Boolean Composability of Constraints and Control Synthesis for Multi-Robot Systems via Nonsmooth Control Barrier Functions

Paul Glotfelter, Jorge Cortés, and Magnus Egerstedt

***Abstract*— From mobile sensor networks to autonomous transportation systems, multi-agent solutions have been proposed to accomplish a variety of tasks. However, these systems typically require satisfaction of multiple constraints, such as safety or connectivity maintenance, while completing their primary objectives. Barrier functions represent one method to enforce such constraints via forward set invariance, and this work extends recent results on Boolean composition of nonsmooth barrier functions by explicitly addressing controlled systems, resulting in nonsmooth control barrier functions. The presented results permit a discontinuous controller, which is particularly amenable to control synthesis, and this paper develops an almost-active gradient for Boolean compositions of nonsmooth control barrier functions, which, when included as a constraint to a quadratic program, yields a valid controller. To verify these theoretical findings, the experimental results encode a series of constraints and synthesize a controller for a leader-follow team of mobile robots in real time.**

I. Introduction

Multi-agent systems have emerged as a method for accomplishing complex tasks, from mobile sensor networks to coverage control [1], [2], [3], [4]. However, the usage of these systems typically introduces a number of constraints that must be respected, such as collision avoidance and connectivity maintenance [5]. For example, consider a team of leader and follower robots. The leaders must traverse an obstacle-covered workspace to pre-specified goal positions, and each follower must stay close to one of the leaders. Additionally, all robots must avoid inter-agent collisions and obstacle collisions. In fact, this example motivates the objective of this work: synthesize a controller that satisfies these constraints in the context of a pre-specified objective. As such, this work encompasses two main theoretical aspects: constraints and control synthesis.

The above-mentioned constraints, among others, may be encoded as forward-set-invariance requirements, and barrier functions represent one method to enforce this property [6], [7]. Barrier functions have been applied to a variety of practical challenges, including avionics and remote-access robotics testbeds [8], [9]. However, composing multiple set-based constraints (e.g., connectivity and collision avoidance) typically involves set intersections and unions, which generally result in nonsmooth functions. Accordingly, a single smooth barrier function may not effectively encode these requirements.

Previous work on Nonsmooth Barrier Functions (NBFs) has expanded this theory to include nonsmooth functions [10]. Tools from nonsmooth analysis, as in [11], [12], [13], enable a Boolean logic system for these NBFs via $\max$ and $\min$ operators, which encapsulate set unions and intersections, respectively. However, this previous work does not explicitly consider controlled systems, which becomes necessary for control synthesis.

A related body of prior work has shown that control synthesis via Quadratic Programs (QPs) can be used to minimally modify an existing controller such that a barrier function remains valid [6], [14], [15], [16], [17], and such methods have seen success on large-scale multi-robot systems but have yet to be extended to Boolean composition of NBFs [8]. In particular, this technique involves taking a derivative along trajectories, considering a sufficient rate function, and generating an inequality constraint for a QP. If this constraint is satisfied at all points, then the forward-set-invariance property holds. However, the nonsmooth case correspondingly requires a generalized derivative, which results in a discontinuous constraint; as such, a discontinuous control inputs may result from this process and must be considered.

By combining and extending the previous work on Boolean composition of NBFs and control synthesis via QPs, this paper develops a constraint satisfaction and control synthesis framework that can be deployed onto multi-robot systems. As in [16], [8], [17], this framework can operate in real time and can be combined with existing controllers. For validation, the proposed framework solves the aforementioned leader-follower problem.

To enable the above-mentioned framework, this paper provides the following theoretical results. Considering a class of control-affine systems and allowing discontinuities in the control input, this work formulates NBFs with respect to this system, resulting in Nonsmooth Control Barrier Functions (NCBFs), and extends the results on NBFs to NCBFs using the techniques from [11], [13], [18], [10].

Next, we focus on providing a system of Boolean logic with Boolean NCBFs (BNCBFs), which are Boolean combinations of NCBFs. This framework leverages the work in [10] and extends it by explicitly considering discontinuous control inputs. This formulation supports the main result of this work: the development of an almost-active gradient for BNCBFs that is suited for control synthesis via a QP. The main result proves that this object, when used as a

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constraint to a QP, provides a validating, though potentially discontinuous, controller.

The paper is organized as follows. Sec. II introduces the problem statement and offers some background material for differential inclusions and nonsmooth analysis. Sec. III develops NCBFs, notes some Boolean composability requirements, formulates the almost-active gradient, and constructs a control synthesis algorithm via a QP, providing the main results of this paper. Accordingly, Sec. IV shows the deployment of a Boolean NCBF onto a multi-robot system with leader-follower constraints, and Sec. V concludes the paper.

II. Problem Statement and Background Material

This section presents the particular application that this paper seeks to solve and discusses relevant background material, including differential inclusions, nonsmooth analysis, NBFs, and analysis along trajectories for nonsmooth functions.

A. Notation

For $k > 0$, the abbreviation $[k]$ represents the set $\{1, \dots, k\}$. The notation $\mathbb{R}_{\geq 0}$ corresponds to the set of nonnegative real numbers; $a.e.$ means almost everywhere in the sense of Lebesgue measure. The function $\langle \cdot, \cdot \rangle$ symbolizes the inner product of two vectors. The operation co represents the convex hull of a set. A function $\alpha : \mathbb{R} \to \mathbb{R}$ is extended class-$\mathcal{K}$ if α is continuous, strictly increasing, and $\alpha(0) = 0$. An extended class-$\mathcal{K}$ function is class-$\mathcal{K}$ when restricted to $\mathbb{R}_{\geq 0}$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is class-$\mathcal{KL}$ if it is class-$\mathcal{K}$ in its first argument and, for each fixed r, $\beta(r, \cdot)$ is continuous, strictly decreasing, and $\lim_{s\to\infty} \beta(r, s) = 0$.

B. Problem Statement

As a motivating example, consider a leader-follower team of N robots with planar states $x_i \in \mathbb{R}^2$, $i \in [N]$, where the leaders must perform a task; but, at the same time, all robots must satisfy a collection of constraints. For example, the robots must not collide. Pairwise, the inequality

$$\|x_i - x_j\| \geq \delta_{col}$$

encodes this constraint, for some $\delta_{col} > 0$. Furthermore, the function

$$h_{ij}(x_i, x_j, \delta_{col}) = \|x_i - x_j\|^2 - \delta_{col}^2, \; i, j \in [N],$$

captures this inequality (i.e., consider $h_{ij}(x_i, x_j, \delta_{col}) \geq 0$).

The robots must also avoid collisions with a fixed number, O, of obstacles, which can be captured by the function

$$h_{ij}(x_i, o_j, \delta_{obs}), \; i \in [N], j \in [O], \tag{1}$$

where the fixed value $o_j \in \mathbb{R}^2$ represents the known location of the obstacle and $\delta_{obs} > 0$ indicates the size of the obstacle.

The subset of leader robots, denoted by $N_L \subset [N]$, must travel to pre-specified goal points $x_{i,g} \in \mathbb{R}^2$, $i \in N_L$. The rest of the robots, the followers, denoted by $N_F \subset [N]$, must remain close to one of the leaders. Pairwise, the inequality

$$\|x_i - x_j\| \leq \delta_{con}, \; i \in N_L, j \in N_F,$$

represents this criterion. In terms of (1), the function

$$-h_{ij}(x_i, x_j, \delta_{con})$$

encapsulates this connectivity constraints. This symbol represents a Boolean $\neg$ (NOT) operation.

Using these pairwise constraints and a system of Boolean logic, the above barrier functions may be composed to satisfy the system-wide constraints. In particular, the Boolean compositions

$$h_{col} = \bigwedge_{i=1}^{N-1} \bigwedge_{j=i+1}^{N} h_{ij}(\cdot, \cdot, \delta_{col}), \; h_{obs} = \bigwedge_{i=1}^{N} \bigwedge_{j\in O} h_{ij}(\cdot, o_j, \delta_{col})$$

encapsulate all of the collision constraints, where the large $\wedge$ symbol refers to Boolean $\wedge$ (AND) and the large $\vee$ symbol refers to Boolean $\vee$ (OR). Similarly, the Boolean composition

$$h_{con} = \bigwedge_{i\in N_F} \bigvee_{j\in N_L} \neg h_{ij}(\cdot, \cdot, \delta_{con})$$

captures the followers' connectivity constraint to the leaders, and taking

$$h = h_{col} \wedge h_{obs} \wedge h_{con} \tag{2}$$

yields the all-encompassing constraint.

The resulting function begs the question: how does one enforce the Boolean compositions encoded by (2)? As such, the main contribution of this paper shows how to synthesize an appropriate controller from (2) for a team of mobile robots by considering it as an NCBF.

C. Differential Inclusions

In this work, the differential inclusion

$$\dot{x}(t) \in F(x(t)), \; x(0) = x_0 \tag{3}$$

becomes of interest, where $\mathcal{D} \subset \mathbb{R}^n$ is an open, connected set and $F : \mathcal{D} \subset \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is an upper semi-continuous, nomempty, compact, convex set-valued map. These conditions ensure that Carathéodory solutions to the differential inclusion exist [19]. A set-valued map $G : Y \subset \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is upper semi-continuous if for every $y \in Y$ and every $\epsilon > 0$ there exists a $\delta > 0$ such that $G(z) \subset G(y) + B(0, \epsilon)$, for every $z \in B(y, \delta)$; and a Carathéodory solution is an absolutely continuous function $x : [0, t_1] \to \mathcal{D} \subset \mathbb{R}^n$ such that $\dot{x}(t) \in F(x(t))$ almost everywhere on the interval $[0, t_1] \ni t$ and $x(0) = x_0$. For a comprehensive survey of set-valued maps and discontinuous differential equations, see [19].

More specifically, we consider control-affine systems of the form

$$\dot{x}(t) = f(x(t)) + g(x(t))u(x(t)), \tag{4}$$

where $f : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathcal{D} \subset \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ are continuous. The controller $u : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}^m$ is assumed to be a feedback control; however, this treatment only requires u to be measurable and locally bounded.

To create a system for which solutions exist, a discontinuous dynamical system, such as in (4), can be turned into a differential inclusion via Filippov's operator

$$\begin{aligned}&\dot{x}(t) \in K[f+gu](x(t)) = \operatorname{co} L[f+gu](x(t)) \\ &= \operatorname{co}\{\lim_{i\to\infty} f(x_i) + g(x_i)u(x_i) : x_i \to x(t),\ x_i \notin S_f, S\},\end{aligned} \tag{5}$$

where S_f is a particular zero-measure set that depends on the system and S is any zero-measure set. The resulting set-valued map, $K[f+gu] : \mathcal{D} \subset \mathbb{R}^n \to 2^{\mathbb{R}^n}$ satisfies the aforementioned sufficient conditions to permit existence of solutions to (3) with $F = K[f+gu]$.

D. Nonsmooth Analysis

This articles utilizes $-$, $\max$, and $\min$ functions for Boolean composition of barrier functions, as in [10]. These operations generally introduce points of nondifferentiability. Fortunately, nonsmooth analysis objects, such as the generalized gradient, have been developed to handle these functions.

Definition 1 ([11, Theorem 2.5.1]): Let f be Lipschitz near x', and suppose S is any set of Lebesgue measure zero in $\mathbb{R}^n$. Then, the generalized gradient of a function $\partial f(x')$ is

$$\partial f(x') = \operatorname{co}\{\lim_{i\to\infty} \nabla f(x_i) : x_i \to x', x_i \notin S, \Omega_f\},$$

where Ω_f represents the zero-measure set where f is nondifferentiable. •

Combining the generalized gradient and a differential inclusion into a set-valued inner product

$$\begin{aligned}&\langle \partial h(x'), F(x')\rangle = \\ &\{a \in \mathbb{R} : \exists v \in F(x'), \exists z \in \partial h(x')\ \langle v, z\rangle = a\}\end{aligned} \tag{6}$$

becomes useful for analysis along Carathéodory solutions.

However, as the eventual goal in this work considers robotic systems, the computational burden of calculating the generalized gradient becomes relevant. Moreover, calculating the set-valued inner product in (6) also becomes a concern.

Toward alleviating these computational issues, the work in [11] develops an extensive calculus for $\min$ and $\max$ functions, which enable the Boolean composition of NBFs. In such cases, the generalized gradients of the components functions (i.e., ∂f_i) may be explicitly known, making $\partial f(\cdot)$ straightforward to compute. Note that the following proposition has been modified to fit the terminology of this work.

Proposition 1 ([11, Proposition 2.3.12]): Let $\{f_i\}$ be a finite collection of functions ($i = 1, 2, \ldots, k$) Lipschitz near x'. Then, the function f defined by

$$f(x') = \max_{i\in[k]}\{f_i(x')\}$$

is Lipschitz near x' as well. Let $I(x')$ denote the set of indices i for which $f_i(x') = f(x')$. Then,

$$\partial f(x') \subset \operatorname{co}\{\partial f_i(x') : i \in I(x')\}.$$

•

E. Forward Invariance and Nonsmooth Barrier Functions

Barrier functions focus on guaranteeing forward invariance of a set (i.e., all Carathéodory solutions that start in the set stay in the set). In particular, given a continuous function $h : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}$, the so-called safe set is defined as

$$\mathcal{C} = \{x' \in \mathcal{D} : h(x') \geq 0\}.$$

As such, a candidate NBF is defined as follows. Note that the definition from [10] has been modified to fit the terminology of this work

Definition 2 ([10, Definition 3]): A locally Lipschitz function $h : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}$, where $\mathcal{D}$ is an open, connected set, is a *candidate NBF (NCBF)* if the set $\mathcal{C}$ is nonempty. •

The goal becomes to ensure forward invariance of $\mathcal{C}$. This articles considers a set $\mathcal{C}$ to be forward invariant with respect to a differential inclusion (e.g., (3)) if every Carathéodory solution starting in $\mathcal{C}$ remains in $\mathcal{C}$. That is,

$$x_0 \in \mathcal{C} \implies x(t) \in \mathcal{C}, \forall t \in [0, t_1].$$

Sometimes this property is referred to as strong forward invariance, owing to the nonuniqueness of solutions to (3). As such, a valid NBF is defined as follows.

Definition 3 ([10, Definition 4]): A candidate NBF $h : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}$ is a *valid NBF* for (3) if $x_0 \in \mathcal{C}$ implies that there exists a class-$\mathcal{KL}$ function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that

$$h(x(t)) \geq \beta(h(x_0), t), \forall t \in [0, t_1],$$

for all Carathéodory solutions $x : [0, t_1] \to \mathcal{D} \subset \mathbb{R}^n$ starting from x_0. •

Note that, by definition of class-$\mathcal{KL}$ functions, $h(x(t)) \geq 0$, for all $t \in [0, t_1]$. Thus, $\mathcal{C}$ is forward invariant. For NBFs, the work in [10] shows the following result, which has been modified to fit the notation of this paper.

Theorem 1 ([10, Theorem 3]): Let $h : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}$ be a candidate NBF. If there exists a locally Lipschitz extended class-$\mathcal{K}$ function $\alpha : \mathbb{R} \to \mathbb{R}$ such that

$$\min\langle \partial h(x'), F(x')\rangle \geq -\alpha(h(x')), \forall x' \in \mathcal{D},$$

then h is a valid NBF for (3).

F. Calculating the Set-Valued Inner Product

Even though Prop. 1 simplifies the calculation of the generalized gradient, validating NBFs still requires consideration of the set-valued inner product in (6). However, directly constructing the set in (6) may be computationally prohibitive, as $\partial h(x')$ and $F(x')$ are convex, compact sets. Previous work has addressed this issue, formulating results that simplify the analysis and computation of this set-valued inner product. The notation of the following theorem has been modified to fit this paper.

Theorem 2 ([10, Theorem 4]): Let $h : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}$ be a candidate NBF, and let $\Phi_1, \Phi_2 : \mathcal{D} \subset \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be set-valued maps such that

$$\partial h(x') \subset \operatorname{co}\Phi_1(x'),\ F(x') \subset \operatorname{co}\Phi_2(x')$$

for all $x' \in \mathcal{D}$. If there exists a locally Lipschitz extended class-$\mathcal{K}$ function $\alpha : \mathbb{R} \to \mathbb{R}$ such that for every $x' \in \mathcal{D}$, $z \in \Phi_1(x')$, and $v \in \Phi_2(x')$,

$$\langle z\,, v \rangle \geq -\alpha(h(x')),$$

then h is a valid NBF for (3).

III. Nonsmooth Control Barrier Functions

This section contains the main results of this work: formulating Nonsmooth Control Barrier Functions (NCBFs); providing a Boolean logic system for them, resulting in Boolean NCBFs (BNCBFs); and addressing control synthesis. As such, we identify composability requirements for BNCBFs, which make them amenable to control synthesis via a QP. Using these requirements, the almost-active gradient is formulated. This object, when used as a constraint to a QP, ensures that the resulting controller validates the BNCBF.

A. Boolean Nonsmooth Control Barrier Functions

This section defines NCBFs and BNCBFs. In particular, the definition of NCBFs ensures validation via Thm. 1, guaranteeing the desired forward invariance property.

Definition 4: A candidate NCBF $h : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}$ is a *valid NCBF* for (5) if there exists a locally Lipschitz extended class-$\mathcal{K}$ function $\alpha : \mathbb{R} \to \mathbb{R}$ and a measurable, locally bounded $u : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\min\langle \partial h(x')\,, K[f + gu](x')\rangle \geq -\alpha(h(x')),\ \forall x' \in \mathcal{D}.$$

•

We next provide a system of logic for NCBFs to create BNCBFs and note some relevant regularity assumptions for BNCBFs, toward control synthesis in Sec. III-B.

Definition 5: For a pair of candidate NCBFs, $h_1, h_2 : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}$, a *Boolean Nonsmooth Control Barrier Function (BNCBF)* is given by

$$\begin{aligned} h(x') &= \min\{h_1(x'), h_2(x')\} \coloneqq h_1 \wedge h_2 \\ h(x') &= \max\{h_1(x'), h_2(x')\} \coloneqq h_1 \vee h_2 \\ h(x') &= -h_1(x') \coloneqq \neg h_1, \end{aligned}$$

at each $x' \in \mathcal{D}$. •

In general, a BNCBF $h : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}$ can be comprised of a finite number of component functions with the above-noted Boolean operators. In this case, h is denoted

$$h = \mathcal{B}(h_1, \dots, h_k),$$

where $\mathcal{B}$ represents a Boolean logic expression containing the operators in Def. 5. An important class of BNCBFs are those composed of smooth functions.

Definition 6: A candidate BNCBF $h : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}$ defined by $h = \mathcal{B}(h_1, \dots, h_k)$ is *smoothly composed* if each component candidate NCBF $h_i : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable. •

Def. 6 implies, from Prop. 1, that, at a point $x' \in \mathcal{D}$, $\partial h(x') \subset \text{co}\{\nabla h_i(x') : i \in I \subset [k]\}$, for some appropriate index set I, where ∇ denotes the usual gradient. This encapsulating set becomes particularly important when synthesizing controllers with a QP in Sec. III-B.

B. Control Synthesis via Quadratic Programs

To enable control synthesis via a QP, this section defines some useful objects. These tools capture the composability requirements outlined in Sec. III and ensure that synthesized controllers validate the requisite BNCBF.

To motive the following discussion, consider the following argument. When validating NCBFs, the inequality

$$\langle \partial h(x')\,, K[f + gu](x')\rangle \geq -\alpha(h(x')) \tag{7}$$

must be satisfied for every $x' \in \mathcal{D}$. As such, the behavior of the controller around the point x' becomes crucial. Moreover, (7) combines all possible directions between the dynamics and the generalized gradient. As such, any active function in $\partial h(x')$, where h is a BNCBF, must be included in a neighborhood of x'. This criterion motivates the following developments.

Definition 7: Given $\epsilon > 0$ and two candidate NCBFs $h_1, h_2 : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}$, the *almost-active set of functions* for a candidate BNCBF given by $h = h_1 \wedge h_2$ or $h = h_1 \vee h_2$ is defined at each $x' \in \mathcal{D}$ as

$$I_\epsilon(x') = \{i : |h_i(x') - h(x')| \leq \epsilon\}.$$

The *almost-active gradient* of a BNCBF, denoted by $\partial_\epsilon h : \mathcal{D} \subset \mathbb{R}^n \to 2^{\mathbb{R}^n}$, at a point $x' \in \mathcal{D}$ is

$$\partial_\epsilon h(x') = \text{co} \bigcup_{i \in I_\epsilon(x')} \partial h_i(x').$$

•

The following results shows that QPs with an almost-active-gradient constraint generate validating controllers for smoothly composed BNCBFs. To do so, the behavior of the almost-active gradient becomes relevant.

Lemma 1: Let $h : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}$ be a candidate BNCBF as in Def. 7, and let $\epsilon > 0$. At every $x \in \mathcal{D}$, if $h_i(x) = h(x)$, then there exists $\delta > 0$ such that the almost-active set of functions satisfies $i \in I_\epsilon(x')$, for all $x' \in B(x, \delta)$.

Proof. Let $x \in \mathcal{D}$, and let i be such that $h_i(x) = h(x)$. By continuity of h_i, h there exists $\delta > 0$ such that

$$|h_i(x') - h_i(x)| \leq \epsilon/2,\ |h(x') - h(x)| \leq \epsilon/2,$$

for all $x' \in B(x, \delta)$. Then,

$$\begin{aligned} |h_i(x') - h(x')| &= |h_i(x') - h(x') - h_i(x) + h(x)| \\ &\leq |h_i(x') - h_i(x)| + |h(x) - h(x')| \leq \epsilon. \end{aligned}$$

Therefore, $i \in I_\epsilon(x')$, for all $x' \in B(x, \delta)$. □

Applying Lem. 1 on a smoothly composed BNCBF yields the main result on controllers resulting from QPs with the almost-active gradient as a constraint.

Theorem 3: Let $h : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}$ be a smoothly composed candidate BNCBF, as in Def. 7. If there exists $\epsilon > 0$ and a

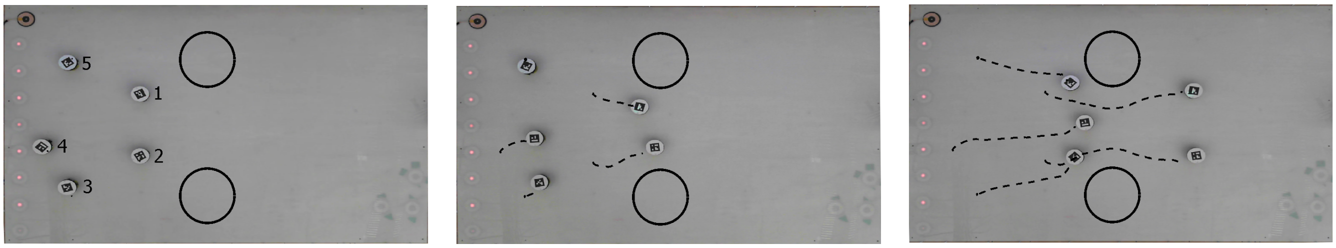

Fig. 1. A group of 5 differential-drive robots in the Robotarium execute the experiment detailed in Sec. IV. In particular, all robots avoid inter-agent collisions and obstacle collisions; each of the three follower robots maintain connectivity to one of the leader robots; and the leader robots successfully achieve their pre-specified goal position. These results show that the synthesized controller satisfies the constraints and completes the pre-existing objective.

locally Lipschitz extended class-$\mathcal{K}$ function $\alpha : \mathbb{R} \to \mathbb{R}$ such that the Quadratic Program (QP)

$$\begin{aligned} u^*(x') \in \underset{u \in \mathbb{R}^m}{\arg\min}\, & u^\top A(x') u + b(x')^\top u \\ \text{s.t. } & \langle \partial_\epsilon h(x') \,, f(x') + g(x')u \rangle \geq -\alpha(h(x')), \end{aligned}$$

with $A : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}^{m \times m}$, $b : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}^m$ continuous, has a solution for every $x' \in \mathcal{D}$ and u^* is measurable and locally bounded, then h is a valid NCBF for (5).

Proof. Let $x' \in \mathcal{D}$. Since h is smoothly composed,

$$\partial h(x') \subset \text{co}\{\nabla h_i(x') : i \in I(x')\},$$

by Prop. 1. By Thm. 2, showing that

$$\langle \nabla h_i(x') \,, L[f + gu] \rangle \geq -\alpha(h(x'))$$

for each $i \in I(x')$ suffices to achieve the desired result. Take $i \in I(x')$. By definition, $h(x') = h_i(x')$, so applying Lem. 1 implies that there exists $\delta > 0$ such that $i \in I_\epsilon(z)$, for all $z \in B(x', \delta)$. As such, u^* satisfies

$$\langle \nabla h_i(z) \,, f(z) + g(z) u^*(z) \rangle \geq -\alpha(h(z)),$$

for all $z \in B(x', \delta)$.

Let $v \in L[f + gu]$. Then, there exists a sequence $x_j \to x'$ such that $f(x_j) + g(x_j)u^*(x_j) \to v$. Moreover, the existence of the limit implies that the same limit holds for any subsequence. Since $x_j \to x'$, there exists a k such that $\|x_j - x'\| \leq \delta$ for all $j \geq k$ so, reusing notation, consider a subsequence $x_j \to x'$ with $j \geq k$.

Because ∇h_i, α, and $\langle \cdot \,, \cdot \rangle$ are continuous

$$\begin{aligned} &\langle \nabla h_i(x') \,, v \rangle + \alpha(h(x')) = \\ &\langle \lim_{j\to\infty} \nabla h_i(x_j) \,, \lim_{j\to\infty} (f(x_j) + g(x_j)u^*(x_j)) \rangle + \lim_{j\to\infty} \alpha(h(x_j)) \\ &= \lim_{j\to\infty} \langle \nabla h_i(x_j) \,, f(x_j) + g(x_j)u^*(x_j) \rangle + \lim_{j\to\infty} \alpha(h(x_j)) \\ &= \lim_{j\to\infty} (\langle \nabla h_i(x_j) \,, f(x_j) + g(x_j)u^*(x_j) \rangle + \alpha(h(x_j))) \\ &= \lim_{j\to\infty} a_j, \end{aligned}$$

where $a_j = \langle \nabla h_i(x_j) \,, f(x_j) + g(x_j)u^*(x_j) \rangle + \alpha(h(x_j))$. By assumption, $a_j \geq 0$ for all j, since $\|x_j - x'\| \leq \delta$; therefore,

$$\lim_{j\to\infty} a_j \geq 0,$$

implying that

$$\langle \nabla h_i(x') \,, v \rangle \geq -\alpha(h(x'))$$

and completing the proof. □

The experimental results in Sec. IV rely on a slightly generalized version of Thm. 3, which is not given in this work. The exact proof of this result would involve a generalization of the almost-active set of functions and Lem. 1 to BNCBFs with nested component functions. However, Thm. 3 lays a significant portion of the groundwork for such a result.

IV. Experimental Results

This experiment solves the problem posed in Sec. II-B, utilizing the same notation. Consider $N = 5$ robots with planar states and dynamics

$$\dot{x}_i = u_i.$$

This experiment also references the ensemble state $x \in \mathbb{R}^{2N}$ with input $u \in \mathbb{R}^{2N}$. Moreover, $N_L = \{1, 2\}$ and $N_F = \{3, 4, 5\}$.

Robots 1 and 2 travel from a specified initial condition to a pre-specified goal point $x_{i,g} \in \mathbb{R}^2$ with the controller

$$u_{i,nom}(x_i) = x_{i,g} - x_i,$$

for $i \in N_L$. Meanwhile, robots 3, 4, and 5 must remain close to either the first or the second robot. While traveling, all robots must avoid collisions with each other and a pair of obstacles with known location. The BNCBF in Sec. II-B (i.e., h) captures these constraints.

This experiment solves the QP indicated in Thm. 3. Since h is smoothly composed, calculating the almost-active gradient involves only the gradient of $h_{i,j}$, which is

$$\nabla_{x_i} h_{i,j}(x_i, x_j, \cdot) = 2(x_i - x_j) = -\nabla_{x_j} h_{i,j}(x_i, x_j, \cdot). \quad (8)$$

As required, these gradients are continuous, and the requisite QP, in the format of Thm. 3, is

$$\begin{aligned} u^*(x) \in \underset{u \in \mathbb{R}^{2N}}{\arg\min}\, & u^\top u - 2u_{nom}^\top(x) u \\ \text{s.t. } & \langle \partial_\epsilon h(x) \,, u \rangle \geq -\gamma h(x)^3, \end{aligned}$$

where $\gamma > 0$ and $h(x) \to h(x)^3$ is the selected extended class-$\mathcal{K}$ function. In this case, $\partial_\epsilon h(x)$ can be calculated by considering the active expressions and substituting an

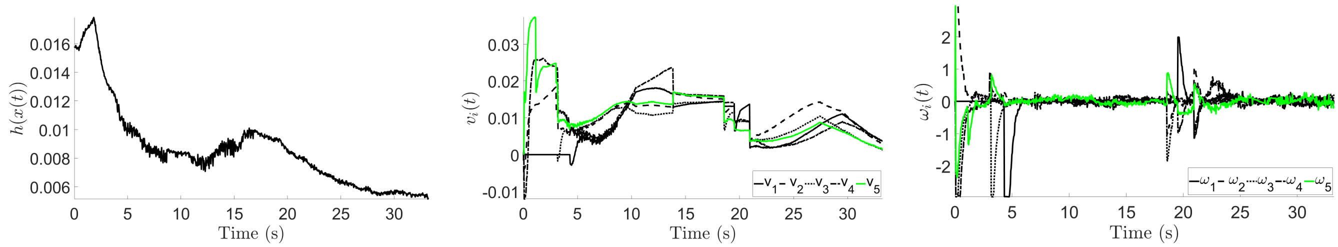

Fig. 2. Numerical results from the team of mobile robots. The left image displays the value of the BNCBF over the course of the experiment. Because the value is always positive, all constraints are satisfied. The middle and right images display the synthesized linear and angular velocities of the mobile robots. Though discontinuous, these control inputs ensure that the constraints are met and the objective is completed.

appropriate gradient in (8). Note that the above QP minimizes the objective function $\|u - u_{nom}\|^2$, ensuring that u^* respects the leaders' primary objective. The parameters for this experiment were chosen as

$$\delta_{con} = 0.35,\ \delta_{obs} = 0.1,\ \delta_{col} = 0.08,\ \gamma = 1000, \epsilon = 0.007.$$

For deployment, this experiment utilizes the Robotarium, a remotely accessible swarm robotics testbed [8]. The differential-drive robots utilized in the Robotarium have nonlinear unicycle dynamics, which are controlled by linear and angular velocity. However, the single-integrator model may be mapped onto such a system using a number of techniques, and this experiment employs the transformation in [4].

Fig. 1 displays the resulting trajectories of the robots under the controller u^*. Due to the minimally invasive QP formulation and the results of Sec. III-B, the team of robots complete the objective while respecting the desired constraints. In particular, Fig. 2 indicates that the BNCBF, h, remains positive over the course of the experiment, implying that the synthesized controller respects all of the constraints. Additionally, the leader robots successfully achieve their prespecified goal positions.

Fig. 2 displays the linear and angular velocities of the robots during the experiment. As expected, the control inputs are discontinuous. However, as predicted by the results of Sec. III-B, the synthesized controller still ensures that the BNCBF remains positive, meaning that all constraints are satisfied.

V. Conclusion

This paper proposed and theoretically validated a framework for constraint composition and control synthesis. Composition of these constraints was obtained through Boolean operators, and their application resulted in nonsmooth functions. As such, this paper presented nonsmooth control barrier functions, which were formulated with respect to controlled systems. Accordingly, we developed an almost-active gradient for nonsmooth functions, and, when included as a constraint to a quadratic program, this object permitted the synthesis of discontinuous but valid controllers. Experimental results on a leader-follower team of mobile robots demonstrated the efficacy of these results.

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