On The Matlis Duals of Local Cohomology Modules

Gennady Lyubeznik and Tuğba Yıldırım

Abstract

Let $(R, \mathfrak{m})$ be a Noetherian regular local ring containing a field of characteristic $p > 0$ and I a nonzero ideal of R. In this short note, we prove that if $\mathrm{H}^i_I(R) \neq 0$, then $\mathrm{Supp}_R(D(\mathrm{H}^i_I(R))) = \mathrm{Spec}(R)$.

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1 Introduction

Let $(R, \mathfrak{m})$ be a Noetherian local commutative ring with unity, I an ideal of R and $E := E_R(R/\mathfrak{m})$ an R-injective hull of the residue field $R/\mathfrak{m}$. Then for any R-module M, we denote by $\mathrm{H}^i_I(M)$ the i-th local cohomology module of M supported in I and by $\mathrm{D}(M) := \mathrm{Hom}_R(M, E)$ the Matlis dual of M.

Suppose now that $\mathrm{H}^i_I(R) = 0$ for all $i \neq c$ and let $\boldsymbol{x}= \{x_1, x_2, ..., x_c\}$ be a regular sequence in I. Hellus [[3], Corollary 1.1.4] proved that I is a set theoretic complete intersection ideal defined by x_i if and only if x_i form a $\mathrm{D}(\mathrm{H}^c_I(R))$-regular sequence. Motivated by this result, Hellus studied the associated primes of Matlis duals of the top local cohomology modules and conjectured the following equality:

$$\mathrm{Ass}_R(D(\mathrm{H}^c_{(x_1,x_2,\cdots,x_c)}(R))) = \{\mathfrak{p} \in Spec(R) \mid \mathrm{H}^c_{(x_1,x_2,\cdots,x_c)}(R/\mathfrak{p}) \neq 0\}$$

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It has been shown that this conjecture holds true in many cases; see eg. [2], [5], [6], [7].
Furthermore, Hellus proved that the above conjecture is equivalent to the following condition [[3], Theorem 1.2.3]:

- If $(R, \mathfrak{m})$ is a Noetherian local domain, $c \geq 1$ and $x_1, x_2, \cdots, x_c \in R$, then the implication
$\mathrm{H}^c_{(x_1,x_2,\cdots,x_c)}(R) \neq 0 \implies 0 \in \mathrm{Ass}_R(D(\mathrm{H}^c_{(x_1,x_2,\cdots,x_c)}(R))$
holds.

We conjecture that if R is regular, then the above implication holds for all non-zero ideals independently of the number of generators, i.e.

Conjecture 1. *Let $(R, \mathfrak{m})$ be a Noetherian regular local ring and I be a non-zero ideal of R. If $\mathrm{H}^i_I(R) \neq 0$, then $0 \in \mathrm{Ass}_R(D(\mathrm{H}^i_I(R)))$.*

Note that Conjecture 1 is not true for non-regular rings. For a concrete example of a Noetherian local ring $(A, \mathfrak{m})$ of dimension > 1 such that $\mathrm{H}^1_{\mathfrak{m}}(A) = A/\mathfrak{m}$, hence $0 \notin \mathrm{Ass}_R(D(\mathrm{H}^1_{\mathfrak{m}}(A)))$, see [[1], Example 2.4]. The authors would like to thank M. Asgarzadeh for bringing this example to our attention.
We prove the following:

Theorem 1.1. *Let $(R, \mathfrak{m})$ be a complete Noetherian regular local ring containing a field of characteristic $p > 0$ and $\mathcal{M}$ be an F-finite module such that $0 \notin \mathrm{Ass}(\mathcal{M})$. Then $0 \in \mathrm{Ass}(D(\mathcal{M}))$.*

We would like to point out that $0 \notin \mathrm{Ass}(\mathcal{M})$ is a necessary condition of Theorem 1.1. Indeed, R itself is an F-finite module and $0 \in \mathrm{Ass}(R)$ but $0 \notin \mathrm{Ass}(D(R)) = \mathrm{Ass}(E) = \{\mathfrak{m}\}$.

As an immediate consequence of Theorem 1.1, we obtained the main result of this paper which establishes Conjecture 1 in the equicharacteristic $p > 0$ case:

Corollary 1.2. *Let $(R, \mathfrak{m})$ be a Noetherian regular local ring containing a field of characteristic $p > 0$ and I a non-zero ideal of R. If $\mathrm{H}^i_I(R) \neq 0$, then $\mathrm{Supp}_R(D(\mathrm{H}^i_I(R))) = \mathrm{Spec}(R)$.*

2 Preliminaries

Throughout, R is a commutative Noetherian regular ring containing a field of characteristic $p > 0$.
Let $R^{'}$ be the additive group of R regarded as an R- bi-module with the usual left action and with the right R- action defined by $r^{'}r = r^p r^{'}$ for all $r \in R$ and $r^{'} \in R^{'}$. The Frobenius functor

$$F : R - mod \longrightarrow R - mod$$

of Peskine-Szpiro [10] is defined by

$$F(M) = R^{'} \otimes_R M$$

$$F(M \xrightarrow{h} N) = (R^{'} \otimes_R M \xrightarrow{id \otimes_R h} R^{'} \otimes_R N)$$

for all R-modules M and all R-module homomorphisms h, where $F(M)$ acquires its R-module structure via the left R-module structure on $R^{'}$.
The iteration of a Frobenius functor on R leads one to the iterated Frobenius functors $F^i(-)$ which are defined for all $i \geq 1$ recursively by $F^1(-) = F(-)$ and $F^{i+1} = F \circ F^i(-)$ for all $i \geq 1$.
Note that the Frobenius functor $F(-)$ is exact [[8], Theorem 2.1]; $F(R) \cong R$ and for any ideal I of R, $F(R/I) = R/I^{[p]}$, where $I^{[p]}$ is the ideal of R generated by p-th powers of all elements of I [[10], I.1.3d].
Note also that for any Artinian module N, $F(D(N)) = D(F(N))$ [[9], Lemma 4.1] and so $R = F(R) = F(D(E)) = D(F(E))$ implies $F(E) = E$. Then it follows from Remark 1.0.(f) of [9] that for any finitely generated R-module M, $F(D(M)) = D(F(M))$.
Now, for an R-module M, define a Frobenius map $\psi_M : M \longrightarrow F(M)$ on M by $\psi_M(m) := 1 \otimes m \in F(M)$ for all $m \in M$. It is worth pointing out that if $\mathrm{ann}(m) = I \subseteq R$, then $\mathrm{ann}(\psi_M(m)) = I^p$.
An $\mathcal{F}$-module $\mathcal{M}$ is an R- module equipped with R-module isomorphism $\theta : \mathcal{M} \longrightarrow F(\mathcal{M})$ which we call the structure morphism.
A generating morphism of an $\mathcal{F}$ module $\mathcal{M}$ is an R-module homomorphism $\beta : M \longrightarrow F(M)$, where M is some R-module, such that $\mathcal{M}$ is the limit of the inductive system in the top row of the commutative diagram

$$\begin{tikzcd}
M \arrow[r, "\beta"] \arrow[d, "\beta"'] & F(M) \arrow[r, "F(\beta)"] \arrow[d, "F(\beta)"] & F^2(M) \arrow[r, "F^2(\beta)"] \arrow[d, "F^2(\beta)"] & \cdots \\
F(M) \arrow[r, "F(\beta)"] & F^2(M) \arrow[r, "F^2(\beta)"] & F^3(M) \arrow[r, "F^3(\beta)"] & \cdots
\end{tikzcd}$$

and $\theta : \mathcal{M} \longrightarrow F(\mathcal{M})$, the structure isomorphism of $\mathcal{M}$, is induced by the vertical arrows in this diagram.
If β is an injective map, then the exactness of F implies that all maps in the direct limit system are injective, so that M injects into $\mathcal{M}$. In this case, we shall refer to β as a root morphism of $\mathcal{M}$, and M as a root of $\mathcal{M}$. If $\mathcal{M}$ is an $\mathcal{F}$-module possesing a root morphism $\beta : M \longrightarrow \mathcal{M}$ with M finitely generated, then we say that $\mathcal{M}$ is $\mathcal{F}$-finite. In particular, R, with any F-module structure, is an $\mathcal{F}$-finite module.

3 Proofs

We begin this section by giving the proof of Theorem 1.1:

Proof of Theorem 1.1. Since $\mathcal{M}$ is an F-finite R module, it follows from Proposition 2.3. of [9] that there exists a root morphism $\beta : M \to F(M)$ such that

$$\mathcal{M} = \varinjlim(M \xrightarrow{\beta} F(M) \xrightarrow{F(\beta)} F^2(M) \xrightarrow{F^2(\beta)} \cdots).$$

Then applying Matlis dual functor $\mathrm{D}(-) = \mathrm{Hom_R(\text{-}, E(R/\mathfrak{m}))}$ to $\mathcal{M}$, we obtain

$$D(\mathcal{M}) = \varprojlim(D(M) \xleftarrow{D(\beta)} D(F(M)) \xleftarrow{D(F(\beta))} D(F^2(M)) \xleftarrow{D(F^2(\beta))} \cdots).$$

But then since Frobenius functor commutes with $D(-)$, we can write $D(\mathcal{M})$ as

$$D(\mathcal{M}) = \varprojlim(N \xleftarrow{\alpha} F(N) \xleftarrow{F(\alpha)} F^2(N) \xleftarrow{F^2(\alpha)} \cdots),$$

where $N = \mathrm{D}(M)$ and $\alpha = \mathrm{D}(\beta)$. Note that since β is injective and F is exact, $F^k(\alpha)$ is surjective for all $k \geq 0$. Note also that since M is a finitely generated R-module, N is Artinian and so are all $F^k(N)$, $k > 0$.
On the other hand, since $0 \notin \mathrm{Ass}(\mathcal{M})$, $I = \mathrm{Ann}(M) = \mathrm{Ann}(N)$ is a nonzero ideal of R. Then it follows that $\mathrm{Ann}(F(N)) = I^{[p]}$ and so $\mathrm{Ker}(\alpha : F(N) \to N) \neq 0$.
Now we claim that there exists an element $n^{'} = (n^{'}_0, n^{'}_1, \cdots, n^{'}_k, \cdots) \in \mathrm{D}(\mathcal{M})$ such that $n^{'}_k \in F^k(N)$ and $F^{k-1}(\alpha)(n^{'}_k) = n^{'}_{k-1}$ and with the property that $\mathrm{ann}(n'_k) \subseteq \mathfrak{m}^k$ for all $k \geq 4$. To construct such an element, first let $b_1 \in$

$\mathrm{Soc}(\mathrm{Ker}(\alpha)) \subseteq F(N)$, where $\mathrm{Soc}(\mathrm{Ker}(\alpha)) := \mathrm{Ann}_{\mathrm{Ker}(\alpha)}(\mathfrak{m})$ denotes the socle of $\mathrm{Ker}(\alpha)$ and define b_k, for all $k \geq 2$, inductively as the image of b_{k-1} under the Frobenius map (defined in the preceding section) on $F^{k-1}(N)$, that is $b_k := \psi_{F^{k-1}(N)}(b_{k-1}) = 1 \otimes b_{k-1} \in F^k(N)$. Then by induction on k (considering that $\mathrm{ann}(b_1) = \mathfrak{m}$ and $\mathrm{ann}(x) = I$ implies $\mathrm{ann}(\psi(x)) = I^{[p]}$), we have $\mathrm{ann}(b_k) = \mathfrak{m}^{[p^{k-1}]}$. On the other hand, since $b_1 \in \mathrm{Ker}(\alpha) := \mathrm{Ker}(F^0(\alpha))$, an easy induction argument shows that $b_k \in \mathrm{Ker}(F^{k-1}(\alpha))$ for all $k \geq 0$. For if $b_{k-1} \in \mathrm{Ker}(F^{k-2}(\alpha))$, then $F^{k-1}(\alpha)(b_k) = F^{k-1}(\alpha)(1 \otimes b_{k-1}) = 1 \otimes F^{k-2}(\alpha)(b_{k-1}) = 0$.
Let now n'_0 be an element of N and, for every $1 \leq k \leq 3$, choose $n'_k \in F^k(N)$ such that $n'_{k-1} = F^{k-1}(\alpha)(n'_k)$. For $k \geq 4$, define n_k in such a way that $F^{k-1}(\alpha)(n_k) = n'_{k-1}$. Then, either $\mathrm{ann}(n_k) \subseteq \mathfrak{m}^k$ or $\mathrm{ann}(n_k + b_k) \subseteq \mathfrak{m}^k$. Indeed, if $\mathrm{ann}(n_k + b_k) \nsubseteq \mathfrak{m}^k$, there exists an element $y \in \mathfrak{m} \setminus \mathfrak{m}^k$ such that $y(n_k + b_k) = 0$. Then it follows that $\mathrm{ann}(n_k) \subseteq \mathrm{ann}(yn_k) = \mathrm{ann}(yb_k)$. Since $y \in \mathfrak{m} \setminus \mathfrak{m}^k$ and $\mathrm{ann}(b_k) = \mathfrak{m}^{[p^{k-1}]}$, we have $\mathrm{ann}(n_k) \subseteq \mathrm{ann}(yb_k) \subseteq m^{p^{k-1}-k}$. To prove the fact that $\mathrm{ann}(yb_k) \subseteq m^{p^{k-1}-k}$, suppose on the contrary that there exists an element $z \in \mathrm{ann}(yb_k)$ such that $z \notin \mathfrak{m}^{p^{k-1}-k}$. Then clearly, $yz \in \mathrm{ann}\, b_k$. On the other hand as $R \cong \kappa[[X_1, ..., X_n]]$, $\kappa \cong R/\mathfrak{m}$ a field of characteristic $p > 0$, and $y \notin \mathfrak{m}^k$ and $z \notin \mathfrak{m}^{p^{k-1}-k}$, we may write

$$y = \underbrace{\Sigma_{i=1}^{k-1} \alpha_i \mathbf{X}^i}_{f} + \underbrace{\Sigma_{i=k}^{\infty} \alpha_i \mathbf{X}^i}_{f'}$$
$$z = \underbrace{\Sigma_{j=1}^{p^{k-1}-k-1} \beta_j \mathbf{X}^j}_{g} + \underbrace{\Sigma_{j=p^{k-1}-k}^{\infty} \beta_j \mathbf{X}^j}_{g'}$$

where f and g are non-zero polynomials over $\kappa[X_1, ..., X_n]$, $\mathbf{X}^l = X_1^{l_1} X_2^{l_2} \cdots X_n^{l_n}$ for any positive integer $l = l_1 + l_2 + ... + l_n$. Then $yz = fg + fg' + gf' + g'f'$. Since $\kappa[X_1, ..., X_n]$ is an integral domain and f and g are non-zero, so is fg and clearly $0 \neq \deg(fg) \leq p^{k-1} - k - 1 + k - 1 = p^{k-1} - 2$ from which it follows that $yz \notin \mathfrak{m}^{p^{k-1}}$. But then this contradicts the fact that $yz \in \mathrm{ann}(b_k) = \mathfrak{m}^{[p^{k-1}]}$. Hence $\mathrm{ann}(n_k) \subseteq \mathrm{ann}(yb_k) \subseteq m^{p^{k-1}-k}$, as desired. Now since $p^{k-1} - k \geq k$ for all $k \geq 4$, we have $\mathrm{ann}(n_k) \subseteq \mathfrak{m}^{p^{k-1}-k} \subseteq \mathfrak{m}^k$. Then define

$$n'_k = \begin{cases} n_k, & \text{if } \mathrm{ann}(n_k) \subseteq \mathfrak{m}^k, \\ n_k + b_k, & \text{otherwise.} \end{cases}$$

Clearly, $n' = (n'_0, n'_1, \cdots, n'_k, \cdots) \in \mathrm{D}(\mathcal{M})$ and $\mathrm{ann}(n'_k) \subseteq \mathfrak{m}^k$ for all $k \geq 4$. This proves the claim.

Finally, $\operatorname{ann}(n^{'}) = 0$ for if $z \in \operatorname{ann}(n^{'})$, then $z \in \operatorname{ann}(n_k^{'})$ for all $k \geq 0$ which implies that $z \in \bigcap_{n \in \mathbb{N}} \mathfrak{m}^n = \{0\}$. This completes the proof of Theorem 1.1. $\square$

The proof of Corollary 1.2 is an immediate consequence of Theorem 1.1:

Proof of Corollary 1.2. Without loss of generality, we may, and do, assume that R is complete [[3], Remark 4.1.1]. Since R is an F-finite module, so are its all local cohomology modules and since $0 \notin \operatorname{Ass}_R(\mathrm{H}^i_I(R))$ for any nonzero ideal I of R, the result follows from Theorem 1.1. $\square$

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Department of Mathematics, University of Minnesota, 127 Vincent Hall, 206 Church St., Minneapolis, MN 55455, USA
e-mail: gennady@math.umn.edu

Department of Mathematics, Istanbul Technical University, Maslak, 34469, Istanbul, Turkey.
e-mail: tugbayildirim@itu.edu.tr