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Sequences of high rank lattices with large systole containing a fixed genus surface group

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Abstract. In this paper we exhibit sequences of torsion-free lattices (both uniform and non-uniform) that have arbitrarily large systole, but all containing a thin surface subgroup of fixed genus.

Contents

1. Introduction

It is well-known that there are many notable differences between lattices in rank 1 semi-simple Lie groups and lattices in higher rank (≥ 2) semi-simple Lie groups. The purpose of this note is to provide another example of this in the context of the surface subgroup structure.

To motivate this, we recall that a powerful consequence of negative curvature in the setting of torsion-free uniform lattices in rank 1 semi-simple Lie groups is [1, Theorem 5.1], which, given a quotient of the symmetric space by such a lattice, provides an estimate for a lower bound of the genus of a

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surface subgroup in terms of the systole (i.e. the length of the shortest closed geodesic). In particular, if the length of the systole $\to \infty$, then the genus must $\to \infty$. A similar result is also known to hold for non-compact finite volume hyperbolic 3-manifolds (see [2, Section 4]). Our main result provides a striking contrast to this, answering a question posed to the authors by M. Belolipetsky.

We introduce the following notation: If $\Gamma < \mathrm{SL}(3,\mathbb{R})$ is a lattice, we denote by $\mathrm{sys}(\Gamma)$ the systole of the locally symmetric space $\Gamma\backslash \mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$.

Theorem 1.1. *Let $\Lambda < \mathrm{SL}(3,\mathbb{R})$ be a non-uniform lattice that is not commensurable with $\mathrm{SL}(3,\mathbb{Z})$.*

Then Λ is commensurable with a sequence of torsion-free groups Γ_j with $\mathrm{sys}(\Gamma_j) \to \infty$, and each Γ_j contains a thin surface subgroup of fixed genus.

This result follows from our main result, Theorem, 3.1, which provides a representative set of these lattices with the property that each lattice in this set contains a thin genus 3 surface subgroup and where the systole can be made arbitrarily large. Moreover, it is not difficult to show that our construction implies that, on subsequencing, these genus 3 surface groups are not mapping class group equivalent.

The geometric picture of these surfaces appears to be like an analogue of the surfaces constructed in [7], where the flat tori in question come from the fact the lattice has rank > 1.

The exclusion of the commensurability class determined by $\mathrm{SL}(3,\mathbb{Z})$ is a consequence of the method of proof, and it seems very likely that the result also holds for lattices in this commensurability class. For example, it is proved in [11] that for every genus $g \geq 2$, $\mathrm{SL}(3,\mathbb{Z})$ contains infinitely distinct commensurability classes of thin surface subgroups of genus g.

We can also prove a similar statement for infinitely many uniform lattices.

Theorem 1.2. *There are infinitely many incommensurable uniform lattices $\Lambda < \mathrm{SL}(3,\mathbb{R})$ such that Λ is commensurable with a sequence of torsion-free groups Γ_j with $\mathrm{sys}(\Gamma_j) \to \infty$, and each Γ_j contains a thin surface subgroup of fixed genus.*

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2. Preliminaries: Lattices in $\mathrm{SL}(3,\mathbb{R})$ and their systoles

2.1. A family of arithmetic lattices. It is well-known that all lattices in $\mathrm{SL}(3,\mathbb{R})$ arithmetic, and we briefly recall one construction of arithmetic lattices in $\mathrm{SL}(3,\mathbb{R})$. We refer the reader to [12] or [9], [10] for more details.

Let F be a totally real algebraic number field with ring of integers $\mathcal{O}_F$, and suppose that $a_1, a_2, a_3, t \in F$ are such that

- $t, a_i > 0$ for $i = 1, 2, 3$.
- $L = F(\sqrt{t})$ with ring of integers $\mathcal{O}$.

- τ is the non-trivial Galois automorphism of L over F.
- At the non-identity embeddings $\sigma : F \to \mathbb{R}$, we have $\sigma(t), \sigma(a_1) < 0$ and $\sigma(a_i) > 0$ for $i = 2, 3$.

Define $J = \mathrm{diag}(-a_1, a_2, a_3)$ which we view as a Hermitian form on $V = L^3$. Note that at the identity place of F, J has signature $(2, 1)$, whilst at the non-identity places, our assumption above shows that

$$J^\sigma = \mathrm{diag}(-\sigma(a_1), \sigma(a_2), \sigma(a_3))$$

has signature $(3, 0)$.

For a matrix $X = (x_{ij}) \in \mathrm{SL}(3, \mathrm{L})$ define $X^* = (\tau(x_{ij}))^t$ and define:

$$\mathrm{SU(J; L}, \tau) = \{\mathrm{X} \in \mathrm{SL}(3, \mathrm{L}) : \mathrm{X^*.J.X = J}\}$$

The *integral special unitary group* defines an arithmetic lattice of $\mathrm{SL}(3, \mathbb{R})$ given by:

$$\Lambda = \mathrm{SU(J; }\mathcal{O}, \tau) = \{\mathrm{X} \in \mathrm{SL}(3, \mathcal{O}) : \mathrm{X^*.J.X = J}\}$$

Note that if $P^*.J.P = J'$ is an L-equivalent Hermitian form, then a standard argument clearing denominators shows that $P^{-1}\Lambda P$ is commensurable with $\mathrm{SU(J'; }\mathcal{O}, \tau)$. Summarizing this discussion we have (see [12, Chapter 6.6] or [9], [10]):

Proposition 2.1. *In the notation above, there is a unique L-equivalence class of Hermitian forms equivalent to J and this determines a unique commensurability class of groups (up to conjugation) commensurable with Λ.*

Moreover, Λ is non-uniform if and only if $F = \mathbb{Q}$ and in this case each real quadratic field $L = \mathbb{Q}(\sqrt{d})$ determines a unique commensurability class of lattices up to conjugacy.

2.2. Systoles. Let $X = \mathrm{SL}(3, \mathbb{R})/\mathrm{SO}(3)$ and let $\Gamma < \mathrm{SL}(3, \mathbb{R})$ be a torsion-free arithmetic lattice with associated locally symmetric space $X_\Gamma = \Gamma\backslash X$. The space X (and hence the quotient spaces X_Γ) come equipped with a natural metric induced by the Killing Form. On comparing with the case of $\mathrm{SL}(2, \mathbb{R})$, it is often convenient to scale this metric so that lengths of closed geodesics in X_Γ relate to translation lengths of semisimple elements in a particular way as we now briefly discuss (see [8] or [16, Section 8] for more details).

Closed geodesics in X_Γ correspond to conjugacy classes of semisimple elements in Γ, and every semisimple element γ has a decomposition $\gamma = \gamma_h\gamma_e$ where its hyperbolic part γ_h has all positive real eigenvalues and its elliptic part γ_e has eigenvalues that lie on the unit circle. Let $\{a_1, a_2, a_3\}$ denote the eigenvalues of γ (so that $\{|a_1|, |a_2|, |a_3|\}$ are the eigenvalues of γ_h), then with the normalization of the metric noted above, γ acts on X by translating along a geodesic axis through a distance $\ell(\gamma)$ where

$$\ell(\gamma) = \sqrt{2((\log|a_1|)^2 + (\log|a_2|)^2 + (\log|a_3|)^2)}.$$

It will suffice for us to note that for a sequence of semisimple elements $\{\gamma_n\}$ with $|\mathrm{tr}(\gamma_\mathrm{n})| \to \infty$ then $\ell(\gamma_n) \to \infty$. Although we do not need it, we note

that a sharper version of this growth can be proved using some routine calculus (see [8, Theorem 3.1,Proposition 3.5]):

Theorem 2.2. *For $\gamma \in \mathrm{SL}(3,\mathbb{R})$ a semisimple element with $|\mathrm{tr}(\gamma)| \geq 1$, then*

$$\ell(\gamma) \geq \sqrt{2}\mathrm{arccosh}(\max\{1, |\mathrm{tr}(\gamma)|/3\}).$$

3. The non-uniform case.

3.1. Hitchin representations. The starting point of our construction is from [9] and [10]. These papers describe a 2-parameter family of discrete faithful representations of the $(3,4,4)$ triangle group

$$\Delta = \Delta(3,4,4) =< a, b \mid a^3 = b^4 = (a.b)^4 = 1 >$$

into $\mathrm{SL}(3,\mathbb{R})$; this generality is not required here and we recall only the version that specializes the parameters $u = v$, and we denote this family of representations by ρ_v:

$$\rho_v(a) = \begin{pmatrix} 1 & 1 & -\left(1+v+\sqrt{(v-7)(1+v)}\right)/4 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

and

$$\rho_v(b) = \begin{pmatrix} 1 & 0 & (3-v-\sqrt{(v-7)(1+v)})/4 \\ (1+v-\sqrt{(v-7)(1+v)})/2 & 1 & -1 \\ (-3+v-\sqrt{(v-7)(1+v)})/2 & 0 & -1 \end{pmatrix}.$$

The point $v = 7$ corresponds to the hyperbolic structure coming from the discrete faithful representation into $\mathrm{SO}(2,1) \subset \mathrm{SL}(3,\mathbb{R})$. As described in [9], the family of representations ρ_v for $v \geq 7$ have characters lying on the Hitchin component of Δ and so are faithful, discrete, and Zariski dense away from ρ_7 (see [6] and [3]).

As is described in [9] and [10], one can choose $v \in \mathbb{Z}$ so that $L = F(\sqrt{(v-7)(1+v)})$ is a real quadratic extension, $F = \mathbb{Q}(\sqrt{d})$ for some square free positive integer d. Throughout, we let $\mathcal{O}_d$ denote the ring of integers of F.

An easy computation (see [10]) now shows that there is a natural Hermitian form, $J_d(v)$ preserved by $\rho_v(\Delta)$ in the sense described in §2, where $\tau : \mathbb{Q}(\sqrt{d}) \to \mathbb{Q}(\sqrt{d})$ is the non-trivial Galois automorphism. In this notation we will show:

Theorem 3.1. *For every square free positive integer d, there are infinitely many rational primes p (which depend on d) so that the following holds:*

(1) *The principal congruence subgroups*

$$\Lambda_d(p) = \ker\{\mathrm{SL}(3,\mathcal{O}_\mathrm{d}) \to \mathrm{SL}(3,\mathcal{O}_\mathrm{d}/\mathrm{p}\mathcal{O}_\mathrm{d})\}$$

are torsion-free and contain a surface subgroup of genus 3.

(2) *Let $J_d(v)$ be the Hermitian forms described above and let $\Lambda_d(v) = \mathrm{SU}(\mathrm{J_d(v)}; \mathcal{O}_\mathrm{d}, \tau)$. Then for the infinitely many rational primes constructed in part 1., the subgroups $\Gamma_d(p) = \Lambda_d(p) \cap \Lambda_d(v)$ are torsion-free and contain a surface subgroup of genus 3.*

As we now explain, Theorem 3.1 easily implies Theorem 1.1. Fixing d, the discussion in §2.2 relating trace to translation length applied to semisimple elements in the sequence of principal congruence subgroups constructed in Theorem 3.1(1) shows that the translation lengths of semisimple elements in $\Lambda_d(p)$, and hence also in $\Gamma_d(p) \to \infty$. Using the relationship between translation lengths and lengths of closed geodesics described in §2 it follows that $\mathrm{sys}(\Gamma_\mathrm{d}(\mathrm{p})) \to \infty$ as required.

Note that since the forms $J_d(v)$ vary with v, the subgroups $\Lambda_d(v)$ do not lie in a fixed non-uniform lattice in $\mathrm{SL}(3, \mathbb{R})$. However, by Proposition 2.1, they do lie in a fixed commensurability class (up to conjugacy).

3.2. A finite representation. Key to the proof of Theorem 3.1 is the following lemma.

Lemma 3.2. *The group $\rho_{-1}(\Delta)$ is isomorphic to the symmetric group S_4, with the kernel defining a genus 3 surface group.*

Proof. One can check directly that $\rho_{-1}(\Delta)$ is a group of matrices of order 24. (It is not used, but one can show easily that the group is isomorphic to S_4). Moreover, $\rho_{-1}(a)$ has order 3, and $\rho_{-1}(b)$ and $\rho_{-1}(ab)$ both have order 4, and so since all elements of finite order in Δ are conjugate into the cyclic subgroups generated by a, b and $a.b$, it follows that all the torsion in Δ injects. In particular, the kernel is torsion-free and a simple Euler characteristic computation shows that it corresponds to a genus three surface group. □

We may now roughly sketch the relevance of Lemma 3.2 in proving Theorem 3.1, deferring the technical details to the next subsection: We will show using Pell's equation, that, for each fixed square-free d, we can find a $v \in \mathbb{Z}$ greater than 7 so that the coefficients of the representation ρ_v lies in $\mathcal{O}_d$ and in addition find a rational prime p so that ρ_v is congruent to ρ_{-1} modulo $p\mathcal{O}_d$. This now represents the crux of the matter: Since $v > 7$, the image group $\rho_v(\Delta)$ is a discrete and faithful representation of Δ, while the condition on prime reduction guarantees that this image contains a subgroup of index 24, (i.e. a genus three surface group) lying in $\Lambda_d(p)$, and hence by intersecting with $\Lambda_d(v)$, in $\Gamma_d(p)$.

3.3. The details. There is a mild technical point that the representations we construct have matrix entries with some denominators divisible by 2, however it is shown in [9] that traces of elements do lie in $\mathcal{O}_d$ and this suffices in all the arguments that follow. With a view to the objectives

described above, we set $v = -1 + K$, giving

$$\rho_K(a) = \begin{pmatrix} 1 & 1 & \frac{1}{4}\left(-K - \sqrt{(K-8)K}\right) \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\rho_K(b) = \begin{pmatrix} 1 & 0 & \frac{1}{4}\left(-K - \sqrt{(K-8)K} + 4\right) \\ \frac{1}{2}\left(K - \sqrt{(K-8)K}\right) & 1 & -1 \\ \frac{1}{2}\left(K - \sqrt{(K-8)K} - 4\right) & 0 & -1 \end{pmatrix}$$

Denote by a_0 and b_0 the matrices obtained by taking $K = 0$, which by Lemma 3.2 generate a group isomorphic to S_4. In this language, we show that for d fixed, we may arrange that the representations ρ_K have image in Λ_d (as in Theorem 3.1), and moreover those K's can also be arranged so that there are infinitely many choices of prime p for which $\rho_K(a) \equiv a_0$ and $\rho_K(b) \equiv b_0$ modulo p.

We wish to solve $K(K-8) = dW^2$, and completing the square on the left hand side we see that this holds if

$$((K-4)/4)^2 - d \cdot (W/4)^2 = 1$$

Taking $u = x_1 + \sqrt{d} \cdot y_1$ to be the fundamental solution of the Pell's equation $x^2 - dy^2 = 1$ and writing $u_k = u^k = x_k + \sqrt{d} \cdot y_k$ generates all positive solutions to the given Pell's equation.

Each u_k is a unit in $\mathcal{O}_d$ and so we can generate solutions lying in the lattice Λ_d by taking $K = 4x_k + 4$. Note the presence of the multiple of 4 provides the mechanism to clear the denominators in the representations ρ_K, which we now denote by ρ_k and display below:

$$\rho_k(a) = \begin{pmatrix} 1 & 1 & \left(-1 - x - \sqrt{x^2-1}\right) \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\rho_k(b) = \begin{pmatrix} 1 & 0 & \left(-x - \sqrt{x^2-1}\right) \\ 2 + 2x - 2\sqrt{x^2-1} & 1 & -1 \\ 2x - 2\sqrt{x^2-1} & 0 & -1 \end{pmatrix}$$

By inspection, to achieve the matrix equalities $\rho_k(a) - a_0 \equiv 0$ and $\rho_k(b) - b_0 \equiv 0$ modulo a prime p, it is necessary and sufficient that $x + 1 \equiv 0$ modulo p. Note that if p is any odd prime dividing $x+1$ and not dividing d, then since the radical $x^2 - 1$ is arranged to be $d \cdot W^2$ and the only prime which could divide $x - 1$ and $x + 1$ is 2, it follows that p^2 divides $x + 1$.

Thus, to obtain infinitely many principal congruence subgroups, we need to show that we can find infinitely many odd primes dividing the terms of the sequence $x_k + 1$; taking $K = 4x_k + 4$ will give the associated sequence

of genus three surface groups. This is done by understanding properties of solutions to Pell's equation that we now discuss.

In the notation established above, we recall that the terms x_k and y_k can also be described as

$$x_k = \frac{1}{2}(u^k + (u^{-1})^k) \text{ and } y_k = \frac{1}{2\sqrt{d}}(u^k - (u^{-1})^k).$$

The key fact here is now the following. Recall that if $\{a_n\}$ is a sequence of positive integers, a prime p is defined to be a *primitive prime divisor* of the term a_n if $p|a_n$ but p does not divide a_m for $m < n$. In [5], it is shown that if a and b (not necessarily rational integers) are such that $a + b$ and ab are non-zero relatively prime rational integers, then one can exhibit primitive prime divisors for the sequences $a^n \pm b^n$ for large enough n (see also [4] for a more comprehensive modern treatment and a value of n). Applying this to $a = u$ and $b = 1/u$ we deduce that as $k \to \infty$ we may find a sequence of primitive prime divisors $p_{n(k)} \to \infty$ with $p_{n(k)}|x_{n(k)}$.

Suppressing the subsequence, for $k \geq 1$ we set

$$M_k = \begin{pmatrix} 0 & 1 \\ -1 & 2x_k \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$$

with characteristic polynomials having as roots the units u_k and its Galois conjugate. Consider the primes p_k exhibited as primitive prime divisors, and consider M_k reduced modulo p_k. Visibly the matrix M_k^2 is congruent to $-I$ modulo p_k. Since $u_{2k} = u_k^2$, M_{2k} has the same eigenvalues as M_k^2, and it follows that $x_{2k} + 1 \equiv 0$ modulo p_k. Thus, we choose $K = 4(x_{2k} + 1)$ and use the prime p_k to complete the proof of Theorem 3.1. □

4. The uniform case: Proof of Theorem 1.2.

The uniform case requires an analysis of solutions to Pell's equations in a number field setting similar to that used in the proof of Theorem 3.1. To that end we fix attention on the Pell's equation $x^2 - \delta y^2 = 1$ where $\delta \in \mathcal{O}_d$ (which, as above, is the ring of integers of the real quadratic field $\mathbb{Q}(\sqrt{d})$) is a non-square. If we insist that $\delta > 0$ and $\sigma(\delta) < 0$ for σ the non-trivial Galois automorphism, then [14] guarantees that the Pell's equation has infinitely many solutions and that positive solutions can again be parametrized as:

$$x_k = \frac{1}{2}(u^k + (u^{-1})^k) \text{ and } y_k = \frac{1}{2\sqrt{\delta}}(u^k - (u^{-1})^k),$$

where $u = x_1 + \sqrt{\delta} \cdot y_1 > 0$ is a unit in the quartic field $\mathbb{Q}(\sqrt{\delta})$. Note that by assumption on δ this quartic field has a pair of real embeddings and one pair of complex conjugate embeddings.

We also need a version of primitive (prime) divisors in the number field setting, and for this we appeal to a result of Schinzel [17] which requires the following definition. Let A and B be algebraic integers in a number field K such that the principal ideals $< A >$ and $< B >$ are coprime (i.e. the integral ideal generated by A and B is the whole ring of integers) and in addition, A/B is not a root of unity. A prime ideal $\mathcal{P}$ of K is called a *primitive divisor* of $A^n - B^n$ if $\mathcal{P}|A^n - B^n$ but $\mathcal{P}$ does not divide $A^m - B^m$ for all positive integers $m < n$.

Theorem 4.1 (Schinzel)**.** *In the notation above, there is an effectively computable constant n_0, depending only on the degree of K, such that $A^n - B^n$ has a primitive divisor for all $n > n_0$.*

We will apply this in the following situation: $A = u$ and $B = u^{-1}$, and since u is a unit, A and B are vacuously coprime. In addition $A/B = u^2$ is a non-trivial unit in a field with real embeddings, and so the only roots of unity are ± 1. However, $u^2 \neq 1$ by construction of u. Hence Theorem 4.1 can be applied. Indeed, we can apply Theorem 4.1 to

$$u^{2k} - (u^{-1})^{2k} = (u^k - (u^{-1})^k)(u^k + (u^{-1})^k)$$

and deduce a primitive divisor for $u^k + (u^{-1})^k$ for all sufficiently large k; i.e. a primitive divisor for x_k for all sufficiently large k.

The proof of Theorem 1.2 follows the line of argument of the the proof of Theorem 3.1, using solutions to Pell's equation over real quadratic fields to construct values of x which embed Δ into a uniform lattice. This needs a little more care, as we now describe.

As already noted, [10] constructs a Hermitian form J_k preserved by $\rho_k(\Delta)$, where we will take $F = \mathbb{Q}(\sqrt{d})$, $L = F(\sqrt{x^2 - 1})$ a quadratic extension and $\mathcal{O}$ will denote the ring of integers of L. The Pell equation we consider has the form $x^2 - \delta y^2 = 1$ where $\delta \in \mathcal{O}_d$ and the form J_k is L-equivalent to a diagonal form, which can be shown to be diag$\{1, -4x_k - 2, -4x_k - 2\}$ (see [10]). Given this, to arrange that $\mathrm{SU}(\mathrm{J_k}; \mathcal{O}, \tau)$ is a uniform lattice in $\mathrm{SL}(3, \mathbb{R})$ we need to ensure that if $\sigma : F \to \mathbb{R}$ is the non-trivial Galois embedding, then

$$\sigma(x_k^2 - 1) < 0 \text{ and } \sigma(-4x_k - 2) < 0.$$

Summarizing this discussion we have shown:

Proposition 4.2. *Suppose that $x_k > 1$ comes from a solution to the Pell equation $x^2 - \delta y^2 = 1$ with $\sigma(x_k) \in (-1, -1/2)$ then $\rho_k(\Delta)$ is contained in a uniform lattice* $\mathrm{SU}(\mathrm{J_k}; \mathcal{O}, \tau)$.

Before describing how to construct infinitely many such x_k, we continue with a discussion of the end of the proof. As in the proof of Theorem 3.1, we need to further arrange that there are infinitely many choice of prime ideals $\mathcal{P} \subset \mathcal{O}$ for which $\rho_k(a) \equiv a_0$ and $\rho_k(b) \equiv b_0$ modulo $\mathcal{P}$. Given this,

it follows from Lemma 3.2 once again that there is a genus 3 surface group in each of the groups $\Gamma(\mathcal{P})$. As in the non-uniform case, this will be done using the theory of primitive divisors that we explain in detail below.

Arranging the x_k: We choose δ to be a fundamental unit of norm -1 in $\mathbb{Q}(\sqrt{d})$. Now this does not always exist, but there are infinitely many values of d for which this does occur, for example if $d = n^2+4$, then $(n+\sqrt{d})/2$ is a unit of norm -1. Note that Nagell [13] proved that there are infinitely many n so that the resulting d is square-free. The reason for this choice is that with this hypothesis, not both of δ and $\sigma(\delta)$ can be positive (resp. negative) and so we can apply Niven's result [14] to produce infinitely many solutions to the Pell equation $x^2 - \delta y^2 = 1$. Given this, we take $u = x_1 + \sqrt{\delta} \cdot y_1$ to be the fundamental solution of the equation $x^2 - \delta y^2 = 1$ and write $u_k = u^k = x_k + \sqrt{\delta} \cdot y_k$. As noted above, $\mathbb{Q}(\sqrt{\delta})$ has degree 4 over $\mathbb{Q}$, having 2 real embeddings and one pair of complex conjugate embeddings. The solutions u_k are units in this field, and indeed are Salem numbers as can be seen as follows.

By construction the four Galois conjugates of u_k are: $u_k > 1$, $1/u_k = x_k - \sqrt{\delta} \cdot y_k < 1$ and $\sigma(x_k) \pm \sqrt{\sigma(\delta)} \cdot \sigma(y_k)$. By hypothesis $\sigma(\delta) < 0$ and so the latter two roots are imaginary lying on the unit circle; i.e. u_k is a Salem number. Our next lemma together with Proposition 4.2 completes the proof of Theorem 3.1 using the analysis of the analogous matrix M_{2k} as done at the end of the proof of Theorem 1.1.

Lemma 4.3. *In the notation above, for infinitely many choices of k we can simultaneously arrange:*

(1) $\sigma(x_{2k}) \in (-1, -1/2)$, *and*
(2) *after subsequencing we can find prime ideals $\mathcal{P}_k \subset \mathcal{O}$ such that $x_{2k} + 1 \equiv 0$ modulo $\mathcal{P}_k$.*

Proof. For the first part, from above, consider the Galois conjugate root of u given by $v = \sigma(x_1) + \sqrt{\sigma(\delta)} \cdot \sigma(y_1)$ lying on the unit circle. Writing $v = e^{i\theta}$, we note that θ cannot be a rational multiple of 2π. For if this were the case, v would be a root of unity, which it is not since it has a real Galois conjugate. Similar statements hold for v^k.

Then $\sigma(x_{2k}) = \cos(2k\theta)$ is dense in the interval $(-1, 1)$, and so we can arrange a sequence of values of k so that (1) holds.

For the second part, we consider those terms x_{2k} given by part (1). Applying Theorem 4.1 to $u^{2k} + (u^{-1})^{2k}$ as described above, produces the primitive divisors needed. □

5. Examples

Non-uniform example: When $d = 2$, a fundamental solution to $x^2 - 2y^2 = 1$ is given by $u_1 = 3 + 2\sqrt{2}$. Then $u_1^2 = 17 + 12\sqrt{2}$ and note that $17 + 1$ is divisble by $p_1 = 3$. Hence $\Gamma_2(3) < \Lambda_2(71)$ contains a genus 3 surface group.

Continuing, $u_1^3 = 99 + 70\sqrt{2}$, $u_1^6 = 19601 + 13860\sqrt{2}$ and $19601 + 1 = 2 \cdot 3^4 \cdot 11^2$, is divisible by $p_3 = 11$. Hence $\Gamma_2(11) < \Lambda_2(78407)$ contains the a genus 3 surface group.

Uniform example: Take $\delta = 1+\sqrt{2}$ a unit of norm -1. A basic solution to the Pell equation $x^2 - \delta y^2 = 1$ is given by $u_1 = (1+\sqrt{2}) + \sqrt{\delta}(\sqrt{2})$. On doing the calculations described in the proof we find that $x_8 = 51137 + 36160\sqrt{2}$ whose Galois conjugate is $-0.962 \in (-1, -1/2)$. Now $x_4 = 113 + 80\sqrt{2}$, and the $\mathbb{Q}(\sqrt{2})$ norm of x_4 is -31. A simple calculation shows that the norm of $x_8 + 1$ is divisible by 31^2.

There are two prime ideals of norm 31 in $\mathbb{Q}(\sqrt{2})$, namely $< 113 \pm 80\sqrt{2} >$, and in the field $K = \mathbb{Q}(\sqrt{1+\sqrt{2}})$ there are three prime ideals dividing 31, two of norm 31 and one of norm 31^2. Letting $\mathcal{P}$ denote the ideal of norm 31^2, it can be checked using Pari [15] that $\mathcal{P}$ divides $< x_4 >$ and hence we construct a principal congruence subgroup of $\mathrm{SU}(\mathrm{J}_8; \mathcal{O}, \tau)$ containing a genus 3 surface group.

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