_i(v_q(N, R)) + \bar{\mu} N v_q(N-1, R) \\ & + \bar{\phi}(R_{\max} - R) v_q(N, R+1) \\ & + \bar{\omega} R G^B(v_q(N, R)) + \Bigg[\bar{\mu}\left(\left\lfloor \frac{R_{\max}}{\bar{r}} \right\rfloor - N\right) + \bar{\phi} R \\ & + \bar{\omega}(R_{\max} - R)\Bigg] v_q(N, R)\Bigg\} \Bigg/ \Gamma, \end{aligned}$$

where $\Gamma = \sum_{i=1}^{M} \lambda_i + \bar{\mu}\lfloor R_{\max}/\bar{r} \rfloor + (\bar{\phi} + \bar{\omega}) R_{\max}$;

$$H_i(v_q(N, R)) = \begin{cases} \max_{\ell}[p_{i\ell}(v_q(N+1, R) + w_{i\ell}) + (1 - p_{i\ell}) v_q(N, R)] & \text{if } N\bar{r} + r_i \le R, \\ v_q(N, R) & \text{if } N\bar{r} + r_i > R, \end{cases}$$

where $w_{i\ell} = u_{i\ell}/\mu_i$ for all $i = 1, \ldots, M$ and $\ell = 1, \ldots, L$; and

$$G^B(v_q(N, R)) = \begin{cases} v_q\left(\left\lfloor \frac{R-1}{\bar{r}} \right\rfloor, R-1\right) - K\left(N - \left\lfloor \frac{R-1}{\bar{r}} \right\rfloor\right) & \text{if } N\bar{r} > R-1, \\ v_q(N, R-1) & \text{if } N\bar{r} \le R-1. \end{cases}$$

3. Set

$$M_q = \max_{(N,R)\in\mathcal{S}} (v_{q+1}(N, R) - v_q(N, R)) \quad \text{and}$$

$$m_q = \min_{(N,R)\in\mathcal{S}} (v_{q+1}(N, R) - v_q(N, R)).$$

If $0 \le (M_q - m_q)/m_q \le \varepsilon$, go to step 4. Otherwise increment q by 1 and return to step 2.

4. For each $(N, R) \in \mathcal{S}$ and i, $i = 1, \ldots, M$: if $N\bar{r} + r_i \le R$, choose

$$\begin{aligned} u_i^{\varepsilon}(N, R) \in \arg\max_{\ell} [& p_{i\ell}(v_{q+1}(N+1, R) + w_{i\ell}) \\ & + (1 - p_{i\ell}) v_{q+1}(N, R)]; \end{aligned}$$

else write "capacity not available" Stop.

5.4. Myopic and Optimal Static Pricing Policies

These heuristics consider state-dependent (or dynamic) pricing policies. We also consider the performance of two *static* pricing policies, myopic and optimal static pricing, in which the same price is offered to all customers of a given class; that is, the price is not dependent on the system state. Under the myopic policy, the SP determines whether there is sufficient capacity to admit the arriving class i customer. If not, the reject price (u_{iL}) is posted. If so, the SP posts the myopic price, which is the price that maximizes the immediate expected discounted revenue earned from admitting the customer. The myopic policy is a useful benchmark for evaluating the proposed dynamic pricing heuristics because it is easy to compute and to implement and has been studied in the previous literature.

In addition, we evaluate the performance of the optimal static pricing policy, in which we find the optimal constant price to charge to each customer class. Unfortunately, finding the optimal static pricing policy is computationally challenging. The previous literature on optimal static pricing policies (e.g., Gayon et al. (2009)) generally considers just a single customer class, which requires optimization over a single dimension. In our setting, however, the static pricing policy must specify a price for each customer class. To find the optimal static pricing policy, we perform an exhaustive search, in which the long-run average value of a given pricing policy is evaluated using simulation as described in Section 6. When searching for the optimal price for customer class i, to ensure fair comparison with the dynamic pricing policies, we start with the price set $\mathcal{U}_i$, and then we exclude from this set all prices that are less than the myopic price for class i. If we let $\mathcal{U}_i^S$ denote the set of remaining prices for class i, then the optimal static pricing policy has search space with size $|\mathcal{U}_1^S| \times |\mathcal{U}_2^S| \times \ldots \times |\mathcal{U}_M^S|$, which can be quite large.

As for the N- and (N, R)-heuristic policies, the static policies specify only a pricing policy. Hence, for implementation, we assume the bandwidth-based drop-off policy is used.

6. Experiments

In this section, we present the results of a numerical study, which we conducted to (i) obtain additional insights, beyond the analytical results provided herein, on the behavior of the optimal prices (Section 6.2); (ii) test the performance of the static myopic pricing policy relative to the optimal policy (Section 6.3); (iii) test the performance of the proposed dynamic pricing heuristics relative to the optimal dynamic pricing policy (Section 6.4) and the optimal static pricing policy (Section 6.5); and (iv) evaluate the value of using the optimal drop-off policy compared with the bandwidth-based drop-off policy (Section 6.6).

The analytical results presented in this paper were derived in the context of a discounted MDP. However, because our MDP is unichain, all of the results also hold for the analogous long-run average-value problems. Unichain implies that, given the use of any stationary policy, there exists a state $(\mathbf{N}, \mathbf{R}) = (0, \ldots, 0)$ to which there is a positive probability of returning within a finite number of transitions starting from any initial state. When $\gamma = 0$ in our MDP formulations, the VIA can be used to determine the optimal long-run average value (see section 8.5 in Puterman 1994). In the literature, the average-value criterion is generally preferred when conducting numerical experiments because average value does not depend on the initial state, and thus, the numerical results are easier to interpret. Therefore, in this section, we present numerical results using the average-value criterion.

To evaluate the value function under the heuristics, we relied on simulation. Although the proposed heuristics can be used to find the dynamic pricing policy for the aggregate state space (which we use as our heuristic policy), the resulting value function does not represent the long-run average profit from implementing the heuristic in the complete state-space setting. Thus, we simulate the implementation of the heuristic dynamic pricing policies in the full state-space setting to evaluate the value function. We ran each simulation for a run length of 30 years with 40 replications. We report the *simulated average value*, which is the average of the long-run average value estimates across the 40 replications.

6.1. Experiment Configurations

We first describe the three sets of experiments we conducted. In the first, we set the number of the channel and customer classes such that the state space of the MDP is relatively small, and thus, VIA can successfully find the optimal policy. We use this experiment to study the behavior of the optimal prices and to compare the performance of the optimal policy and the heuristic policies. In the second, we set the channel and customer classes to larger values, which results in a larger state space, making it impossible to solve the VIA to find the optimal policy. We use this experiment to compare the performance of the proposed dynamic pricing heuristic policy with that of the (static) myopic policy and to determine when the proposed heuristic

Table 1. Channel Characteristics in Experiment 1

	Channel class 1			Channel class 2		
Cases	$R_{\max,1}$	$1/\phi_1$	$1/\omega_1$	$R_{\max,2}$	$1/\phi_2$	$1/\omega_2$
I	1	1	3	5	8	3
II	1	4	3	5	4	3
III	1	8	3	5	1	3

Table 2. Customer Characteristics in Experiment 1

Sensitivity parameters	Customer class 1	Customer class 2
$\Delta_1 \in \{0.25, 0.5, 1\}$	$r_1 = 0.5$	$r_2 = \Delta_1 r_1 \in \{0.125, 0.25, 0.5\}$
$\Delta_2 \in \{0.1, 0.25, 0.5\}$	$\lambda_1 = 10$	$\lambda_2 = \Delta_2 \lambda_1 \in \{1, 2.5, 5\}$
$\Delta_3 \in \{0.5, 1, 1.5\}$	$\mu_1 = 5$	$\mu_2 = \Delta_3 \mu_1 \in \{2.5, 5, 7.5\}$
$\Delta_4 \in \{1, 2, 10\}$	$K = \max\{2^{\text{nd}}$ largest w_{1j}, 2^{nd} largest $w_{2j}\} + \Delta_4$	
$\Delta_5 \in \{0.1, 0.5, 1\}$	$\beta_1 = 3$	$\beta_2 = \Delta_5 \beta_1 \in \{0.3, 1.5, 3\}$

offers the most value. In the third, we consider a realistically sized problem in which there are 32 distinct customer classes. We use this experiment to further investigate the performance of the proposed dynamic pricing heuristic.

In all three sets of experiments, the price set is $\mathcal{U} = \{1.0, 1.1, \ldots, 5.9, 6.0\}$. To model the price sensitivities, we use the same approach as GS; that is, we calculate the price acceptance probabilities as $p_{i\ell}(w_{i\ell}) = ((w_{i,\max} - w_{i\ell})/(w_{i,\max} - w_{i,\min}))^{\beta_i}$, where $w_{i,\max} = \max_\ell w_{i\ell}$ and $w_{i,\min} = \min_\ell w_{i\ell}$. Here, larger β_i implies that customer class i is more price sensitive.

6.1.1. Experiment 1. This set of experiments uses configurations with $D = 2$ channel classes and $M = 2$ customer classes. We define three cases for the channel classes, which we label as I, II, and III. As shown in Table 1, the expected availability times of the channels, $1/\omega_1$ and $1/\omega_2$, are identical for all three cases. However, the expected unavailability time in channel class 1, $1/\phi_1$, increases from case I to case III, and the opposite holds for $1/\phi_2$. Also, in all three cases, $R_{\max,2}$ is larger than $R_{\max,1}$. As a result, the expected capacity increases from case I to III. This allows us to understand how the performance of the optimal and heuristic policies changes as the expected capacity increases.

To enable sensitivity analysis, we defined a set of sensitivity parameters, denoted by Δ_j, for $j = 1, \ldots, 5$. In Experiment 1, we considered the Δ_j values given in the first column of Table 2. The parameters for customer class 1 were fixed to the configuration presented in the second column of the table. The parameters for customer class 2 were varied according to the formulas given in the third column of the table. Thus, we created a set of configurations by changing the properties of customer class 2 while the properties of customer class 1 remain fixed. The configuration in which the $(\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5) = (0.5, 0.5, 1.5, 1, 0.5)$ is referred to as the *customer base-case configuration*.

A full-factorial design based on the parameter values presented in Tables 1 and 2 gives 729 distinct configurations for Experiment 1. Because the state spaces of these configurations are reasonably small, the VIA can be used to find the optimal policy.

6.1.2. Experiment 2. These experiments use configurations with $D = 3$ channel classes and $M = 5$ customer classes. The channel class characteristics are shown in Table 3.

The parameters for the five customer classes are shown in Table 4. We do not vary $\mathbf{r}$ in the experiments. However, the arrival rates, $\boldsymbol{\lambda}$, service rates, $\boldsymbol{\mu}$, and price sensitivities, $\boldsymbol{\beta}$, are assigned in either ascending or descending order as specified in Table 4. For example, in the ascending setting for $\boldsymbol{\lambda}$, class 1 is assigned an arrival rate equal to 16 and class 5 is assigned an arrival rate equal to 20. In contrast, in the descending setting for $\boldsymbol{\lambda}$, class 1 is assigned an arrival rate equal to 20 and class 5 is assigned an arrival rate equal to 16. The overall service rates are also controlled by $\Delta_3 \in \{4, 2, 1, 0.5\}$, where smaller values of Δ_3 represent a smaller ratio of capacity to demand rate. The drop-off fee is generated in a similar way as for Experiment 1 except that there are five customer classes, $i = 1, \ldots, 5$, and we fix $\Delta_4 = 1$. We consider a full factorial design for the customer class characteristics. Thus, in total, Experiment 2 consists of a set of 32 $(= 2 \times 4 \times 2 \times 2)$ configurations.

6.1.3. Experiment 3. In practice, it is likely that the number of customer classes will be significantly larger than the five classes considered in Experiment 2. Therefore, we constructed a single configuration with a more realistic number of distinct customer classes. Customer classes are defined according to the customers' bandwidth requirement, arrival rate, service rate, and price sensitivity. We considered $r_i \in \{0.0625, 0.125, 0.25, 0.5\}$, $\lambda_i \in \{10, 50\}$, $\mu_i \in \{12.5, 37.5\}$, and $\beta_i \in \{0.9, 9\}$. Our experiment used the $M = 32$ $(= 4 \times 2 \times 2 \times 2)$ customer classes that result from considering all possible combinations of these parameter values. In this experiment, we used the $D = 3$ channel classes given in Table 3.

Table 3. Channel Characteristics in Experiments 2 and 3

Parameters	Channel class 1	Channel class 2	Channel class 3
$R_{\max}$	1	5	4
$1/\omega$	3	8	4
$1/\phi$	8	3	4

Table 4. Customer Characteristics in Experiment 2

Parameters	Ascending	Descending
r	[0.9, 0.7, 0.5, 0.3, 0.1]	
λ	[16, 17, 18, 19, 20]	[20, 19, 18, 17, 16]
μ	$\Delta_3 \times$ [1, 2, 3, 4, 5]	$\Delta_3 \times$ [5, 4, 3, 2, 1]
B	[1, 1.5, 2, 2.5, 3]	[3, 2.5, 2, 1.5, 1]

6.2. Properties of the Optimal Dynamic Pricing Policies

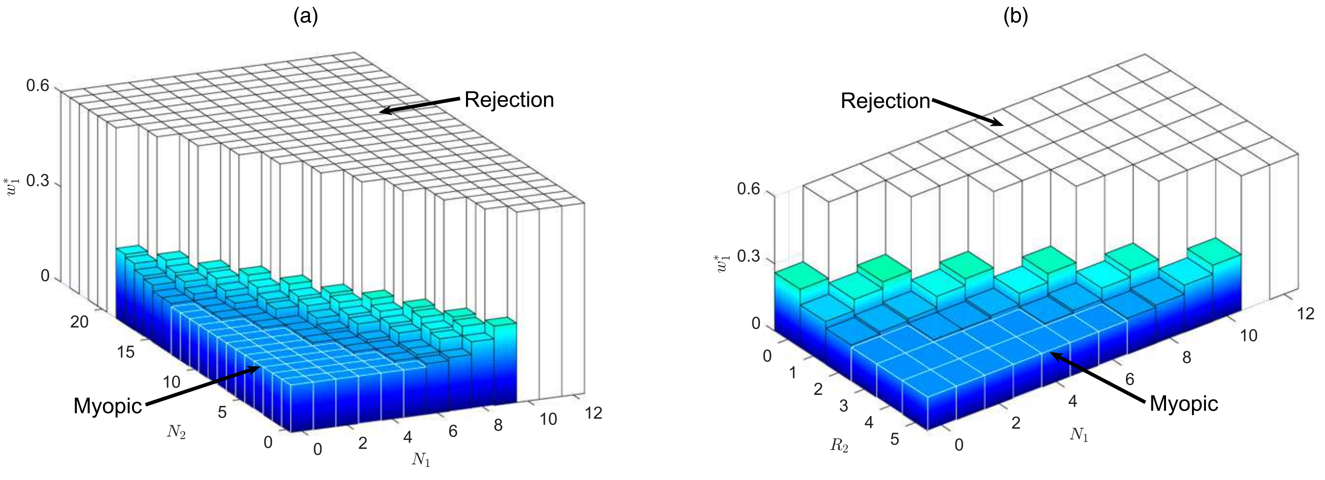
In Figure 1, we show how the optimal discounted price for customer class 1, w_1^*, changes as the state changes for the customer base-case configuration and channel case I configuration in Experiment 1. For example, in Figure 1(a), we see how w_1^* changes as N_1 and N_2 change when $R_1 = 1$ and $R_2 = 4$. When N_1 and N_2 are large relative to R_1 and R_2, the reject price is posted. This is because either (i) the remaining capacity is not sufficient to admit the arriving class 1 customer or (ii) the remaining capacity is sufficient, but the optimal policy reserves that remaining capacity for a potentially more profitable future customer. The lowest price posted is the myopic price, consistent with Corollary 1.

Although Figure 1(a) and (c) suggest that the optimal prices are nondecreasing in N_1 and N_2, the results of other configurations indicate that this does not always hold:

Observation 1. The optimal discounted prices are usually, but not always, nondecreasing in the number of customers that have been admitted to the system.

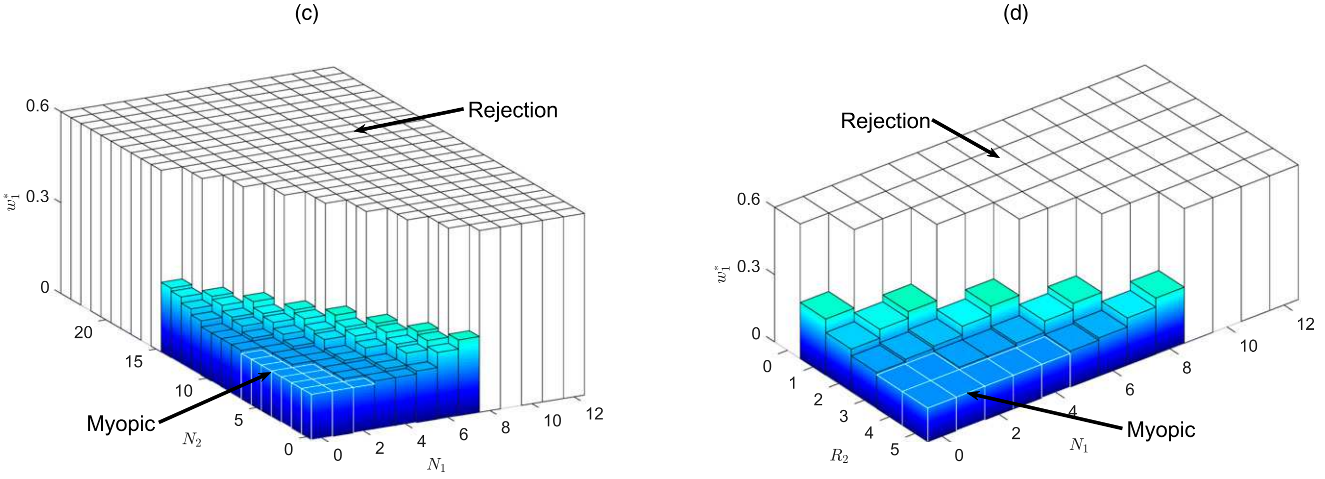
If we look at all 729 configurations that we tested, we find 486 configurations for which w_i^*, in some states, decreases when N_i increases. These violations occur for

Figure 1. (Color online) Optimal Discounted Prices w_1^* (Customer Base-Case and Channel Case I Configuration in Experiment 1)



w_1^* by N_1 and N_2 $(R_1 = 1, R_2 = 4)$

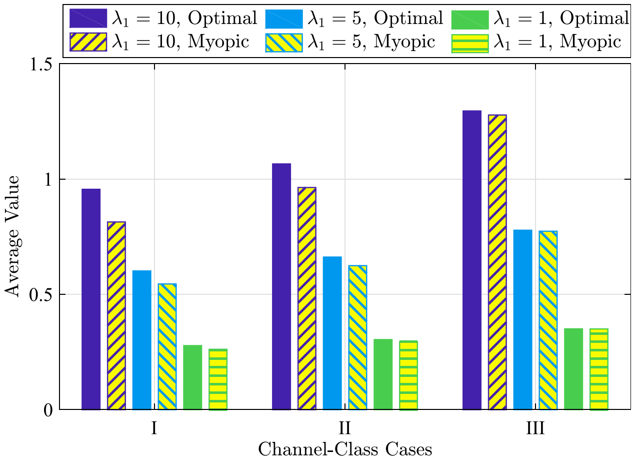
w_1^* by N_1 and R_2 $(N_2 = 2, R_1 = 1)$



w_1^* by N_1 and N_2 $(R_1 = 0, R_2 = 4)$

w_1^* by N_1 and R_2 $(N_2 = 2, R_1 = 0)$

Figure 2. (Color online) Simulated Average Values Across 40 Replications (Customer Base-Case Configuration in Experiment 1)



configurations in which $r_1 > r_2$, and they always occur for customer class 2. When we look at the specific violations, we observe that w_2^* in states where, for example, $N_2 = 0$ is sometimes larger than that in a state with $N_2 = 1$. We believe this is mainly a result of the difference in the bandwidth requirements of the two customer classes although the different service rates may also play a role as explained in Theorem 3. For example, suppose $r_1 = 2r_2$. In this case, when N_2 is an even number, such as $N_2 = 0$, accepting a customer from class 2 may in the future take the model to a state in which the remaining capacity is enough only to admit a class 2 customer but not enough to admit a class 1 customer. A similar issue does not apply when N_2 is odd, such as $N_2 = 1$. Örmeci and Burnetas (2004) find similar violations of monotonicity for loss systems in which jobs arrive in batches of random size when partial batch acceptance is not allowed and the number of servers is fixed. Specifically, they find examples in which the system may reject arriving jobs to wait for a job with a batch size that will just make use of all of the available resources (servers).

Corollary 2 indicates that the optimal discounted prices are nonincreasing in R_d; that is, $w_i^*(N, \mathbf{R}) \geq w_i^*(N, \mathbf{R} + \mathbf{e}_d^D)$ for the case of *similar* customer classes. Our numerical experiments indicate that this result holds in general as shown in Figure 1(b) and (d):

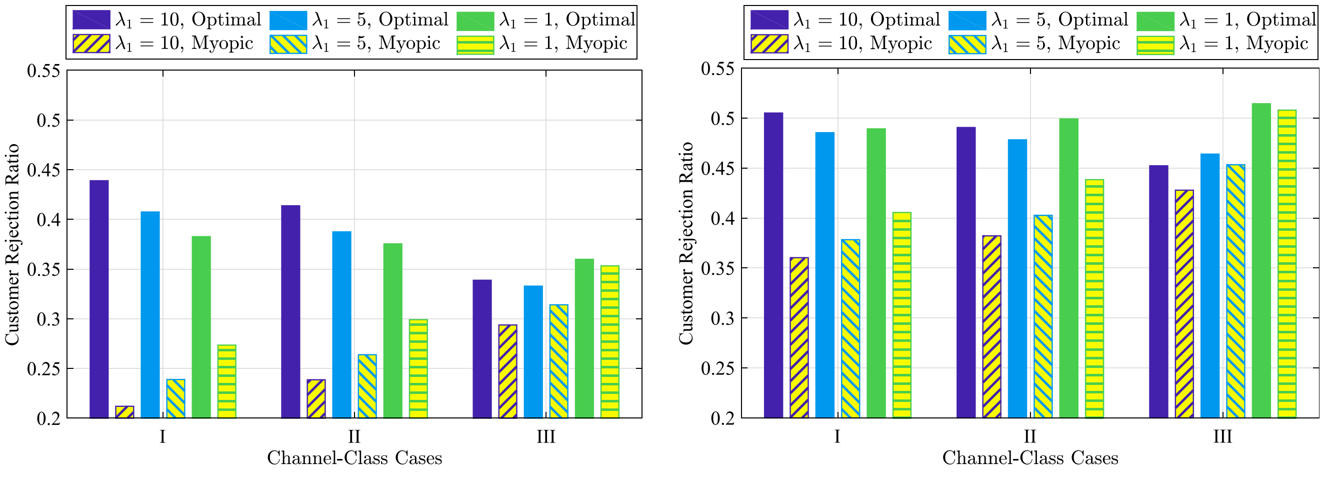
Observation 2. The optimal discounted prices are nonincreasing in the number of available channels.

Finally, the figures in Online Appendix G show how the optimal average values change as each sensitivity parameter, Δ_j, changes in Experiment 1. The results are consistent with the structural results proven in Section 4.

6.3. Performance of the Myopic Pricing Policy

We next consider the performance of the myopic policy relative to the optimal policy. Figure 2 presents the simulated average values of the optimal and myopic policies for the customer base-case configuration in Experiment 1 as the arrival rate of customer class 1, λ_1, decreases and the channel–class case changes. As expected, the average value of the optimal policy is always larger than the average value of the myopic policy. In addition, Observation 3 provides insights that are critical for the dynamic spectrum access application setting, in which scarcity of capacity (bandwidth) is likely to be a significant issue.

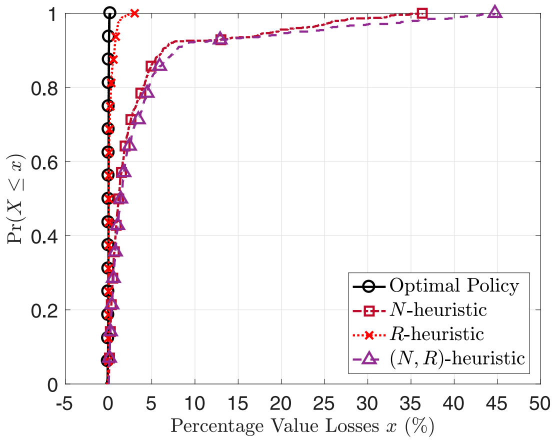
Figure 3. (Color online) Rejection Ratios Averaged Across 40 Replications (Customer Base-Case Configuration in Experiment 1)



Customer class 1

Customer class 2

Figure 4. (Color online) Empirical CDF of Percentage Value Loss in Experiment 1 (Across 40 Replications)



Observation 3. The myopic policy offers performance similar to that of the optimal policy when the customer arrival rate is small and capacity is high (recall that the expected capacity increases from case I to III). Performance degradation from the myopic policy is largest when the ratio of capacity to demand rate is low.

Figure 2 shows the customer base-case configuration. When all configurations are considered, the percentage value loss of the myopic policy with respect to the optimal policy ranges from 1.1% to 385.3% with an average of 36.2%.

To explain the performance gap between the optimal and myopic policies, we analyzed the customer rejection ratios for the optimal and myopic policies as shown in Figure 3. The customer rejection ratio is the percentage of customers that reject the posted price when there is sufficient capacity in the system. We have the following observation:

Observation 4. The customer rejection ratios for the myopic policy are always smaller than those for the optimal policy. The rejection ratios for the myopic policy decrease and deviate more from those of the optimal policy as the customer arrival rate increases and as capacity decreases (recall that the expected capacity increases from case I to III).

The intuition behind these results is that the myopic policy always posts the myopic price if capacity exists to admit the arriving customer. This leads to lower rejection ratios, which ultimately leads to a high rate of customer drop-off as well as a high rate of no-capacity rejection, that is, rejections that occur when there is no capacity available in the system to accept the arriving customer. In contrast, the optimal policy posts prices that are higher than the myopic prices when doing so is necessary to prevent congestion and future customer drop-offs. This leads to higher rejection ratios and also higher profits.

6.4. Performance of the Proposed Dynamic Pricing Heuristics

We next compare the performance of the three dynamic pricing heuristics, the N-, R-, and (N, R)-heuristics for Experiment 1. The average value estimated from the simulation of the heuristic policies is compared with the optimal average value, $\bar{v}$, obtained from the VIA. We report two kinds of percentage value losses. When the optimal policy is simulated, the percentage value loss, X, is defined as $X = (\bar{v} - \bar{V}_{\text{opt}})/\bar{v}$, where $\bar{V}_{\text{opt}}$ is the average value obtained from simulation of the optimal policy. This percentage value loss shows the accuracy of the simulation model. In contrast, when one of the heuristic policies is simulated, the percentage value loss, X, is defined as $X = (\bar{v} - \bar{V}_{\text{heu}})/\bar{v}$, where $\bar{V}_{\text{heu}}$ is the average value obtained from simulation of the heuristic policy. This percentage value loss shows the performance of the evaluated heuristic compared with optimal policy. We compute the percentage value loss for each of the 729 configurations and use these 729 observed values to draw an empirical cumulative distribution function (CDF), that is, $\Pr(X \leq x)$.

The empirical CDFs for the optimal policy and the heuristics are shown in Figure 4. For the optimal policy, the percentage error induced by random simulations is close to zero, which validates the simulation model. The CDFs of the heuristics are always to the right of that for the optimal policy. The differences between the CDFs of the heuristics and the optimal policy show the value loss resulting from aggregating customer and/or channel classes.

Observation 5. The performance of the R-heuristic is close to that of the optimal policy; that is, aggregation of channel classes results in only a slight performance degradation. The N- and (N, R)-heuristics result in more significant performance degradation. Although they provide similar performance in most configurations, in 10% of the configurations, the (N, R)-heuristic results in larger loss than the N-heuristic because it aggregates channel classes, and the N-heuristic does not. Overall, the results indicate that aggregating customer classes results in a greater loss in value than aggregating channel classes.

In Online Appendix H, the CDFs for each of the heuristics are analyzed for six key problem parameters to demonstrate which problem parameters have the greatest impact on the performance of the heuristics. The following observation summarizes the results:

Observation 6. The performance of the R-heuristic is robust to the changes in the key parameters. The

Table 5. Percentage Value Gain of (N, R)-Heuristic over Myopic Policy in Experiment 2 (Across 40 Replications)

$\Delta_3 = 4$		$\Delta_3 = 2$		$\Delta_3 = 1$		$\Delta_3 = 0.5$	
Average	Standard error	Average	Standard error	Average	Standard error	Average	Standard error
0.3%	0.7%	5.2%	2.2%	31.3%	5.5%	998.7%	723.3%

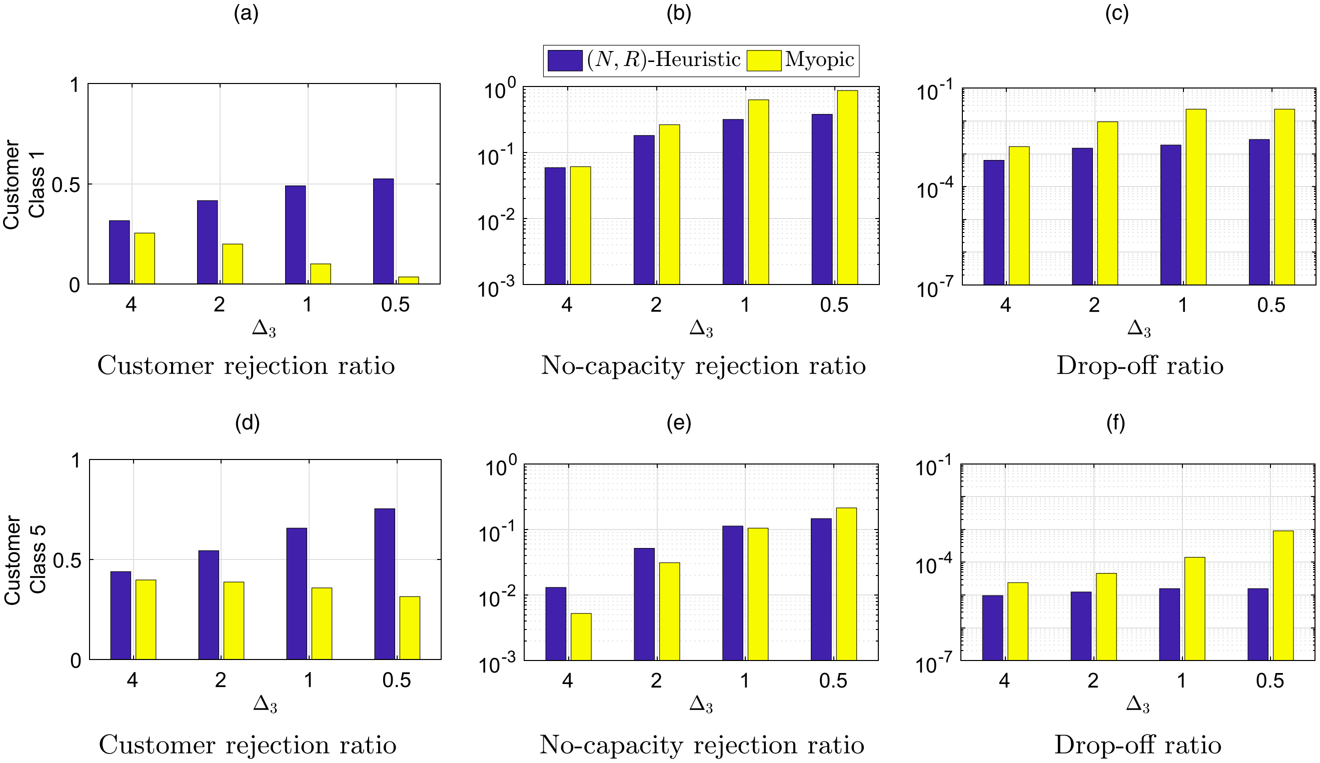
performance of the N- and (N, R)-heuristics is not affected by the service rate, arrival rate, and price sensitivity. However, the heuristic performance degrades when the bandwidth requirement for class 2 customers is small (which implies that there is a significant difference in bandwidth requirements for the classes), when the drop-off fee is large, and in channel–class case I (which represents low expected capacity).

Although Observation 5 states that the R-heuristic outperforms the N- and (N, R)-heuristics, it is also important to consider computational complexity. In practice, it is likely that there will be many customer classes, which will lead to a high-dimension state space. In that case, channel-class aggregation may not be sufficient to enable implementation of VIA; that is, customer-class aggregation may also be needed to reduce the size of the state space to a feasible level. Thus, in the remainder of the paper, we focus on the performance of the (N, R)-heuristic, which combines both customer- and channel-class aggregation.

We next evaluate the performance of the (N, R)-heuristic for Experiment 2, which has a larger state space. For these configurations, it is computationally infeasible to solve VIA to find the optimal policy. Therefore, to evaluate the performance of the (N, R)-heuristic in Experiment 2, we compare it with the non-state-dependent myopic policy. We report the percentage value gain of the heuristic policy over the myopic policy, which is defined as $(\bar{V}_{\text{heu}} - \bar{V}_{\text{myo}})/\bar{V}_{\text{myo}}$, where $\bar{V}_{\text{heu}}$ and $\bar{V}_{\text{myo}}$ are the simulated average values of the (N,R)-heuristic and myopic policies, respectively. Table 5 shows the average of this percentage value gain for each of the four levels of capacity (Δ_3) in Experiment 2. The table also reports the corresponding standard errors. Our main results are as follows:

Observation 7. Using the (N, R)-heuristic pricing policy adds the most value relative to the myopic policy

Figure 5. (Color online) Customer Rejection Ratio, No-Capacity Rejection Ratio, and Drop-off Ratio Averaged Across 40 Replications in Experiment 2



Note. The configuration of all parameters are in descending order in Table 4.

when capacity is scarce (Δ_3 small), and the value of using the (N, R)-heuristic is minimal when capacity is abundant (Δ_3 large).

The dynamic (N, R)-heuristic and the static myopic policy differ significantly in terms of how they manage capacity. This can be seen by considering the customer rejection ratio, no-capacity rejection ratio, and drop-off ratio in Experiment 2 as shown in Figure 5. The figure shows results for customer classes 1 and 5; a similar pattern holds for the other classes. Figure 5(b), (c), (e), and (f) use a logarithmic scale to clearly show the results.

Observation 8. As the ratio of capacity to demand rate decreases (as Δ_3 decreases),

- Under the myopic policy, the reject prices are offered more often because of a lack of capacity. This leads to an increase in no-capacity rejection ratios, a decrease in customer rejection ratios, and an increase in the drop-off ratios.
- Under the (N, R)-heuristic, higher prices are posted, leading to higher customer rejection ratios. Because more customers are rejecting these higher prices at admission, the no-capacity rejection ratio and the drop-off ratio are less than those for the myopic policy. Overall, a higher value is achieved.

Observations 7 and 8 demonstrate the importance of dynamically managing pricing and admission by using the dynamic (state-dependent) (N, R)-heuristic policy rather than the static myopic policy, particularly in settings with scarce capacity.

Finally, we also compared the performance of the (N, R)- and myopic heuristics for Experiment 3, which has a realistically sized state space. We found that the simulated average value of the (N, R)-heuristic was 22.3% higher than that of myopic policy.

6.5. Performance of the Optimal Static Pricing Policy

Finally, we evaluate the performance of the optimal static pricing policy. As described in Section 5.4, finding the optimal static prices is computationally challenging. Therefore, we determined the optimal static pricing policy for 27 of the 729 experimental configurations defined in Experiment 1. For these problems, we found that the percentage value loss for each heuristic, relative to the optimal average value, was as follows: 1.69% for N-heuristic, 0.51% for the R-heuristic, 2.42% for the (N, R)-heuristic, 30.84% for the myopic policy, and 10.23% for the optimal static pricing policy. In all 27 configurations, all of the dynamic pricing heuristics outperformed the optimal static pricing policy. From these experiments, we have the following observation:

Observation 9. When capacity is scarce, using dynamic pricing to manage admission of customers becomes essential.

Dynamic pricing adds value relative to static pricing when customers are highly price sensitive because dynamic pricing enables the SP to offer lower prices when the remaining capacity is sufficiently large. Overall, we conclude that, in practical settings for which capacity is expected to be tight, the dynamic pricing heuristics provide both better performance and reduced computational time compared with the optimal static pricing policy.

6.6. Comparison of Optimal and Bandwidth-Based Drop-Off Functions

As discussed in Section 4, the optimal drop-off policy is not identical to the bandwidth-based drop-off policy. However, the latter is easier to compute and to implement than the former. In addition, the proposed (N, R)-heuristic uses the bandwidth-based drop-off policy for implementation. Therefore, we used Experiment 1 to study how much the SP could lose from implementing the bandwidth-based drop-off policy rather than the optimal drop-off policy. To do so, we compared the average value when the MDP in Section 3.2 is solved using the optimal drop-off policy compared with when the MDP is solved using the bandwidth-based drop-off policy. The results are summarized in the following observation:

Observation 10. The percentage value loss from using the bandwidth-based drop-off policy rather than the optimal drop-off policy is less than 0.4% for each of the 729 configurations in Experiment 1.

7. Conclusion

Inspired by new developments in dynamic spectrum access solutions for improved management of wireless networks, we study the optimal dynamic pricing policy for a service provider who operates in a setting in which both demand (customers wanting to access the Internet) and capacity (available bandwidth) are stochastic. We characterize the behavior of the optimal pricing policy as a function of the system state (number of customers in the system and number of available channels) and of the input parameters. Further, for the setting in which both customer arrivals and capacity are stochastic, we have demonstrated the value of using a dynamic, that is, state-dependent, pricing policy compared with static, that is, non-state-dependent, policies. The previous literature, that is, GS, has studied systems with stochastic customer arrivals and fixed capacity. This previous work has focused on analytically and numerically characterizing the conditions under which static myopic policies perform optimally or close to optimally. As in this previous work, we find that using the optimal dynamic pricing policy rather than the myopic policy becomes more critical when capacity is scarce. In addition, when capacity is scarce, the optimal pricing policy outperforms the myopic policy in terms of the no-capacity rejection ratios and customer drop-off ratios, which can be viewed as measures of quality of service and access.

Compared with the setting studied in GS, our problem setting is more dynamic; that is, our setting has dynamic capacity as channels become available and unavailable for use. Therefore, identifying good state-dependent heuristic pricing policies, that is, policies that adjust the price based on the actual capacity as well as the number of customers currently being served, is of greater value for our problem setting than for the GS setting with constant capacity. Thus, we develop and propose the (N, R)-heuristic in which the pricing policy is dependent on the aggregate capacity as well as the aggregate number of customers being served. Although still state-dependent, this policy is computationally much more tractable and easier to implement in practical settings than both the optimal dynamic pricing policy and the optimal static pricing policy. We find that the (N, R)-heuristic can provide a significant performance improvement relative to the myopic and optimal static pricing policies, particularly when capacity is scarce. In the dynamic spectrum access application, scarcity of capacity (bandwidth) is likely to be a significant issue. Thus, in practice, the dynamic heuristic pricing policy is likely to provide significantly improved profits for the service provider as well as improved quality of service (lower drop-off ratios) and access (lower no-capacity rejection ratios), compared with the static pricing policies.

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