CLASSIFYING BRAIDINGS ON FUSION CATEGORIES

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Abstract. We show that braidings on a fusion category $\mathcal{C}$ correspond to certain fusion subcategories of the center of $\mathcal{C}$ transversal to the canonical Lagrangian algebra. This allows to classify braidings on non-degenerate braided fusion categories and on those dual to the categories of group-graded vector spaces.

1. Introduction

Throughout this article we work over an algebraically closed field k of characteristic 0.

In general, a fusion category $\mathcal{C}$ may have several different braidings or no braidings at all. For example, if $\mathcal{C} = \mathsf{Vec}_G$, the category of finite-dimensional k-vectors spaces graded by a finite abelian group G, then braidings on $\mathcal{C}$ are parameterized by bilinear forms on G. If G is non-Abelian then of course Vec_G does not admit any braidings.

The goal of this note is to give a convenient parameterization of braidings on an arbitrary fusion category $\mathcal{C}$. We introduce the notion of transversality between algebras and subcategories of a braided fusion category. Then we show that the set of braidings on $\mathcal{C}$ is in bijection with the set of fusion subcategories $\mathcal{B}$ of the center $\mathcal{Z}(\mathcal{C})$ such that $\mathrm{FPdim}(\mathcal{B}) = \mathrm{FPdim}(\mathcal{C})$ and $\mathcal{B}$ is transversal to the canonical Lagrangian algebra of $\mathcal{Z}(\mathcal{C})$. In several interesting situations it is possible to give an explicit parameterization of such subcategories. We do this in two cases: (1) for fusion categories $\mathcal{C}$ admitting a non-degenerate braiding and (2) for group-theoretical categories. In the latter case the parameterization is given in terms of the subgroup lattice of a group and can be conveniently used in concrete computations.

The paper is organized as follows. Section 2 contains some background information and a categorical analogue of Goursat's lemma (Theorem 2.2) for subcategories of tensor products of fusion categories. In Section 3 we introduce transversal pairs of algebras and subcategories and characterize braidings in these terms. In Section 4 we classify braidings on a fusion category $\mathcal{B}$ that already admits a non-degenerate braiding (Theorem 4.1) and consider several examples. We show that with respect to any other braiding the symmetric center of $\mathcal{B}$ remains pointed. In Section 5 we classify braidings on group-theoretical fusion categories (dual to the category Vec_G). As an application we parameterize braidings on the Drinfeld center of Vec_G.

Acknowledgements. The author thanks Costel-Gabriel Bontea and Alexei Davydov for useful discussions. The author is grateful to the referee for her or his useful comments. This

material is based upon work supported by the National Science Foundation under grant DMS-1801198 and by the NSA grant H98230-16-1-0008.

2. Preliminaries

2.1. **Fusion categories.** We refer the reader to [EGNO] for a general theory of tensor categories and to [DGNO] for braided fusion categories.

A *fusion category* over k is a k-linear semisimple rigid tensor category with finitely many isomorphism classes of simple objects, finite-dimensional Hom-spaces, and a simple unit object $\mathbf{1}$. By a *fusion subcategory* of a fusion category $\mathcal{C}$ we always mean a full tensor subcategory. An example of subcategory is the maximal pointed subcategory $\mathcal{C}_{pt} \subset \mathcal{C}$ generated by invertible objects of $\mathcal{C}$. We say that $\mathcal{C}$ is *pointed* if $\mathcal{C} = \mathcal{C}_{pt}$.

We denote Vec the fusion category of finite-dimensional k-vector spaces.

For a fusion category $\mathcal{C}$ let $\mathcal{O}(\mathcal{C})$ denote the set of isomorphism classes of simple objects.

Let G be a finite group. A grading of $\mathcal{C}$ by G is a map $\deg : \mathcal{O}(\mathcal{C}) \to G$ with the following property: for any simple objects X, Y, $Z \in \mathcal{C}$ such that $X \otimes Y$ contains Z one has $\deg Z = \deg X \cdot \deg Y$. We will identify a grading with the corresponding decomposition

$$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g, \tag{1}$$

where $\mathcal{C}_g$ is the full additive subcategory of $\mathcal{C}$ generated by simple objects of degree $g \in G$. The subcategory $\mathcal{C}_1$ is called the *trivial component* of the grading. The grading is called *faithful* if $\deg : \mathcal{O}(\mathcal{C}) \to G$ is surjective.

For any fusion category $\mathcal{C}$ there is a *universal grading* $\mathcal{O}(\mathcal{C}) \to U(\mathcal{C})$ [GN], where $U(\mathcal{C})$ is the *universal grading group* of $\mathcal{C}$. Any grading of $\mathcal{C}$ comes from a quotient of $U(\mathcal{C})$. The trivial component of the universal grading is the adjoint fusion subcategory $\mathcal{C}_{ad} \subset \mathcal{C}$ generated by objects $X \otimes X^*$, $X \in \mathcal{O}(\mathcal{C})$.

2.2. **Fiber products of fusion categories.** Let $\mathcal{C}$, $\mathcal{D}$ be fusion categories graded by the same group G. The *fiber product* of $\mathcal{C}$ and $\mathcal{D}$ is the fusion category

$$\mathcal{C} \boxtimes_G \mathcal{D} := \bigoplus_{g \in G} \mathcal{C}_g \boxtimes \mathcal{D}_g. \tag{2}$$

Here $\boxtimes$ denotes Deligne's tensor product of abelian categories. Clearly, $\mathcal{C} \boxtimes_G \mathcal{D}$ is a fusion subcategory of $\mathcal{C} \boxtimes \mathcal{D}$ graded by G. The trivial component of this grading is $\mathcal{C}_1 \boxtimes \mathcal{D}_1$. When the gradings of $\mathcal{C}$ and $\mathcal{D}$ are faithful one has

$$\mathrm{FPdim}(\mathcal{C} \boxtimes_G \mathcal{D}) = \frac{\mathrm{FPdim}(\mathcal{C})\mathrm{FPdim}(\mathcal{D})}{|G|}. \tag{3}$$

2.3. **Goursat's Lemma for subcategories of the tensor product.** Let $\mathcal{C}$, $\mathcal{D}$ be fusion categories.

Definition 2.1. A *subcategory datum* for $\mathcal{C} \boxtimes \mathcal{D}$ consists of a pair $\mathcal{E} \subset \mathcal{C}$ and $\mathcal{F} \subset \mathcal{D}$ of fusion subcategories, a group G, and fixed faithful gradings of $\mathcal{E}$ and $\mathcal{F}$ by G.

We will identify subcategory data $(\mathcal{E}, \mathcal{F}, G)$ and $(\mathcal{E}, \mathcal{F}, G')$ if there is an isomorphism $\alpha : G \xrightarrow{\sim} G'$ such that $\mathcal{E}_g = \mathcal{E}_{\alpha(g)}$ and $\mathcal{F}_g = \mathcal{F}_{\alpha(g)}$. When no confusion is likely we will denote a subcategory datum simply by $(\mathcal{E}, \mathcal{F}, G)$ omitting the grading maps.

Given a subcategory datum $(\mathcal{E}, \mathcal{F}, G)$ we can form a fusion subcategory

$$\mathcal{S}(\mathcal{E}, \mathcal{F}, G) := \mathcal{E} \boxtimes_G \mathcal{F} \subset \mathcal{C} \boxtimes \mathcal{D}. \tag{4}$$

It turns out that $\mathcal{S}(\mathcal{E}, \mathcal{F}, G)$ is a typical example of a fusion subcategory of $\mathcal{E} \boxtimes \mathcal{F}$. The following theorem is a categorical analogue of the well known Goursat's Lemma in group theory.

Theorem 2.2. *Let $\mathcal{C}$, $\mathcal{D}$ be fusion categories. The assignment*

$$(\mathcal{E}, \mathcal{F}, G) \mapsto \mathcal{S}(\mathcal{E}, \mathcal{F}, G) \tag{5}$$

is a bijection between the set of subcategory data for $\mathcal{C} \boxtimes \mathcal{D}$ and the set of fusion subcategories of $\mathcal{C} \boxtimes \mathcal{D}$.

Proof. We need to show that every fusion subcategory $\mathcal{S} \subset \mathcal{C} \boxtimes \mathcal{D}$ is equal to some $\mathcal{S}(\mathcal{E}, \mathcal{F}, G)$ for a unique choice of $(\mathcal{E}, \mathcal{F}, G)$.

Let $\mathcal{E} \subset \mathcal{C}$ be a fusion subcategory generated by all $X \in \mathcal{O}(\mathcal{C})$ such that $X \boxtimes Y \in \mathcal{S}$ for some non-zero $Y \in \mathcal{D}$. Similarly, let $\mathcal{F} \subset \mathcal{D}$ be a fusion subcategory generated by all $Y \in \mathcal{O}(\mathcal{D})$ such that $X \boxtimes Y \in \mathcal{S}$ for some non-zero $X \in \mathcal{C}$.

Let

$$\tilde{\mathcal{E}} := \mathcal{S} \cap (\mathcal{C} \boxtimes \mathsf{Vec}) \subset \mathcal{E} \text{ and } \tilde{\mathcal{F}} := \mathcal{S} \cap (\mathsf{Vec} \boxtimes \mathcal{D}) \subset \mathcal{F}. \tag{6}$$

If $X \in \mathcal{O}(\mathcal{C})$ and $Y \in \mathcal{O}(\mathcal{D})$ are such that $X \boxtimes Y \in \mathcal{S}$ then $(X^* \otimes X) \boxtimes \mathbf{1}$ and $\mathbf{1} \boxtimes (Y^* \otimes Y)$ are objects of $\mathcal{S}$. This means that $\mathcal{E}_{ad} \subset \tilde{\mathcal{E}}$ and $\mathcal{F}_{ad} \subset \tilde{\mathcal{F}}$. Let $H_{\mathcal{E}} \subset U(\mathcal{E})$ and $H_{\mathcal{F}} \subset U(\mathcal{F})$ be the subgroups of the universal groups corresponding to $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{F}}$. We claim that these subgroups are normal. Indeed, let $X \in \mathcal{O}(\tilde{\mathcal{E}})$ and $V \in \mathcal{O}(\mathcal{E})$. Then $X \boxtimes \mathbf{1} \in \mathcal{S}$ and $V \boxtimes U \in \mathcal{S}$ for some $U \in \mathcal{O}(\mathcal{D})$. So $(V^* \boxtimes U^*) \otimes (X \boxtimes \mathbf{1}) \otimes (V \boxtimes U) = (V^* \otimes X \otimes V) \boxtimes (U^* \otimes U) \in \mathcal{S}$ and $V^* \otimes X \otimes V \in \tilde{\mathcal{E}}$. This implies $gxg^{-1} \in H_{\mathcal{E}}$ for all $x \in H_{\mathcal{E}}$ and $g \in U(\mathcal{E})$. Thus, $H_{\mathcal{E}} \subset U(\mathcal{E})$ is normal. Similarly, $H_{\mathcal{F}} \subset U(\mathcal{F})$ is normal.

Hence, subcategories $\mathcal{E}$ and $\mathcal{F}$ have faithful gradings $\deg_{\mathcal{E}} : \mathcal{O}(\mathcal{E}) \to U(\mathcal{E})/H_{\mathcal{E}} =: G_{\mathcal{E}}$ and $\deg_{\mathcal{F}} : \mathcal{O}(\mathcal{F}) \to U(\mathcal{F})/H_{\mathcal{F}} =: G_{\mathcal{F}}$ with trivial components $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{F}}$, respectively.

Let $X \in \mathcal{O}(\mathcal{C})$ and Y_1, $Y_2 \in \mathcal{O}(\mathcal{D})$ be such that that $X \boxtimes Y_1$, $X \boxtimes Y_2 \in \mathcal{S}$. Then $\mathbf{1} \boxtimes (Y_1^* \otimes Y_2)$ is a subobject of $(X \boxtimes Y_1)^* \otimes (X \boxtimes Y_2)$ and so belongs to $\mathcal{S}$. Therefore, $Y_1^* \otimes Y_2 \in \tilde{\mathcal{F}}$, so $\deg_{\mathcal{F}}(Y_1) = \deg_{\mathcal{F}}(Y_2)$. Similarly, if X_1, $X_2 \in \mathcal{O}(\mathcal{C})$ and $Y \in \mathcal{O}(\mathcal{D})$ are such that that $X_1 \boxtimes Y$, $X_2 \boxtimes Y \in \mathcal{S}$ then $\deg_{\mathcal{E}}(X_1) = \deg_{\mathcal{E}}(X_2)$.

Therefore, there is a well-defined isomorphism $f : G_{\mathcal{E}} \to G_{\mathcal{F}}$ such that $f(\deg_{\mathcal{E}}(X)) = \deg_{\mathcal{F}}(Y)$ for all $X \in \mathcal{O}(\mathcal{C})$ and $Y \in \mathcal{O}(\mathcal{D})$ such that $X \boxtimes Y \in \mathcal{O}(\mathcal{S})$. This means that $\mathcal{S}$ is a fiber product of $\mathcal{E}$ and $\mathcal{F}$.

It is clear that subcategories $\mathcal{E}$, $\mathcal{F}$ and their gradings are invariants of $\mathcal{S}$. □

Remark 2.3. Let $(\mathcal{E}_1, \mathcal{F}_1, G_1)$ and $(\mathcal{E}_2, \mathcal{F}_2, G_2)$ be subcategory data for $\mathcal{C} \boxtimes \mathcal{D}$. Then

$$\mathcal{S}(\mathcal{E}_1, \mathcal{F}_1, G_1) \cap \mathcal{S}(\mathcal{E}_2, \mathcal{F}_2, G_2) \cong (\mathcal{E}_1 \cap \mathcal{E}_2) \boxtimes_{G_1 \times G_2} (\mathcal{F}_1 \cap \mathcal{F}_2), \tag{7}$$

where the gradings of $\mathcal{E}_1 \cap \mathcal{E}_2$ and $\mathcal{F}_1 \cap \mathcal{F}_2$ by $G_1 \times G_2$ are such that the (g_1, g_2) components are $(\mathcal{E}_1)_{g_1} \cap (\mathcal{E}_2)_{g_2}$ and $(\mathcal{F}_1)_{g_1} \cap (\mathcal{F}_2)_{g_2}$, respectively (note that these gradings are not faithful in general).

2.4. **Braided fusion categories and their gradings.** Let $\mathcal{B}$ be a braided fusion category with a braiding $c_{X,Y} : X \otimes Y \to Y \otimes X$. Two objects X, Y of $\mathcal{B}$ *centralize* each other if $c_{Y,X} \circ c_{X,Y} = \mathrm{id}_{X \otimes Y}$ and *projectively centralize* each other if $c_{Y,X} \circ c_{X,Y} = \lambda \mathrm{id}_{X \otimes Y}$ for some scalar $\lambda \in k$. For a fusion subcategory $\mathcal{D} \subset \mathcal{B}$ its *centralizer* is

$$\mathcal{D}' = \{Y \in \mathcal{B} \mid Y \text{ centralizes each } X \in \mathcal{D}\}.$$

The symmetric center of $\mathcal{B}$ is $\mathcal{Z}_{sym}(\mathcal{B}) := \mathcal{B} \cap \mathcal{B}'$. We say that $\mathcal{B}$ is *non-degenerate* if $\mathcal{Z}_{sym}(\mathcal{B}) = \mathsf{Vec}$.

For a non-degenerate $\mathcal{B}$ there is a canonical non-degenerate bimultiplicative pairing

$$\langle\, ,\, \rangle : \mathcal{O}(\mathcal{B}_{pt}) \times U(\mathcal{B}) \to k^\times \tag{8}$$

defined by $c_{Y,X} c_{X,Y} = \langle X, g \rangle \mathrm{id}_{X \otimes Y}$ for all $X \in \mathcal{O}(\mathcal{B}_{pt})$ and $Y \in \mathcal{B}_g$, $g \in U(\mathcal{B})$. See [DGNO, 3.3.4] for details.

Proposition 2.4. *Let $\mathcal{B}$ be a non-degenerate braided fusion category and let $\mathcal{D} \subset \mathcal{B}$ be a fusion subcategory with a faithful grading*

$$\mathcal{D} = \bigoplus_{g \in G} \mathcal{D}_g,$$

where G is an Abelian group. The centralizer of the trivial component $\mathcal{D}_1$ of $\mathcal{D}$ admits a faithful grading

$$\mathcal{D}_1' = \bigoplus_{\phi \in \widehat{G}} (\mathcal{D}_1')_\phi,$$

where $\widehat{G}$ is the group of characters of G and

$$(\mathcal{D}'_1)_\phi = \{X \in \mathcal{B} \mid c_{Y,X} \circ c_{X,Y} = \phi(g) id_{X\otimes Y}, \text{ for all } Y \in \mathcal{D}_g,\ g \in G\}.$$

The trivial component of this grading is $\mathcal{D}'$.

Proof. It follows from [DGNO, 3.3] that a simple object X belongs to $\mathcal{D}'_1$ if and only if projectively centralizes every simple $Y \in \mathcal{D}$, i.e., $c_{Y,X} \circ c_{X,Y} = \lambda_Y \mathrm{id}_{X\otimes Y}$ for some $\lambda_Y \in k^\times$. Furthermore, if Y_1, Y_2 are simple objects lying in $\mathcal{D}_g$ then $\lambda_{Y_1} = \lambda_{Y_2}$. Let us denote the latter scalar by $\phi_X(g)$. It follows from the braiding axioms that the assignment

$$\mathcal{O}(\mathcal{D}'_1) \to \widehat{G} : X \mapsto \phi_X$$

is a grading of $\mathcal{D}'_1$ by $\widehat{G}$.

The fact that the trivial component is $\mathcal{D}'$ and the faithfulness of grading follow from the non-degeneracy of $\mathcal{B}$. □

2.5. **Lagrangian algebras in the center.** For any fusion category $\mathcal{C}$ let $\mathcal{Z}(\mathcal{C})$ denote its Drinfeld center.

Let $\mathcal{B}$ be a braided fusion category. A *Lagrangian* algebra in $\mathcal{B}$ is a commutative separable algebra A in $\mathcal{B}$ such that $\mathsf{Hom}_{\mathcal{B}}(A, \mathbf{1}) \cong k$ and $\mathrm{FPdim}(A)^2 = \mathrm{FPdim}(\mathcal{B})$.

Let $I : \mathcal{C} \to \mathcal{Z}(\mathcal{C})$ denote the adjoint of the forgetful functor $F : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$. Then $I(\mathbf{1})$ is a canonical Lagrangian algebra in $\mathcal{Z}(\mathcal{C})$.

It was explained in [DMNO] that any braided equivalence $a : \mathcal{Z}(\mathcal{C}) \xrightarrow{\sim} \mathcal{B}$ gives rise to a Lagrangian algebra $A = a(I(\mathbf{1}))$ in $\mathcal{B}$. Conversely, given a Lagrangian algebra $A \in \mathcal{B}$ there is a braided tensor equivalence $\mathcal{Z}(\mathcal{B}_A) \xrightarrow{\sim} \mathcal{B}$, where $\mathcal{B}_A$ denotes the fusion category of A-modules in $\mathcal{B}$.

3. Subcategories transversal to a Lagrangian algebra

Definition 3.1. Let $\mathcal{C}$ be a fusion category, let $\mathcal{B} \subset \mathcal{C}$ be a fusion subcategory, and let A be an algebra in $\mathcal{C}$. We will assume that $\mathsf{Hom}_{\mathcal{C}}(A, \mathbf{1}) \cong k$, i.e., that A is a *connected* algebra. We say that $\mathcal{B}$ is *transversal* to A if

$$\mathsf{Hom}_{\mathcal{C}}(X, A) = \mathsf{Hom}_{\mathcal{C}}(X, \mathbf{1}) \tag{9}$$

for all $X \in \mathcal{B}$.

In other words, $\mathcal{B}$ is transversal to A if and only if $\mathsf{Hom}_{\mathcal{C}}(X, A) = 0$ for all non-identity $X \in \mathcal{O}(\mathcal{B})$.

Theorem 3.2. *Let $\mathcal{C}$ be a fusion category and let $A := I(\mathbf{1})$ be the canonical Lagrangian algebra in $\mathcal{Z}(\mathcal{C})$. Braidings on $\mathcal{C}$ are in bijection with fusion subcategories $\mathcal{B} \subset \mathcal{Z}(\mathcal{C})$ transversal to A and such that $FPdim(\mathcal{B}) = FPdim(\mathcal{C})$.*

Proof. It is well known that braidings on a fusion category $\mathcal{C}$ are in bijection with sections of the forgetful functor $F : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$, i.e., with embeddings $\iota : \mathcal{C} \to \mathcal{Z}(\mathcal{C})$ such that $F \circ \iota = \mathrm{id}_{\mathcal{C}}$. The latter correspond to fusion subcategories $\mathcal{B} \subset \mathcal{Z}(\mathcal{C})$ such that the restriction $F|_{\mathcal{B}} : \mathcal{B} \to \mathcal{C}$ is an equivalence. This is equivalent to $\mathrm{FPdim}(\mathcal{B}) = \mathrm{FPdim}(\mathcal{C})$ and $F|_{\mathcal{B}} : \mathcal{B} \to \mathcal{C}$ being injective, i.e., fully faithful.

Note that F is identified with the functor of taking free A-modules:

$$\mathcal{Z}(\mathcal{C}) \to \mathcal{Z}(\mathcal{C})_A \cong \mathcal{C} : Z \mapsto A \otimes Z.$$

Observe that

$$\mathsf{Hom}_{\mathcal{C}}(F(Z), \mathbf{1}) \cong \mathsf{Hom}_{\mathcal{Z}(\mathcal{C})_A}(A \otimes Z, A) \cong \mathsf{Hom}_{\mathcal{Z}(\mathcal{C})}(Z, A),$$

for all $Z \in \mathcal{Z}(\mathcal{C})$. The injectivity of $F|_{\mathcal{B}} : \mathcal{B} \to \mathcal{C}$ is equivalent to $\mathsf{Hom}_{\mathcal{C}}(F(Z), \mathbf{1}) = \mathsf{Hom}_{\mathcal{B}}(Z, \mathbf{1})$ for all $Z \in \mathcal{B}$ and, hence, to A and $\mathcal{B}$ being transversal. □

4. Braidings on non-degenerate fusion categories

4.1. **Classification of braidings.** Let $\mathcal{B}$ be a fusion category with a non-degenerate braiding $c = \{c_{X,Y}\}$.

Any grading of a fusion category $\mathcal{C}$ by a group G determines a homomorphism

$$h_{\mathcal{C}} : \mathcal{O}(\mathcal{C}_{pt}) \to G.$$

Theorem 4.1. *The braidings on $\mathcal{B}$ are in bijection with subcategory data $(\mathcal{E}, \mathcal{F}, G)$ such that $\mathcal{E} \vee \mathcal{F} = \mathcal{B}$, $\mathcal{E} \cap \mathcal{F}$ is pointed, and $h_{\mathcal{E}} + h_{\mathcal{F}} : \mathcal{O}(\mathcal{F} \cap \mathcal{E}) \to G$ is an isomorphism. Here $\mathcal{E} \vee \mathcal{F}$ denotes the fusion subcategory of $\mathcal{B}$ generated by $\mathcal{E}$ and $\mathcal{F}$.*

Proof. We will use the characterization of braidings from Theorem 3.2.

Since $\mathcal{B}$ is non-degenerate, we have $\mathcal{Z}(\mathcal{B}) \cong \mathcal{B} \boxtimes \mathcal{B}^{\mathrm{rev}}$, where $\mathcal{B}^{\mathrm{rev}}$ denotes $\mathcal{B}$ equipped with the reverse braiding $c^{\mathrm{rev}}_{X,Y} := c^{-1}_{Y,X}$. The forgetful functor $F : \mathcal{Z}(\mathcal{B}) \to \mathcal{B}$ is identified with the tensor multiplication $\mathcal{B} \boxtimes \mathcal{B}^{\mathrm{rev}} \to \mathcal{B}$ and the canonical Lagrangian algebra in $\mathcal{Z}(\mathcal{B})$ is

$$A = \oplus_{X \in \mathcal{O}(\mathcal{B})} X^* \boxtimes X.$$

The notion of a subcategory datum for a tensor product of fusion categories was introduced in Definition 2.1. Suppose that $\mathcal{S}(\mathcal{E}, \mathcal{F}, G)$ is transversal to A and is such that $\mathrm{FPdim}(\mathcal{S}(\mathcal{E}, \mathcal{F}, G)) = \mathrm{FPdim}(\mathcal{B})$. Since the restriction of F on $\mathcal{S}(\mathcal{E}, \mathcal{F}, G)$ is injective we must have

$$\mathrm{FPdim}(F(\mathcal{S}(\mathcal{E}, \mathcal{F}, G))) = \mathrm{FPdim}(\mathcal{B}).$$

On the other hand, $\mathrm{FPdim}(F(\mathcal{S}(\mathcal{E}, \mathcal{F}, G))) \leq \mathrm{FPdim}(\mathcal{E} \vee \mathcal{F})$, so $\mathcal{E} \vee \mathcal{F} = \mathcal{B}$.

Using [DGNO, Lemma 3.38] we get

$$\mathrm{FPdim}(\mathcal{S}(\mathcal{E}, \mathcal{F}, G)) = \frac{\mathrm{FPdim}(\mathcal{E})\mathrm{FPdim}(\mathcal{F})}{|G|} = \frac{\mathrm{FPdim}(\mathcal{E} \vee \mathcal{F})\mathrm{FPdim}(\mathcal{E} \cap \mathcal{F})}{|G|}. \tag{10}$$

It follows from (10) that $\mathrm{FPdim}(\mathcal{E}\cap\mathcal{F}) = |G|$. If X is a non-zero simple object in $\mathcal{E}_g\cap\mathcal{F}_h$ then $X\otimes X^*\in\mathcal{E}_1\cap\mathcal{F}_1$. It follows that $X\otimes X^* = \mathbf{1}$ (since other possibilities contradict the transversality of $\mathcal{S}(\mathcal{E},\mathcal{F},G)$ and A). Thus, X is invertible and $\mathcal{E}\cap\mathcal{F}$ is pointed. For any non-identity $g\in G$ we must have $\mathcal{E}_g\cap\mathcal{F}_{g^{-1}} = 0$. This is equivalent to the injectivity of $h_\mathcal{E} + h_\mathcal{F}$. Indeed, otherwise there is a nonzero $X\in\mathcal{E}_g$ such that $X^*\in\mathcal{F}_g$ and $X\boxtimes X^*\in\mathcal{S}(\mathcal{E},\mathcal{F},G)$, contradicting the transversality assumption.

Since $|\mathcal{O}(\mathcal{E}\cap\mathcal{F})| = |G|$, $h_\mathcal{E} + h_\mathcal{F}$ is an isomorphism.

Conversely, suppose that a datum $(\mathcal{E},\mathcal{F},G)$ satisfies conditions in the statement of the theorem. By (10), $\mathrm{FPdim}(\mathcal{S}(\mathcal{E},\mathcal{F},G)) = \mathrm{FPdim}(\mathcal{B})$. We have $\mathcal{E}_g\cap\mathcal{F}_{g^{-1}} = 0$ for all $g\in G$, $g\neq e$. Thus, $\mathcal{S}(\mathcal{E},\mathcal{F},G)$ contains no simple objects of the form $X^*\boxtimes X$ for $X\neq 1$, i.e., $\mathcal{S}(\mathcal{E},\mathcal{F},G)$ is transversal to A. □

Remark 4.2. Under the conditions of Theorem 4.1, we have $\mathcal{B}\cong\mathcal{E}\boxtimes_G\mathcal{F}$ (as a fusion category) and the corresponding braiding $\tilde{c}$ is given by

$$\tilde{c}_{X_1\boxtimes Y_1, X_2\boxtimes Y_2} = c_{X_1,X_2}\boxtimes c^{-1}_{Y_2,Y_1}$$

for all $X_1\boxtimes Y_1$, $X_2\boxtimes Y_2$ in $\mathcal{B}$.

Corollary 4.3. *Let $(\mathcal{E},\mathcal{F},G)$ be a subcategory datum for $\mathcal{B}\boxtimes\mathcal{B}^{rev}$. Then*

$$\mathcal{S}(\mathcal{E},\mathcal{F},G)' = \bigoplus_{\phi\in\widehat{G}} (\mathcal{E}_1')_\phi\boxtimes(\mathcal{F}_1)'_{\phi^{-1}}, \tag{11}$$

where the $\widehat{G}$-gradings on $\mathcal{E}_1'$ and $\mathcal{F}_1'$ are defined as in Proposition 2.4.

Proof. For all objects V, W let us denote $\beta_{V,W} := c_{W,V}\circ c_{V,W}$.

Let $X\boxtimes Y$ be an object of $\mathcal{B}\boxtimes\mathcal{B}$ and let $X_g\boxtimes Y_g$ be an object of $\mathcal{S}(\mathcal{E},\mathcal{F},G)_g$, $g\in G$. Then

$$\beta_{X\boxtimes Y, X_g\boxtimes Y_g} = \beta_{X,X_g}\boxtimes\beta_{Y,Y_g}$$

and so $X\boxtimes Y$ centralizes $X_g\boxtimes Y_g$ if and only if β_{X,X_g} and β_{Y,Y_g} are mutually inverse scalars. This means that X projectively centralizes $\mathcal{E}$ and centralizes $\mathcal{E}_1$ (respectively, Y projectively centralizes $\mathcal{F}$ and centralizes $\mathcal{F}_1$). Thus,

$$\mathcal{S}(\mathcal{E},\mathcal{F},G)' = \mathcal{E}_1'\boxtimes_{\widehat{G}}\mathcal{F}_1' = \bigoplus_{\phi\in\widehat{G}} (\mathcal{E}_1')_\phi\boxtimes(\mathcal{F}_1)'_{\phi^{-1}},$$

as required. □

Let $\mathcal{B}(\mathcal{F},\mathcal{E},G)$ denote the braided fusion category (with underlying fusion category $\mathcal{B}$) corresponding to the datum $(\mathcal{E},\mathcal{F},G)$ from Theorem 4.1.

Corollary 4.4. *We have $\mathcal{B}(\mathcal{E},\mathcal{F},G)^{rev}\cong\mathcal{B}(\mathcal{E}_1',\mathcal{F}_1',\widehat{G})$, where the fiber product of $\mathcal{E}_1'$ and $\mathcal{F}_1'$ is as in* (11).

Corollary 4.5. *The symmetric center of $\mathcal{B}(\mathcal{F}, \mathcal{E}, G)$ has a (not necessarily faithful) grading*

$$\mathcal{Z}_{sym}(\mathcal{B}(\mathcal{F}, \mathcal{E}, G)) = \bigoplus_{(g,\phi)\in G\times\widehat{G}} \mathcal{Z}_{sym}(\mathcal{B}(\mathcal{F}, \mathcal{E}, G))_{(g,\phi)},$$

where $\mathcal{Z}_{sym}(\mathcal{B}(\mathcal{F}, \mathcal{E}, G))_{(g,\phi)} \cong (\mathcal{E}_g \cap (\mathcal{E}_1)'_\phi) \boxtimes (\mathcal{F}_g \cap (\mathcal{F}_1)'_{\phi^{-1}})$. In particular, $\mathcal{Z}_{sym}(\mathcal{B}(\mathcal{F}, \mathcal{E}, G))$ is pointed.

Proof. The formula for homogeneous components follows from Corollary 4.3. The trivial component of the grading of $\mathcal{Z}_{sym}(\mathcal{B}(\mathcal{F}, \mathcal{E}, G))$ is contained in $\mathcal{E}_1 \boxtimes \mathcal{F}_1$ and so it is equivalent to Vec. Hence, $\mathcal{Z}_{sym}(\mathcal{B}(\mathcal{F}, \mathcal{E}, G))$ is pointed. □

Remark 4.6. Corollary 4.5 means that if $\mathcal{B}$ has a non-degenerate braiding then other braidings on $\mathcal{B}$ cannot be "too symmetric" as the symmetric center remains pointed. Conversely, if $\mathcal{B}$ has a braiding such that $\mathcal{Z}_{sym}(\mathcal{B})$ is not pointed, then no non-degenerate braidings on $\mathcal{B}$ can exist. In particular, $\mathsf{Rep}(G)$ for a non-abelian G does not admit any non-degenerate braidings (equivalently, there are no modular category structures on $\mathsf{Rep}(G)$).

Proposition 4.7. *Let $\mathcal{B}$ be a fusion category that admits a non-degenerate braiding. Then all non-degenerate braidings on $\mathcal{B}$ correspond to data $(\mathcal{E}, \mathcal{F}, G)$ such that*

$$\mathcal{B} \cap \big((\mathcal{E}_g \cap \mathcal{F}_\phi) \boxtimes (\mathcal{F}_g \cap \mathcal{E}_{\phi^{-1}})\big) = \begin{cases} \mathsf{Vec} & \text{if } g = 1, \phi = 1 \\ 0 & \text{otherwise,} \end{cases} \tag{12}$$

where we use identification $\mathcal{B} = \mathcal{E} \boxtimes_G \mathcal{F} \subset \mathcal{E} \boxtimes \mathcal{F}$.

Proof. Follows Corollary 4.5. □

4.2. **Braidings on unpointed categories.** Let $\mathcal{B}$ be a fusion category with non-degenerate braiding. Suppose that $\mathcal{B}_{pt} = \mathsf{Vec}$, i.e., $\mathcal{B}$ is *unpoitned*. It was shown in [Mu] that in this case there is factorization of $\mathcal{B}$ into a direct product of prime subcategories:

$$\mathcal{B} = \mathcal{B}_1 \boxtimes \cdots \boxtimes \mathcal{B}_n, \tag{13}$$

which is unique up to a permutation of factors.

Corollary 4.8. *Let $\mathcal{B}$ be a fusion category such that $\mathcal{B}_{pt} = \mathsf{Vec}$. Suppose that $\mathcal{B}$ admits a non-degenerate braiding. Let* (13) *be the prime factorization of $\mathcal{B}$. Then all braidings on $\mathcal{B}$ are non-degenerate and there are precisely 2^n such braidings. The corresponding braided fusion categories are*

$$\mathcal{B} = \mathcal{B}_1^{\pm} \boxtimes \cdots \boxtimes \mathcal{B}_n^{\pm}, \tag{14}$$

where $\mathcal{B}_i^+ = \mathcal{B}_i$ and $\mathcal{B}_i^- = \mathcal{B}_i^{rev}$ for $i = 1, \dots, n$.

Proof. Since, $\mathcal{B}$ is unpointed, according to Remark 4.2 we have $\mathcal{B} \cong \mathcal{E} \boxtimes \mathcal{F}$ as a fusion category. We claim that $\mathcal{E}$ and $\mathcal{F}$ centralize each other with respect to the original braiding of $\mathcal{B}$. Indeed, for all $X \in \mathcal{O}(\mathcal{E})$ and $X \in \mathcal{O}(\mathcal{F})$ the object $X \boxtimes Y$ is simple and, therefore,

$$c_{Y,X} \circ c_{X,Y} = \lambda_{X,Y} \mathrm{id}_{X \boxtimes Y}, \qquad \lambda_{X,Y} \in k^\times.$$

It follows that the map

$$\mathcal{O}(\mathcal{E} \boxtimes \mathcal{F}) \to k^\times : X \boxtimes Y \mapsto \lambda_{X,Y}$$

is a grading of $\mathcal{E} \boxtimes \mathcal{F}$. But $U(\mathcal{E} \boxtimes \mathcal{F}) \cong \mathcal{O}(\widehat{(\mathcal{E} \boxtimes \mathcal{F})_{pt}})$ is trivial, and so $\lambda_{X,Y} = 1$ for all X, Y, which proves the claim. It follows that $\mathcal{E}$ and $\mathcal{F}$ must be non-degenerate subcategories of $\mathcal{B}$. By [DMNO, Section 2.2] there is a subset $J \subset \{1, \dots, n\}$ such that $\mathcal{E} = \bigoplus_{i \in J} \mathcal{B}_i$ and $\mathcal{F} = \bigoplus_{i \notin J} \mathcal{B}_i$. This implies the statement. □

4.3. **Gauging.** Let $\mathcal{B}$ be a non-degenerate braided fusion category with a braiding $c_{X,Y} : X \otimes Y \to Y \otimes X$. A *gauging* of $\mathcal{B}$ is the following procedure of changing the braiding by a bilinear form $b : U(\mathcal{B}) \times U(\mathcal{B}) \to k^\times$. A new braiding $\tilde{c}_{X,Y} : X \otimes Y \to Y \otimes X$ is defined by

$$\tilde{c}_{X,Y} = b(\deg(X), \deg(Y))\, c_{X,Y},$$

for all $X, Y \in \mathcal{O}(\mathcal{B})$, where deg denotes the degree of a simple object with respect to the universal grading. By definition, gaugings of a given braiding form a torsor over the group of bilinear forms on $U(\mathcal{B})$.

The corresponding embedding $\mathcal{B} \to \mathcal{Z}(\mathcal{B}) = \mathcal{B} \boxtimes \mathcal{B}^{\mathrm{rev}}$ is given by $X \mapsto (X \otimes V_X) \boxtimes V_X^*$ for all $X \in \mathcal{O}(\mathcal{B})$, where $V_X \in \mathcal{O}(\mathcal{B}_{pt})$ is determined by the condition

$$\langle V_X, y \rangle = b(\deg(X), y), \quad \text{for all } y \in U(\mathcal{B}).$$

Here $\langle\ ,\ \rangle : \mathcal{O}(\mathcal{B}_{pt}) \times \widehat{U(\mathcal{B})} \to k^\times$ denotes the canonical pairing (8).

In this situation $\mathcal{E} = \mathcal{B}$, $\mathcal{F}_1 = \mathsf{Vec}$ (so that $\mathcal{F} \subset \mathcal{B}_{pt}$), and $G \subset \mathcal{O}(\mathcal{B}_{pt})$ is the image of the homomorphism $U(\mathcal{B}) \to \mathcal{O}(\mathcal{B}_{pt}) : X \mapsto V_X$.

Conversely, if a datum $(\mathcal{E}, \mathcal{F}, G)$ from Theorem 4.1 is such that $\mathcal{E} = \mathcal{B}$ and $\mathcal{F}_1 = \mathsf{Vec}$ (respectively, $\mathcal{F} = \mathcal{B}$ and $\mathcal{E}_1 = \mathsf{Vec}$) then the corresponding braiding is a gauging of the original braiding of $\mathcal{B}$ (respectively, of the reverse braiding).

In the next two examples for a finite group G we denote by $\mathcal{Z}(G)$ the center of Vec_G.

Example 4.9. (This result was independently obtained by Costel-Gabriel Bontea using different techniques). Let $\mathcal{B} := \mathcal{Z}(S_n)$, $n \geq 3$, where S_n denotes the symmetric group on n symbols. Observe that $\mathcal{B}$ has a unique maximal fusion subcategory $\mathcal{B}_{ad}$, which is the subcategory of vector bundles supported on the alternating subgroup A_n. Thus, in any presentation $\mathcal{B} = \mathcal{E} \boxtimes_G \mathcal{F}$ either $\mathcal{E} = \mathcal{B}$ or $\mathcal{F} = \mathcal{B}$. Since $U(\mathcal{B}) = \mathbb{Z}_2$ we must have $G = \{1\}$ or $G = \mathbb{Z}_2$. If $\mathcal{E} = \mathcal{B}$ then $\mathcal{F} = \mathsf{Vec}$ or $\mathcal{F} = \mathcal{B}_{pt}$ (note that $\mathrm{FPdim}(\mathcal{B}_{pt}) = 2$). The first

possibility gives the standard braiding of $\mathcal{B}$, while the second gives its gauging with respect to the $\mathbb{Z}_2$-grading of $\mathcal{B}$. The situation when $\mathcal{F} = \mathcal{B}$ is completely similar.

Hence, $\mathcal{B}$ has 4 different braidings: the usual braiding of the center, its reverse, and their gaugings with respect to the $\mathbb{Z}_2$-grading of $\mathcal{Z}(S_n)$. The corresponding data are: $(\mathcal{B}, \mathsf{Vec}, 1)$, $(\mathsf{Vec}, \mathcal{B}, 1)$, $(\mathcal{B}, \mathcal{B}_{pt}, \mathbb{Z}_2)$, and $(\mathcal{B}_{pt}, \mathcal{B}, \mathbb{Z}_2)$, respectively.

Example 4.10. Let G be a non-abelian group of order 8, i.e., G is either the dihedral group or the quaternion group. Let $\mathcal{B} = \mathcal{Z}(G)$. We claim that every braiding of $\mathcal{B}$ is a gauging of either its standard braiding or its reverse. The structure of $\mathcal{Z}(G)$ was studied in detail by various authors including [GMN, MN]. One has $U(\mathcal{B}) = \mathbb{Z}_2^3$ (so in particular, the standard braiding of $\mathcal{B}$ has $2^9 = 512$ different gaugings!) The trivial component of the universal grading is $\mathcal{B}_{pt} = \mathcal{B}_{ad}$, this is a pointed Lagrangian subcategory of the Frobenius-Perron dimension 8. Furthermore, for any non-pointed fusion subcategory $\mathcal{E} \subset \mathcal{B}$ its adjoint subcategory $\mathcal{E}_{ad}$ contains at least 4 invertible objects. In any presentation $\mathcal{B} = \mathcal{E} \boxtimes_G \mathcal{F}$ satisfying the conditions of Theorem 4.1 one of the subcategories $\mathcal{E}$, $\mathcal{F}$ must be non-pointed and and another must be pointed. Indeed, if both are pointed then so is $\mathcal{B}$, a contradiction. If both are non-pointed then $\mathrm{FPdim}(\mathcal{E}_1 \cap \mathcal{F}_1) \geq 2$, a contradiction.

Suppose that $\mathcal{E}$ is non-pointed. Then $\mathcal{F}$ is a pointed fusion subcategory of $\mathcal{B}$ with $\mathrm{FPdim}(\mathcal{B}) = 1$, 2, 4 or 8.

If $\mathrm{FPdim}(\mathcal{F}) = 1$ then we get the standard braiding of $\mathcal{B}$.

If $\mathrm{FPdim}(\mathcal{F}) = 2$ then either $G = \mathbb{Z}_2$ and the corresponding braiding is a gauging of the standard one, or $G = \{1\}$ and $\mathcal{B} = \mathcal{E} \boxtimes \mathcal{F}$. The latter is impossible since in this case $\mathrm{FPdim}(\mathcal{E}) = 32$ and $\mathcal{E}$ contains $\mathcal{B}_{pt}$ and, hence, $\mathcal{F}$.

If $\mathrm{FPdim}(\mathcal{F}) = 4$ then either $G = \mathbb{Z}_4$ the corresponding braiding is a gauging of the standard one, or $G = \mathbb{Z}_2$ and so $\mathrm{FPdim}(\mathcal{E}) = 32$ and $\mathrm{FPdim}(\mathcal{E}_1 \cap \mathcal{F}_1) = 2$, a contradiction, or $G = \{1\}$ and $\mathcal{B} = \mathcal{E} \boxtimes \mathcal{F}$ which is impossible.

Finally, if $\mathrm{FPdim}(\mathcal{F}) = 8$ then we must have $G = \mathbb{Z}_2^3$ since otherwise we again have $\mathrm{FPdim}(\mathcal{E}_1 \cap \mathcal{F}_1) \geq 2$, which contradicts conditions of Theorem 4.1. So in this case $\mathcal{F}_1 = \mathsf{Vec}$ and the grading of $\mathcal{B}$ is a gauging of the standard one.

Thus, if $\mathcal{E}$ is non-pointed then the corresponding grading is always a gauging of the standard one. Switching $\mathcal{E}$ and $\mathcal{F}$ will give gaugings of the reverse brading.

5. Braidings on group-theoretical categories

Let G be a finite group. Let us denote $\mathcal{C}(G) = \mathsf{Vec}_G$ and $\mathcal{Z}(G) := \mathcal{Z}(\mathsf{Vec}_G) = \mathcal{C}(G)^G$.

5.1. **Lagrangian algebras in the center of Vec_G.** It is well known that $\mathcal{Z}(G)$ is identified with the category of G-equivariant vector bundles on G. The isomorphism classes of simple

objects of $\mathcal{Z}(G)$ are parameterized by pairs (K, π), where $K \subset G$ is a conjugacy class and π is the isomorphism class an irreducible representation of the centralizer $C_G(g_K)$ of $g_K \in K$. The corresponding object $V(K, \pi) = \oplus_{g \in K} V(K, \pi)_g$ is the vector bundle supported on K whose equivariant structure restricted to $C_G(g_K)$ acts by π on $V(K, \pi)_{g_K}$.

Recall that equivalence classes of indecomposable $\mathcal{C}(G)$-module categories are parameterized by conjugacy classes of pairs (H, μ), where H is a subgroup of G and $\mu \in H^2(G, k^\times)$. The module category $\mathcal{M}(H, \mu)$ corresponding to (H, μ) is the category of modules over the twisted group algebra $k_\mu[H]$ in $\mathcal{C}(G)$. It can be identified with a certain category of H-invariant vector bundles on G.

Let $\mathcal{Z}(G;\, H) = \mathsf{Vec}_G^H$ be the category of H-equivariant objects in Vec_G. We have $\mathcal{Z}(H) \subset \mathcal{Z}(G;\, H)$. There is an obvious forgetful functor $F_H : \mathcal{Z}(G) \to \mathcal{Z}(G;\, H)$. Let $I_H : \mathcal{Z}(G;\, H) \to \mathcal{Z}(G)$ denote its adjoint.

The following construction was given in [D2]. The twisted group algebra $k_\mu[H]$ is a Lagrangian algebra in $\mathcal{Z}(H)$ with the obvious grading and the H-equivariant structure given by

$$k_\mu[H] \to g k_\mu[H] g^{-1} : x \mapsto \varepsilon_g(x) g x g^{-1}, \quad \text{where } \varepsilon_g(x) = \frac{\mu(gxg^{-1},\, g)}{\mu(g,\, x)}, \qquad g, x \in H. \tag{15}$$

Here we abuse notation and identify the cohomology class μ with a 2-cocycle representing it. Note that

$$\frac{\varepsilon_g(x)\varepsilon_g(y)}{\varepsilon_g(xy)} = \frac{\mu(gxg^{-1},\, gyg^{-1})}{\mu(x,\, y)}, \qquad g,\, x,\, y \in H. \tag{16}$$

In particular, ε_{g_K} restricts to a linear character of $C_G(g_K)$. As an object of $\mathcal{Z}(H)$,

$$k_\mu[H] \cong \bigoplus_K V(K,\, \varepsilon_{g_K}). \tag{17}$$

Let $A(H, \mu) \in \mathcal{Z}(G)$ be the Lagrangian algebra corresponding to the $\mathcal{C}(G)$-module category $\mathcal{M}(H, \mu)$. It was shown in [D2, Section 3.4] that

$$A(H,\, \mu) \cong I_H(k_\mu[H]). \tag{18}$$

Here $k_\mu[H] \in \mathcal{Z}(H)$ is considered as an algebra in $\mathcal{Z}(G;\, H)$.

5.2. **Transversality criterion and parameterization of braidings.** Tensor subcategories of $\mathcal{Z}(G)$ were classified in [NNW]. They are in bijection with triples (L, M, B), where

(T1) L and M are normal subgroups of G commuting with each other,

(T2) $B : L \times M \to k^\times$ is a G-invariant bicharacter.

The corresponding subcategory $\mathcal{S}_G(L, M, B)$ consists of vector bundles supported on L and such that the restriction of their G-equivariant structure on M is the scalar multiplication by $B(g, -)$ for all $g \in L$. Equivalently, simple objects of $\mathcal{S}_G(L, M, B)$ are objects $V(K, \pi)$,

where K is a conjugacy class contained in L and π is contained in the induced representation $\mathsf{Ind}_M^{C(g_K)} B(g_K, -)$. We have

$$\mathrm{FPdim}(\mathcal{S}_G(L, M, B)) = |L|[G : M]. \tag{19}$$

We denote by $\widehat{B} : L \to \widehat{M}$ the group homomorphism associated to B.

Let μ be a 2-cocycle on G with values in $k^\times$. The map $\mathsf{Alt}(\mu) : C_G(M) \times M \to k^\times$ defined by

$$\mathsf{Alt}(\mu)(g, x) = \frac{\mu(x, g)}{\mu(g, x)}, \qquad g \in C_G(M),\, x \in M. \tag{20}$$

is bimultiplicative and G-invariant. We have

$$\mathsf{Alt}(\mu)(g, x) = \varepsilon_g(x)$$

for all $g \in C_G(M)$, $x \in M$.

Lemma 5.1. *The subcategory $\mathcal{S}_G(L, M, B) \subset \mathcal{Z}(G)$ is transversal to the Lagrangian algebra $k_\mu[G]$ if and only if $\frac{B}{\mathsf{Alt}(\mu)} : L \times M \to k^\times$ is non-degenerate in the second argument, i.e., for all $g \in L$, $g \neq 1$, there is $x \in M$ such that*

$$\frac{B}{\mathsf{Alt}(\mu)}(g, x) \neq 1. \tag{21}$$

Proof. By (17) the transversality is equivalent to the condition

$$\mathsf{Hom}_{C_G(g_K)}(\varepsilon_{g_K}, \mathsf{Ind}_M^{C_G(g_K)} \widehat{B}(g_K)) = 0$$

for all non-identity conjugacy classes $K \subset L$. By the Frobenius reciprocity this is equivalent to

$$\mathsf{Hom}_M(\varepsilon_{g_K}|_M, \widehat{B}(g_K)) = 0, \qquad K \subset L,\, K \neq \{1\},$$

i.e., $\varepsilon_{g_K}|_M \neq \widehat{B}(g_K)$ for all non-identity K. This condition means that for each g_K with $K \subset L$ ($K \neq \{1\}$) there is $x \in M$ such that

$$\frac{\mu(x, g_K)}{\mu(g_K, x)} \neq B(g_K, x).$$

Using the G-invariance of B and $\mathsf{Alt}(\mu)$ we get the result. □

Theorem 5.2. *Braidings on $\mathcal{C}(G)^*_{\mathcal{M}(H,\mu)}$ are in bijection with triples (L, M, B) satisfying (T1), (T2) and the following conditions:*

(i) *$LH = MH = G$,*
(ii) *the restriction of $\frac{B}{\mathsf{Alt}(\mu)}$ on $(L \cap H) \times (M \cap H)$ is non-degenerate.*

Proof. By Theorem 3.2 braidings on $\mathcal{C}(G)^*_{\mathcal{M}(H,\mu)}$ are parameterized by fusion subcategories $\mathcal{S}_G(L, M, B) \subset \mathcal{Z}(G)$ of the Frobenius-Perron dimension $|G|$ transversal to $A(H, \mu)$.

The above dimension condition is equivalent to $|L| = |M|$ by (19). Note that this condition follows from (i) and (ii).

In view of (18) we see that a necessary condition for the above transversality is that the restriction of the forgetful functor $F_H : \mathcal{Z}(G) \to \mathcal{Z}(G;\, H)$ to $\mathcal{S}_G(L, M, B)$ is injective. The latter condition is equivalent to transversality of $\mathsf{Rep}(G/M)$ and the function algebra $\mathrm{Fun}(G/H,\, k)$ in $\mathsf{Rep}(G)$, in other words, to $\mathsf{Hom}_G(\mathsf{Ind}_M^G\, k^\times,\, \mathsf{Ind}_H^G\, k^\times) = k$. Here $k^\times$ denotes the trivial module. By the Mackey restriction formula the latter is equivalent to $MH = G$.

If the above condition is satisfied, the transversality of $\mathcal{S}_G(L, M, B)$ and $A(H,\, \mu)$ in $\mathcal{Z}(G)$ is equivalent to the transversality of $F_H(\mathcal{S}_G(L, M, B))$ and $k_\mu[H]$ in $\mathcal{Z}(G;\, H)$.

In this case we have

$$F_H(\mathcal{S}_G(L, M, B)) \cap \mathcal{Z}(H) = \mathcal{S}_H(L \cap H,\, M \cap H,\, B|_{(L\cap H)\times(M\cap H)}).$$

Now we can apply Lemma 5.1 (with G replaced by H). The transversality of the subcategory $\mathcal{S}_H(L \cap H,\, M \cap H,\, B|_{(L\cap H)\times(M\cap H)})$ and the algebra $k_\mu[H]$ is equivalent to the injectivity of the corresponding homomorphism $L \cap H \to \widehat{M \cap H}$, whence $|L \cap H| \le |M \cap H|$. This implies

$$|LH| = \frac{|L||H|}{|L \cap H|} \ge \frac{|M||H|}{|M \cap H|} = |MH| = |G|,$$

so that $LH = G$, $|L \cap H| = |M \cap H|$ and, hence, $\frac{B}{\mathsf{Alt}(\mu)}|_{(L\cap H)\times(M\cap H)}$ is non-degenerate. □

Example 5.3. Let G be a non-abelian group and let $H \subset G$ be a subgroup such that the only normal subgroup N of G such that $HN = G$ is G itself. Then $\mathcal{C}(G)^*_{\mathcal{M}(H,\,\mu)}$ does not admit a braiding. In particular, if G is simple non-abelian and $H \neq G$ then $\mathcal{C}(G)^*_{\mathcal{M}(H,\,\mu)}$ does not admit a braiding.

Remark 5.4.
(i) Masuoka [Ma] showed that that certain self-dual non-commutative and non-cocommutative semisimple Hopf algebras of dimension p^3, where p is an odd prime, admit no quasi-triangular structures, thus giving examples of group-theoretical fusion categories that do not admit any braiding. These examples, however, are not of the form considered in Theorem 5.2 (to obtain them one has to generalize Theorem 5.2 by replacing Vec_G with Vec_G^ω for a non-trivial 3-cocycle ω).

(ii) An absence of braidings on certain group-theoretical categories associated to exact factorizations of almost simple groups was established by Natale [N].

Example 5.5. The category $\mathcal{C}(G)^*_{\mathcal{M}(G,\,1)}$ is equivalent to $\mathsf{Rep}(G)$. In this case Theorem 5.2 says that braidings on $\mathsf{Rep}(G)$ are in bijections of triples $(L,\, M,\, B)$, where L and M are normal subgroups of G commuting with each other and $B : L \times M \to k^\times$ is a non-degenerate G-invariant bilinear form (note that these conditions on B imply that L and M must be Abelian). This classification was obtained by Davydov [D1].

Example 5.6. The category $\mathcal{Z}(G)$ is equivalent to $\mathcal{C}(G\times G^{\rm op})^*_{\mathcal{M}(D,1)}$, where $G^{\rm op}$ is the group with the opposite multiplication and $D=\{(g,\,g^{-1})\mid g\in G\}$. Braidings on this category are parameterized by triples $(L,\,M,\,B)$, where L, M are normal subgroups of $G\times G^{\rm op}$ and $B: L\times M\to k^\times$ is a $G\times G^{\rm op}$-invariant bilinear form such that the following conditions are satisfied:

(i) L and M commute with each other,
(ii) $LD=MD=G\times G^{op}$, and
(iii) the restriction of B on $(L\cap D)\times(M\cap D)$ is non-degenerate.

The standard braiding of $\mathcal{Z}(G)$ corresponds to $L=G\times 1$, $M=1\times G^{op}$, and trivial B.

This parameterization is an alternative to the description of quasitriangular structures on the Drinfeld double of G given by Keilberg [K].

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