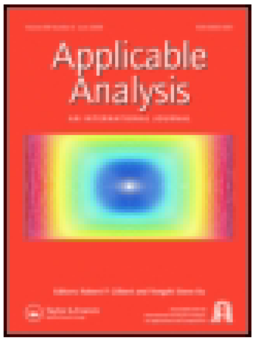

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ABSTRACT
In this paper, we study the normal mode solutions of 3D incompressible viscous fluid flow models. The obtained theoretical results are then applied to analyze several time-stepping schemes for the numerical solutions of the 3D incompressible fluid flow models.

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1. Introduction

Many types of flow motions can be described by using various forms of incompressible viscous flow models [1]. In particular, the incompressible Navier–Stokes equations (NSEs) have been widely used to model many important applications [2] such as ocean currents [1], water flow in a pipe [3], water pollution, and blood flow in vessels. The NSEs are useful because they describe the physics of many phenomena of scientific and engineering interest. The NSEs for a viscous, incompressible flow read as

$$\frac{\partial \vec{v}}{\partial t} + \nabla p = -\vec{v} \cdot \nabla \vec{v} + \nu \nabla^2 \vec{v} + \vec{f} \quad (\vec{x} \in \Omega); \tag{1}$$

$$\nabla \cdot \vec{v} = 0 \quad (\vec{x} \in \Omega). \tag{2}$$

Here, $\vec{v}(\vec{x}, t)$ is the velocity field at $(\vec{x}, t)$; $p(\vec{x}, t)$ is the pressure, ν is the kinematic viscosity, $f(\vec{x}, t)$ is an external force and Ω is a bounded domain with piecewise smooth boundary $\partial\Omega$. The derivation of the NSEs can be found in many references such as [4–9]. However, the theoretical analysis of NSEs in the three-dimensional case is notoriously difficult [4, 5, 9–12]. Many researchers resort to numerical investigation of incompressible viscous flow models [6, 13–19]. Among various numerical algorithms for NSEs, the nonlinear term is usually treated explicitly while the reaction–diffusion term is solved implicitly. The pressure term can be viewed as a Lagrange multiplier or can be solved by using a deduced pressure Poisson equation [13, 15, 18]. Nevertheless, various numerical algorithms may lead to instability, inaccuracy, and artificial numerical boundary layers. In the literature, a powerful tool called normal mode analysis is frequently used to analyze the stability and the errors of various

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numerical methods. Such an analysis provides a convenient framework for the analysis of numerical methods to solve incompressible flow problems. In the normal mode analysis, we first linearize the nonlinear model (if the original model is nonlinear), then we find all possible eigenfunctions (so-called *normal modes*) of the eigenvalue problem associated with the linearized model. The general operator theory shows that all normal modes form a basis of a properly selected function space for the solutions so that the solution of the linearized model is a superposition of the normal modes. Therefore, we only need to test the stability of each normal mode solution in order to test it for the arbitrary solution of the problem. Our focus is the 3D NSEs because the increase in dimension will lead to essential difficulties on both theoretical aspect and numerical aspect. Correspondingly, the analysis is more difficult. In this paper, we start with the analysis of a quasi-one dimensional Stokes model. We derive the normal mode solution in a detailed way. Then, we investigate the stability and accuracy of several time-stepping schemes for the solution of incompressible viscous flow models.

The rest of the paper is organized as follows: in Section 2 we derive the normal mode solutions to a one-dimensional linearized model of NSEs. In Section 3, we perform the normal mode analysis to the Backward Euler time differencing method and Crank–Nicolson time differencing method. We study the stability and accuracy of solutions of a splitting method in the Section 4. Concluding remarks are given in Section 5.

2. Normal mode solutions

The linearized model of NSEs without the external force is the following three-dimensional Stokes equations :

$$\frac{\partial \vec{v}}{\partial t} + \nabla p = \nu \nabla^2 \vec{v} \quad (\vec{x} \in \Omega); \tag{3}$$

$$\nabla \cdot \vec{v} = 0 \quad (\vec{x} \in \Omega). \tag{4}$$

A one-dimensional reduced linear model that embodies the essential features of the incompressibility and viscous terms of the Navier–Stokes equations can be obtained by considering the solution of the form

$$\vec{v} = (u(x,t), v(x,t), w(x,t))\mathrm{e}^{i(ky+lz)}, \qquad p = p(x,t)\mathrm{e}^{i(ky+lz)}$$

for the three-dimensional Stokes equations with homogeneous Dirichlet boundary conditions along x -direction:

$$\vec{v}(\vec{x},t) = 0 \quad (\text{if } x = -1 \text{ and } x = 1). \tag{5}$$

To proceed normal mode analysis, as stated as above, we assume that the solution $(\vec{v}, p)$ is periodic along y and z directions with k and l being the corresponding wave numbers. Denoting $D = \partial/\partial x$, then equations satisfied by (u, v, w, p) are

$$\frac{\partial u}{\partial t} = -Dp + \nu[D^2 - (k^2 + l^2)]u; \tag{6}$$

$$\frac{\partial v}{\partial t} = -ikp + \nu[D^2 - (k^2 + l^2)]v; \tag{7}$$

$$\frac{\partial w}{\partial t} = -ilp + \nu[D^2 - (k^2 + l^2)]w; \tag{8}$$

$$Du + ikv + ilw = 0. \tag{9}$$

Here and there after, without losing generality, we set the domain

$$\Omega = [-1,1] \times [-\pi,\pi] \times [-\pi,\pi] = \{(x,y,z) \mid -1 \le x \le 1;\ -\pi \le y \le \pi;\ -\pi \le z \le \pi\}.$$

The homogeneous Dirichlet boundary conditions then become

$$u(\pm 1,t) = v(\pm 1,t) = w(\pm 1,t) = 0, \tag{10}$$

and the wave numbers k and l become integers.

We notice that $(u,v,w,p) = (0,0,0,p(t))$ is always a solution of the problem (3)–(5) for any single variable function $p(t)$. This type of solutions are called *trivial solutions*. In this paper, we are only interested in nontrivial solutions of the problem (3)–(5).

If the nontrivial solution of the boundary-value problem (6)–(10) is in separable form:

$$u(x,t) = \hat{u}(x)U(t), \quad v(x,t) = \hat{v}(x)V(t), \quad w(x,t) = \hat{w}(x)W(t), \quad p(x,t) = \hat{p}(x)P(t),$$

then we have the following proposition, which actually states that the time frequency is a negative number σ and ensures the solution is stable.

Proposition 2.1: *Without losing generality, we assume that*

$$U(0) = V(0) = W(0) = P(0) = 1.$$

Then, there exists a real constant $\sigma < 0$ *such that*

$$U(t) = V(t) = W(t) = P(t) = e^{\sigma t}.$$

Proof: We break the proof into two steps: in first step we prove the existence of the real constant σ and then we prove that σ must be negative for a nontrivial solution in the second step.

After substituting the separate forms of $u(x,t)$, $v(x,t)$, $w(x,t)$ and $p(x,t)$ into (6)–(10), we obtain

$$\hat{u}\frac{dU}{dt} = (-D\hat{p})P + \nu[D^2 - (k^2 + l^2)]\hat{u}U; \tag{11}$$

$$\hat{v}\frac{dV}{dt} = -ik\hat{p}P + \nu[D^2 - (k^2 + l^2)]\hat{v}V; \tag{12}$$

$$\hat{w}\frac{dW}{dt} = -il\hat{p}P + \nu[D^2 - (k^2 + l^2)]\hat{w}W; \tag{13}$$

$$(D\hat{u})U + ik\hat{v}V + il\hat{w}W = 0. \tag{14}$$

Applying the operator D on Equation (11), we have

$$(D\hat{u})\frac{dU}{dt} = (-D^2\hat{p})P + \nu[D^2 - (k^2 + l^2)](D\hat{u})U. \tag{15}$$

Multiplying ik to Equation (12) and il to Equation (13) respectively, we obtain

$$(ik\hat{v})\frac{dV}{dt} = k^2\hat{p}P + \nu[D^2 - (k^2 + l^2)](ik\hat{v})V; \tag{16}$$

$$(il\hat{w})\frac{dW}{dt} = l^2\hat{p}P + \nu[D^2 - (k^2 + l^2)](il\hat{w})W. \tag{17}$$

Differentiating Equation (14) with respect to the variable t, we get

$$(D\hat{u})\frac{\mathrm{d}U}{\mathrm{d}t}+ik\hat{v}\frac{\mathrm{d}V}{\mathrm{d}t}+il\hat{w}\frac{\mathrm{d}W}{\mathrm{d}t}=0. \tag{18}$$

We denote

$$r=\sqrt{k^2+l^2}. \tag{19}$$

Applying (14) and (18) on the sum of (15), (16) and (17), we obtain

$$[(D^2-r^2)\hat{p}]P(t)=0,$$

which implies

$$(D^2-r^2)\hat{p}=0. \tag{20}$$

Thus, we have

$$\hat{p}(x)=c_1\sinh rx+c_2\cosh rx. \tag{21}$$

■

Applying (D^2-r^2) on (11), we get

$$(D^2-r^2)\hat{u}\frac{\mathrm{d}U}{\mathrm{d}t}=\nu(D^2-r^2)^2\hat{u}U,$$

which implies

$$\frac{\frac{\mathrm{d}U}{\mathrm{d}t}}{U}=\frac{\nu(D^2-r^2)^2\hat{u}}{(D^2-r^2)\hat{u}}=\sigma_1. \tag{22}$$

Here, σ_1 must be a constant since the left-hand side of the above equation is a function depending on t only while the right-hand side of the above equation is a function depending on x only. Combining with $U(0)=1$, we obtain

$$U(t)=\mathrm{e}^{\sigma_1 t}.$$

Applying (D^2-r^2) on (12), we get

$$(D^2-r^2)\hat{v}\frac{\mathrm{d}V}{\mathrm{d}t}=\nu(D^2-r^2)^2\hat{v}V,$$

which implies that

$$\frac{\frac{\mathrm{d}V}{\mathrm{d}t}}{V}=\frac{\nu(D^2-r^2)^2\hat{v}}{(D^2-r^2)\hat{v}}=\sigma_2$$

with σ_2 being a constant. Combining with $V(0)=1$, we obtain

$$V(t)=\mathrm{e}^{\sigma_2 t}.$$

Similarly, we obtain

$$\frac{\frac{\mathrm{d}W}{\mathrm{d}t}}{W}=\frac{\nu(D^2-r^2)^2\hat{w}}{(D^2-r^2)\hat{w}}=\sigma_3$$

and

$$W(t)=\mathrm{e}^{\sigma_3 t}. \tag{23}$$

Substituting V and W into (12) and (13) respectively, we obtain

$$e^{(\sigma_2-\sigma_3)t} = \frac{k}{l}\frac{\sigma_3\hat{w} - \nu(D^2 - r^2)\hat{w}}{\sigma_2\hat{v} - \nu(D^2 - r^2)\hat{v}},$$

which implies that $\sigma_2 = \sigma_3 \equiv \sigma$ since the right hand of the above equation is independent of t. Similarly, after multiplying ik on (11) and applying D on (12), we obtain

$$e^{(\sigma_1-\sigma_2)t} = \frac{\sigma_2 D\hat{v} - \nu(D^2 - r^2)D\hat{v}}{ik(\sigma_1\hat{u} - \nu(D^2 - r^2)\hat{u})},$$

which implies that $\sigma_2 = \sigma_1 = \sigma$ since the right hand of the above equation is independent of t.

Therefore, we complete the proof for the existence of the real constant σ.

Now we show that $\sigma < 0$ in the nontrivial solution of the form

$$u(x,t) = \hat{u}(x)e^{\sigma t}, \quad v(x,t) = \hat{v}(x)e^{\sigma t}, \quad w(x,t) = \hat{w}(x)e^{\sigma t}, \quad p(x,t) = \hat{p}(x)e^{\sigma t} \tag{24}$$

to the boundary-value problem (6)–(10). This kind of solutions are called *normal modes*. Substituting (24) into (6)–(10), we have

$$\sigma\hat{u} = -D\hat{p} + \nu[D^2 - r^2]\hat{u}; \tag{25}$$

$$\sigma\hat{v} = -ik\hat{p} + \nu[D^2 - r^2]\hat{v}; \tag{26}$$

$$\sigma\hat{w} = -il\hat{p} + \nu[D^2 - r^2]\hat{w}; \tag{27}$$

$$D\hat{u} + ik\hat{v} + il\hat{w} = 0. \tag{28}$$

The homogeneous Dirichlet boundary conditions become

$$\hat{u}(\pm 1) = \hat{v}(\pm 1) = \hat{w}(\pm 1) = 0. \tag{29}$$

From (28) and (29), we obtain

$$D\hat{u}(\pm 1) = 0. \tag{30}$$

It is easy to show that if $r = \sqrt{k^2 + l^2} = 0$, then there are only trivial solutions to the problem (3)–(5). Therefore, we can assume that $r \geq 1$ from now on.

Applying (ikD) on (26) and (27), then adding the resulting equations, we obtain

$$-\sigma D^2\hat{u} = r^2 D\hat{p} - \nu[D^2 - r^2]D^2\hat{u}, \tag{31}$$

which leads to the following equation after combining with (25):

$$\nu[D^2 - r^2]^2\hat{u} = \sigma[D^2 - r^2]\hat{u}. \tag{32}$$

We can rewrite (32) as

$$(D^2 - r^2)(D^2 - r^2 - \sigma/\nu)\hat{u} = 0.$$

The corresponding characteristic equation is

$$(m^2 - r^2)(m^2 - r^2 - \sigma/\nu) = 0.$$

Defining

$$\mu = \sqrt{-\sigma/\nu - r^2},$$

then we can show that σ must be negative by proving the following stronger statement.

Claim: $r^2 + \sigma/\nu < 0$, and therefore $\mu = \sqrt{-(\sigma/\nu + r^2)} > 0$ and $\sigma = -\nu(r^2 + \mu^2) < 0$.

Proof: We assert that $r^2 + \sigma/\nu < 0$ and therefore $\mu > 0$. We use the argument by contradiction to prove this statement. To proceed, we discuss the following two cases. ■

Case (1): $r^2 + \sigma/\nu > 0$.
Let $\tilde{\mu} = \sqrt{r^2 + \sigma/\nu}$. If $\sigma \neq 0$ then the general solution of (32) is

$$\begin{aligned}\hat{u} &= c_1 \cosh(rx) + c_2 \sinh(rx) + c_3 \cosh(\tilde{\mu}x) + c_4 \sinh(\tilde{\mu}x),\\ D\hat{u} &= r(c_1 \sinh(rx) + c_2 \cosh(rx)) + \tilde{\mu}(c_3 \sinh(\tilde{\mu}x) + c_4 \cosh(\tilde{\mu}x)).\end{aligned}$$

The boundary conditions give use that

$$\hat{u}(\pm 1) = 0, \quad D\hat{u}(\pm 1) = 0.$$

This implies that

$$\begin{pmatrix} \cosh r & \sinh r \\ \cosh r & -\sinh r \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -\cosh\tilde{\mu} & -\sinh\tilde{\mu} \\ -\cosh\tilde{\mu} & \sinh\tilde{\mu} \end{pmatrix} \begin{pmatrix} c_3 \\ c_4 \end{pmatrix} \tag{33}$$

and

$$\begin{pmatrix} r\sinh r & r\cosh r \\ -r\sinh r & r\cosh r \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -\tilde{\mu}\sinh\tilde{\mu} & -\tilde{\mu}\cosh\tilde{\mu} \\ \tilde{\mu}\sinh\tilde{\mu} & -\tilde{\mu}\cosh\tilde{\mu} \end{pmatrix} \begin{pmatrix} c_3 \\ c_4 \end{pmatrix}. \tag{34}$$

More clearly, from Equation (33), we have

$$c_1 = -\frac{\cosh\tilde{\mu}}{\cosh r} c_3, \quad c_2 = -\frac{\sinh\tilde{\mu}}{\sinh r} c_4,$$

from Equation (34), we see that

$$c_1 = -\frac{\tilde{\mu}\sinh\tilde{\mu}}{r\sinh r} c_3, \quad c_2 = -\frac{\tilde{\mu}\cosh\tilde{\mu}}{r\cosh r} c_4.$$

If $c_3 \neq 0$, then $\tilde{\mu}\tanh\tilde{\mu} = r\tanh r$.
If $c_4 \neq 0$, then $\tanh\tilde{\mu}/\tilde{\mu} = \tanh r/r$.
Since $x\tanh x$ is strictly increasing on the interval $[0, \infty)$ and $\tanh x/x$ is strictly decreasing on $[0, \infty)$, there must hold $\tilde{\mu} = r$, which implies that $\sigma = 0$. This is a contradiction!
If $\sigma = 0$, then the general solution of (32) is

$$\hat{u} = c_1 \sinh rx + c_2 \cosh rx + x(c_3 \sinh rx + c_4 \cosh rx).$$

By a similar argument we can derive a contradiction. Therefore, in **Case (1)**, there is only a trivial solution $c_1 = c_2 = c_3 = c_4 = 0$ for (32).

Case (2): $r^2 + \sigma/\nu = 0$.
The general solution of (32) is

$$\hat{u} = c_1 \sinh rx + c_2 \cosh rx + c_3 x + c_4.$$

By a similar argument, we can prove that there is only trivial solution $c_1 = c_2 = c_3 = c_4 = 0$ for (32) for **Case (2)**. Thus the Claim has been proved. Therefore the proof of Proposition 2.1 is completed.

From Proposition 2.1, we only need to consider the following ODE with $\mu > 0$.

$$\nu(D^2 - r^2)^2\hat{u} = \sigma(D^2 - r^2)\hat{u}, \tag{35}$$

and $\hat{u}$ satisfies the boundary conditions

$$\hat{u}(\pm 1) = 0, \quad D\hat{u}(\pm 1) = 0. \tag{36}$$

We notice that σ is actually an eigenvalue and the nonzero function $\hat{u}$ is the associated eigenfunction to the boundary value problem (35)–(36).

The general solution of (35) is

$$\hat{u} = c_1 \cosh(rx) + c_2 \sinh(rx) + c_3 \cos(\mu x) + c_4 \sin(\mu x).$$

Therefore,

$$D\hat{u} = c_1 r \sinh(rx) + c_2 r \cosh(rx) - c_3 \mu \sin(\mu x) + c_4 \mu \cos(\mu x).$$

Applying the boundary conditions (36), we obtain

$$\begin{pmatrix} \cosh r & \sinh r \\ \cosh r & -\sinh r \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -\cos\mu & -\sin\mu \\ -\cos\mu & \sin\mu \end{pmatrix} \begin{pmatrix} c_3 \\ c_4 \end{pmatrix}$$

and

$$\begin{pmatrix} r\sinh r & r\cosh r \\ -r\sinh r & r\cosh r \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \mu\sin\mu & -\mu\cos\mu \\ -\mu\sin\mu & -\mu\cos\mu \end{pmatrix} \begin{pmatrix} c_3 \\ c_4 \end{pmatrix}.$$

We only discuss the nontrivial solutions. Notice that if $c_4 = 0$ and $c_3 \neq 0$, then $-\mu\tan\mu = r\tanh r$; if $c_4 \neq 0$ and $c_3 = 0$, then $\mu\cot\mu = r\coth r$; while $c_4 \neq 0, c_3 \neq 0$ implies a contradiction. Thus, we have the following conclusion.

Proposition 2.2: *The boundary value problem* (35)–(36) *has only the following two type of nontrivial solutions for $r \geq 1$.*

Case 1 (even solution): $\mu = \sqrt{-r^2 - \sigma/\nu}$ *satisfies*

$$-\mu\tan\mu = r\tanh r,$$

and the corresponding eigenvalue $\sigma = -\nu(\mu^2 + r^2) < 0$ and the associated eigenfunction is a constant multiple of

$$\hat{u} = \cos\mu\cosh rx - \cosh r\cos\mu x.$$

Correspondingly,

$$\hat{v} = ik\left[\frac{\cos\mu}{r}\sinh rx + \frac{\mu\cosh r}{r^2}\sin\mu x\right],$$

$$\hat{w} = il\left[\frac{\cos\mu}{r}\sinh rx + \frac{\mu\cosh r}{r^2}\sin\mu x\right],$$

$$\hat{p} = -\frac{\sigma\cos\mu}{r}\sinh rx.$$

Case 2 (odd solution): $\mu = \sqrt{-r^2 - \sigma/\nu}$ *satisfies*

$$\mu\cot\mu = r\coth r,$$

and the corresponding eigenvalue $\sigma = -\nu(\mu^2 + r^2) < 0$ *and the associated eigenfunction is a constant multiple of*

$$\hat{u} = \sin\mu \sinh rx - \sinh r \sin\mu x.$$

Correspondingly,

$$\hat{v} = ik\left[\frac{\sin\mu}{r}\cosh rx - \frac{\mu\sinh r}{r^2}\cos\mu x\right],$$
$$\hat{w} = il\left[\frac{\sin\mu}{r}\cosh rx - \frac{\mu\sinh r}{r^2}\cos\mu x\right],$$
$$\hat{p} = -\frac{\sigma\sin\mu}{r}\cosh rx.$$

Remark 2.3: (1) For given $r = \sqrt{k^2 + l^2} \geq 1$, by a simple analysis we know that on each interval $((\pi/2) + (k-1)\pi, (\pi/2) + k\pi)$, $k = 1, 2, \ldots$, there is a unique μ_k such that

$$-\mu_k \tan\mu_k = r\tanh r,$$

and the eigenvalue $\sigma_k = -\nu(\mu_k^2 + r^2) < 0$, the associated eigenfunction $\hat{u}$ is denoted by u_k.

(2) For given $r \geq 1$, on each interval $(k\pi, (k+1)\pi)$, $k = 1, 2, \ldots$, there is a unique $\tilde{\mu}_k$ such that

$$\tilde{\mu}_k \cot\tilde{\mu}_k = r\coth r,$$

and the eigenvalue $\tilde{\sigma}_k = -\nu(\tilde{\mu}_k^2 + r^2) < 0$, the associated eigenfunction $\hat{u}$ is denoted by $\tilde{u}_k$.

(3) If we define the function space

$$W_0^2[-1,1] = \{u \in H^2[-1,1] : u(\pm 1) = Du(\pm 1) = 0\},$$

then the general operator theory shows that the set of all eigenfunctions $\{u_k, \tilde{u}_k : k = 1, 2, 3, \ldots\}$ forms a basis of $W_0^2[-1,1]$ (see [6]).

3. Semidiscrete implicit coupled methods

In this section, we are seeking normal mode solutions to some semidiscrete equations which are analogues to the normal mode solutions of the continuous model problem in Section 2. The corresponding normal modes are of the form

$$(u^n(x), v^n(x), w^n(x), p^n(x)) = K^n(\tilde{u}(x), \tilde{v}(x), \tilde{w}(x), \tilde{p}(x)), \tag{37}$$

where K is the amplification factor and n denotes the n-th time step. The semidiscrete approximation is stable if $|K| \leq 1 + \mathcal{O}(\Delta t)$ for all normal modes and is unstable otherwise [6]. The accuracy of the particular semidiscrete methods can be studied by computing the exponential growth rate $\tilde{\sigma}$ defined by

$$K^n = \exp(\tilde{\sigma}\, n\Delta t),\ \tilde{\sigma} = \frac{\ln K}{\Delta t}. \tag{38}$$

The error $\tilde{\sigma} - \sigma$, where σ is given by (35)–(36), measures the time discretization error.

3.1. Backward Euler time differencing

If the system (6)–(10) is approximated by the backward Euler time differencing scheme, denoting $r = \sqrt{k^2 + l^2}$, then

$$\frac{u^{n+1} - u^n}{\Delta t} = -D\,p^{n+1} + \nu\,[D^2 - r^2]\,u^{n+1}, \tag{39}$$

$$\frac{u^{n+1} - v^n}{\Delta t} = -ik\,p^{n+1} + \nu\,[D^2 - r^2]\,v^{n+1}, \tag{40}$$

$$\frac{w^{n+1} - w^n}{\Delta t} = -il\,p^{n+1} + \nu\,[D^2 - r^2]\,w^{n+1}, \tag{41}$$

$$Du^{n+1} + ik\,v^{n+1} + il\,w^{n+1} = 0, \tag{42}$$

$$u^{n+1}(\pm 1) = v^{n+1}(\pm 1) = w^{n+1}(\pm 1) = 0, \tag{43}$$

which imply

$$Du^{n+1}(\pm 1) = 0. \tag{44}$$

Applying D on (39), we have

$$\frac{Du^{n+1} - Du^n}{\Delta t} = -D^2 p^{n+1} + \nu\,[D^2 - r^2]\,(Du^{n+1}). \tag{45}$$

Multiplying ik to (40), we see that

$$\frac{iku^{n+1} - ikv^n}{\Delta t} = k^2 p^{n+1} + \nu\,[D^2 - r^2]\,(ik\,v^{n+1}). \tag{46}$$

Multiplying il to (41), we have

$$\frac{ilw^{n+1} - ilw^n}{\Delta t} = l^2 p^{n+1} + \nu\,[D^2 - r^2]\,(il\,w^{n+1}). \tag{47}$$

Adding (45)–(47) together and applying (42), we obtain

$$(D^2 - r^2)p^{n+1} = 0.$$

Applying $(D^2 - r^2)$ on (37), we obtain

$$\frac{(D^2 - r^2)(u^{n+1} - u^n)}{\Delta t} = -D(D^2 - r^2)\,p^{n+1} + \nu\,(D^2 - r^2)^2\,u^{n+1}, \tag{48}$$

$$\frac{(D^2 - r^2)(u^{n+1} - u^n)}{\Delta t} = \nu\,(D^2 - r^2)^2\,u^{n+1}, \tag{49}$$

$$u^{n+1}(\pm 1) = Du^{n+1}(\pm 1) = 0. \tag{50}$$

Substituting $u^n = K^n\,\tilde{u}$, $u^{n+1} = K^{n+1}\,\tilde{u}$ into (49), we obtain

$$\frac{K-1}{K\Delta t}(D^2 - r^2)\tilde{u} = \nu(D^2 - r^2)^2\tilde{u},$$

$$\tilde{u}(\pm 1) = D\tilde{u}(\pm 1) = 0. \tag{51}$$

Let

$$\sigma = \frac{K-1}{K\Delta t} \Rightarrow K = \frac{1}{1-\sigma\Delta t}.$$

We have

$$\sigma(D^2 - r^2)\tilde{u} = \nu(D^2 - r^2)^2\tilde{u},$$

which is the same as (35). The time dependance of (u^n, v^n, w^n, p^n) is proportional to $K^n = \exp(\tilde{\sigma}\, n\Delta t)$, where

$$\tilde{\sigma} = \frac{\ln K}{\Delta t} = \frac{-\ln(1-\sigma\Delta t)}{\Delta t}.$$

Since

$$\ln\,(1-x) = -\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}$$

for small Δt, we have

$$-\ln(1-\sigma\Delta t) = \sum_{n=0}^{\infty}\frac{(\sigma\Delta t)^{n+1}}{n+1} = \sigma\Delta t + \frac{1}{2}\sigma^2\Delta t^2 + \frac{1}{3}\sigma^3\Delta t^3 + \cdots.$$

Thus

$$\tilde{\sigma} = \sigma + \frac{1}{2}\sigma^2\Delta t + \frac{1}{3}\sigma^3\Delta t^2 + \cdots,$$

which implies that

$$\tilde{\sigma} - \sigma = \mathcal{O}(\Delta t).$$

So the exponential growth rate $\tilde{\sigma}$ for the Backward Euler scheme is in error of the order $\mathcal{O}(\Delta t)$. Noting that $\sigma < 0$, we have $K < 1$, and therefore this numerical scheme is unconditionally stable. Thus, the normal analysis demonstrates the stability property of this implicit semidiscrete scheme. Moreover, this scheme is convergent if $\Delta t \to 0$.

3.2. Crank–Nicolson time differencing

If the system (6)–(10) is approximated by Crank–Nicolson scheme, then we have

$$\frac{u^{n+1}-u^n}{\Delta t} = -Dp^{n+\frac{1}{2}} + \frac{1}{2}\nu[D^2 - r^2]\,(u^{n+1}+u^n), \tag{52}$$

$$\frac{v^{n+1}-v^n}{\Delta t} = -ik\,p^{n+\frac{1}{2}} + \frac{1}{2}\nu[D^2 - r^2]\,(v^{n+1}+v^n), \tag{53}$$

$$\frac{w^{n+1}-w^n}{\Delta t} = -il\,p^{n+\frac{1}{2}} + \frac{1}{2}\nu[D^2 - r^2]\,(w^{n+1}+w^n), \tag{54}$$

$$Du^{n+1} + ik\,v^{n+1} + il\,w^{n+1} = 0, \tag{55}$$

with

$$u^{n+1}(\pm 1) = v^{n+1}(\pm 1) = w^{n+1}(\pm 1) = 0\,. \tag{56}$$

We will show that this numerical scheme is both unconditionally stable and accurate to the order of $\mathcal{O}(\Delta t^2)$. Indeed, we can use a similar method to derive that

$$(D^2 - r^2)p^{n+\frac{1}{2}} = 0.$$

Applying $(D^2 - r^2)$ on (52),we obtain

$$\frac{(D^2 - r^2)(u^{n+1} - u^n)}{\Delta t} = \frac{1}{2}\nu(D^2 - r^2)^2(u^{n+1} + u^n). \tag{57}$$

Let

$$u^n = K^n\tilde{u}, \quad v^n = K^n\tilde{v}, \quad w^n = K^n\tilde{w},$$

substituting them into (57), we have

$$\frac{(K-1)}{\Delta t}(D^2 - r^2)\tilde{u} = \frac{(K+1)\nu}{2}(D^2 - r^2)^2\tilde{u}.$$

Let

$$\sigma = \frac{2(K-1)}{\Delta t(K+1)},$$

we have

$$\sigma(D^2 - r^2)\tilde{u} = \nu(D^2 - r^2)^2\tilde{u},$$

which is the same as (35) where we have $\sigma < 0$. Let

$$\sigma = \frac{2(K-1)}{\Delta t(K+1)} \Rightarrow K = \frac{2 + \sigma\Delta t}{2 - \sigma\Delta t} = \frac{1 + \frac{1}{2}\sigma\Delta t}{1 - \frac{1}{2}\sigma\Delta t} < 1.$$

Setting

$$K^n = \exp(\tilde{\sigma}\; n\Delta t),$$

we have

$$\tilde{\sigma} = \frac{\ln\, K}{\Delta t} = \frac{\ln(1 + \frac{1}{2}\sigma\Delta t) - \ln(1 - \frac{1}{2}\sigma\Delta t)}{\Delta t},$$

$$\ln(1 + \frac{1}{2}\sigma\Delta t) = \sum_{n=0}^{\infty}\frac{(-1)^n(\frac{1}{2}\sigma\Delta t)^{n+1}}{n+1},$$

$$-\ln(1 - \frac{1}{2}\sigma\Delta t) = \sum_{n=0}^{\infty}\frac{(\frac{1}{2}\sigma\Delta t)^{n+1}}{n+1},$$

$$\tilde{\sigma} = \frac{1}{\Delta t}\sum_{n=0}^{\infty}[(-1)^n + 1]\frac{(\frac{1}{2}\sigma\Delta t)^{n+1}}{n+1}$$

$$= \frac{2}{\Delta t}\sum_{m=0}^{\infty}\frac{\sigma^{2m+1}\Delta t^{2m+1}}{(2m+1)2^{2m+1}} = \sum_{m=0}^{\infty}\frac{\sigma^{2m+1}\Delta t^{2m}}{(2m+1)2^{2m}}$$

$$= \sigma + \frac{\sigma^3}{12}(\Delta t)^2 + \frac{\sigma^5}{80}(\Delta t)^4 + \cdots\cdot = \sigma + \mathcal{O}(\Delta t^2)$$

which implies

$$\tilde{\sigma} - \sigma = \mathcal{O}(\Delta t^2).$$

As $\sigma < 0$, we see that $\tilde{\sigma} < 0$ and $|K| < 1$. Thus, the exponential growth rate $\tilde{\sigma}$ for the Crank–Nicolson scheme is in error of the order $\mathcal{O}(\Delta t^2)$ and this scheme is unconditionally stable.

4. Splitting method

In this section, we consider the velocity-pressure splitting with the normal velocity boundary conditions [15, 18]. The time-differencing scheme involves the following two split time steps.

The first step involves solution of the inviscid equation:

$$\frac{u^* - u^n}{\Delta t} = -Dp^{n+1}, \tag{58}$$

$$\frac{v^* - v^n}{\Delta t} = -ik\, p^{n+1}, \tag{59}$$

$$\frac{w^* - w^n}{\Delta t} = -il\, p^{n+1}, \tag{60}$$

$$Du^* + ikv^* + ilw^* = 0, \tag{61}$$

$$u^*(\pm 1) = v^*(\pm 1) = w^*(\pm 1) = 0. \tag{62}$$

The second step involves the solution of the viscous equation

$$\frac{u^{n+1} - u^*}{\Delta t} = \nu(D^2 - r^2)\, u^{n+1}, \tag{63}$$

$$\frac{u^{n+1} - v^*}{\Delta t} = \nu(D^2 - r^2)\, v^{n+1}, \tag{64}$$

$$\frac{w^{n+1} - w^*}{\Delta t} = \nu(D^2 - r^2)\, w^{n+1}, \tag{65}$$

$$u^{n+1}(\pm 1) = v^{n+1}(\pm 1) = w^{n+1}(\pm 1) = 0. \tag{66}$$

Remark: In the above splitting algorithm, (58)–(66), $u^n,\ v^n,\ w^n$ do not satisfy the incompressibility constraint, although the intermediate variables u^*, v^*, w^* do.

4.1. The computation of $\tilde{u}$ and $\tilde{u}^*$

To analyze the splitting method, we firstly derive the results for the normal modes of the above system. The general conclusion then follow from the completeness of the normal modes.

Let

$$(u^n,\ v^n, w^n,\ p^n) = K^n(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{p}) \tag{67}$$

and

$$(u^*, v^*, w^*) = K^n(\tilde{u}^*, \tilde{v}^*, \tilde{w}^*). \tag{68}$$

Substituting (67) and (68) into (58)–(66) gives

$$\tilde{u}^* - \tilde{u} = -(K\Delta t)D\tilde{p}, \tag{69}$$

$$\tilde{v}^* - \tilde{v} = -ik\,(K\Delta t)\tilde{p}, \tag{70}$$

$$\tilde{w}^* - \tilde{w} = -il(K\Delta t)\tilde{p}, \tag{71}$$

$$D\tilde{u}^* + ik\tilde{v}^* + il\tilde{w}^* = 0, \tag{72}$$

$$\tilde{u}^*(\pm 1) = \tilde{v}^*(\pm 1) = \tilde{w}^*(\pm 1) = 0, \tag{73}$$

$$(D^2 - r^2 - \frac{1}{\nu \Delta t})\,\tilde{u} = -\frac{1}{K\nu\Delta t}\tilde{u}^*, \tag{74}$$

$$(D^2 - r^2 - \frac{1}{\nu \Delta t})\,\tilde{u} = -\frac{1}{K\nu\Delta t}\tilde{v}^*, \tag{75}$$

$$(D^2 - r^2 - \frac{1}{\nu \Delta t})\,\tilde{u} = -\frac{1}{K\nu\Delta t}\tilde{w}^*, \tag{76}$$

$$\tilde{u}(\pm 1) = \tilde{v}(\pm 1) = \tilde{w}(\pm 1) = 0. \tag{77}$$

Applying D on (69), multiplying ik on (70) and multiplying il on (71), then adding them together, we have

$$(K\Delta t)(D^2 - r^2)\,\tilde{p} = D\tilde{u} + ik\tilde{v} + il\tilde{w}. \tag{78}$$

Applying D on (74), multiplying ik on (75) and multiplying il on (76), then adding them together, we see that

$$(D^2 - r^2 - \frac{1}{\nu\Delta t})(D\tilde{u} + ik\tilde{v} + il\tilde{w}) = 0. \tag{79}$$

(78) and (79) imply that

$$(D^2 - r^2)(D^2 - r^2 - \frac{1}{\nu\Delta t})\,\tilde{p} = 0. \tag{80}$$

Applying $(D^2 - r^2)(D^2 - r^2 - \frac{1}{\nu\Delta t})$ on (69) gives

$$(D^2 - r^2)\left(D^2 - r^2 - \frac{1}{\nu\Delta t}\right)\tilde{u}^* = (D^2 - r^2)\left(D^2 - r^2 - \frac{1}{\nu\Delta t}\right)\tilde{u} = -\frac{(D^2 - r^2)}{K\nu\Delta t}\tilde{u}^*,$$

$$(D^2 - r^2)^2\tilde{u}^* = \frac{K-1}{K\nu\Delta t}(D^2 - r^2)\tilde{u}^*. \tag{81}$$

(72) and (73) imply

$$D\tilde{u}^*(\pm 1) = \tilde{u}^*(\pm 1) = 0. \tag{82}$$

The characteristic equation of (81)

$$(m^2 - r^2)^2 = \frac{K-1}{K\nu\Delta t}(m^2 - r^2) \Rightarrow\ m = \pm r, \pm\sqrt{-r^2 - \frac{K-1}{K\nu\Delta t}}.$$

Let

$$\tilde{\mu} = \left(-r^2 - \frac{K-1}{K\nu\Delta t}\right)^{1/2} > 0.$$

The general solution of Equation (81) can be expressed as

$$\begin{aligned}
\tilde{u}^* &= c_1\ \cosh rx + c_2\ \sinh rx + c_3\ \cos\ \tilde{\mu}x + c_4\ \sin\tilde{\mu}x,\\
D\tilde{u}^* &= c_1\, r\ \sinh rx + c_2\, r\ \cosh rx - c_3\,\tilde{\mu}\sin\tilde{\mu}x + c_4\,\tilde{\mu}\cos\tilde{\mu}x,\\
\tilde{u}^*(1) = 0 &\Rightarrow c_1\ \cosh r + c_2 \sinh\ r + c_3\ \cos\tilde{\mu} + c_4\ \sin\tilde{\mu} = 0,\\
\tilde{u}^*(-1) = 0 &\Rightarrow c_1\ \cosh r - c_2\ \sinh\ r + c_3\ \cos\tilde{\mu} - c_4\ \sin\tilde{\mu} = 0,\\
D\tilde{u}^*(1) = 0 &\Rightarrow c_1\, r\ \sinh r + c_2\, r\ \cosh r - c_3\,\tilde{\mu}\sin\tilde{\mu} + c_4\,\tilde{\mu}\cos\tilde{\mu} = 0,\\
D\tilde{u}^*(-1) = 0 &\Rightarrow -c_1 r\ \sinh r + c_2\, r\cosh r + c_3\,\tilde{\mu}\sin\tilde{\mu} + c_4\,\tilde{\mu}\cos\tilde{\mu} = 0,
\end{aligned}$$

which imply that

$$c_1 = -\frac{\cos\tilde{\mu}}{\cosh r}c_3, \quad c_2 = -\frac{\sin\tilde{\mu}}{\sinh r}c_4,$$
$$c_1 = -\frac{\tilde{\mu}\sin\tilde{\mu}}{r\sinh r}c_3, \quad c_2 = -\frac{\tilde{\mu}\cos\tilde{\mu}}{r\cosh r}c_4.$$

By a similar analysis to that in Section 2, we obtain the boundary value problem (81)–(84) has only the following two types of solutions.

(i) Even solution: The eigenvalue $\tilde{\mu}$ satisfies

$$-\tilde{\mu}\tan\tilde{\mu} = r\tanh r,$$

and the corresponding eigenfunction is a constant multiple of

$$\tilde{u}^* = \cos\tilde{\mu}\cosh rx - \cosh r\cos\tilde{\mu}x.$$

(ii) **Odd solution:** The eigenvalue $\tilde{\mu}$ satisfies

$$\tilde{\mu}\cot\tilde{\mu} = r\coth r,$$

and the corresponding eigenfunction is a constant multiple of

$$\tilde{u}^* = \sin\tilde{\mu}\ \sinh rx - \sinh r\sin\tilde{\mu}x.$$

Let

$$\sigma = \frac{K-1}{K\Delta t},$$

then (81) becomes

$$\nu(D^2 - r^2)^2\tilde{u}^* = \sigma(D^2 - r^2)\tilde{u}^*,$$

which is exactly the same as (35). Since

$$\tilde{\mu} = \left(-r^2 - \frac{\sigma}{\nu}\right)^{1/2} = \mu,$$

we see that

$$K = \frac{1}{1-\sigma\Delta t} = e^{\tilde{\sigma}\Delta t}.$$

The leading behavior σ of the growth rate $\tilde{\sigma}$ agrees with that of the exact solution:

$$\tilde{\sigma} = \frac{\ln K}{\Delta t} = -\frac{\ln(1-\sigma\Delta t)}{\Delta t}$$
$$= \sigma + \frac{1}{2}\sigma^2\Delta t + \frac{1}{3}\sigma^3\Delta t^3 + \cdots, \tag{83}$$

which implies that

$$\tilde{\sigma} - \sigma = \mathcal{O}(\Delta t).$$

Let

$$\lambda = \left(r^2 + \frac{1}{\nu\Delta t}\right)^{1/2} > 0.$$

Equation (74) becomes

$$(D^2 - \lambda^2)\tilde{u} = -\frac{1}{K\nu\Delta t}\tilde{u}^*,$$

$$\tilde{u}(\pm 1) = 0. \tag{84}$$

We consider two cases:
Case (i):

$$\tilde{u}^* = \cos\mu \cosh rx - \cosh r \cos \mu x$$

is even.

Noting that $r \neq \lambda$ and $r \neq \mu$, we see that the general solution of (84) is

$$\tilde{u} = c_1 \cosh \lambda x + c_2 \sinh \lambda x + \tilde{u}_p.$$

Here,

$$\begin{aligned}
\tilde{u}_p &= A \cosh rx + B \cos \mu x,\\
D\tilde{u}_p &= Ar \sinh rx - B\mu \sin \mu x,\\
D^2\tilde{u}_p &= Ar^2 \cosh rx - B\mu^2 \cos \mu x,\\
(D^2 - \lambda^2)\tilde{u}_p &= A(r^2 - \lambda^2) \cosh rx - B(\mu^2 + \lambda^2) \cos \mu x,\\
&= -\frac{\cos \mu}{K \nu \Delta t} \cosh rx + \frac{\cosh r}{K \nu \Delta t} \cos \mu x,
\end{aligned} \tag{85}$$

which implies that

$$\begin{aligned}
A &= \frac{\cos\mu}{K\nu\Delta t(\lambda^2 - r^2)},\\
B &= -\frac{\cosh r}{K\nu\Delta t(\lambda^2 + \mu^2)},\\
\tilde{u}_p &= \frac{\cos\mu \cosh rx}{K\nu\Delta t(\lambda^2 - r^2)} - \frac{\cosh r \cos \mu x}{K\nu\Delta t(\lambda^2 + \mu^2)}.
\end{aligned}$$

Noting that

$$K = \frac{1}{1 - \sigma\Delta t} \Rightarrow K(1 - \sigma\Delta t) = 1.$$

Thus,

$$\tilde{u} = c_1 \cosh \lambda x + c_2 \sinh \lambda x + \frac{\cos\mu \cosh rx}{K\nu\Delta t(\lambda^2 - r^2)} - \frac{\cosh r \cos \mu x}{K\nu\Delta t(\lambda^2 + \mu^2)}.$$

From

$$\lambda^2 - r^2 = \frac{1}{\nu\Delta t}, \qquad \lambda^2 + \mu^2 = \frac{1}{\nu\Delta t} - \frac{\sigma}{\nu},$$

we obtain

$$\tilde{u} = c_1 \cosh \lambda x + c_2 \sinh \lambda x + (1 - \sigma\Delta t) \cos\mu \cosh rx - \cosh r \cos \mu x.$$

Applying the boundary conditions:

$$\begin{aligned}
\tilde{u}(1) = 0 &\Rightarrow c_1 \cosh\lambda + c_2 \sinh\lambda - \sigma\Delta t \cos\mu \cosh r = 0,\\
\tilde{u}(-1) = 0 &\Rightarrow c_1 \cosh\lambda - c_2 \sinh\lambda - \sigma\Delta t \cos\mu \cosh r = 0,
\end{aligned}$$

we have

$$c_1 = \frac{\sigma\Delta t \cos\mu \cosh r}{\cosh\lambda}, \quad c_2 = 0.$$

Thus,

$$\tilde{u} = \sigma \Delta t \cos \mu \left(\cosh r \frac{\cosh \lambda x}{\cosh \lambda} - \cosh rx \right) + (\cos \mu \cosh rx - \cosh r \cos \mu x)$$

is also even.
Case (ii): $\tilde{u}^* = \sin \mu \sinh rx - \sinh r \sin \mu x$ is odd.

$$\tilde{u} = c_1 \cosh \lambda x + c_2 \sinh \lambda x + \tilde{u}_p,$$

where

$$\begin{aligned}
\tilde{u}_p &= A \sinh rx + B \sin \mu x, \\
D\tilde{u}_p &= Ar \cosh rx + B\mu \cos \mu x, \\
D^2\tilde{u}_p &= Ar^2 \sinh rx - B\mu^2 \sin \mu x, \\
(D^2 - \lambda^2)\tilde{u}_p &= -\frac{1}{K\nu\Delta t}\tilde{u}^* = -\frac{\sin \mu}{K\nu\Delta t} \sinh rx + \frac{\sinh r}{K\nu\Delta t} \sin \mu x, \\
&\Rightarrow A(r^2 - \lambda^2) \sinh rx - B(\mu^2 + \lambda^2) \sin \mu x = -\frac{\sin \mu}{K\nu\Delta t} \sinh rx + \frac{\sinh r}{K\nu\Delta t} \sin \mu x \\
&\Rightarrow A = \frac{\sin \mu}{K}, \ B = -\sinh r. \\
&\Rightarrow \tilde{u} = c_1 \cosh \lambda x + c_2 \sinh \lambda x + \frac{\sin \mu \sinh rx}{K} - \sinh r \sin \mu x.
\end{aligned}$$

Applying the boundary condition,

$$\begin{aligned}
\tilde{u}(1) = 0 &\Rightarrow c_1 \cosh \lambda + c_2 \sinh \lambda = -\frac{\sin \mu \sinh r}{K} + \sin \mu \sinh r, \\
\tilde{u}(-1) = 0 &\Rightarrow c_1 \cosh \lambda - c_2 \sinh \lambda = \frac{\sin \mu \sinh r}{K} - \sinh r \sin \mu.
\end{aligned}$$

We have

$$c_1 = 0, \ c_2 = \frac{\sigma \Delta t \sin \mu \sinh r}{\sinh \lambda}.$$

Thus,

$$\tilde{u} = \sigma \Delta t \sin \mu \left(\sinh r \frac{\sinh \lambda x}{\sinh \lambda} - \sinh rx \right) + (\sin \mu \sinh rx - \sinh r \sin \mu x)$$

is also odd.

Since $\tilde{u}^*$ is the exact normal mode solution, we can compute the error $\tilde{u} - \tilde{u}^*$ as follows.
Case (i): $\tilde{u}^* = \cos \mu \cosh rx - \cosh r \cos \mu x$,

$$\tilde{u} = \sigma \Delta t \cos \mu \left(\cosh r \frac{\cosh \lambda x}{\cosh \lambda} - \cosh rx \right) + (\cos \mu \cosh rx - \cosh r \cos \mu x),$$

which implies that

$$\tilde{u} - \tilde{u}^* = \sigma \Delta t \cos \mu \left(\cosh r \frac{\cosh \lambda x}{\cosh \lambda} - \cosh rx \right).$$

Obviously, we have the error estimate:

$$\tilde{u} - \tilde{u}^* = \mathcal{O}(\Delta t).$$

The error is uniformly bounded with respect to x in $[-1,1]$ as $\Delta t \to 0$ because

$$\lambda = \left(r^2 + \frac{1}{\nu \Delta t}\right)^{\frac{1}{2}} \to \infty \quad \text{as } \Delta t \to 0,$$

$$|x| < 1 \Rightarrow \frac{\cosh \lambda\, x}{\cosh \lambda} \to 0 \quad \text{as } \lambda \to \infty.$$

Case (ii):$\tilde{u}^* = \sin\mu \ \sinh\ rx - \sinh\ r \sin\mu\ x,$

$$\tilde{u} = \sigma \Delta t \sin\mu \left(\sinh r \frac{\sinh \lambda x}{\sinh \lambda} - \sinh rx\right) + (\sin\mu \sinh rx - \sinh r \sin \mu x),$$

which implies that

$$\tilde{u} - \tilde{u}^* = \mathcal{O}(\Delta t).$$

The error is uniformly bounded with respect to x in $[-1,1]$ as $\Delta t \to 0$ because

$$\lambda = \left(r^2 + \frac{1}{\nu \Delta t}\right)^{\frac{1}{2}} \to \infty \quad \text{as } \Delta t \to 0,$$

$$|x| < 1 \Rightarrow \frac{\sinh \lambda\, x}{\sinh \lambda} \to 0 \quad \text{as } \lambda \to \infty.$$

4.2. The Computation of $\tilde{p}$

Case (i): $\tilde{u}^*$ is even.
From (69),

$$D\tilde{p} = \frac{\tilde{u} - \tilde{u}^*}{K\Delta t} = \frac{\sigma}{K}\frac{\cos\mu}{\cosh\lambda}[\cosh\ r \cosh \lambda\, x - \cosh\lambda \cosh\ rx]$$

$$\Rightarrow \tilde{p} = \frac{\sigma}{K}\frac{\cos\mu}{\cosh\lambda}\left[\frac{\cosh\ r}{\lambda}\sinh\lambda\, x - \frac{\cosh\lambda}{r}\ \sinh\ rx\right] + c,$$

where c is a constant. From (81):

$$(D^2 - r^2)(D^2 - \lambda^2)\tilde{p} = 0.$$

$$\Rightarrow \tilde{p} = c_1\ \cosh\ rx + c_2\ \sinh\ rx + c_3\ \cosh\lambda\, x + c_4\ \sinh\lambda\, x.$$

Thus, we see that $c = 0$ and

$$\tilde{p} = \frac{\sigma \cos\mu}{\cosh\lambda}\left[\frac{\cosh\ r}{\lambda}\sinh\lambda\, x - \frac{\cosh\lambda}{r}\ \sinh\ rx\right](1 - \sigma\Delta t).$$

Notice that

$$\hat{p} = -\frac{\sigma\cos\mu}{r}\sinh\ rx,$$

$$\tilde{p} - \hat{p} = \frac{\sigma\cos\mu}{\cosh\lambda}\frac{\cosh\ r}{\lambda}\sinh\lambda\, x - \frac{\Delta t \sigma^2\cos\mu}{\cosh\lambda}\left[\frac{\cosh\ r}{\lambda}\sinh\lambda\, x - \frac{\cosh\lambda}{r}\ \sinh\ rx\right].$$

Moreover,

$$\lambda = \left(r^2 + \frac{1}{\nu \Delta t} \right)^{1/2} = \mathcal{O}(\Delta t^{-(1/2)}) \Rightarrow \frac{1}{\lambda} = \mathcal{O}(\Delta t^{1/2})$$

Because $\sinh \lambda\, x / \cosh \lambda$ is uniformly bounded on $[-1, 1]$ as $\Delta t \to 0$, we have

$$\tilde{p} - \hat{p} = \mathcal{O}(\Delta t^{1/2}).$$

Case (ii): $\tilde{u}^*$ is odd.

$$D\tilde{p} = \frac{\tilde{u} - \tilde{u}^*}{K \Delta t} = \frac{\sigma}{K} \left[\sinh\, r \sinh \lambda\, x - \sinh \lambda \sinh\, rx \right] \frac{\sin \mu}{\sinh \lambda}.$$

From (80),

$$\tilde{p} = c_1\ \cosh\ rx + c_2\ \sinh\ rx + c_3\ \cosh \lambda\, x + c_4\ \sinh \lambda\, x.$$

Thus,

$$\tilde{p} = \frac{\sigma \sin \mu}{\sinh \lambda} \left[\frac{\sinh\, r}{\lambda} \cosh \lambda\, x - \frac{\sinh \lambda}{r}\ \cosh\ rx \right] (1 - \sigma \Delta t),$$

$$\hat{p} = -\frac{\sigma \sin \mu}{r} \cosh\ rx,$$

$$\tilde{p} - \hat{p} = \frac{\sigma \sin \mu \sinh\, r}{\lambda} \frac{\cosh \lambda}{\sinh \lambda} - \frac{\Delta t \sigma^2 \sin \mu}{\sinh \lambda} \left[\frac{\sinh\, r}{\lambda} \cosh \lambda\, x - \frac{\sinh \lambda}{r}\ \cosh\ rx \right].$$

We see that

$$\tilde{p} - \hat{p} = \mathcal{O}(\Delta t^{1/2}).$$

Similarly, one can compute $\tilde{v}, \tilde{v}^*, \tilde{w}, \tilde{w}^*$. As the derivations are similar, we omit the details.

5. Concluding remarks

In this work, we derive the normal mode solution of a 3D linearized incompressible fluid flow model. Then, we apply the results to analyze some implicit time stepping schemes including the Backward Euler and the Crank–Nicolson schemes, as well as the splitting method. By using the normal mode analysis, we rigorously prove that both the Backward Euler scheme and Crank–Nicolson schemes for the 3D Stokes equations are unconditionally stable; The time errors of the Backward Euler scheme and the Crank–Nicolson scheme are of the order $\mathcal{O}(\Delta t)$ and $\mathcal{O}(\Delta t^2)$, respectively. Moreover, based on the normal mode analysis, we give the estimates of error orders of each variable and the intermediate variables for the splitting method.

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