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CONSTRUCTIVE APPROXIMATION

On Uniform Convergence of Diagonal Multipoint Padé Approximants for Entire Functions

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Abstract We prove that for most entire functions f in the sense of category, a strong form of the Baker–Gammel–Wills conjecture holds. More precisely, there is an infinite sequence $\mathcal{S}$ of positive integers n, such that given any $r > 0$, and multipoint Padé approximants R_n to f with interpolation points in $\{z : |z| \le r\}$, $\{R_n\}_{n\in S}$ converges locally uniformly to f in the plane. The sequence $\mathcal{S}$ does not depend on r, or on the interpolation points. For entire functions with smooth rapidly decreasing coefficients, full diagonal sequences of multipoint Padé approximants converge.

Keywords Padé approximation · Multipoint Padé approximants · Spurious poles

Mathematics Subject Classification 41A21 · 41A20 · 30E10

1 Introduction

Let D be an open connected subset of $\mathbb{C}$, and let $f : D \to \mathbb{C}$ be analytic. Given $n \ge 0$ and $2n+1$ not necessarily distinct points $\Lambda_n = \{z_j\}_{j=1}^{2n+1}$ in D, and

$$\omega_n(z) = \omega_n(\Lambda_n, z) = \prod_{j=1}^{2n+1} (z - z_j),$$

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the (n, n) multipoint Padé approximant to f with interpolation set Λ_n is a rational function

$$R_n(\Lambda_n, z) = \frac{p_n(\Lambda_n, z)}{q_n(\Lambda_n, z)},$$

or more simply,

$$R_n(z) = \frac{p_n(z)}{q_n(z)},$$

where p_n and q_n are polynomials of degree $\leq n$, with q_n not identically zero, such that

$$\frac{f(z)\, q_n(z) - p_n(z)}{\omega_n(z)}$$

is analytic in D. The special case where all $z_j = 0$ gives the Padé approximant $[n/n](z)$. It is easily seen that R_n exists and is unique, though p_n and q_n are not separately unique.

The convergence of Padé and multipoint Padé approximants is a much studied subject, with uniform convergence established for large classes of special functions. One of the pitfalls of the method is the appearance of *spurious poles*, namely poles that do not reflect the analytic properties of the interpolated function f [1,2,5,8,12, 14,22–24,27,29–31,33]. For this reason, the most general results, such as the Nuttall–Pommerenke theorem, often involve convergence in capacity, rather than uniform convergence. In 1961, Baker, Gammel, and Wills nevertheless conjectured that at least a subsequence of the diagonal Padé sequence converges locally uniformly. Throughout this paper,

$$B_r = \{z : |z| < r\}, r > 0.$$

Baker–Gammel–Wills Conjecture
Let f be meromorphic in B_1 and analytic at 0*. Then there is a subsequence $\{[n/n]\}_{n\in\mathcal{S}}$ of $\{[n/n]\}_{n\geq 1}$ that converges uniformly to f in compact subsets of B_1 omitting poles of f.*

The author showed in 2001 [24] that the conjecture is false, by considering the Rogers–Ramanujan function with a nonstandard value of q on the unit circle. While this provided a meromorphic counterexample, A.P. Buslaev quickly followed [6] with an analytic counterexample, formed from an algebraic function, and then showed that even the Rogers–Ramanujan function provides an analytic counterexample [7]. Baker [3] subsequently noted that for these counterexamples, just two subsequences together provide locally uniform convergence in the unit ball. He went on to conjecture that a patchwork of finitely many subsequences can provide locally uniform convergence for functions meromorphic in the ball [4].

One of the unsolved issues is whether the Baker–Gammel–Wills conjecture is valid for entire functions, or perhaps even functions meromorphic in the plane. To date, there is still no counterexample. The author proved [19] that the Baker–Gammel–Wills conjecture is true for most entire functions in the sense of category.

In this paper, we shall show that a stronger form of the conjecture, allowing interpolation points in any compact set, with the same subsequence, is true for most entire functions in the sense of category. Accordingly, let $\mathcal{A}$ denote the space of entire functions, with metric defined in terms of power series coefficients: if

$$f(z)=\sum_{j=0}^{\infty}a_jz^j \text{ and } g(z)=\sum_{j=0}^{\infty}b_jz^j,$$

then define

$$d^*(f,g)=\sup_{j\geq 0}\left|a_j-b_j\right|^{1/\max\{j,1\}}. \tag{1.1}$$

Convergence in this space is equivalent to uniform convergence in compact sets. Recall that a subset of $\mathcal{A}$ is of the first category if it is a countable union of nowhere dense sets. As such, it is small in the sense of category. Recall too that an F_σ set is a countable union of closed sets.

Theorem 1.1
There is an F_σ subset $\mathcal{E}$ of $\mathcal{A}$ of the first category, such that for $f\in\mathcal{A}\backslash\mathcal{E}$, there is an infinite subsequence $\mathcal{S}$ of positive integers with the following property: given any $r>0$ and for $n\in\mathcal{S}$, multipoint Padé approximants R_n to f of type (n,n) formed from interpolation points $\Lambda_n\subset B_r$, we have

$$\lim_{n\to\infty,n\in\mathcal{S}}R_n(z)=f(z) \tag{1.2}$$

uniformly in compact subsets of the plane.

Observe that while $\mathcal{S}$ depends on f, it does not depend on the ball B_r in which the interpolation points lie. As far as the author is aware, e^z is the only function for which diagonal rational interpolants with interpolation points in any compact set (and that are not restricted to include complex conjugate interpolation points) has been proven to converge locally uniformly [10,32]. For Markov–Stieltjes functions, convergence of diagonal multipoint Padé approximants, with interpolation points symmetric about the real axis, has been investigated in [9,13].

We also prove some more explicit results when the Maclaurin series coefficients decay rapidly and/or smoothly:

Theorem 1.2
Assume that

$$f(z)=\sum_{j=0}^{\infty}a_jz^j, \tag{1.3}$$

where $a_j\neq 0$ for $j\geq 0$ and for some fixed J, and for $j\geq J$,

$$\left|\frac{a_{j-1}a_{j+1}}{a_j^2}\right|\leq\chi^2, \tag{1.4}$$

where $\chi<\rho_0$ and $\rho_0=0.4559\ldots$ is the positive root of the equation

$$\sum_{j=1}^{\infty}\rho^{j^2}=\frac{1}{2}. \tag{1.5}$$

Let $r > 0$. For $n \geq 1$, let R_n denote a multipoint Padé approximant to f formed with interpolation points in B_r. Then uniformly in compact subsets of the plane,

$$\lim_{n \to \infty} R_n(z) = f(z).$$

Theorem 1.3
Assume that f is given by (1.3), where $a_j \neq 0$ for $j \geq 0$ and for some $|q| < 1$,

$$\lim_{j \to \infty} \frac{a_{j-1} a_{j+1}}{a_j^2} = q. \tag{1.6}$$

Then the conclusion of Theorem 1.2 remains valid.

We note that Theorems 1.2 and 1.3 were proved for the special case of Padé approximants in [20,21]. In [20], the slightly more general condition $\chi \leq \rho_0$ was allowed. We note also that Theorems 1.2 and 1.3 and the results of [25] show that given $s > r > 0$, then for large enough n, $f - R_n$ formed from interpolation points in B_r, has exactly $2n + 1$ zeros, counting multiplicity, in B_s. Related results dealing with smooth Maclaurin series coefficients appear in [11,15,16,28]. Without smoothness but with more rapid decay, we prove convergence of a subsequence:

Theorem 1.4
Assume that for f given by (1.3),

$$\limsup_{j \to \infty} \left| a_j \right|^{1/j^2} < \frac{1}{3}. \tag{1.7}$$

Then there is a subsequence $\mathcal{S}$ of integers with the property (1.2) as described in Theorem 1.1.

Theorem 1.1 suggests a stronger form of the Baker–Gammel–Wills conjecture for entire functions:

Conjecture 1.5
Let f be entire. Then there is an infinite subsequence $\mathcal{S}$ of positive integers with the following property: given any $r > 0$ and for $n \in \mathcal{S}$, multipoint Padé approximants R_n to f of type (n, n) formed from interpolation points $\Lambda_n \subset B_r$, we have (1.2).

We close this section with more notation, firstly, finite differences: given distinct $z_1, z_2, z_3, \ldots$, define $f[z_1] = f(z_1)$,

$$f[z_1, z_2] = \frac{f(z_2) - f(z_1)}{z_2 - z_1};$$

and recursively, for $r \geq 2$,

$$f[z_1, \ldots, z_{r+1}] = \frac{f[z_1, \ldots, z_{r-1}, z_{r+1}] - f[z_1, \ldots, z_{r-1}, z_r]}{z_{r+1} - z_r}.$$

When points coalesce, that is, not all $\{z_j\}$ are distinct, the finite difference is defined as the appropriate derivative. We also set

$$f_{i,j} = f\left[z_{i+1}, z_{i+2}, \ldots, z_{j+1}\right]. \tag{1.8}$$

We shall make extensive use of the following formula for the denominator in $R_n(\Lambda_n, z)$ when $\Lambda_n = \{z_j\}_{j=1}^{2n+1}$ [2, p. 339]:

$$\det \begin{bmatrix} f_{n,n+1} & f_{n,n+2} & \cdots & f_{n,2n} & \prod_{k=1}^{n}(z-z_k) \\ f_{n-1,n+1} & f_{n-1,n+2} & \cdots & f_{n-1,2n} & \prod_{k=1}^{n-1}(z-z_k) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{0,n+1} & f_{0,n+2} & \cdots & f_{0,2n} & 1 \end{bmatrix}.$$

It is valid as long as this last determinant is not identically 0. By row and column swaps, we can recast it (absorbing a sign change into the numerator polynomial) as

$$q_n(\Lambda_n, z) = \det \begin{bmatrix} 1 & f_{0,n+1} & f_{0,n+2} & \cdots & f_{0,2n} \\ z-z_1 & f_{1,n+1} & f_{1,n+2} & \cdots & f_{1,2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \prod_{k=1}^{n-1}(z-z_k) & f_{n-1,n+1} & f_{n-1,n+2} & \cdots & f_{n-1,2n} \\ \prod_{k=1}^{n}(z-z_k) & f_{n,n+1} & f_{n,n+2} & \cdots & f_{n,2n} \end{bmatrix}. \tag{1.9}$$

Throughout this paper, we assume that f is entire, not a polynomial, and has Maclaurin series given by (1.3). The paper is organized as follows: we prove Theorems 1.1 and 1.4 in Sect. 2, Theorem 1.2 in Sect. 3, and Theorem 1.3 in Sect. 4.

2 Proof of Theorems 1.1 and 1.4

We begin by bounding coefficients of nonpolynomial entire functions, given in (1.3), much as in [19]. Let

$$K = \max\{1, |a_0|\}.$$

Define an increasing sequence of integers

$$0 = j_0 < j_1 < j_2 < \cdots$$

and positive numbers $\{\rho_j\}$ as follows: first, choose $j_1 \geq 1$ such that

$$\rho_{j_1}^{-1} = \left(\frac{|a_{j_1}|}{K}\right)^{1/j_1} = \max\left\{\left(\frac{|a_j|}{K}\right)^{1/j} : j \geq 1\right\}.$$

Having defined $\rho_{j_1}, \ldots, \rho_{j_k}$, define $\rho_{j_{k+1}}$ by

$$\rho_{j_{k+1}}^{-1} = \left|\frac{a_{j_{k+1}}}{a_{j_k}}\right|^{\frac{1}{j_{k+1}-j_k}} = \max\left\{\left|\frac{a_j}{a_{j_k}}\right|^{\frac{1}{j-j_k}} : j > j_k\right\}.$$

If there is more than one choice of j_k, choose the largest one. Define

$$\rho_n = \rho_{j_{k+1}} \text{ for } j_k + 1 \leq n \leq j_{k+1} \text{ and } k \geq 0.$$

Lemma 2.1

(a)

$$|a_n| \leq K / \prod_{\ell=1}^{n} \rho_\ell, \textit{ for } n \geq 0, \tag{2.1}$$

with equality when

$$n = j_k \textit{ for some } k \geq 1.$$

(b) $\{\rho_k\}$ *is an increasing sequence with limit* ∞.

(c) *If* $n = j_k$ *for some* $k \geq 1$, *then for* $r \geq -n$,

$$\left|\frac{a_{n+r}}{a_n}\right| \leq \begin{cases} \rho_{n+1}^{-r}, & r \geq 0, \\ \rho_n^{-r}, & r < 0. \end{cases} \tag{2.2}$$

Proof

(a) Suppose that k is given, and $j_k + 1 \leq n \leq j_{k+1}$. Then by definition of j_{k+1},

$$\left|\frac{a_n}{a_{j_k}}\right|^{\frac{1}{n-j_k}} \leq \rho_{j_{k+1}}^{-1}$$
$$\Rightarrow |a_n| \leq \left|a_{j_k}\right| \rho_{j_{k+1}}^{-(n-j_k)} = \left|a_{j_k}\right| / \prod_{\ell=j_k+1}^{n} \rho_\ell.$$

We have equality if $n = j_{k+1}$. Applying this inequality recursively to $a_{j_k}, a_{j_{k-1}}, \ldots$, we obtain (2.1), with equality if n equals some j_k, for some $k \geq 1$.

(b) Now as $j_{k+2} > j_{k+1}$,

$$\rho_{j_{k+1}}^{-1} = \left|\frac{a_{j_{k+1}}}{a_{j_k}}\right|^{\frac{1}{j_{k+1}-j_k}} > \left|\frac{a_{j_{k+2}}}{a_{j_k}}\right|^{\frac{1}{j_{k+2}-j_k}}$$
$$= \left|\frac{a_{j_{k+2}}}{a_{j_{k+1}}}\right|^{\frac{1}{j_{k+2}-j_k}} \left|\frac{a_{j_{k+1}}}{a_{j_k}}\right|^{\frac{1}{j_{k+2}-j_k}}$$

$$\Rightarrow \left|\frac{a_{j_{k+1}}}{a_{j_k}}\right|^{\frac{1}{j_{k+1}-j_k}-\frac{1}{j_{k+2}-j_k}} > \left|\frac{a_{j_{k+2}}}{a_{j_{k+1}}}\right|^{\frac{1}{j_{k+2}-j_k}}$$

$$\Rightarrow \left|\frac{a_{j_{k+1}}}{a_{j_k}}\right|^{\frac{j_{k+2}-j_{k+1}}{j_{k+1}-j_k}} > \left|\frac{a_{j_{k+2}}}{a_{j_{k+1}}}\right|$$

$$\Rightarrow \left|\frac{a_{j_{k+1}}}{a_{j_k}}\right|^{\frac{1}{j_{k+1}-j_k}} > \left|\frac{a_{j_{k+2}}}{a_{j_{k+1}}}\right|^{\frac{1}{j_{k+2}-j_{k+1}}}$$

$$\Rightarrow \rho_{j_{k+1}}^{-1} > \rho_{j_{k+2}}^{-1}.$$

The monotonicity of $\{\rho_n\}$ follows, and the fact that f is entire forces them to have limit ∞.

(c) If $r \geq 0$, we have (using that there is equality in (2.1) for $n = j_k$),

$$\left|\frac{a_{n+r}}{a_n}\right| \leq \prod_{\ell=n+1}^{n+r} \rho_\ell^{-1} \leq \rho_{n+1}^{-r}.$$

If $r < 0$, we instead have

$$\left|\frac{a_{n+r}}{a_n}\right| \leq \prod_{\ell=n+r}^{n-1} \rho_\ell \leq \rho_n^{-r}.$$

□

We shall frequently use a series expansion for finite differences. Assume $R > 0$, $\ell \geq 1$, and $z_1, z_2, \ldots, z_\ell \in \overline{B_R}$. Then by the contour integral representation for finite differences [26, p. 11], if $\Gamma = \{t : |t| = S\}$, where $S > R$, then

$$\begin{aligned} f[z_1, z_2, \ldots, z_\ell] &= \frac{1}{2\pi i} \int_\Gamma \frac{f(t)}{\prod_{k=1}^{\ell}(t - z_k)} dt \\ &= \frac{1}{2\pi i} \int_\Gamma \frac{f(t)}{t^\ell} \sum_{i_1 \cdots i_\ell \geq 0} \frac{z_1^{i_1} z_2^{i_2} \cdots z_\ell^{i_\ell}}{t^{i_1+i_2+..+i_\ell}} dt \\ &= \sum_{i_1 \cdots i_\ell \geq 0} z_1^{i_1} z_2^{i_2} \cdots z_\ell^{i_\ell} a_{\ell-1+i_1+i_2+..+i_\ell}. \end{aligned} \tag{2.3}$$

Lemma 2.2
Assume that $n = j_i$ for some $i \geq 1$, where the $\{j_i\}$ are as above. Assume that all $|z_j| \leq R$ and $\rho_n > R$.

(a) *Then for $0 \leq j, k \leq n$, with the notation (1.8),*

$$(\rho_n \rho_{n+1})^{\frac{k-j}{2}} \left|\frac{f_{j,n+k}}{a_n}\right| \leq \left(\frac{\rho_n}{\rho_{n+1}}\right)^{\frac{|k-j|}{2}} (1 - R/\rho_n)^{-(n+1+k-j)}. \tag{2.4}$$

(b) *For* $0 \le j \le n$,

$$\left| \frac{f_{j,n+j}}{a_n} - 1 \right| \le (1 - R/\rho_{n+1})^{-(n+1)} - 1. \tag{2.5}$$

Proof

(a) Let $\ell = n + 1 + k - j$. From the series (2.3) above,

$$\begin{aligned} \left| \frac{f_{j,n+k}}{a_n} \right| &= \left| \sum_{i_1 \cdots i_\ell \ge 0} z_{j+1}^{i_1} z_{j+2}^{i_2} \cdots z_{j+\ell}^{i_\ell} \frac{a_{\ell-1+i_1+i_2+\cdots+i_\ell}}{a_n} \right| \\ &\le \sum_{s=0}^{\infty} \left| \frac{a_{\ell-1+s}}{a_n} \right| R^s \sum_{\substack{i_1 \cdots i_\ell \ge 0, \\ i_1+i_2+\cdots+i_\ell = s}} 1 \\ &= \sum_{s=0}^{\infty} \left| \frac{a_{\ell-1+s}}{a_n} \right| R^s \left| \binom{-\ell}{s} \right|. \end{aligned} \tag{2.6}$$

If $k \ge j$, then all indices $\ell - 1 + s \ge n$, so (2.2) gives

$$\begin{aligned} \left| \frac{f_{j,n+k}}{a_n} \right| &\le \sum_{s=0}^{\infty} \rho_{n+1}^{-(\ell-1-n+s)} R^s \left| \binom{-\ell}{s} \right| \\ &= \rho_{n+1}^{-(\ell-1-n)} (1 - R/\rho_{n+1})^{-\ell}. \end{aligned}$$

Then

$$(\rho_n \rho_{n+1})^{\frac{k-j}{2}} \left| \frac{f_{j,n+k}}{a_n} \right| \le \left(\frac{\rho_n}{\rho_{n+1}} \right)^{\frac{k-j}{2}} (1 - R/\rho_{n+1})^{-(n+1+k-j)}. \tag{2.7}$$

If $k < j$, then we split into those indices $<$ and $\ge n$, and use the appropriate inequalities in (2.2):

$$\begin{aligned} \left| \frac{f_{j,n+k}}{a_n} \right| &\le \sum_{s=0}^{j-k-1} \rho_n^{-(\ell-1-n+s)} R^s \left| \binom{-\ell}{s} \right| + \sum_{s=j-k}^{\infty} \rho_{n+1}^{-(\ell-1-n+s)} R^s \left| \binom{-\ell}{s} \right| \\ &\le \sum_{s=0}^{j-k-1} \rho_n^{-(k-j+s)} R^s \left| \binom{-\ell}{s} \right| + \sum_{s=j-k}^{\infty} \rho_n^{-(k-j+s)} R^s \left| \binom{-\ell}{s} \right| \\ &= \rho_n^{-(k-j)} (1 - R/\rho_n)^{-\ell}. \end{aligned}$$

Then

$$(\rho_n \rho_{n+1})^{\frac{k-j}{2}} \left| \frac{f_{j,n+k}}{a_n} \right| \leq \left(\frac{\rho_n}{\rho_{n+1}} \right)^{\frac{|k-j|}{2}} (1 - R/\rho_n)^{-(n+1+k-j)} .$$

Since $\rho_n \leq \rho_{n+1}$, this and (2.7) give (2.4).

(b) From (2.6) with $\ell = n + 1$,

$$\left| \frac{f_{j,n+j}}{a_n} - 1 \right| \leq \sum_{s=1}^{\infty} \left| \frac{a_{n+s}}{a_n} \right| R^s \left| \binom{-n-1}{s} \right|$$
$$\leq \sum_{s=1}^{\infty} (R/\rho_{n+1})^s \left| \binom{-n-1}{s} \right| = (1 - R/\rho_{n+1})^{-(n+1)} - 1.$$

□

In estimating the denominators, we need the notion of diagonal dominance: a matrix

$$B = \left[b_{jk} \right]_{1 \leq j,k \leq n}$$

is called *diagonally dominant* if for all $1 \leq j \leq n$, we have

$$\left| b_{jj} \right| > \sum_{k=1, k \neq j}^{n} \left| b_{jk} \right| .$$

We shall use the basic fact that a diagonally dominant matrix has nonzero determinant [17, p. 373].

Lemma 2.3
Assume that for some $\varepsilon \in \left(0, \frac{1}{2}\right)$, *infinite sequence of integers* $\mathcal{S}$, *and* $n = j_i, i \in \mathcal{S}$,

$$\frac{\rho_n}{\rho_{n+1}} \leq \frac{1}{9} (1 - \varepsilon) \tag{2.8}$$

and

$$\lim_{n = j_i, i \in \mathcal{S}} \frac{\rho_n}{n} = \infty. \tag{2.9}$$

Then for any $R > 0$ *and all* $\Lambda_n \subset \overline{B_R}$, *we have for large enough* $n = j_i$, $i \in \mathcal{S}$,

$$\inf_{|z| \leq R} |q_n (\Lambda_n, z)| > 0. \tag{2.10}$$

Proof
Let us assume that the integers $\{j_i\} = \{j_i (f)\}$ and $\{\rho_n\}$ are chosen as above. Assume that $n = j_i$ for some i. We use (1.9):

$$q_n (\Lambda_n, z) = \det [\mathbf{c}\, B] ,$$

where **c** is an $(n+1)\times 1$ column vector and B is an $(n+1)\times n$ matrix:

$$\mathbf{c}=\left[\prod_{\ell=1}^{j-1}(z-z_\ell)\right]_{1\le j\le n+1};\ B=\left[f_{j-1,n+k-1}\right]_{\substack{1\le j\le n+1;\\ 2\le k\le n+1}}.$$

We divide the 2nd, 3rd, …, $(n+1)$st column in $[\mathbf{c}\ B]$ by a_n and multiply the jth row by $(\rho_{n+1}\rho_n)^{-j/2}$ and kth column by $(\rho_{n+1}\rho_n)^{k/2}$ for all j,k. Then we obtain

$$q_n(\Lambda_n,z)\,a_n^{-n}=\det\left[\hat{\mathbf{c}}\ \hat{B}\right],$$

where

$$\hat{\mathbf{c}}=\left[(\rho_{n+1}\rho_n)^{(1-j)/2}\prod_{\ell=1}^{j-1}(z-z_\ell)\right]_{1\le j\le n+1};$$
$$\hat{B}=\left[(\rho_{n+1}\rho_n)^{(k-j)/2}\frac{f_{j-1,n+k-1}}{a_n}\right]_{\substack{1\le j\le n+1;\\ 2\le k\le n+1}}.$$

Let $R>0$, $|z|\le R$, and all $|z_j|\le R$. We now show that this last matrix $\left[\hat{\mathbf{c}}\ \hat{B}\right]$ is diagonally dominant for $n=j_i$ and i large enough. Consider the jth row. If $j=1$, its diagonal element is 1. For $j\ge 2$, the diagonal element is

$$\frac{f_{j-1,n+j-1}}{a_n}=1+\varepsilon_{j,n}, \tag{2.11}$$

where by the previous lemma and by (2.9),

$$\left|\varepsilon_{j,n}\right|\le(1-R/\rho_{n+1})^{-n}-1\to 0\text{ as }n\to\infty \tag{2.12}$$

uniformly in j. Now consider the sum of the absolute values of the nondiagonal elements in the jth row of $\left[\hat{\mathbf{c}}\ \hat{B}\right]$, namely

$$\tau_j:=(\rho_{n+1}\rho_n)^{(1-j)/2}\left|\prod_{\ell=1}^{j-1}(z-z_\ell)\right|$$
$$+\left(\sum_{k=2}^{j-1}+\sum_{k=j+1}^{n+1}\right)(\rho_{n+1}\rho_n)^{(k-j)/2}\left|\frac{f_{j-1,n+k-1}}{a_n}\right|.$$

Of course, if $j = 1$, the first term and first sum are omitted. Using that $|z| \le R$, and all $|z_k| \le R$, and (2.4), we continue this as

$$\begin{aligned}
\tau_j &\le \left(\frac{2R}{(\rho_{n+1}\rho_n)^{1/2}}\right)^{j-1} + \left(\sum_{k=2}^{j-1} + \sum_{k=j+1}^{n+1}\right)\left(\frac{\rho_n}{\rho_{n+1}}\right)^{\frac{|k-j|}{2}} (1 - R/\rho_n)^{-(n+1+k-j)} \\
&\le \frac{2R}{(\rho_{n+1}\rho_n)^{1/2}} + (1 - R/\rho_n)^{-2n-2}\, 2\sum_{\ell=1}^{\infty}\left(\frac{\rho_n}{\rho_{n+1}}\right)^{\ell/2} \\
&= \frac{2R}{(\rho_{n+1}\rho_n)^{1/2}} + (1 - R/\rho_n)^{-2n-2} \frac{2\left(\frac{\rho_n}{\rho_{n+1}}\right)^{1/2}}{1 - \left(\frac{\rho_n}{\rho_{n+1}}\right)^{1/2}} \\
&< o(1) + (1 + o(1)) \frac{\frac{2}{3}(1-\varepsilon)^{1/2}}{1 - \frac{1}{3}(1-\varepsilon)^{1/2}} < (1 + o(1))(1-\varepsilon)^{1/2},
\end{aligned}$$

by (2.8) and (2.9). In view of (2.11), (2.12), we have diagonal dominance of $\left[\hat{\mathbf{c}}\ \hat{B}\right]$, and then (2.10) follows. □

Proof of Theorem 1.4
Now for $n = j_i$, for some $i \ge 1$, we have

$$\begin{aligned}
|a_n| &= K/\prod_{\ell=1}^{n} \rho_\ell \ge K/\rho_n^n \\
&\Rightarrow \liminf_{n\to\infty, n=j_i} \rho_n^{1/n} \ge \liminf_{n\to\infty, n=j_i} K^{1/n^2}/|a_n|^{1/n^2} \ge 3/(1-\varepsilon)^{1/2}
\end{aligned}$$

for some $\varepsilon > 0$, by (1.7). Thus ρ_{j_i} grows roughly at least as fast as $\left(3(1-\varepsilon)^{1/2}\right)^{j_i}$. Next, for $n = j_i$,

$$\begin{aligned}
|a_n| &= \frac{K}{\rho_1^n}\prod_{\ell=1}^{n}\frac{\rho_1}{\rho_\ell} = \frac{K}{\rho_1^n}\prod_{\ell=2}^{n}\left(\frac{\rho_1}{\rho_2}\frac{\rho_2}{\rho_3}\frac{\rho_3}{\rho_4}\cdots\frac{\rho_{\ell-1}}{\rho_\ell}\right) \\
&= \frac{K}{\rho_1^n}\left(\frac{\rho_1}{\rho_2}\right)^{n-1}\left(\frac{\rho_2}{\rho_3}\right)^{n-2}\left(\frac{\rho_3}{\rho_4}\right)^{n-3}\cdots\frac{\rho_{n-1}}{\rho_n}.
\end{aligned}$$

Then from (1.7), for some $\varepsilon > 0$,

$$\begin{aligned}
\frac{(1-\varepsilon)^{1/2}}{3} &\ge \limsup_{n\to\infty} |a_n|^{1/n^2} \\
&\ge \liminf_{n\to\infty}\left(\frac{\rho_n}{\rho_{n+1}}\right)^{[(n-1)+(n-2)+(n-3)\cdots+1]/n^2} \\
&= \liminf_{n\to\infty}\left(\frac{\rho_n}{\rho_{n+1}}\right)^{1/2}.
\end{aligned}$$

So for some infinite sequence of integers $\mathcal{S}$,

$$\frac{\rho_n}{\rho_{n+1}} \leq \frac{1-\varepsilon}{9},\ n \in \mathcal{S}.$$

So we have (2.8). But since $\rho_{n+1} = \rho_n$ unless $n = j_i$ and $n+1 = j_i + 1$ for some i, so, as above, $\rho_n = \rho_{j_i}$ grows at least as fast as $(3\,(1-\varepsilon))^{j_i}$. Then (2.9) also follows. Now let $S > R > 0$. It follows from the previous lemma that for $z \in B_S$ and $\Lambda_n \subset B_R$, we have for large enough $n \in \mathcal{S}$ that $|q_n\,(\Lambda_n, z)| > 0$, so $R_n\,(\Lambda_n, z)$ has no poles in B_S. Then the uniform convergence in compact subsets of B_S follows easily from the contour integral error formula for multipoint Padé approximation [2]. □

We turn to the proof of Theorem 1.1 and first introduce some notation. Recall that $\mathcal{A}$ denotes the space of entire functions, with metric defined by (1.1). Given $R > 0$, $n \geq 1$, we let

$$\mathcal{B}_{n,R} = \left\{ f \in \mathcal{A} : q_n\,(\Lambda_n, z) \text{ has full degree } n \text{ and no zeros in } \overline{B_R} \text{ whenever } \Lambda_n \subset \overline{B_R} \right\}.$$

Also, let

$$\mathcal{C}_{n,R} = \bigcup_{j=n}^{\infty} \mathcal{B}_{j,R}.$$

Lemma 2.4
For each n and $R > 0$, $\mathcal{C}_{n,R}$ is open and dense in $\mathcal{A}$.

Proof
We first show that each $\mathcal{B}_{n,R}$ is open, and then the openness of $\mathcal{C}_{n,R}$ follows. Fix an n and $f \in \mathcal{B}_{n,R}$. Since we need to indicate dependence of the multipoint Padé denominators on f, we use the notation $q_n\,(f, \Lambda_n, z)$ in this proof only. By compactness, and the continuity of $q_n\,(f, \Lambda_n, z)$ in Λ_n as long as it has full degree [as follows from (1.8), (1.9)], we see that

$$\min \left\{ |q_n\,(f, \Lambda_n, z)| : z \in \overline{B_R}, \Lambda_n \subset \overline{B_R} \right\} > 0.$$

Moreover, by our hypothesis that the denominators have full degree, all their leading coefficients are nonzero, and then also from (1.9).

$$\min \left\{ \left| \det \left[f_{j-1,n+k} \right]_{1 \leq j,k \leq n} \right| : \Lambda_n \subset \overline{B_R} \right\} > 0.$$

If we consider entire g with $d^*\,(f, g)$ small enough, then the Maclaurin series coefficients of g will be as close to those of f as we please, and consequently all finite differences $g_{j-1,n+k}$ will be close to the corresponding differences for f. Then we can ensure that also

$$\min \left\{ \left| \det \left[g_{j-1,n+k} \right]_{1 \leq j,k \leq n} \right| : \Lambda_n \subset \overline{B_R} \right\} > 0.$$

By the same token, considering the formula (1.9), we can also ensure that

$$\min\left\{|q_n(g,\Lambda_n,z)| : z\in\overline{B_R},\ \Lambda_n\subset\overline{B_R}\right\}>0.$$

Thus also $g\in\mathcal{B}_{n,R}$. So each $\mathcal{B}_{n,R}$ is open, and hence each $\mathcal{C}_{n,R}$ is open.
The denseness is somewhat more difficult. Fix now some $f\in\mathcal{A}$. We shall construct $g\in\mathcal{C}_{n,R}$ with $d^*(f,g)$ as small as we please. Let us assume that the integers $\{j_i\}=\{j_i(f)\}$ and $\{\rho_n\}=\{\rho_n(f)\}$ are chosen as above. Choose now some large positive integer i_0. Note that by choice of our $\{\rho_j\}$ above,

$$|a_n|\le K/\prod_{j=1}^{n}\rho_j(f)\ \text{ for } n\le j_{i_0}(f),$$

and we have equality when $n=j_k$, some $1\le k\le i_0$. Now define $\hat{\rho}_j$ for $j>j_{i_0}$ by

$$\hat{\rho}_j=10^{j-j_{i_0}}\rho_{j_{i_0}}.$$

Also define for $n>j_{i_0}$,

$$\hat{a}_n=a_{j_{i_0}}/\prod_{j=j_{i_0}+1}^{n}\hat{\rho}_j=a_{j_{i_0}}\rho_{j_{i_0}}^{-(n-j_{i_0})}10^{-(n-j_{i_0})(n-j_{i_0}+1)/2}.$$

and

$$g(z)=\sum_{j=0}^{j_{i_0}}a_jz^j+\sum_{j=j_{i_0}+1}^{\infty}\hat{a}_jz^j.$$

Then g is entire (of order 0), and

$$d^*(f,g)\le\sup\left\{\left|a_n-a_{j_{i_0}}\rho_{j_{i_0}}^{-(n-j_{i_0})}10^{-(n-j_{i_0})(n-j_{i_0}+1)/2}\right|^{1/n}:n>j_{i_0}\right\}.$$

Straightforward estimation shows that by choosing i_0 large enough, this can be made as small as we please. Next, as g and f have the same series coefficients up to the coefficient of $z^{j_{i_0}}$, we see that

$$j_k(g)=j_k(f),\ k\le i_0\ \text{and}\ \rho_n(g)=\rho_n(f),\ n\le j_{i_0}.$$

Next, if $n>m\ge j_{i_0}$, we see that

$$\begin{aligned}\left(\frac{|\hat{a}_n|}{|\hat{a}_m|}\right)^{1/(n-m)}&=\left(\prod_{j=m+1}^{n}\left(\rho_{j_{i_0}}^{-1}10^{-(j-j_{i_0})}\right)\right)^{1/(n-m)}\\&=\rho_{j_{i_0}}^{-1}10^{j_{i_0}}\left(10^{-n(n+1)/2+m(m+1)/2}\right)^{1/(n-m)}\end{aligned}$$

$$= \rho_{j_{i_0}}^{-1} 10^{j_{i_0}} 10^{-(n-m)(n+m+1)/(2(n-m))}$$
$$= \rho_{j_{i_0}}^{-1} 10^{j_{i_0}-(n+m+1)/2},$$

which is maximal for a given m and $n > m$, if and only if $n = m + 1$. It follows easily that

$$j_{i_0+k}(g) = j_{i_0}(f) + k, k \geq 1,$$

and for $n > j_{i_0}$,

$$\rho_n(g) = \hat{\rho}_n = 10^{n-j_{i_0}} \rho_{j_{i_0}}.$$

Then for $n \geq j_{i_0}$,

$$\frac{\rho_{n+1}(g)}{\rho_n(g)} = 10$$

and

$$\lim_{n\to\infty} \rho_n(g)^{1/n} = 10.$$

It then follows from Lemma 2.3 that for large enough n and all $\Lambda_n \subset \overline{B_R}$, $q_n(g, \Lambda_n, z)$ has no zeros in $\overline{B_R}$. Thus $g \in \mathcal{B}_{n,R}$ for all large enough n, and in particular, $g \in \mathcal{C}_{n,R}$, while g may be made as close to the given f as we please, by choosing i_0 large enough. As $f \in \mathcal{A}$ is arbitrary, so $\mathcal{C}_{n,R}$ is dense. □

Proof of Theorem 1.1
Let

$$\mathcal{C} = \bigcap_{\ell=1}^{\infty} \bigcap_{n=1}^{\infty} \mathcal{C}_{n,\ell}$$

and

$$\mathcal{E} = \mathcal{A} \backslash \mathcal{C} = \bigcup_{\ell=1}^{\infty} \bigcup_{n=1}^{\infty} \mathcal{A} \backslash \mathcal{C}_{n,\ell}.$$

Here since $\mathcal{C}_{n,\ell}$ is open and dense, $\mathcal{E} = \mathcal{A} \backslash \mathcal{C}$ is a countable union of closed nowhere dense sets and is an F_σ set. Next if $f \in \mathcal{A} \backslash \mathcal{E}$, then $f \in \mathcal{C}$, so $f \in \mathcal{C}_{n,\ell}$ for all n, ℓ . Then we can choose an increasing sequence of integers $\{n_\ell\}_{\ell \geq 1}$ such that $f \in \mathcal{B}_{n_\ell,\ell}$ for $\ell \geq 1$. Then $q_{n_\ell}\left(\Lambda_{n_\ell}, z\right)$ has full degree n and no zeros in $\overline{B_\ell}$ whenever $\Lambda_{n_\ell} \subset \overline{B_\ell}$. This gives the desired uniform convergence of $\left\{R_{n_\ell}\left(\Lambda_{n_\ell}, \cdot\right)\right\}_{\ell \geq 1}$ whenever $\Lambda_{n_\ell} \subset \overline{B_R}$, for some $R > 0$. □

3 Proof of Theorem 1.2

We proceed partly as in [20]. *Let* $a_j \neq 0$, $j \geq 1$, *and*

$$\sigma_j = a_{j-1} a_{j+1} / a_j^2, \; j \geq 1. \tag{3.1}$$

For integers $t \geq -n + 1$, we let

$$r_{n,t} = \frac{a_{n+t}}{a_n} \left(\frac{a_n}{a_{n+1}} \sigma_n^{1/2} \right)^t. \tag{3.2}$$

Lemma 3.1

(a)
$$r_{n,t} = \sigma_n^{|t|/2} \prod_{\ell=1}^{|t|-1} \sigma_{n+\ell \operatorname{sign}(t)}^{|t|-\ell}. \tag{3.3}$$

(b) *Assume that for* $j \geq J$,
$$\left|\sigma_j\right| \leq \chi < 1. \tag{3.4}$$
Then for $n \geq 1$ *and* $t \geq -n-1$ *such that* $\min\{n, n+t+1\} \geq J$,
$$\left|r_{n,t}\right| \leq \chi^{t^2/2}. \tag{3.5}$$

(c) *If* (3.4) *holds for* $j \geq J$, *then for some* $C_0 = C_0(J)$, *and all* $n \geq J$, *and* $t = 0, \pm 1, \pm 2, \ldots$ *such that* $n+t > 0$,
$$\left|r_{n,t}\right| \leq C_0 \chi^{t^2/2}. \tag{3.6}$$

Proof

(a) If $t > 0$, we use
$$\frac{a_{j+1}}{a_j} = \sigma_j \frac{a_j}{a_{j-1}},$$
so
$$\begin{aligned} \frac{a_{n+t}}{a_n} &= \prod_{k=0}^{t-1} \frac{a_{n+k+1}}{a_{n+k}} \\ &= \prod_{k=0}^{t-1} \left(\sigma_{n+k}\sigma_{n+k-1} \cdots \sigma_{n+1} \frac{a_{n+1}}{a_n} \right) \\ &= \left(\frac{a_{n+1}}{a_n} \right)^t \sigma_{n+1}^{t-1} \sigma_{n+2}^{t-2} \cdots \sigma_{n+t-1}. \end{aligned} \tag{3.7}$$

If $t < 0$, we use
$$\frac{a_{j-1}}{a_j} = \sigma_j \frac{a_j}{a_{j+1}},$$
so
$$\begin{aligned} \frac{a_{n+t}}{a_n} &= \prod_{k=1}^{|t|} \frac{a_{n-k}}{a_{n-k+1}} \\ &= \prod_{k=1}^{|t|} \left(\sigma_{n-k+1}\sigma_{n-k+2} \cdots \sigma_n \frac{a_n}{a_{n+1}} \right) \end{aligned}$$

$$= \left(\frac{a_n}{a_{n+1}}\right)^{|t|} \sigma_n^{|t|} \sigma_{n-1}^{|t|-1} \cdots \sigma_{n+1-|t|}.$$

This and (3.7) easily give the identity (3.3).

(b) This follows directly from (a), as $n + \ell\operatorname{sign}(t) \geq \min\{n, n+t+1\} \geq J$ in the product, so we can apply the bound (3.4).

(c) Assume

$$C_1 = \sup_{j \geq 1} \left|\sigma_j\right|.$$

This also follows directly from (a). Indeed, using (3.4),

$$\left|r_{n,t}\right| \leq \chi^{|t|/2} \left(\prod_{\substack{1 \leq \ell \leq |t|-1; \\ n+\ell\operatorname{sign}(t) \geq J}} \chi^{|t|-\ell} \right) \prod_{\substack{1 \leq \ell \leq |t|-1; \\ n+\ell\operatorname{sign}(t) < J}} C_1^{|t|-\ell},$$

which easily gives the result, as there are $O\left(J^2\right)$ factors of C_1 arising from ℓ where $n + \ell\operatorname{sign}(t) < J$.

□

Next for given n and $j \geq 0$, $k \geq 0$, let

$$b_{j,k} = \frac{f_{j,n+k}}{a_n} \left(\frac{a_n}{a_{n+1}} \sigma_n^{1/2}\right)^{k-j}, \tag{3.8}$$

where $f_{j,n+k}$ is the divided difference in (1.8).

Lemma 3.2
Assume all $\left|z_j\right| \leq R$. Assume also that (3.4) holds for $j \geq J$.

(a) *There exists M depending only on R, χ, J, such that for $0 \leq j, k \leq n$, satisfying*

$$n + k - j \geq M, \tag{3.9}$$

we have for some constant C_2 depending only on R, χ, J,

$$\left|b_{j,k}\right| \leq \chi^{(k-j)^2/2} \left\{1 + C_2 (n+1+k-j) \chi^{n+1+k-j}\right\}. \tag{3.10}$$

(b) *For $n \geq M$ and $0 \leq j \leq n$,*

$$\left|b_{j,j} - 1\right| \leq C_2 n \chi^n. \tag{3.11}$$

(c) *We have for some constant C_4 depending only on R, χ, J, and all $0 \leq j, k \leq n$,*

$$\left|b_{j,k}\right| \leq C_4 \chi^{(k-j)^2/2}. \tag{3.12}$$

Proof

(a) We may assume that $M > J$. Let $\ell = n + 1 + k - j$. Using (2.6),

$$\left|\frac{f_{j,n+k}}{a_n}\right| \leq \sum_{s=0}^{\infty} \left|\frac{a_{\ell-1+s}}{a_n}\right| R^s \left|\binom{-\ell}{s}\right|.$$

Then as all $\ell - 1 + s \geq J$, we can apply our definitions (3.2), (3.8), and the bound (3.5) and deduce

$$\begin{aligned}
\left|b_{j,k}\right| &\leq \sum_{s=0}^{\infty} \left(\left|\frac{a_n}{a_{n+1}}\right| \sigma_n^{1/2}\right)^{-s} \left|r_{n,k-j+s}\right| R^s \left|\binom{-\ell}{s}\right| \\
&\leq \sum_{s=0}^{\infty} \left(\left|\frac{a_n}{a_{n+1}}\right| \sigma_n^{1/2}\right)^{-s} \chi^{(k-j+s)^2/2} R^s \left|\binom{-\ell}{s}\right| \\
&= \chi^{(k-j)^2/2} \sum_{s=0}^{\infty} \left(\left|\frac{a_{n+1}}{a_n}\right| \sigma_n^{-1/2} \chi^{k-j} R\right)^s \chi^{s^2/2} \left|\binom{-\ell}{s}\right| \\
&\leq \chi^{(k-j)^2/2} \sum_{s=0}^{\infty} \left(\left|\frac{a_{n+1}}{a_n}\right| \sigma_n^{-1/2} \chi^{k-j+1/2} R\right)^s \left|\binom{-\ell}{s}\right|.
\end{aligned}$$

Here

$$\begin{aligned}
&\left|\frac{a_{n+1}}{a_n}\right| \sigma_n^{-1/2} \chi^{k-j+1/2} R \\
&\quad = \sigma_n \sigma_{n-1} \sigma_{n-2} \cdots \sigma_J \left|\frac{a_J}{a_{J-1}}\right| \sigma_n^{-1/2} \chi^{k-j+1/2} R \\
&\quad \leq \chi^{n+1+k-j-J} \left|\frac{a_J}{a_{J-1}}\right| R \\
&\quad = C_3 \chi^{\ell},
\end{aligned} \tag{3.13}$$

where $C_3 = \chi^{-J} \left|\frac{a_J}{a_{J+1}}\right| R$. So

$$\begin{aligned}
\left|b_{j,k}\right| &\leq \chi^{(k-j)^2/2} \sum_{s=0}^{\infty} \left(C_3 \chi^{\ell}\right)^s \left|\binom{-\ell}{s}\right| \\
&\leq \chi^{(k-j)^2/2} \left(1 - C_3 \chi^{\ell}\right)^{-\ell},
\end{aligned}$$

provided $C_3\chi^\ell < 1$. We now use the inequality

$$(1-x)^{-\ell} - 1 \le 2\ell x \text{ for } x \in \left[0, \frac{1}{2\ell}\right],\ \ell \ge 2. \tag{3.14}$$

This gives

$$\left|b_{j,k}\right| \le \chi^{(k-j)^2/2}\left\{1 + 2C_3\ell\chi^\ell\right\},$$

provided

$$C_3\chi^\ell \le \frac{1}{2\ell}. \tag{3.15}$$

As $\chi < 1$, this last inequality holds for $\ell = n+1+k-j \ge M$ and some threshhold M.

(b) Here, proceeding as in (a), we see that for $n \ge J$, with $\ell = n+1$,

$$\begin{aligned}
\left|\frac{f_{j,n+j}}{a_n} - 1\right| &\le \sum_{s=1}^{\infty}\left|\frac{a_{\ell-1+s}}{a_n}\right| R^s \left|\binom{-\ell}{s}\right| \\
&\le \sum_{s=1}^{\infty}\left(\left|\frac{a_n}{a_{n+1}}\right|\sigma_n^{1/2}\right)^{-s}\left|r_{n,s}\right| R^s \left|\binom{-\ell}{s}\right| \\
&\le \sum_{s=1}^{\infty}\left(\left|\frac{a_n}{a_{n+1}}\right|\sigma_n^{1/2}\right)^{-s}\chi^{s^2/2} R^s \left|\binom{-\ell}{s}\right| \\
&\le \sum_{s=1}^{\infty}\left(\left|\frac{a_{n+1}}{a_n}\right|\sigma_n^{-1/2} R\chi^{1/2}\right)^{s}\left|\binom{-\ell}{s}\right| \\
&\le \left(1 - C_3\chi^n R\right)^{-n-1} - 1,
\end{aligned}$$

as above. Using (3.14), we continue this for $n \ge M$, and uniformly in j, as

$$\left|b_{j,j} - 1\right| \le C_2 n\chi^n.$$

(c) As above, we let $\ell = n+1+k-j$. If $n+1+k-j \ge M$, we can apply (a). So assume $\ell = n+1+k-j < M$. Now

$$\left|\binom{-\ell}{s}\right| = \binom{\ell+s-1}{s} \le 2^{\max\{\ell+s-1,0\}} \le 2^{\ell+s}.$$

Proceeding as in (a), but using Lemma 3.1 (c),

$$\begin{aligned}
\left|b_{j,k}\right| &\le \sum_{s=0}^{\infty}\left(\left|\frac{a_n}{a_{n+1}}\right|\sigma_n^{1/2}\right)^{-s}\left|r_{n,k-j+s}\right| R^s \left|\binom{-\ell}{s}\right| \\
&\le C_0\sum_{s=0}^{\infty}\left(\left|\frac{a_n}{a_{n+1}}\right|\sigma_n^{1/2}\right)^{-s}\chi^{(k-j+s)^2/2} R^s 2^{\ell+s}
\end{aligned}$$

$$\leq C_0 2^\ell \chi^{(k-j)^2/2} \sum_{s=0}^{\infty} \left(2 \left| \frac{a_{n+1}}{a_n} \right| \sigma_n^{-1/2} \chi^{k-j+1/2} R \right)^s \chi^{(s^2-s)/2}.$$

As in (3.13), and since $0 \leq \ell = n + 1 + k - j < M$,

$$2 \left| \frac{a_{n+1}}{a_n} \right| \sigma_n^{-1/2} \chi^{k-j+1/2} R \leq 2R\chi^{n+k-j-J+1} \left| \frac{a_J}{a_{J-1}} \right| \leq 2R\chi^{-J} \left| \frac{a_J}{a_{J-1}} \right| = 2C_3,$$

so

$$\left| b_{j,k} \right| \leq C_0 2^M \chi^{(k-j)^2/2} \sum_{s=0}^{\infty} (2C_3)^s \chi^{(s^2-s)/2}.$$

Then (3.12) follows. □

Lemma 3.3
Assume that (1.4) *holds with* $\chi < \rho_0$, *where* $\rho_0 = 0.4559\ldots$ *is the root of* (1.5). *For any* $R > 0$ *and* $\Lambda_n \subset \overline{B_R}$, *we have for large enough* n,

$$\inf_{|z| \leq R} |q_n (\Lambda_n, z)| > 0. \tag{3.16}$$

Proof
We use (1.9):

$$q_n (\Lambda_n, z) = \det [\mathbf{c}\, B],$$

where $\mathbf{c}$ is a column vector and B is an $(n+1) \times n$ matrix:

$$\mathbf{c} = \left[\prod_{\ell=1}^{j-1} (z - z_\ell) \right]_{1 \leq \ell \leq n+1} ; B = \left[f_{j-1,n+k-1} \right]_{\substack{1 \leq j \leq n+1; \\ 2 \leq k \leq n+1}} .$$

We divide the 2nd, 3rd, ..., $(n+1)$st column by a_n, and multiply the jth row by $\left(\frac{a_n}{a_{n+1}} \sigma_n^{1/2} \right)^{-j}$ and kth column by $\left(\frac{a_n}{a_{n+1}} \sigma_n^{1/2} \right)^k$, for all j, k. Then we obtain

$$q_n (\Lambda_n, z)\, a_n^{-n} = \det \left[\hat{\mathbf{c}}\, \hat{B} \right], \tag{3.17}$$

where

$$\hat{\mathbf{c}} = \left[\left(\frac{a_n}{a_{n+1}} \sigma_n^{1/2} \right)^{1-j} \prod_{\ell=1}^{j-1} (z - z_\ell) \right]_{1 \leq j \leq n+1} ;$$

and after an index change,

$$\hat{B} = \left[\left(\frac{a_n}{a_{n+1}} \sigma_n^{1/2} \right)^{k-j} \frac{f_{j,n+k}}{a_n} \right]_{\substack{0 \leq j \leq n; \\ 1 \leq k \leq n}} = \left[b_{j,k} \right]_{\substack{0 \leq j \leq n; \\ 1 \leq k \leq n}} . \tag{3.18}$$

Here we are using the notation (3.8). Let $R > 0$, $|z| \le R$, and all $\left|z_j\right| \le R$. We now show that $\left[\hat{\mathbf{c}}\ \hat{B}\right]$ is diagonally dominant. Consider the $(j+1)$st row, where $0 \le j \le n$. If $j = 0$, its diagonal element is 1. For $j \ge 1$, the diagonal element is

$$b_{j,j} = 1 + \varepsilon_{j,n}, \tag{3.19}$$

where by Lemma 3.2 (b),

$$\left|\varepsilon_{j,n}\right| \le C_2 n \chi^n \to 0 \text{ as } n \to \infty, \tag{3.20}$$

uniformly in j. Now consider the sum of the absolute values of the nondiagonal elements in the $(j+1)$st row, namely

$$\tau_j = \left(\left|\frac{a_{n+1}}{a_n}\sigma_n^{-1/2}\right|\right)^j \left|\prod_{\ell=1}^{j}(z - z_\ell)\right| + \left(\sum_{k=1}^{j-1} + \sum_{k=j+1}^{n}\right)\left|b_{j,k}\right|.$$

Of course, if $j = 1$, the first term and first sum are omitted. First let us assume that

$$n - j \ge \left[\frac{2\log n}{|\log \chi|}\right] = \Delta_n,$$

say. Then for large enough n, we have (3.9) for all terms in the sum, so can then estimate

$$\begin{aligned}
\tau_j &\le \left(\left|\frac{a_{n+1}}{a_n}\sigma_n^{-1/2}\right| 2R\right)^j \\
&\quad + \left(\sum_{k=1}^{j-1} + \sum_{k=j+1}^{n+1}\right) \chi^{(k-j)^2/2}\left\{1 + C_2\,(n+1+k-j)\,\chi^{n+1+k-j}\right\} \\
&\le o(1) + (1 + o(1))\, 2\sum_{\ell=1}^{\infty} \chi^{\ell^2/2} \le 1 - \varepsilon,
\end{aligned}$$

for some small enough ε. Recall that $\chi < \rho_0$, where ρ_0 is the root of (1.5). Using (3.19) and (3.20), we see then that the $(j+1, j+1)$ element in the $(j+1)$st row has absolute value larger than τ_j, as required for diagonal dominance. We still have to handle those terms for which $n - j \le \Delta_n$. Here most of τ_j can be estimated as above:

$$\begin{aligned}
&\left(\left|\frac{a_{n+1}}{a_n}\sigma_n^{-1/2}\right| 2R\right)^j + \sum_{k=\Delta_n+1}^{n+1} \chi^{(k-j)^2/2}\left\{1 + C_2\,(n+1+k-j)\,\chi^{n+1+k-j}\right\} \\
&\le 1 - \varepsilon.
\end{aligned}$$

Next, for $2 \leq k \leq \Delta_n$ and $n - j \leq \Delta_n$, we have $j - k \geq n - 2\Delta_n$, so

$$\sum_{k=2}^{\Delta_n} \left|b_{j,k}\right| \leq C_3 \sum_{k=2}^{\Delta_n} \chi^{(k-j)^2/2} \leq C_3 \Delta_n \chi^{(n-2\Delta_n)^2/2} \to 0,$$

as $n \to \infty$, by Lemma 3.2 (c). Again we obtain diagonal dominance. □

Proof of Theorem 1.2
Given any $r > 0$, the interpolants $R_n(\Lambda_n, z)$ have no poles in B_r for large enough n. Then as above, the locally uniform convergence follows. □

4 Proof of Theorem 1.3

This case is more delicate than the proof of Theorem 1.2. We have to multiply by a suitable matrix before proving diagonal dominance. Accordingly for $q \in \mathbb{C}$ and $n \geq 1$, let

$$A_n(q) = \left[q^{(k-j)^2/2}\right]_{1 \leq j,k \leq n}.$$

The determinant of this matrix can be reduced to that of a Vandermonde matrix by multiplying rows and columns by suitable factors. It is known that (see, e.g., [21, p. 326])

$$\det(A_n(q)) = \prod_{j=1}^{n-1} \left(1 - q^j\right)^{n-j}.$$

When this matrix is nonsingular, its inverse admits uniform bounds on its entries. More precisely, the (k, ℓ) entry in $A_n(q)^{-1}$ admits the bound

$$\left|\left(A_n(q)^{-1}\right)_{k\ell}\right| \leq S\,|q|^{|\ell-k|/2}, \tag{4.1}$$

where

$$S = 2 \prod_{j=1}^{\infty} \left(\frac{1 + |q|^j}{1 - |q|^j}\right)^2.$$

See [21, Lemma 2.1, pp. 326-327]. For integers $t \geq -n + 1$, we define $r_{n,t}$ by (3.2). We begin with bounds and asymptotics for $r_{n,t}$:

Lemma 4.1
Let $L \geq 1$. We have for $|t| \leq L$,

$$r_{n,t} = q^{t^2/2}\left(1 + \varepsilon_{n,t}\right),$$

where

$$\max_{|t|\leq L} \left|\varepsilon_{n,t}\right| \to 0 \text{ as } n \to \infty. \tag{4.2}$$

Proof

Since the total number of σ factors in the right-hand side of (3.3) is $|t|/2 + (|t|-1) + (|t|-2) + \cdots + 1 = |t|^2/2$, the assertion follows from our hypothesis that $\sigma_m \to q$ as $m \to \infty$. □

Next for given n and $j \geq 0$, $n+k \geq 0$, define $b_{j,k}$ by (3.8).

Lemma 4.2

Let $L \geq 1$ and $\chi \in (|q|, 1)$. For $1 \leq j, k \leq n$, we can write

$$b_{j,k} = q^{(k-j)^2/2} + \delta_{j,k}, \tag{4.3}$$

where

(a) *if $|j-k| \leq L$,*

$$\eta_L^* = \max_{|j-k|\leq L} \left|\delta_{j,k}/q^{(k-j)^2/2}\right| \to 0 \text{ as } n \to \infty. \tag{4.4}$$

(b) *for all $0 \leq j, k \leq n$,*

$$\left|\delta_{j,k}\right| \leq C_4 \chi^{(k-j)^2/2}, \tag{4.5}$$

where C_4 is independent of n, j, k and of L above.

Proof

(a) Let $\ell = n+1+k-j$, where $|j-k| \leq L$. As in (2.6),

$$\begin{aligned}\left|\frac{f_{j,n+k}}{a_n} - \frac{a_{\ell-1}}{a_n}\right| &= \left|\frac{a_{\ell-1}}{a_n} \sum_{i_1\cdots i_\ell \geq 0 \text{ with at least one } i_j \geq 1} z_{j+1}^{i_1} z_{j+2}^{i_2} \cdots z_{j+\ell}^{i_\ell} \frac{a_{\ell-1+i_1+i_2+\cdots+i_\ell}}{a_{\ell-1}}\right| \\ &\leq \left|\frac{a_{n+k-j}}{a_n}\right| \sum_{s=1}^{\infty} R^s \left|\frac{a_{\ell-1+s}}{a_{\ell-1}}\right| \left|\binom{-\ell}{s}\right|.\end{aligned}$$

Proceeding as in Lemma 3.2 and using Lemma 3.1 (c),

$$\begin{aligned}\left|b_{j,k} - r_{n,k-j}\right| &\leq \left|r_{n,k-j}\right| \sum_{s=1}^{\infty} R^s \left|\frac{a_\ell}{a_{\ell-1}} \sigma_{\ell-1}^{-1/2}\right|^s \left|r_{\ell-1,s}\right| \left|\binom{-\ell}{s}\right| \\ &\leq \left|r_{n,k-j}\right| \sum_{s=1}^{\infty} R^s \left|\frac{a_\ell}{a_{\ell-1}} \sigma_{\ell-1}^{-1/2}\right|^s C_4 \chi^{s^2/2} \left|\binom{-\ell}{s}\right| \\ &\leq \left|r_{n,k-j}\right| C_4 \left\{\left(1 - R\left|\frac{a_\ell}{a_{\ell-1}} \sigma_{\ell-1}^{-1/2}\right| \chi^{1/2}\right)^{-\ell} - 1\right\} \\ &\leq \left|r_{n,k-j}\right| C_5 (n+k-j) \chi^{n+k-j} \\ &\leq C_6 \left|r_{n,k-j}\right| n \chi^n,\end{aligned}$$

by (3.14). Then Lemma 4.1 gives

$$\begin{aligned}\left|\delta_{j,k}\right| &= \left|b_{j,k} - q^{(k-j)^2/2}\right| \\ &\leq \left|b_{j,k} - r_{n,k-j}\right| + \left|r_{n,k-j} - q^{(k-j)^2/2}\right| \\ &\leq \left|r_{n,k-j}\right| \left\{C_6 n \chi^n + C_7 \max_{|t|\leq L} \left|\varepsilon_{n,t}\right|\right\} \\ &\leq \eta_L^* \left|q\right|^{(k-j)^2/2},\end{aligned}$$

where $\eta_L^* \to 0$ as $L \to \infty$.
(b) Choose J such that $\left|\sigma_j\right| \leq \chi$ for $j \geq J$. Note that J is independent of L in (a). Using Lemma 3.2 (c), for all $0 \leq j, k \leq n$,

$$\left|b_{j,k}\right| \leq C_4 \chi^{(k-j)^2/2},$$

so

$$\left|\delta_{j,k}\right| \leq \left|b_{j,k}\right| + |q|^{(k-j)^2/2} \leq (C_4 + 1)\, \chi^{(k-j)^2/2}.$$

□

Proof of Theorem 1.3
It suffices to show that given $R > 0$ and $\Lambda_n \subset B_R$, $n \geq 1$, then for large enough n,

$$\inf_{z \in B_R} \left|q_n\left(\Lambda_n, z\right)\right| > 0.$$

We use (3.17), namely

$$q_n\left(\Lambda_n, z\right) a_n^{-n} = \det\left[\hat{\mathbf{c}}\ \hat{B}\right], \tag{4.6}$$

where $\left[\hat{\mathbf{c}}\ \hat{B}\right]$ is given by (3.18). We partition the column $\hat{\mathbf{c}}$ and matrix $\hat{B}$ as follows:

$$q_n\left(\Lambda_n, z\right) a_n^{-n} = \det\begin{bmatrix} 1 & \mathbf{f} \\ \mathbf{d} & A_n(q) + \Delta \end{bmatrix},$$

where $\mathbf{d}$ is an n by 1 column vector; $\mathbf{f}$ is an n by 1 row vector; and Δ is an n by n matrix with "small" entries. Thus

$$\mathbf{d} = \left[\left(\frac{a_n}{a_{n+1}}\sigma_n^{1/2}\right)^{-j} \prod_{\ell=1}^{j} \left(z - z_\ell\right)\right]_{1 \leq j \leq n};$$

$$\mathbf{f} = \left[\left(\frac{a_n}{a_{n+1}}\sigma_n^{1/2}\right)^{k} \frac{f_{0,n+k}}{a_n}\right]_{1 \leq k \leq n}^{T} = \left[b_{0,k}\right]_{1 \leq k \leq n};$$

$$\Delta = \left[\left(\frac{a_n}{a_{n+1}} \sigma_n^{1/2} \right)^{k-j} \frac{f_{j,n+k}}{a_n} \right]_{1 \le j,k \le n} - A_n(q)$$
$$= \left[b_{j,k} \right]_{1 \le j,k \le n} - A_n(q) = \left[\delta_{j,k} \right]_{1 \le j,k \le n}. \tag{4.7}$$

We multiply the determinant above by

$$\det \begin{bmatrix} 1 & -\mathbf{f} A_n(q)^{-1} \\ \mathbf{0} & A_n(q)^{-1} \end{bmatrix} = \det A_n(q)^{-1}.$$

We then see that

$$\begin{aligned} & q_n(\Lambda_n, z)\, a_n^{-n} (\det A_n(q))^{-1} \\ &= \det \begin{bmatrix} 1 & \mathbf{f} \\ \mathbf{d} & A_n(q) + \Delta \end{bmatrix} \begin{bmatrix} 1 & -\mathbf{f} A_n(q)^{-1} \\ \mathbf{0} & A_n(q)^{-1} \end{bmatrix} \\ &= \det \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{d} & -\mathbf{df} A_n(q)^{-1} + I + \Delta A_n(q)^{-1} \end{bmatrix} \\ &= \det \left[I - \mathbf{df} A_n(q)^{-1} + \Delta A_n(q)^{-1} \right]. \end{aligned} \tag{4.8}$$

We shall show the matrix in this last determinant is diagonally dominant. First,

$$\begin{aligned} & \mathbf{df} A_n(q)^{-1} \\ &= \left[\left(\frac{a_n}{a_{n+1}} \sigma_n^{1/2} \right)^{-j} \left(\prod_{\ell=1}^{j} (z - z_\ell) \right) b_{0,k} \right]_{1 \le j,k \le n} A_n(q)^{-1}, \end{aligned}$$

so the sum of the absolute values of elements in the jth row of the matrix $\mathbf{df} A_n(q)^{-1}$ is, using (4.5) and (4.1),

$$\begin{aligned} & \left| \left(\frac{a_n}{a_{n+1}} \sigma_n^{1/2} \right)^{-j} \prod_{\ell=1}^{j} (z - z_\ell) \right| \left| \sum_{\ell=1}^{n} \sum_{k=1}^{n} b_{0,k} \left(A_n(q)^{-1} \right)_{k\ell} \right| \\ & \le S C_3 \left(2R \left| \frac{a_{n+1}}{a_n} \right| \right)^j \sum_{\ell=1}^{n} \sum_{k=1}^{n} \chi^{k^2/2} |q|^{|k-\ell|/2} \\ & \le 2 S C_3 \left(1 - |q|^{1/2} \right)^{-1} \left(\sum_{k=1}^{\infty} \chi^{k^2/2} \right) \left(2R \left| \frac{a_{n+1}}{a_n} \right| \right)^j \\ & \le C_4 \left(2R \left| \frac{a_{n+1}}{a_n} \right| \right)^j. \end{aligned} \tag{4.9}$$

Next, the sum of absolute values of elements in the jth row of the matrix $\Delta A_n(q)^{-1}$ is bounded above by

$$
\begin{aligned}
&\sum_{k=1}^{n}\sum_{\ell=1}^{n}\left|\delta_{j,\ell}A_n(q)^{-1}_{\ell,k}\right| \\
&\leq S\sum_{k=1}^{n}\sum_{\ell=1}^{n}\left|\delta_{j,\ell}\right||q|^{|k-\ell|/2} \\
&\leq 2S\left(1-|q|^{1/2}\right)^{-1}\sum_{\ell=1}^{n}\left|\delta_{j,\ell}\right| \\
&\leq 2S\left(1-|q|^{1/2}\right)^{-1}\left\{\eta_L^*\sum_{\ell:|j-\ell|\leq L}|q|^{(j-\ell)^2/2}+C_3\sum_{\ell:|j-\ell|>L}\chi^{(j-\ell)^2/2}\right\} \\
&\leq 2S\left(1-|q|^{1/2}\right)^{-1}\left\{2\eta_L^*\left(1-|q|^{1/2}\right)^{-1}+C_3\chi^{L^2/2}\right\},
\end{aligned}
$$

by (4.2) and (4.5). It is crucial here that C_3 is independent of L. Choosing L large enough and then using that $\eta_L^* \to 0$ as $n \to \infty$, we see that this row sum may be made $< \frac{1}{4}$ for large enough n. Together with (4.9), this shows that the matrix in the determinant in (4.8) is diagonally dominant, and we are done. □

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