Spectral and Combinatorial Properties of Some Algebraically Defined Graphs

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Abstract

Let $k \geq 3$ be an integer, q be a prime power, and $\mathbb{F}_q$ denote the field of q elements. Let $f_i, g_i \in \mathbb{F}_q[X]$, $3 \leq i \leq k$, such that $g_i(-X) = -g_i(X)$. We define a graph $S(k,q) = S(k,q; f_3, g_3, \cdots, f_k, g_k)$ as a graph with the vertex set $\mathbb{F}_q^k$ and edges defined as follows: vertices $a = (a_1, a_2, \ldots, a_k)$ and $b = (b_1, b_2, \ldots, b_k)$ are adjacent if $a_1 \neq b_1$ and the following $k-2$ relations on their components hold:

$$b_i - a_i = g_i(b_1 - a_1) f_i\Big(\frac{b_2 - a_2}{b_1 - a_1}\Big), \quad 3 \leq i \leq k.$$

We show that graphs $S(k,q)$ generalize several recently studied examples of regular expanders and can provide many new such examples.

1 Introduction and Motivation

All graphs in this paper are simple, i.e., undirected, with no loops and no multiple edges. See, e.g., Bollobás [4] for standard terminology. Let $\Gamma = (V, E)$ be a graph

with vertex set V and edge set E. For a subset of vertices A of V, ∂A denotes the set of edges of Γ with one endpoint in A and the other endpoint in $V \setminus A$. The *Cheeger constant* $h(\Gamma)$ (also known as *edge-isoperimetric number* or *expansion ratio*) of Γ, is defined by $h(\Gamma) := \min\Big\{ \frac{|\partial A|}{|A|} : A \subseteq V, 0 < |A| \leq \frac{1}{2}|V| \Big\}$. The graph Γ is d-regular if each vertex is adjacent to exactly d others. An infinite family of *expanders* is an infinite family of regular graphs whose Cheeger constants are uniformly bounded away from 0. More precisely, for $n \geq 1$, let $\Gamma_n = (V_n, E_n)$ be a sequence of graphs such that each Γ_n is d_n-regular and $|V_n| \to \infty$ as $n \to \infty$. We say that the members of the sequence form a *family of expanders* if the corresponding sequence $\big(h(\Gamma_n)\big)$ is bounded away from zero, i.e. there exists a real number $c > 0$ such that $h(\Gamma_n) \geq c$ for all $n \geq 1$. In general, one would like the valency sequence $(d_n)_{n\geq 1}$ to be growing slowly with n, and ideally, to be bounded above by a constant. For examples of families of expanders, their theory and applications, see Davidoff, Sarnak and Valette [8], Hoory, Linial and Wigderson [10], and Krebs and Shaheen [12].

The adjacency matrix $A = A(\Gamma)$ of a graph $\Gamma = (V, E)$ has its rows and columns labeled by V and $A(x, y)$ equals the number of edges between x and y (i.e. 0 or 1). When Γ is simple, the matrix A is symmetric and therefore, its eigenvalues are real numbers. For j between 1 and the order of Γ, let $\lambda_j = \lambda_j(G)$ denote the j-th eigenvalue of A. For an arbitrary graph Γ, it is hard to find or estimate $h(\Gamma)$, and often it is done by using the second-largest eigenvalue $\lambda_2(\Gamma)$ of the adjacency matrix of Γ. If Γ is a connected d-regular graph, then $\frac{1}{2}\big(d - \lambda_2\big) \leq h(\Gamma) \leq \sqrt{d^2 - \lambda_2^2}$. The lower bound was proved by Dodziuk [9] and independently by Alon-Milman [1] and by Alon [2]. In both [1] and [2], the upper bound on $h(\Gamma)$, namely $\sqrt{2d(d - \lambda_2)}$ was provided. Mohar in [20] improved the upper bound to the one above. See [5, 10, 12], for terminology and results on spectral graph theory and connections between eigenvalues and expansion properties of graphs. The difference $d - \lambda_2$ which is present is both sides of this inequality above, also known as the *spectral gap* of Γ, provides an estimate on the expansion ratio of the graph. In particular, for an infinite family of d-regular graphs Γ_n, the sequence $\big(h(\Gamma_n)\big)$ is bounded away from zero if and only if the sequence $\big(d - \lambda_2(\Gamma_n)\big)$ is bounded away from zero. A d-regular

connected graph Γ is called *Ramanujan* if $\lambda_2(\Gamma) \leq 2\sqrt{d-1}$. Alon and Boppana [22] proved that this bound is asymptotically best possible for any infinite family of d-regular graphs and their results imply that for any infinite family of d-regular connected graphs Γ_n, $\lambda_2(\Gamma_n) \geq 2\sqrt{d-1} - o_n(1)$. For functions $f, g : \mathbb{N} \to \mathbb{R}^+$, we write $f = o_n(g)$ if $f(n)/g(n) \to 0$ as $n \to \infty$.

For the rest of the paper, let $q = p^e$, where p is a prime and e is a positive integer. For a sequence of prime powers $(q_m)_{m\geq 1}$, we always assume that $q_m = p_m^{e_m}$, where p_m is a prime and $e_m \geq 1$. Let $\mathbb{F}_q$ be the finite field of q elements and $\mathbb{F}_q^k$ be the cartesian product of k copies of $\mathbb{F}_q$. Clearly, $\mathbb{F}_q^k$ is a vector space of dimension k over $\mathbb{F}_q$. For $2 \leq i \leq k$, let h_i be an arbitrary polynomial in $2i-2$ indeterminants over $\mathbb{F}_q$. We define the bipartite graph $B\Gamma_k = B\Gamma(q; h_2, \ldots, h_k)$, $k \geq 2$, as follows. The vertex set of $B\Gamma_k$ is the disjoint union of two copies of $\mathbb{F}_q^k$, one denoted by P_k and the other by L_k. We define edges of $B\Gamma_k$ by declaring vertices $p = (p_1, p_2, \ldots, p_k) \in P_k$ and $l = (l_1, l_2, \ldots, l_k) \in L_k$ to be adjacent if the following $k-1$ relations on their coordinates hold:

$$p_i + l_i = h_i(p_1, l_1, p_2, l_2, \ldots, p_{i-1}, l_{i-1}),\ i = 2, \ldots, k. \tag{1}$$

The graphs $B\Gamma_k$ were introduced by Lazebnik and Woldar [15], as generalizations of graphs introduced by Lazebnik and Ustimenko in [14] and [16]. For surveys on these graphs and their applications, see [15] and Lazebnik, Sun and Wang [13]. An important basic property of graphs $B\Gamma_k$ (see [15]) is that for every vertex v of $B\Gamma_k$ and every $\alpha \in \mathbb{F}_q$, there exists a unique neighbor of v whose first coordinate is α. This implies that each $B\Gamma_k$ is q-regular, has $2q^k$ vertices and q^{k+1} edges.

The spectral and combinatorial properties of three specializations of graphs $B\Gamma_k$ has received particular attention in recent years. Cioabă, Lazebnik and Li [7] determined the complete spectrum of the *Wenger* graphs $W_k(q) = B\Gamma(q; h_2, \ldots, h_{k+1})$ with $h_i = p_1 l_1^{i-1}$, $2 \leq i \leq k+1$. Cao, Lu, Wan, Wang and Wang [6] determined the eigenvalues of the *linearized Wenger* graphs $L_k(q) = B\Gamma(q; h_2, \ldots, h_{k+1})$ with $h_i = p_1^{p^{i-2}} l_1$, $2 \leq i \leq k+1$, and Yan and Liu [24] determined the multiplicities of the eigenvalues the linearized Wenger graphs. Moorhouse, Sun and Williford [21] studied the spectra of graphs $D(4, q) = B\Gamma(q; p_1 l_1, p_1 l_2, p_2 l_1)$, and in particular,

proved that the second largest eigenvalues of these graphs are bounded from above by $2\sqrt{q}$ (so $D(4,q)$ is 'close' to being Ramanujan).

Let V_1 and V_2 denote the partite sets or color classes of the vertex set of a bipartite graph Γ. The *distance-two graph of* Γ *on* V_1 is the graph having V_1 as its vertex set with the adjacency defined as follows: two vertices $x \neq y \in V_1$ are adjacent if there exists a vertex $z \in V_2$ adjacent to both x and to y in Γ (which is equivalent of saying that x and y are at distance two in Γ). If Γ is d-regular and contains no 4-cycles, then $\Gamma^{(2)}$ is a $d(d-1)$-regular simple graph. There is simple connection between the eigenvalues of Γ and the eigenvalues of $\Gamma^{(2)}$ (see, e.g., [7]): every eigenvalue λ of $\Gamma^{(2)}$ with multiplicity m, corresponds to a pair of eigenvalues $\pm\sqrt{\lambda+d}$ of Γ, each with multiplicity m (or a single eigenvalue 0 of multiplicity $2m$ in case $\lambda = -d$).

This relation between the spectra of q-regular bipartite graph Γ and its $q(q-1)$-regular distance-two graph $\Gamma^{(2)}$ has been utilized in each of the papers [7, 6, 21] in order to find or to bound the second-largest eigenvalue of Γ, and then use this information to claim the expansion property of Γ. In each of these cases, $\Gamma^{(2)}$ turned out to be a Cayley graph of a group, that allowed to use representation theory to compute its spectrum. In [7, 6] the group turned out be abelian, as in [21] it was not for odd q.

The main motivation behind the construction below is to directly generalize the defining systems of equations for $W_k^{(2)}(q)$ and of $L_k^{(2)}(q)$, thereby obtaining a family of $q(q-1)$-regular Cayley graphs of an abelian group. The adverb *directly* used in the previous sentence was to stress that the graphs we build are not necessarily distance-two graphs of q-regular bipartite graphs Γ. Examples when they are not will be discussed in Remark 1 of Section 7.

2 Main Results

In this section, we define the main object of this paper, the family of graphs $S(k,q)$ and we describe our main results. Let k be an integer, $k \geq 3$. Let $f_i, g_i \in \mathbb{F}_q[X]$,

$3 \le i \le k$, be $2(k-2)$ polynomials of degrees at most $q-1$ such that $g_i(-X) = -g_i(X)$ for each i. We define $S(k,q) = S(k,q; f_3, g_3, \cdots, f_k, g_k)$ as the graph with the vertex set $\mathbb{F}_q^k$ and edges defined as follows: $a = (a_1, a_2, \ldots, a_k)$ is adjacent to $b = (b_1, b_2, \ldots, b_k)$ if $a_1 \neq b_1$ and the following $k-2$ relations on their coordinates hold:

$$b_i - a_i = g_i(b_1 - a_1) f_i\Big(\frac{b_2 - a_2}{b_1 - a_1}\Big), \quad 3 \le i \le k. \tag{2}$$

Clearly, the requirement $g_i(-X) = -g_i(X)$ is used for the definition of the adjacency in $S(k,q)$ to be symmetric. One can easily see that $S(k,q)$ is a Cayley graph with the *underlying group* G being the additive group of the vector space $\mathbb{F}_q^k$ with *generating set*

$$\big\{\big(a, au, g_3(a)f_3(u), \cdots, g_k(a)f_k(u)\big) \mid a \in \mathbb{F}_q^*, u \in \mathbb{F}_q\big\}.$$

This implies that $S(k,q)$ is vertex transitive of degree $q(q-1)$.

Note that for $f_i = X^{i-1}$ and $g_i = X$, $3 \le i \le k+1$, $S(k+1,q) = W_k^{(2)}(q)$ is the distance-two graph of the Wenger graphs $W_k(q)$ on lines and for $f_i = X^{p^{i-2}}$ and $g_i = X$, $3 \le i \le k+1$, $S(k+1,q) = L_k^{(2)}(q)$ is the distance-two graph of the linearized Wenger graphs $L_k(q)$ on lines.

In order to present our results, we need a few more notation. For any $\alpha \in \mathbb{F}_q$, let $Tr(\alpha) = \alpha + \alpha^p + \cdots + \alpha^{p^{e-1}}$ be the *trace of* α *over* $\mathbb{F}_p$. It is known that $Tr(\alpha) \in \mathbb{F}_p$. For any element $\beta \in \mathbb{F}_p$, let β^* denote the unique integer such that $0 \le \beta^* < p$ and the residue class of β^* in $\mathbb{F}_p$ is β. For any complex number c, the expression c^β will mean c^{β^*}. Let $\zeta_p = \exp(\frac{2\pi}{p}i)$ be a complex p-th root of unity. For every $f \in \mathbb{F}_q[X]$, we call $\varepsilon_f = \sum_{x \in \mathbb{F}_q} \zeta_p^{Tr\left(f(x)\right)}$ the *exponential sum of* f.

We are ready to state the main results of this paper.

The following theorem describes the spectrum of the graphs $S(k,q)$.

Theorem 2.1. *Let $k \ge 3$. Then the spectrum of $S(k,q)$ is the multiset $\{\lambda_w \mid w = (w_1, \cdots, w_k) \in \mathbb{F}_q^k\}$, where*

$$\lambda_w = \sum_{a \in \mathbb{F}_q^*, u \in \mathbb{F}_q} \zeta_p^{Tr\left(aw_1 + auw_2 + \sum_{i=3}^{k} g_i(a) f_k(u) w_k\right)}. \tag{3}$$

For a fixed $k \geq 3$, the theorem below provides sufficient conditions for the graphs $S(k,q)$ to form a family of expanders.

Theorem 2.2. *Let $k \geq 3$, $(q_m)_{m\geq 1}$ be an increasing sequence of prime powers, and let*

$$S(k,q_m) = S(k,q_m; f_{3,m}, g_{3,m}, \cdots, f_{k,m}, g_{k,m}).$$

Set $d_f^{(m)} = \max\limits_{3\leq i\leq k} \deg(f_{i,m})$ and $d_g^{(m)} = \max\limits_{3\leq i\leq k} \deg(g_{i,m})$. Suppose $1 \leq d_f^{(m)} = o_m(q_m)$, $d_g^{(m)} = o_m(\sqrt{q_m})$, $1 \leq d_g^{(m)} < p_m$, and for all $m \geq 1$, at least one of the following two conditions is satisfied:

1. *The polynomials $1, X, f_{3,m}, \ldots, f_{k,m}$ are $\mathbb{F}_q$-linearly independent, and $g_{i,m}$ has linear term for all i, $3 \leq i \leq k$.*

2. *The polynomials $f_{3,m}, \ldots, f_{k,m}$ are $\mathbb{F}_q$-linearly independent, and there exists some j, $2 \leq j \leq d_g^{(m)}$, such that each polynomial $g_{i,m}$, $3 \leq i \leq k$, contains a term $c_{i,j}^{(m)} X^j$ with $c_{i,j} \neq 0$.*

Then $S(k,q_m)$ is connected and $\lambda_2\big(S(k,q_m)\big) = o_m(q_m^2)$.

The following two theorems demonstrate that for some specializations of $S(k,q)$, we can obtain stronger upper bounds on their second largest eigenvalues.

Theorem 2.3. *Let q be an odd prime power with $q \equiv 2 \mod 3$, and $4 \leq k \leq q+1$. Let $g_i(X) = X^3$ and $f_i(X) = X^{i-1}$ for each i, $3 \leq i \leq k$. Then $S(k,q)$ is connected, and*

$$\lambda_2\big(S(k,q)\big) = \max\big\{q(k-3), (q-1)M_q\big\},$$

where $M_q = \max\limits_{a,b\in\mathbb{F}_q^*} \varepsilon_{ax^3+bx} \leq 2\sqrt{q}$.

For large k, specifically, when $(q-1)M_q \leq q(k-3)$,

$$\lambda_2\big(S(k,q)\big) = q(k-3) < q(k-2) = \lambda_2\big(W_{k-1}^{(2)}(q)\big).$$

Similarly to Theorem 2.3, when choosing $f_i(X) = X^{p^{i-2}}$, the same f functions as in $L_k^{(2)}(q)$, we obtain the following upper bounds for the second largest eigenvalue.

Theorem 2.4. *Let q be an odd prime power with $q \equiv 2 \mod 3$, and $3 \le k \le e+2$. Let $g_i(X) = X^3$ and $f_i(X) = X^{p^{i-2}}$ for each i, $3 \le i \le k$. Then $S(k,q)$ is connected, and*

$$\lambda_2\big(S(k,q)\big) \le \max\big\{q(p^{k-3}-1), (q-1)M_q\big\},$$

where $M_q = \max\limits_{a,b\in\mathbb{F}_q^*} \varepsilon_{ax^3+bx} \le 2\sqrt{q}$.

For large k, specifically, when $(q-1)M_q \le q(p^{k-3}-1)$,

$$\lambda_2\big(S(k,q)\big) = q(p^{k-3}-1) < q(p^{k-2}-1) = \lambda_2\big(L_{k-1}^{(2)}(q)\big).$$

The paper is organized as follows. In Section 3, we present necessary definitions and results concerning finite fields used in the proofs. In Section 4, we prove Theorem 2.1. In Section 5, we study some sufficient conditions on f_i and g_i for the graph $S(k,q)$ to be connected and have large eigenvalue gap, and prove Theorem 2.2. In Section 6, we prove Theorem 2.3 and Theorem 2.4. We conclude the paper with several remarks in Section 7.

3 Background on finite fields

For definitions and theory of finite fields, see Lidl and Niederreiter [18].

Lemma 3.1 ([18], Ch.5)**.** *If $f(X) = bX + c \in \mathbb{F}_q[X]$ is a polynomial of degree one or less, then*

$$\varepsilon_f = \begin{cases} 0, & \text{if } b \neq 0, \\ q\zeta^{Tr(c)}, & \text{otherwise.} \end{cases}$$

For a general $f \in \mathbb{F}_q[X]$, no explicit expression for the exponential sum ε_f exists. The following theorem provides a good upper bound for the exponential sum ε_f.

Theorem 3.2 (Hasse-Davenport-Weil Bound, [18], Ch.5)**.** *Let $f \in \mathbb{F}_q[X]$ be a polynomial of degree $n \ge 1$. If $gcd(n,q) = 1$, then*

$$|\varepsilon_f| \le (n-1)q^{1/2}.$$

Lemma 3.3. *Suppose that $g \in \mathbb{F}_q[X]$ and $g(-X) = -g(X)$. Then ε_g is a real number.*

Proof. We have that

$$\varepsilon_g = \sum_{a\in\mathbb{F}_q} \zeta_p^{\,Tr(g(a))} = 1 + \sum_{a\in\mathbb{F}_q^*} \zeta_p^{\,Tr(g(a))} = 1 + \frac{1}{2}\sum_{a\in\mathbb{F}_q^*} \left(\zeta_p^{\,Tr(g(a))} + \zeta_p^{\,Tr(g(-a))}\right)$$
$$= 1 + \frac{1}{2}\sum_{a\in\mathbb{F}_q^*} \left(\zeta_p^{\,Tr(g(a))} + \zeta_p^{\,Tr(-g(a))}\right) = 1 + \frac{1}{2}\sum_{a\in\mathbb{F}_q^*} \left(\zeta_p^{\,Tr(g(a))} + \zeta_p^{\,-Tr(g(a))}\right).$$

Since $\zeta_p^{\beta} + \zeta_p^{-\beta} \in \mathbb{R}$ for any $\beta \in \mathbb{F}_p$, it follows that $\varepsilon_g \in \mathbb{R}$. □

4 Spectra of the graphs $S(k, q)$

The proof we present here is based on the same idea as the one in [7]. Namely, computing eigenvalues of Cayley graphs by using the method suggested in Babai [3]. The original completely different (and much longer) proof of Theorem 2.1 that used circulants appears in Sun [23].

Theorem 4.1 ([3])**.** *Let G be a finite group and $S \subseteq G$ such that $1 \notin S$ and $S^{-1} = S$. Let $\{\pi_1, \ldots, \pi_k\}$ be a representative set of irreducible $\mathbb{C}$-representations of G. Suppose that the multiset $\Lambda_i := \{\lambda_{i,1}, \lambda_{i,2}, \ldots, \lambda_{i,n_i}\}$ is the spectrum of the complex $n_i \times n_i$ matrix $\pi_i(S) = \sum_{s\in S} \pi_i(s)$. Then the spectrum of the Cayley graph $X = Cay(G, S)$ is the multiset formed as the union of n_i copies of Λ_i for $i \in \{1, 2, \ldots, k\}$.*

Proof of Theorem 2.1. As we mentioned in Section 2, $S(k, q)$ is a Cayley graph with the underlying group G being the additive group of the vector space $\mathbb{F}_q^k$, and connection set

$$\left\{ \left(a, au, g_3(a)f_3(u), \cdots, g_k(a)f_k(u)\right) \mid a \in \mathbb{F}_q^*, u \in \mathbb{F}_q \right\}.$$

Since G is an abelian group, it follows that the irreducible $\mathbb{C}$-representations of G are linear (see [11], Ch. 2). They are given by

$$\pi_w(v) = [\zeta_p^{\,Tr(w_1v_1+\cdots+w_kv_k)}],$$

where $w = (w_1, \cdots, w_k) \in \mathbb{F}_q^k$ and $v = (v_1, \cdots, v_k) \in \mathbb{F}_q^k$.

Using Theorem 4.1, we conclude that the spectrum of $S(k,q)$ is a multiset formed by all λ_w, $w = (w_1, \cdots, w_k) \in \mathbb{F}_q^k$, of the form:

$$\begin{aligned}\lambda_w &= \sum_{s \in S} \zeta_p^{Tr(w_1 s_1 + \cdots + w_k s_k)} \\ &= \sum_{a \in \mathbb{F}_q^*, u \in \mathbb{F}_q} \zeta_p^{Tr\left(aw_1 + auw_2 + \sum_{i=3}^{k} g_i(a) f_i(u) w_i\right)}.\end{aligned}$$

□

5 Connectivity and expansion of the graphs $S(k,q)$

It is hard to get a closed form of λ_w in (3) for arbitrary f_i and g_i. But if the degrees of the polynomials f_i and g_i satisfy some conditions, we are able to show that the components of the graphs $S(k,q)$ have large eigenvalue gap. For these f_i and g_i, we find sufficient conditions such that the graphs $S(k,q)$ are connected, and hence form a family of expanders.

From now on, for any graph $S(k,q; f_3, g_3, \cdots, f_k, g_k)$, we let $d_g = \max\limits_{3 \le i \le k} \deg(g_i)$ and $d_f = \max\limits_{3 \le i \le k} \deg(f_i)$. We also assume that $d_f \ge 1$ and $d_g \ge 1$. For each i, $3 \le i \le k$, let $c_{i,j}$ be the coefficient of X^j in the polynomial g_i, for any j, $1 \le j \le d_g$, i.e.

$$g_i(X) = c_{i,1}X + c_{i,2}X^2 + \ldots + c_{i,d_g}X^{d_g}.$$

For any $w = (w_1, \cdots, w_k)$ in $\mathbb{F}_q^k$, let N_w be the number of u's in $\mathbb{F}_q$ satisfying the following system

$$\begin{aligned} w_1 + uw_2 + \sum_{i=3}^{k} c_{i,1} f_i(u) w_i &= 0, \\ \sum_{i=3}^{k} c_{i,j} f_i(u) w_i &= 0, \qquad 2 \le j \le d_g \end{aligned} \tag{4}$$

and let S_w be the set of all u's in $\mathbb{F}_q$ such that the following inequality holds for some j, $2 \le j \le d_g$,

$$\sum_{i=3}^{k} c_{i,j} f_i(u) w_i \neq 0. \tag{5}$$

If $d_g = 1$, then system (4) contains only the first equation, and $S_w = \emptyset$.

Lemma 5.1. *Let $k \ge 3$. If $1 \le d_g < p$, then for any $w = (w_1, \cdots, w_k)$ in $\mathbb{F}_q^k$, the eigenvalue λ_w of $S(k,q)$ in (3) is at most*

$$N_w(q-1) + |S_w|[(d_g - 1)\sqrt{q} + 1]. \tag{6}$$

Moreover, $\lambda_w = q(q-1)$ if and only if $N_w = q$.

Proof. Let $w = (w_1, \ldots, w_k) \in \mathbb{F}_q^k$. Using Theorem 2.1, we have

$$\begin{aligned}
\lambda_w &= \sum_{u \in \mathbb{F}_q} \sum_{a \in \mathbb{F}_q^*} \zeta_p^{Tr\left(a(w_1 + uw_2) + \sum_{i=3}^{k} g_i(a) f_i(u) w_i\right)} \\
&= \sum_{u \in \mathbb{F}_q} \sum_{a \in \mathbb{F}_q^*} \zeta_p^{Tr\left(a\left[w_1 + uw_2 + \sum_{i=3}^{k} c_{i,1} f_i(u) w_i\right] + a^2 \sum_{i=3}^{k} c_{i,2} f_i(u) w_i + \cdots + a^{d_g} \sum_{i=3}^{k} c_{i,d_g} f_i(u) w_i\right)} \\
&= \sum_{u \in \mathbb{F}_q} z_u,
\end{aligned}$$

where

$$z_u = \sum_{a \in \mathbb{F}_q^*} \zeta_p^{Tr\left(a\left[w_1 + uw_2 + \sum_{i=3}^{k} c_{i,1} f_i(u) w_i\right] + a^2 \sum_{i=3}^{k} c_{i,2} f_i(u) w_i + \cdots + a^{d_g} \sum_{i=3}^{k} c_{i,d_g} f_i(u) w_i\right)}.$$

If u satisfies (4), then $z_u = q - 1$. If $u \in S_w$, then z_u is an exponential sum of a polynomial of degree at least 2 and at most d_g. By the assumption of the theorem that $d_g < p$ and Weil's bound in Theorem 3.2, it follows that

$$|z_u| \le (d_g - 1)\sqrt{q} + 1.$$

Finally, for the remaining $q - N_w - |S_w|$ elements $u \in \mathbb{F}_q$, we have

$$w_1 + uw_2 + \sum_{i=3}^{k} c_{i,1} f_i(u) w_i \neq 0$$

$$\sum_{i=3}^{k} c_{i,j} f_i(u) w_i = 0, \qquad 2 \le j \le d_g. \tag{7}$$

If $d_g = 1$, then system (7) contains only the first inequality. In both cases, we have $z_u = -1$. Therefore, we have

$$\begin{aligned}
\lambda_w &= N_w(q-1) + \sum_{u \in S_w} z_u + (q - N_w - |S_w|)(-1) \\
&\le (N_w - 1)q + |S_w|[(d_g - 1)\sqrt{q} + 2] \\
&\le N_w(q-1) + |S_w|[(d_g - 1)\sqrt{q} + 1].
\end{aligned}$$

Let us now prove the second statement of the lemma. It is clear that if $N_w = q$, then $|S_w| = 0$ and $\lambda_w = q(q-1)$. For the rest of this proof, we assume that $N_w < q$, and show that $\lambda_w < q(q-1)$.

If $e > 1$, then $(d_g - 1)\sqrt{q} + 1 < q - 1$ as $d_g < p$. Therefore, $\lambda_w < q(q-1)$.

For $e = 1$, we consider the following two cases: $q = p = 2$ and $q = p \ge 3$.

If $q = p = 2$, then $d_g = 1$ as $d_g < p$, and hence $|S_w| = 0$. Therefore, $\lambda_w < q(q-1)$.

If $q = p \ge 3$, then, as λ_w is a real number and $|z_u| \le p - 1$, we have

$$\lambda_w \le |\lambda_w| = |\sum_{u \in \mathbb{F}_p} z_u| \le \sum_{u \in \mathbb{F}_p} |z_u| \le p(p-1),$$

and $\lambda_w = p(p-1)$ if and only if $z_u = p - 1$ for all $u \in \mathbb{F}_p$. The latter condition is equivalent to

$$Tr\Big(a\big[w_1 + uw_2 + \sum_{i=3}^{k} c_{i,1} f_i(u) w_i\big] + a^2 \sum_{i=3}^{k} c_{i,2} f_i(u) w_i + \cdots + a^{d_g} \sum_{i=3}^{k} c_{i,d_g} f_i(u) w_i\Big) = 0$$

for all $u \in \mathbb{F}_p$. For $x \in \mathbb{F}_p$, $Tr(x) = 0$ if and only if $x = 0$. This implies that

$$a\big[w_1 + uw_2 + \sum_{i=3}^{k} c_{i,1} f_i(u) w_i\big] + a^2 \sum_{i=3}^{k} c_{i,2} f_i(u) w_i + \cdots + a^{d_g} \sum_{i=3}^{k} c_{i,d_g} f_i(u) w_i = 0$$

for any $a \in \mathbb{F}_p^*$. Therefore, the polynomial

$$X\big[w_1 + uw_2 + \sum_{i=3}^{k} c_{i,1} f_i(u) w_i\big] + X^2 \sum_{i=3}^{k} c_{i,2} f_i(u) w_i + \cdots + X^{d_g} \sum_{i=3}^{k} c_{i,d_g} f_i(u) w_i,$$

which is over $\mathbb{F}_p$, has p distinct roots in $\mathbb{F}_p$ and is of degree at most d_g, $d_g < p$. Hence, it must be zero polynomial, and so $N_p = p$, a contradiction. Hence, $\lambda_w < p(p-1)$. □

Let $(q_m)_{m\geq 1}$ be an increasing sequence of prime powers. For a fixed k, $k \geq 3$, we consider an infinite family of graphs $S(k, q_m; f_{3,m}, g_{3,m}, \cdots, f_{k,m}, g_{k,m})$. Hence, $|V\big(S(k,q_m)\big)| = q_m^k \to \infty$ when $m \to \infty$. Let $d_f^{(m)} = \max\limits_{3\leq i\leq k} \deg(f_{i,m})$ and $d_g^{(m)} = \max\limits_{3\leq i\leq k} \deg(g_{i,m})$, for each m. In what follows we present conditions on $d_f^{(m)}$ and $d_g^{(m)}$ which imply that the components of these graphs have large eigenvalue gaps.

Theorem 5.2. *Let $(q_m)_{m\geq 1}$ be an increasing sequence of prime powers. Suppose that $d_f^{(m)} \geq 1$ and $1 \leq d_g^{(m)} < p_m$ for all m. Let $\lambda^{(m)}$ be the largest eigenvalue of $S(k, q_m)$ which is not $q_m(q_m-1)$ for any m. Then*

$$\lambda^{(m)} = \max\big(O(d_f^{(m)} q_m), O(d_g^{(m)} q_m^{3/2})\big).$$

Proof. For any $w \in \mathbb{F}_{q_m}^k$, the eigenvalue λ_w of $S(k, q_m)$ is at most

$$N_w(q_m - 1) + |S_w|[(d_g^{(m)} - 1)\sqrt{q_m} + 1],$$

by Lemma 5.1.

It is clear that for any $w \in \mathbb{F}_{q_m}^k$, system (4) has either $N_w = q_m$ solutions or at most $d_f^{(m)}$ solutions with respect to u. If $N_w = q_m$, then $\lambda_w = q_m(q_m - 1)$ by Lemma 5.1. If $N_w < q_m$, then $N_w \leq d_f^{(m)}$. Therefore, we have

$$\begin{aligned}\lambda_w &\leq d_f^{(m)}(q_m - 1) + q_m[(d_g^{(m)} - 1)\sqrt{q_m} + 1] \\ &\leq d_f^{(m)} q_m + d_g^{(m)} q_m^{3/2} = \max\big(O(d_f^{(m)} q_m), O(d_g^{(m)} q_m^{3/2})\big).\end{aligned}$$

□

As an immediate corollary from Theorem 5.2, we have the following theorem.

Theorem 5.3. *Let $(q_m)_{m\geq 1}$ be an increasing sequence of prime powers. Suppose that $1 \leq d_f^{(m)} = o_m(q_m)$, $d_g^{(m)} = o_m(\sqrt{q_m})$ and $1 \leq d_g^{(m)} < p_m$ for all m. Let $\lambda^{(m)}$ be the largest eigenvalue of $S(k, q_m)$ which is not $q_m(q_m-1)$ for any m. Then*

$$\lambda^{(m)} = o_m(q_m^2).$$

Our next theorem provides a sufficient condition for the graph $S(k,q)$ to be connected.

Theorem 5.4. *For $k \geq 3$, let $S(k,q) = S(k,q; f_3, g_3, \cdots, f_k, g_k)$ and $1 \leq d_g < p$. If at least one of the following two conditions is satisfied, then $S(k,q)$ is connected.*

1. *The polynomials $1, X, f_3, \ldots, f_k$ are $\mathbb{F}_q$-linearly independent, and g_i contains a linear term for each i, $3 \leq i \leq k$.*

2. *The polynomials $f_3, \ldots, f_k$ are $\mathbb{F}_q$-linearly independent, and there exists some j, $2 \leq j \leq d_g$, such that each polynomial g_i, $3 \leq i \leq k$, contains a term $c_{i,j}X^j$ with $c_{i,j} \neq 0$.*

Proof. First, notice that the number of components of $S(k,q)$ is equal to the multiplicity of the eigenvalue $q(q-1)$. By Lemma 5.1, this multiplicity is equal to $\big|\{w \in \mathbb{F}_q^k : N_w = q\}\big|$. As the equality $N_w = q$ is equivalent to the statement that system (4) (with respect to u) has q solutions, the set $\{w \in \mathbb{F}_q^k : N_w = q\}$ is a subspace of $\mathbb{F}_q^k$.

Let $v_1 = (1, 0, \cdots, 0)$, $v_2 = (X, 0, \cdots, 0)$, and $v_i = (c_{i,1}f_i, \cdots, c_{i,d_g}f_i)$, $3 \leq i \leq k$. Let $\mathrm{rank}(v_1, v_2, v_3 \cdots, v_k)$ denote the dimension of the subspace generated by $\{v_1, v_2, v_3 \cdots, v_k\}$. Then, we have,

$$\big|\{w \in \mathbb{F}_q^k : N_w = q\}\big| = q^{k - \mathrm{rank}(v_1, v_2, v_3 \cdots, v_k)}.$$

It is clear that if one of the two conditions in the statement of the theorem is satisfied, then $v_1, v_2, v_3, \cdots, v_k$ are $\mathbb{F}_q$-linearly independent, and hence

$$\mathrm{rank}(v_1, v_2, v_3, \cdots, v_k) = k.$$

Therefore, the graph $S(k,q)$ is connected. □

We are ready to prove Theorem 2.2.

Proof of Theorem 2.2. Theorem 2.2 is an immediate corollary of Theorem 5.3 and Theorem 5.4. □

We conclude this section with an example of families of expanders. Their expansion properties follow from Theorem 2.2.

Example 5.5. Fix $k \geq 3$. Choose $(b_n)_{n\geq 1}$ and $(c_n)_{n\geq 1}$ to be two increasing sequences of positive real numbers such that $b_n = o(n)$, and $c_n = o(\sqrt{n})$.

Let $(q_m)_{m\geq 1}$ be an increasing sequence of prime powers such that $b_{q_1} \geq k$.

Let $f_{3,m}, \ldots, f_{k,m}$ be such that $1, X, f_{3,m}, \ldots, f_{k,m}$ are $\mathbb{F}_q$-linearly independent and $1 \leq d_f^{(m)} < b_q$. Let $g_{3,m}, \ldots, g_{k,m}$ be such that $g_{i,m}(-X) = -g_{i,m}(X)$ for each i, the coefficient of X in $g_{i,m}$ is non-zero and $1 \leq d_g^{(m)} < \min(p_m, c_{q_m})$. Then the graphs $S(k, q_m) = S(k, q_m; f_{3,m}, g_{3,m}, \cdots, f_{k,m}, g_{k,m})$, $m \geq 1$, form a family of expanders.

6 Spectra of the graphs $S(k, q)$ for $g_i(X) = X^3$

In this section, we provide some specializations of the graphs $S(k, q)$ for $g_i(X) = X^3$, $3 \leq i \leq k$, and bound or compute their eigenvalues. Our goal is to prove Theorems 2.3 and 2.4.

Lemma 6.1. *Let q be an odd prime power with $q \equiv 2 \mod 3$ and $k \geq 3$. Suppose that $g_i(X) = X^3$ for any i, $3 \leq i \leq k$. For any $w \in \mathbb{F}_q^k$, let T_w be the number of $u \in \mathbb{F}_q$ such that $f_3(u)w_3 + \cdots + f_k(u)w_k = 0$. Then λ_w is either $q(T_w - 1)$ or at most $(q - T_w)M_q$, where $M_q = \max_{a,b\in\mathbb{F}_q^*} \varepsilon_{ax^3+bx} \leq 2\sqrt{q}$.*

Proof. By (3), we have the following,

$$\lambda_w = \sum_{a\in\mathbb{F}_q^*, u\in\mathbb{F}_q} \zeta_p^{Tr\left(a(w_1+uw_2)+a^3\sum_{i=3}^{k} f_i(u)w_i\right)},$$

for any $w = (w_1, \cdots, w_k)$. Let $F(X) = f_3(X)w_3 + \cdots + f_k(X)w_k$.

Case 1: For w of the form $(0, 0, w_3, \cdots, w_k)$, we have:

$$\lambda_w = \sum_{\substack{u\in\mathbb{F}_q \\ F(u)=0}} \sum_{a\in\mathbb{F}_q^*} \zeta_p^{Tr\left(a^3F(u)\right)} + \sum_{\substack{u\in\mathbb{F}_q \\ F(u)\neq 0}} \sum_{a\in\mathbb{F}_q^*} \zeta_p^{Tr\left(a^3F(u)\right)}$$

$$= (q-1)T_w + \sum_{\substack{u\in\mathbb{F}_q \\ F(u)\neq 0}} \sum_{a\in\mathbb{F}_q^*} \zeta_p^{Tr\left(a^3F(u)\right)}.$$

Since $q \equiv 2 \bmod 3$, it follows that $\gcd(q-1,3) = 1$, and $a \mapsto a^3$ defines a bijection of $\mathbb{F}_q$. Therefore the above term $\sum_{a\in\mathbb{F}_q^*} \zeta_p^{Tr\left(a^3F(u)\right)}$ equals -1. Hence,

$$\lambda_w = (q-1)T_w - (q-T_w) = q(T_w - 1).$$

Case 2: For those w of the form $(w_1, 0, w_3, \cdots, w_k)$ with $w_1 \neq 0$, we have,

$$\begin{aligned}
\lambda_w &= \sum_{\substack{u\in\mathbb{F}_q \\ F(u)=0}} \sum_{a\in\mathbb{F}_q^*} \zeta_p^{Tr\left(aw_1+a^3F(u)\right)} + \sum_{\substack{u\in\mathbb{F}_q \\ F(u)\neq 0}} \sum_{a\in\mathbb{F}_q^*} \zeta_p^{Tr\left(aw_1+a^3F(u)\right)} \\
&= -T_w + \sum_{\substack{u\in\mathbb{F}_q \\ F(u)\neq 0}} (\varepsilon_{w_1a+F(u)a^3} - 1) \\
&= -q + \sum_{\substack{u\in\mathbb{F}_q \\ F(u)\neq 0}} \varepsilon_{w_1a+F(u)a^3} \\
&\leq -q + \sum_{\substack{u\in\mathbb{F}_q \\ F(u)\neq 0}} M_q \qquad \text{(by Lemma 3.3, } \varepsilon_{w_1a+F(u)a^3} \text{ is real)} \\
&\leq -q + (q-T_w)M_q < (q-T_w)M_q.
\end{aligned}$$

Case 3: For those w of the form $w = (w_1, w_2, w_3, \cdots, w_k)$ with $w_2 \neq 0$, we have

$$\lambda_w = \sum_{\substack{u\in\mathbb{F}_q \\ F(u)=0 \\ w_1+uw_2=0}} \sum_{a\in\mathbb{F}_q^*} \zeta_p^{Tr(aw_1+auw_2)} + \sum_{\substack{u\in\mathbb{F}_q \\ F(u)=0 \\ w_1+uw_2\neq 0}} \sum_{a\in\mathbb{F}_q^*} \zeta_p^{Tr(aw_1+auw_2)} \tag{8}$$

$$+ \sum_{\substack{u\in\mathbb{F}_q \\ F(u)\neq 0}} \sum_{a\in\mathbb{F}_q^*} \zeta_p^{Tr\left(a(w_1+uw_2)+a^3F(u)\right)}. \tag{9}$$

If $F(-w_1/w_2) = 0$, then the number of $u \in \mathbb{F}_q$ such that $F(u) = 0$ and $w_1 + uw_2 = 0$ is 1, and hence,

$$\lambda_w = (q-1) - (T_w - 1) + \sum_{\substack{u\in\mathbb{F}_q \\ F(u)\neq 0}} \sum_{a\in\mathbb{F}_q^*} \zeta_p^{Tr\left(a(w_1+uw_2)+a^3F(u)\right)}$$

$$
\begin{aligned}
&= q - T_w + \sum_{\substack{u\in\mathbb{F}_q \\ F(u)\neq 0 \\ w_1+uw_2\neq 0}} (\varepsilon_{(w_1+uw_2)a+F(u)a^3} - 1) \\
&= \sum_{\substack{u\in\mathbb{F}_q \\ F(u)\neq 0 \\ w_1+uw_2\neq 0}} \varepsilon_{(w_1+uw_2)a+F(u)a^3} \\
&\leq (q - T_w)M_q.
\end{aligned}
$$

Now assume that $F(-w_1/w_2) \neq 0$. Then, $w_1 + uw_2 \neq 0$ if $F(u) = 0$. Then the first double sum in (8) has no terms, the second double sum in (8) is equal to $T_w(-1)$, and splitting the double sum in (9) into two double sums, we obtain:

$$
\begin{aligned}
\lambda_w &= -T_w + \sum_{\substack{u\in\mathbb{F}_q \\ F(u)\neq 0 \\ w_1+uw_2=0}} \sum_{a\in\mathbb{F}_q^*} \zeta_p^{Tr\left(a(w_1+uw_2)+a^3F(u)\right)} + \sum_{\substack{u\in\mathbb{F}_q \\ F(u)\neq 0 \\ w_1+uw_2\neq 0}} \sum_{a\in\mathbb{F}_q^*} \zeta_p^{Tr\left(a(w_1+uw_2)+a^3F(u)\right)} \\
&= -T_w + \sum_{a\in\mathbb{F}_q^*} \zeta_p^{Tr\left(a^3F(u)\right)} + \sum_{\substack{u\in\mathbb{F}_q \\ F(u)\neq 0 \\ w_1+uw_2\neq 0}} \sum_{a\in\mathbb{F}_q^*} \zeta_p^{Tr\left(a(w_1+uw_2)+a^3F(u)\right)} \\
&= -T_w - 1 + \sum_{\substack{u\in\mathbb{F}_q \\ F(u)\neq 0 \\ w_1+uw_2\neq 0}} (\varepsilon_{(w_1+uw_2)a+F(u)a^3} - 1) \\
&= -q + \sum_{\substack{u\in\mathbb{F}_q \\ F(u)\neq 0 \\ w_1+uw_2\neq 0}} \varepsilon_{(w_1+uw_2)a+F(u)a^3} \\
&\leq -q + (q - T_w - 1)M_q < (q - T_w)M_q.
\end{aligned}
$$

As $q \equiv 2 \mod 3$, we have $\gcd(3, q) = 1$. By Theorem 3.2, $M_q \leq 2\sqrt{q}$, and the lemma is proven. □

Now we prove Theorem 2.3, where $f_i(X) = X^{i-1}$, for any $3 \leq i \leq k$. In this case, we are able to determine their second largest eigenvalues.

Proof of Theorem 2.3. Since $3 \leq k \leq q+1$, it follows that $X^2, X^3, \cdots, X^{k-1}$ are $\mathbb{F}_q$-linearly independent, and hence $S(k, q)$ is connected by Theorem 5.4. For any

$w = (w_1, \cdots, w_k) \in \mathbb{F}_q^k$, let $F(X) = X^2 w_3 + X^3 w_4 + \cdots + X^{k-1} w_k = X^2(w_3 + X w_4 + \cdots + X^{k-3} w_k)$, which implies that T_w (defined in the statement of Lemma 6.1) is either q or between 1 and $k-2$. By Lemma 6.1, we have that if λ_w is not $q(T_w - 1)$, then it is at most $(q - T_w)M_q \leq (q-1)M_q$. Therefore, we obtain:

$$\lambda_2\big(S(k,q)\big) \leq \max\{q(k-3), (q-1)M_q\}.$$

Moreover, if $k \geq 4$, then the above inequality becomes equality. Indeed, for any $w \in \mathbb{F}_q^k$ of the form $w = (0, w_2, 0, w_4, 0, \cdots, 0)$, where $w_2, w_4 \neq 0$, the following holds:

$$\begin{aligned}
\lambda_w &= \sum_{a \in \mathbb{F}_q^*, u \in \mathbb{F}_q} \zeta_p^{\,Tr(auw_2 + a^3 u^3 w_4)} \\
&= \sum_{x \in \mathbb{F}_q} \sum_{\substack{a \in \mathbb{F}_q^*, u \in \mathbb{F}_q \\ au = x}} \zeta_p^{\,Tr(w_2 x + w_4 x^3)} \\
&= \sum_{x \in \mathbb{F}_q} (q-1) \zeta_p^{\,Tr(w_2 x + w_4 x^3)} \\
&= (q-1)\varepsilon_{w_2 x + w_4 x^3}.
\end{aligned}$$

This implies that

$$\max_{\substack{w = (0, w_2, 0, w_4, 0, \cdots, 0) \\ w_2, w_4 \neq 0}} \{\lambda_w\} = (q-1)M_q.$$

Therefore, we have $\lambda_2(S(k,q)) = \max\{q(k-3), (q-1)M_q\}$. As $q \equiv 2 \pmod 3$, by Theorem 3.2, $M_q \leq 2\sqrt{q}$. □

Proof of Theorem 2.4. Since $3 \leq k \leq e+2$, it follows that $X^p, \cdots, X^{p^{k-2}}$ are $\mathbb{F}_q$-linearly independent, and hence $S(k,q)$ is connected by Theorem 5.4. For any $w = (w_1, \cdots, w_k) \in \mathbb{F}_q^k$, let $F(X) = X^p w_3 + \cdots + X^{p^{k-2}} w_k = (X^p) w_3 + \cdots + (X^p)^{p^{k-3}} w_k = Y w_3 + \cdots + Y^{p^{k-3}} w_k$ where $Y = X^p$. Since $a \mapsto a^p$ defines a bijection on $\mathbb{F}_q$, it implies that T_w (defined here as the number of roots of $F(X)$ in $\mathbb{F}_q$), is either q or at most p^{k-3}. The statement of the theorem then follows from Lemma 6.1. □

7 Concluding remarks

In this section, we make some remarks on several specializations of $S(k,q)$ considered in Section 6.

Remark 1. As we mentioned in Section 1, for every q-regular bipartite graph Γ, every eigenvalue of $\Gamma^{(2)}$ should be at least $-q$. For graphs $S(3,q;x^2,x^3)$ for prime q between 5 and 19, and for graphs $S(4,q;x^2,x^3,x^3,x^3)$ for prime q between 5 and 13, our computations show that their smallest eigenvalues are strictly less than $-q$. This implies that these graphs are not distance two graphs of any q-regular bipartite graphs.

Remark 2. In Section 6, we discussed the graphs $S(k,q)$ with $g_i(X)=X^3$. Now assume that $n\geq 1$, and $g_i(X)=X^{2n+1}$ for all i, $3\leq i\leq k$. For these graphs, Lemma 6.1 can be generalized as follows:

Let q be an odd prime power with $q\not\equiv 1 \mod (2n+1)$ and $(2n+1,q)=1$. For any $w\in\mathbb{F}_q^k$, let N_w be the number of $u\in\mathbb{F}_q$ such that $w_3f_3(u)+\cdots+w_kf_k(u)=0$. Then λ_w is either $q(N_w-1)$ or at most $2n(q-N_w)\sqrt{q}$.

In the case when $3\leq k\leq q+1$, $f_i(X)=X^{i-1}$ and $g_i(X)=X^{2n+1}$ for all i, $3\leq i\leq k$, the conclusion of Theorem 2.3 can be stated in a slightly weaker form:

$$\lambda_2\big(S(k,q)\big)\leq \max\{q(k-3),2n(q-1)\sqrt{q}\}.$$

Actually, for fixed q, if k is sufficiently large, $\lambda_2\big(S(k,q)\big)=q(k-3)$ for all $n\geq 1$.

Remark 3. The quantity $M_q=\max\limits_{a,b\in\mathbb{F}_q^*}\varepsilon_{ax^3+bx}$ in Theorem 2.3 and Theorem 2.4 is at most $2\sqrt{q}$ by Weil's bound. From the computational results, $M_q\geq 2\sqrt{q}-2$ for $q\leq 1331$. Interestingly, when $q=5^3$ or 5^5, the Weil's bound is tight.

Remark 4. Let $k\geq 3$ be an integer and let $f_i,g_i\in\mathbb{F}_q[X]$, $3\leq i\leq k+1$, be $2k-2$ polynomials of degree at most $q-1$ such that $g_i(-X)=-g_i(X)$ for each i, $3\leq i\leq k+1$. If $S(k+1,q)=S(k+1,q;f_3,g_3,\ldots,f_k,g_k,f_{k+1},g_{k+1})$ and $S(k,q)=S(k,q;f_3,g_3,\ldots,f_k,g_k)$, then it is not hard to show that $S(k+1,q)$ is a q-cover of $S(k,q)$ (see, e.g., [10, Section 6]). This implies that the spectrum of $S(k+1,q)$ is a submultiset the spectrum of $S(k,q)$ and, in particular, $\lambda_2(S(k+1,q))\geq\lambda_2(S(k,q))$.

Interestingly, in the case when $f_i(X) = X^{i-1}$ and $g_i(X) = X^3$ for each $i \geq 3$, we actually have equality in the inequality above for (q, k) whenever $k < \frac{q-1}{q} M_q + 2$ (immediate from Theorem 2.3).

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