CONSTRUCTIVE THEORY OF FUNCTIONS, Sozopol 2016
(K. Ivanov, G. Nikolov and R. Uluchev, Eds.), pp. 59-73
Prof. Marin Drinov Academic Publishing House, Sofia, 2018

A New Proof of the Atomic Decomposition of Hardy Spaces

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A new proof is given of the atomic decomposition of Hardy spaces H^p, $0 < p \le 1$, in the classical setting on $\mathbb{R}^n$. The new method can be used to establish atomic decomposition of maximal Hardy spaces in general and nonclassical settings.

Keywords and Phrases: Hardy spaces, atomic decomposition.

Mathematics Subject Classification 2010: 42B30.

1. Introduction

The study of the real-variable Hardy spaces H^p, $0 < p \le 1$, on $\mathbb{R}^n$ was pioneered by Stein and Weiss [6] and a major step forward in developing this theory was made by Fefferman and Stein in [3], see also [5]. Since then there has been a great deal of work done on Hardy spaces. The atomic decomposition of H^p was first established by Coifman [1] in dimension $n = 1$ and by Latter [4] in dimensions $n > 1$.

The purpose of this article is to give a new proof of the atomic decomposition of the H^p spaces in the classical setting on $\mathbb{R}^n$. Our method does not use the Calderón-Zygmund decomposition of functions and an approximation of the identity as the classical argument does, see [5]. The main advantage of the new proof over the classical one is that it is amenable to utilization in more general and nonclassical settings. For instance, it is used in [2] for establishing the equivalence of maximal and atomic Hardy spaces in the general setting of a metric measure space with the doubling property and in the presence of a non-negative self-adjoint operator whose heat kernel has Gaussian localization.

We denote by $|x|$ the Euclidean norm of $x \in \mathbb{R}^n$ and by $B(x, \delta)$ the open ball centered at $x \in \mathbb{R}^n$ of radius δ, i.e. $B(x, \delta) := \{y \in \mathbb{R}^n : |x - y| < \delta\}$. Positive

*G. Kerkyacharian has been supported by ANR Forewer.
†P. Petrushev has been supported by NSF Grant DMS-1714369.

constants will be denoted by $c, c_1, \dots$ and they may vary at every occurrence; $a \sim b$ will stand for $c_1 \le a/b \le c_2$.

1.1. Maximal Operators and H^p Spaces

We begin by recalling some basic facts about Hardy spaces on $\mathbb{R}^n$. For a complete account of Hardy spaces we refer the reader to [5].

Given $\varphi \in \mathcal{S}$ with $\mathcal{S}$ being the Schwartz class on $\mathbb{R}^n$ and $f \in \mathcal{S}'$ one defines

$$M_\varphi f(x) := \sup_{t>0} |\varphi_t * f(x)| \text{ with } \varphi_t(x) := t^{-n}\varphi(t^{-1}x),$$

and

$$M^*_{\varphi,a} f(x) := \sup_{t>0} \sup_{\substack{y\in\mathbb{R}^n \\ |x-y|\le at}} |\varphi_t * f(y)|, \qquad a \ge 1.$$

We now recall the grand maximal operator. Write

$$\mathcal{P}_N(\varphi) := \sup_{x\in\mathbb{R}^n} (1+|x|)^N \max_{|\alpha|\le N+1} |\partial^\alpha \varphi(x)|$$

and denote

$$\mathcal{F}_N := \{\varphi \in \mathcal{S} : \mathcal{P}_N(\varphi) \le 1\}.$$

The grand maximal operator $\mathcal{M}_N$ is defined by

$$\mathcal{M}_N f(x) := \sup_{\varphi\in\mathcal{F}_N} M^*_{\varphi,1} f(x), \qquad f \in \mathcal{S}'. \tag{1}$$

It is easy to see that for any $\varphi \in \mathcal{S}$ and $a \ge 1$ one has

$$M^*_{\varphi,a} f(x) \le a^N \mathcal{P}_N(\varphi)\mathcal{M}_N f(x), \qquad f \in \mathcal{S}'. \tag{2}$$

Definition 1. The space H^p, $0 < p \le 1$, is defined as the set of all bounded distributions $f \in \mathcal{S}'$ such that the Poisson maximal function $\sup_{t>0} |P_t * f(x)|$ belongs to L^p; the quasi-norm on H^p is defined by

$$\|f\|_{H^p} := \big\| \sup_{t>0} |P_t * f(\cdot)| \big\|_{L^p}.$$

As is well known the following assertion holds, see [3, 5]:

Proposition 1. *Let $0 < p \le 1$, $a \ge 1$, and assume $\varphi \in \mathcal{S}$ and $\int_{\mathbb{R}^n} \varphi \ne 0$. Then for any $N \ge \lfloor \frac{n}{p} \rfloor + 1$*

$$\|f\|_{H^p} \sim \|M^*_{\varphi,a} f\|_{L^p} \sim \|\mathcal{M}_N f\|_{L^p}, \qquad \forall f \in H^p.$$

1.2. Atomic H^p Spaces

A function $a \in L^\infty(\mathbb{R}^n)$ is called an atom if there exists a ball B such that

(i) $\operatorname{supp} a \subset B$,

(ii) $\|a\|_{L^\infty} \le |B|^{-1/p}$, and

(iii) $\int_{\mathbb{R}^n} x^\alpha a(x)\,dx = 0$ for all α with $|\alpha| \le n(p^{-1}-1)$.

The atomic Hardy space H^p_A, $0 < p \le 1$, is defined as the set of all distributions $f \in \mathcal{S}'$ that can be represented in the form

$$f = \sum_{j=1}^\infty \lambda_j a_j, \quad \text{where} \quad \sum_{j=1}^\infty |\lambda_j|^p < \infty,$$

$\{a_j\}$ are atoms, and the convergence is in $\mathcal{S}'$. Set

$$\|f\|_{H^p_A} := \inf_{f=\sum_j \lambda_j a_j} \Big(\sum_{j=1}^\infty |\lambda_j|^p\Big)^{1/p}, \qquad f \in H^p_A.$$

2. Atomic Decomposition of H^p Spaces

We now come to the main point in this article, that is, to give a new proof of the following classical result [1, 4], see also [5]:

Theorem 1. *For any $0 < p \le 1$ the continuous embedding $H^p \subset H^p_A$ is valid, that is, if $f \in H^p$, then $f \in H^p_A$ and*

$$\|f\|_{H^p_A} \le c\|f\|_{H^p},$$

where $c > 0$ is a constant depending only on p, n. This along with the easy to prove embedding $H^p_A \subset H^p$ leads to $H^p = H^p_A$ and $\|f\|_{H^p} \sim \|f\|_{H^p_A}$ for $f \in H^p$.

Proof. We first derive a simple *decomposition* identity which will play a central role in this proof. For this construction we need the following

Lemma 1. *For any $m \ge 1$ there exists a function $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\varphi \subset B(0,1)$, $\hat\varphi(0) = 1$, and $\partial^\alpha\hat\varphi(0) = 0$ for $0 < |\alpha| \le m$. Here $\hat\varphi$ is the Fourier transform of φ, defined by $\hat\varphi(\xi) := \int_{\mathbb{R}^n} \varphi(x)e^{-ix\cdot\xi}\,dx$.*

Proof. We will construct a function φ with the claimed properties in dimension $n = 1$. Then a properly normalized dilation of $\varphi(x_1)\varphi(x_2)\cdots\varphi(x_n)$ will have the claimed properties on $\mathbb{R}^n$.

For the univariate construction, pick a smooth "bump" ϕ with the following properties: $\phi \in C_0^\infty(\mathbb{R})$, $\operatorname{supp}\phi \subset [-1/4, 1/4]$, $\phi(x) > 0$ for $x \in (-1/4, 1/4)$, and ϕ is even. Let $\Theta(x) := \phi(x+1/2) - \phi(x-1/2)$ for $x \in \mathbb{R}$. Clearly Θ is odd.

We may assume that $m \geq 1$ is even, otherwise we work with $m+1$ instead. Denote $\Delta_h^m := (T_h - T_{-h})^m$, where $T_h f(x) := f(x+h)$.

We define $\varphi(x) := \frac{1}{x}\Delta_h^m \Theta(x)$, where $h = \frac{1}{8m}$. Clearly, $\varphi \in C^\infty(\mathbb{R})$, φ is even, and $\operatorname{supp}\varphi \subset [-\frac{7}{8}, -\frac{1}{8}] \cup [\frac{1}{8}, \frac{7}{8}]$. It is readily seen that for $\nu = 1, 2, \dots, m$

$$\hat{\varphi}^{(\nu)}(\xi) = (-i)^\nu \int_{\mathbb{R}} x^{\nu-1}\Delta_h^m \Theta(x) e^{-i\xi x}\,dx$$

and hence

$$\hat{\varphi}^{(\nu)}(0) = (-i)^\nu \int_{\mathbb{R}} x^{\nu-1}\Delta_h^m \Theta(x)\,dx = (-i)^{\nu+m}\int_{\mathbb{R}} \Theta(x)\Delta_h^m x^{\nu-1}\,dx = 0.$$

On the other hand,

$$\hat{\varphi}(0) = \int_{\mathbb{R}} \varphi(x)\,dx = 2\int_0^\infty x^{-1}\Delta_h^m\Theta(x)\,dx = 2(-1)^m \int_{1/4}^{3/4} \Theta(x)\Delta_h^m x^{-1}\,dx.$$

However, if f is a sufficiently smooth function, then $\Delta_h^m f(x) = (2h)^m f^{(m)}(\xi)$, where $\xi \in (x - mh, x+mh)$. Hence,

$$\Delta_h^m x^{-1} = (2h)^m m!(-1)^m \xi^{-m-1} \quad \text{with} \quad \xi \in (x-mh, x+mh) \subset [1/8, 7/8].$$

Consequently, $\hat{\varphi}(0) \neq 0$ and then $\hat{\varphi}(0)^{-1}\varphi(x)$ has the claimed properties. $\square$

With the aid of the above lemma, we pick $\varphi \in C_0^\infty(\mathbb{R}^n)$ with the following properties: $\operatorname{supp}\varphi \subset B(0,1)$, $\hat{\varphi}(0) = 1$, and $\partial^\alpha\hat{\varphi}(0) = 0$ for $0 < |\alpha| \leq K$, where K is sufficiently large. More precisely, we choose $K \geq n/p$.

Set $\psi(x) := 2^n\varphi(2x) - \varphi(x)$. Then $\hat{\psi}(\xi) = \hat{\varphi}(\xi/2) - \hat{\varphi}(\xi)$. Therefore, $\partial^\alpha\hat{\psi}(0) = 0$ for $|\alpha| \leq K$ which implies $\int_{\mathbb{R}^n} x^\alpha\psi(x)\,dx = 0$ for $|\alpha| \leq K$. We also introduce the function $\tilde{\psi}(x) := 2^n\varphi(2x) + \varphi(x)$. We will use the notation $h_k(x) := 2^{kn}h(2^k x)$.

Clearly, for any $f \in \mathcal{S}'$ we have $f = \lim_{j\to\infty}\varphi_j * \varphi_j * f$ (convergence in $\mathcal{S}'$), which leads to the following representation: For any $j \in \mathbb{Z}$

$$\begin{aligned} f &= \varphi_j * \varphi_j * f + \sum_{k=j}^\infty \big(\varphi_{k+1} * \varphi_{k+1} * f - \varphi_k * \varphi_k * f\big) \\ &= \varphi_j * \varphi_j * f + \sum_{k=j}^\infty \big(\varphi_{k+1} - \varphi_k\big) * \big(\varphi_{k+1} + \varphi_k\big) * f. \end{aligned}$$

Thus we arrive at

$$f = \varphi_j * \varphi_j * f + \sum_{k=j}^\infty \psi_k * \tilde{\psi}_k * f, \quad \forall f \in \mathcal{S}' \;\; \forall j \in \mathbb{Z} \quad (\text{convergence in } \mathcal{S}').$$

Observe that $\operatorname{supp}\psi_k \subset B(0,2^{-k})$ and $\operatorname{supp}\tilde{\psi}_k \subset B(0,2^{-k})$.

In what follows we will utilize the grand maximal operator $\mathcal{M}_N$, defined in (1) with $N := \lfloor \frac{n}{p} \rfloor + 1$. The following claim is an immediate consequence of (2): If $\phi \in \mathcal{S}$, then for any $f \in \mathcal{S}'$, $k \in \mathbb{Z}$, and $x \in \mathbb{R}^n$

$$|\phi_k * f(y)| \le c\mathcal{M}_N f(x) \quad \text{for all} \quad y \in \mathbb{R}^n \;\; \textit{with} \;\; |y-x| \le 2^{-k+1}, \tag{3}$$

where the constant $c > 0$ depends only on $\mathcal{P}_N(\phi)$ and N.

Let $f \in H^p$, $0 < p \le 1$, $f \ne 0$. We define

$$\Omega_r := \{x \in \mathbb{R}^n : \mathcal{M}_N f(x) > 2^r\}, \qquad r \in \mathbb{Z}.$$

Clearly, Ω_r is open, $\Omega_{r+1} \subset \Omega_r$, and $\mathbb{R}^n = \cup_{r\in\mathbb{Z}} \Omega_r$. It is easy to see that

$$\sum_{r\in\mathbb{Z}} 2^{pr}|\Omega_r| \le c\int_{\mathbb{R}^n} \mathcal{M}_N f(x)^p \, dx \le c\|f\|_{H^p}^p. \tag{4}$$

Indeed, we have

$$\begin{aligned}
\sum_{r\in\mathbb{Z}} 2^{pr}|\Omega_r| &= \sum_{r\in\mathbb{Z}} 2^{pr} \sum_{\nu\ge r} |\Omega_\nu \setminus \Omega_{\nu+1}| = \sum_{\nu\in\mathbb{Z}} |\Omega_\nu \setminus \Omega_{\nu+1}| \sum_{r\le\nu} 2^{pr} \\
&\le c_p \sum_{\nu\in\mathbb{Z}} 2^{p\nu}|\Omega_\nu \setminus \Omega_{\nu+1}| \le c_p \sum_{\nu\in\mathbb{Z}} \int_{\Omega_\nu\setminus\Omega_{\nu+1}} \mathcal{M}_N f(x)^p \, dx \\
&= c_p \int_{\mathbb{R}^n} \mathcal{M}_N f(x)^p \, dx.
\end{aligned}$$

From (4) we get $|\Omega_r| \le c\,2^{-pr}\|f\|_{H^p}^p$ for $r \in \mathbb{Z}$. Therefore, for any $r \in \mathbb{Z}$ there exists $J > 0$ such that $\|\varphi_j * \varphi_j * f\|_\infty \le c\,2^r$ for $j < -J$. Consequently, $\|\varphi_j * \varphi_j * f\|_\infty \to 0$ as $j \to -\infty$, which implies

$$f = \lim_{K\to\infty} \sum_{k=-\infty}^{K} \psi_k * \tilde{\psi}_k * f \qquad (\text{convergence in } \mathcal{S}'). \tag{5}$$

Assuming that $\Omega_r \ne \emptyset$ we write

$$E_{rk} := \big\{x \in \Omega_r : \operatorname{dist}(x, \Omega_r^c) > 2^{-k+1}\big\} \setminus \big\{x \in \Omega_{r+1} : \operatorname{dist}(x, \Omega_{r+1}^c) > 2^{-k+1}\big\}.$$

By (4) it follows that $|\Omega_r| < \infty$ and hence there exists $s_r \in \mathbb{Z}$ such that $E_{rs_r} \ne \emptyset$ and $E_{rk} = \emptyset$ for $k < s_r$. Evidently $s_r \le s_{r+1}$. We define

$$F_r(x) := \sum_{k\ge s_r} \int_{E_{rk}} \psi_k(x-y)\tilde{\psi}_k * f(y)\, dy, \qquad x \in \mathbb{R}^n, \; r \in \mathbb{Z}, \tag{6}$$

and more generally

$$F_{r,\kappa_0,\kappa_1}(x) := \sum_{k=\kappa_0}^{\kappa_1} \int_{E_{rk}} \psi_k(x-y)\tilde{\psi}_k * f(y)\, dy, \qquad s_r \le \kappa_0 \le \kappa_1 \le \infty. \tag{7}$$

It will be shown in Lemma 2 below that the functions F_r and F_{r,κ_0,κ_1} are well defined and $F_r, F_{r,\kappa_0,\kappa_1} \in L^\infty$.

Note that $\operatorname{supp}\psi_k \subset B(0, 2^{-k})$ and hence

$$\operatorname{supp}\Big(\int_{E_{rk}} \psi_k(x-y)\tilde{\psi}_k * f(y)\,dy\Big) \subset \big\{x : \operatorname{dist}(x, E_{rk}) < 2^{-k}\big\}. \tag{8}$$

On the other hand, clearly $B(y, 2^{-k+1}) \cap \big(\Omega_r \setminus \Omega_{r+1}\big) \neq \emptyset$ for each $y \in E_{rk}$, and $\mathcal{P}_N(\tilde{\psi}) \le c$. Therefore, see (3), $|\tilde{\psi}_k * f(y)| \le c\,2^r$ for $y \in E_{rk}$, which implies

$$\Big\| \int_E \psi_k(\cdot - y)\tilde{\psi}_k * f(y)\,dy \Big\|_\infty \le c\,2^r, \qquad \forall E \subset E_{rk}. \tag{9}$$

Similarly,

$$\Big\| \int_E \varphi_k(\cdot - y)\tilde{\varphi}_k * f(y)\,dy \Big\|_\infty \le c\,2^r, \qquad \forall E \subset E_{rk}. \tag{10}$$

We collect all we need about the functions F_r and F_{r,κ_0,κ_1} in the following

Lemma 2. (a) *We have*

$$E_{rk} \cap E_{r'k} = \emptyset \quad \text{if } r \neq r' \quad \text{and} \quad \mathbb{R}^n = \cup_{r\in\mathbb{Z}} E_{rk}, \quad \forall k \in \mathbb{Z}. \tag{11}$$

(b) *There exists a constant $c > 0$ such that for any $r \in \mathbb{Z}$ and $s_r \le \kappa_0 \le \kappa_1 \le \infty$*

$$\|F_r\|_\infty \le c\,2^r, \qquad \|F_{r,\kappa_0,\kappa_1}\|_\infty \le c\,2^r. \tag{12}$$

(c) *The series in* (6) *and* (7) *(if $\kappa_1 = \infty$) converge point-wise and in distributional sense.*

(d) *Moreover,*

$$F_r(x) = 0, \qquad \forall x \in \mathbb{R}^n \setminus \Omega_r, \ \forall r \in \mathbb{Z}. \tag{13}$$

Proof. Identities (11) are obvious and (13) follows readily from (8).

We next prove the left-hand side inequality in (12); the proof of the right-hand side inequality is similar and will be omitted. Consider the case when $\Omega_{r+1} \neq \emptyset$ (the case when $\Omega_{r+1} = \emptyset$ is easier). Write

$$U_k := \big\{x \in \Omega_r : \operatorname{dist}(x, \Omega_r^c) > 2^{-k+1}\big\},$$
$$V_k := \big\{x \in \Omega_{r+1} : \operatorname{dist}(x, \Omega_{r+1}^c) > 2^{-k+1}\big\}.$$

Observe that $E_{rk} = U_k \setminus V_k$.

Clearly, (8) implies $|F_r(x)| = 0$ for $x \in \mathbb{R}^n \setminus \cup_{k\ge s_r}\big\{x : \operatorname{dist}(x, E_{rk}) < 2^{-k}\big\}$. We next estimate $|F_r(x)|$ for $x \in \cup_{k\ge s_r}\big\{x : \operatorname{dist}(x, E_{rk}) < 2^{-k}\big\}$. Two cases present themselves here.

Case 1: $x \in \big[\cup_{k\ge s_r}\big\{x : \operatorname{dist}(x, E_{rk}) < 2^{-k}\big\}\big] \cap \Omega_{r+1}$. Then there exist $\nu, \ell \in \mathbb{Z}$ such that

$$x \in (U_{\ell+1} \setminus U_\ell) \cap (V_{\nu+1} \setminus V_\nu). \tag{14}$$

Due to $\Omega_{r+1} \subset \Omega_r$ we have $V_k \subset U_k$, implying $(U_{\ell+1} \setminus U_\ell) \cap (V_{\nu+1} \setminus V_\nu) = \emptyset$ if $\nu < \ell$. We next consider two subcases depending on whether $\nu \geq \ell + 3$ or $\ell \leq \nu \leq \ell + 2$.

(a) Let $\nu \geq \ell + 3$. We claim that (14) yields

$$B(x, 2^{-k}) \cap E_{rk} = \emptyset \quad \text{for} \quad k \geq \nu + 2 \ \text{ or } \ k \leq \ell - 1. \tag{15}$$

Indeed, if $k \geq \nu+2$, then $E_{rk} \subset \Omega_r \setminus V_{\nu+2}$, which implies (15), while if $k \leq \ell-1$, then $E_{rk} \subset U_{\ell-1}$, again implying (15).

We also claim that

$$B(x, 2^{-k}) \subset E_{rk} \quad \text{for} \quad \ell + 2 \leq k \leq \nu - 1. \tag{16}$$

Indeed, clearly

$$(U_{\ell+1} \setminus U_\ell) \cap (V_{\nu+1} \setminus V_\nu) \subset (U_{k-1} \setminus U_\ell) \cap (V_{\nu+1} \setminus V_{k+1}) \subset U_{k-1} \setminus V_{k+1},$$

which implies (16).

From (8) and (15)–(16) it follows that

$$\begin{aligned} F_r(x) = \sum_{k=\ell}^{\nu+1} \int_{E_{rk}} \psi_k(x-y) \tilde{\psi}_k * f(y)\, dy &= \sum_{k=\ell}^{\ell+1} \int_{E_{rk}} \psi_k(x-y) \tilde{\psi}_k * f(y)\, dy \\ + \sum_{k=\ell+2}^{\nu-2} \int_{\mathbb{R}^n} \psi_k(x-y) \tilde{\psi}_k * f(y)\, dy &+ \sum_{k=\nu-1}^{\nu+1} \int_{E_{rk}} \psi_k(x-y) \tilde{\psi}_k * f(y)\, dy. \end{aligned}$$

However,

$$\begin{aligned} &\sum_{k=\ell+2}^{\nu-2} \int_{\mathbb{R}^n} \psi_k(x-y) \tilde{\psi}_k * f(y)\, dy \\ &\quad = \sum_{k=\ell+2}^{\nu-2} \big(\varphi_{k+1} * \varphi_{k+1} * f(x) - \varphi_k * \varphi_k * f(x) \big) \\ &\quad = \varphi_{\nu-1} * \varphi_{\nu-1} * f(x) - \varphi_{\ell+2} * \varphi_{\ell+2} * f(x) \\ &\quad = \int_{E_{r,\nu-1}} \varphi_{\nu-1}(x-y) \varphi_{\nu-1} * f(y)\, dy - \int_{E_{r,\ell+2}} \varphi_{\ell+2}(x-y) \varphi_{\ell+2} * f(y)\, dy. \end{aligned}$$

Combining the above with (9) and (10) we obtain $|F_r(x)| \leq c\, 2^r$.

(b) Let $\ell \leq \nu \leq \ell + 2$. Just as above we have

$$F_r(x) = \sum_{k=\ell}^{\nu+1} \int_{E_{rk}} \psi_k(x-y) \tilde{\psi}_k * f(y)\, dy = \sum_{k=\ell}^{\ell+3} \int_{E_{rk}} \psi_k(x-y) \tilde{\psi}_k * f(y)\, dy.$$

We use (9) to estimate each the above four integrals and obtain $|F_r(x)| \leq c\, 2^r$.

Case 2: $x \in \Omega_r \setminus \Omega_{r+1}$. Then there exists $\ell \geq s_r$ such that

$$x \in (U_{\ell+1} \setminus U_\ell) \cap (\Omega_r \setminus \Omega_{r+1}).$$

Just as in the proof of (15) we have $B(x, 2^{-k}) \cap E_{rk} = \emptyset$ for $k \leq \ell - 1$, and as in the proof of (16) we have

$$(U_{\ell+1} \setminus U_\ell) \cap (\Omega_r \setminus \Omega_{r+1}) \subset U_{k-1} \setminus V_{k+1},$$

which implies $B(x, 2^{-k}) \subset E_{rk}$ for $k \geq \ell + 2$. We use these and (8) to obtain

$$\begin{aligned} F_r(x) &= \sum_{k=\ell}^{\infty} \int_{E_{rk}} \psi_k(x-y)\tilde{\psi}_k * f(y)\, dy \\ &= \sum_{k=\ell}^{\ell+1} \int_{E_{rk}} \psi_k(x-y)\tilde{\psi}_k * f(y)\, dy + \sum_{k=\ell+2}^{\infty} \int_{\mathbb{R}^n} \psi_k(x-y)\tilde{\psi}_k * f(y)\, dy. \end{aligned}$$

For the last sum we have

$$\begin{aligned} &\sum_{k=\ell+2}^{\infty} \int_{\mathbb{R}^n} \psi_k(x-y)\tilde{\psi}_k * f(y)\, dy = \lim_{\nu\to\infty} \sum_{k=\ell+2}^{\nu} \psi_k * \tilde{\psi}_k * f(x) \\ &= \lim_{\nu\to\infty} \big(\varphi_{\nu+1} * \varphi_{\nu+1} * f(x) - \varphi_{\ell+2} * \varphi_{\ell+2} * f(x)\big) \\ &= \lim_{\nu\to\infty} \Big(\int_{E_{r,\nu+1}} \varphi_{\nu+1}(x-y)\varphi_{\nu+1} * f(y)\, dy \\ &\qquad\qquad - \int_{E_{r,\ell+2}} \varphi_{\ell+2}(x-y)\varphi_{\ell+2} * f(y)\, dy \Big). \end{aligned}$$

From the above and (9)-(10) we obtain $|F_r(x)| \leq c\, 2^r$.

The point-wise convergence of the series in (6) follows from above and we similarly establish the point-wise convergence in (7).

The convergence in distributional sense in (6) relies on the following assertion: For every $\phi \in \mathcal{S}$

$$\sum_{k \geq s_r} |\langle g_{rk}, \phi \rangle| < \infty, \quad \text{where} \quad g_{rk}(x) := \int_{E_{rk}} \psi_k(x-y)\tilde{\psi}_k * f(y)\, dy. \tag{17}$$

Here $\langle g_{rk}, \phi \rangle := \int_{\mathbb{R}^n} g_{rk}\overline{\phi}\, dx$. To prove the above we will employ this estimate:

$$\|\tilde{\psi}_k f\|_\infty \leq c\, 2^{kn/p} \|f\|_{H^p}, \qquad k \in \mathbb{Z}. \tag{18}$$

Indeed, using (2) we get

$$\begin{aligned} |\tilde{\psi}_k f(x)|^p &\leq \inf_{y:|x-y|\leq 2^{-k}} \sup_{z:|y-z|\leq 2^{-k}} |\tilde{\psi}_k f(z)|^p \leq \inf_{y:|x-y|\leq 2^{-k}} c\mathcal{M}_N f(y)^p \\ &\leq c|B(x, 2^{-k})|^{-1} \int_{B(x,2^{-k})} \mathcal{M}_N f(y)^p\, dy \leq c\, 2^{kn} \|f\|_{H^p}^p, \end{aligned}$$

and (18) follows.

We will also need the following estimate: For any $\sigma > n$ there exists a constant $c_\sigma > 0$ such that

$$\Big|\int_{\mathbb{R}^n} \psi_k(x-y)\phi(x)\,dx\Big| \le c_\sigma 2^{-k(K+1)}(1+|y|)^{-\sigma}, \qquad y \in \mathbb{R}^n,\ k \ge 0. \tag{19}$$

This is a standard estimate for inner products taking into account that $\phi \in \mathcal{S}$ and $\psi \in C^\infty$, $\operatorname{supp}\psi \subset B(0,1)$, and $\int_{\mathbb{R}^n} x^\alpha \psi(x)\,dx = 0$ for $|\alpha| \le K$.

We now estimate $|\langle g_{rk}, \phi\rangle|$. From (18) and the fact that $\psi \in C_0^\infty(\mathbb{R})$ and $\phi \in \mathcal{S}$ it readily follows that

$$\int_{E_{rk}} \int_{\mathbb{R}^n} |\psi_k(x-y)|\,|\phi(x)|\,|\tilde{\psi}_k f(y)|\,dydx < \infty, \qquad k \ge s_r.$$

Therefore, we can use Fubini's theorem, (18), and (19) to obtain for $k \ge 0$

$$\begin{aligned} |\langle g_{rk}, \phi\rangle| &\le \int_{E_{rk}} \Big|\int_{\mathbb{R}^n} \psi_k(x-y)\phi(x)\,dx\Big| |\tilde{\psi}_k f(y)|\,dy \\ &\le c\,2^{-k(K+1-n/p)}\|f\|_{H^p} \int_{E_{rk}} (1+|y|)^{-\sigma}\,dy \le c\,2^{-k(K+1-n/p)}\|f\|_{H^p}, \end{aligned} \tag{20}$$

which implies (17) because $K \ge n/p$.

Denote $G_\ell := \sum_{k=s_r}^{\ell} g_{rk}$. From the above proof of (b) and (12) we infer that $G_\ell(x) \to F_r(x)$ as $\ell \to \infty$ for $x \in \mathbb{R}^n$ and $\|G_\ell\|_\infty \le c\,2^r < \infty$ for $\ell \ge s_r$. On the other hand, from (17) it follows that the series $\sum_{k \ge s_r} g_{rk}$ converges in distributional sense. By applying the dominated convergence theorem one easily concludes that $F_r = \sum_{k\ge s_r} g_{rk}$ with the convergence in distributional sense. $\square$

We set $F_r := 0$ in the case when $\Omega_r = \emptyset$, $r \in \mathbb{Z}$.

Note that by (11) it follows that

$$\psi_k * \tilde{\psi}_k * f(x) = \int_{\mathbb{R}^n} \psi_k(x-y)\tilde{\psi}_k * f(y)\,dy = \sum_{r\in\mathbb{Z}} \int_{E_{rk}} \psi_k(x-y)\tilde{\psi}_k * f(y)\,dy \tag{21}$$

and using (5) and the definition of F_r in (6) we arrive at

$$f = \sum_{r\in\mathbb{Z}} F_r \ \text{ in } \ \mathcal{S}', \ \text{ i.e. } \ \langle f, \phi\rangle = \sum_{r\in\mathbb{Z}} \langle F_r, \phi\rangle, \qquad \forall \phi \in \mathcal{S}, \tag{22}$$

where the last series converges absolutely. Above $\langle f, \phi\rangle$ denotes the action of f on $\overline{\phi}$. We next provide the needed justification of equality (22).

From (5), (6), (21), and the notation from (17) we obtain for $\phi \in \mathcal{S}$

$$\langle f, \phi\rangle = \sum_k \langle \psi_k * \tilde{\psi}_k * f, \phi\rangle = \sum_k \sum_r \langle g_{rk}, \phi\rangle = \sum_r \sum_k \langle g_{rk}, \phi\rangle = \sum_r \langle F_r, \phi\rangle.$$

Clearly, to justify the above it suffices to show that $\sum_k \sum_r |\langle g_{rk}, \phi\rangle| < \infty$. We split this sum into two:

$$\sum_k \sum_r |\langle g_{rk}, \phi\rangle| = \sum_{k\ge 0} \sum_r |\langle g_{rk}, \phi\rangle| + \sum_{k<0} \sum_r |\langle g_{rk}, \phi\rangle| =: \Sigma_1 + \Sigma_2.$$

To estimate Σ_1 we use (20) and obtain

$$\begin{aligned}\Sigma_1 &\le c\|f\|_{H^p} \sum_{k\ge 0} 2^{-k(K+1-n/p)} \sum_r \int_{E_{rk}} (1+|y|)^{-\sigma}\,dy \\ &\le c\|f\|_{H^p} \sum_{k\ge 0} 2^{-k(K+1-n/p)} \int_{\mathbb{R}^n} (1+|y|)^{-\sigma}\,dy \le c\|f\|_{H^p}.\end{aligned}$$

Here we also used that $K \ge n/p$ and $\sigma > n$.

We estimate Σ_2 in a similar manner, using that $\int_{\mathbb{R}^n} |\psi_k(y)|\,dy \le c < \infty$ and (18). We get

$$\begin{aligned}\Sigma_2 &\le c\|f\|_{H^p} \sum_{k<0} 2^{kn/p} \sum_r \int_{E_{rk}} \int_{\mathbb{R}^n} |\psi_k(x-y)|\,dy|\phi(x)|\,dx \\ &\le c\|f\|_{H^p} \sum_{k<0} 2^{kn/p} \int_{\mathbb{R}^n} (1+|x|)^{-n-1}\,dx \le c\|f\|_{H^p}.\end{aligned}$$

The above estimates of Σ_1 and Σ_2 imply $\sum_k \sum_r |\langle g_{rk}, \phi\rangle| < \infty$, which completes the justification of (22).

Observe that due to $\int_{\mathbb{R}^n} x^\alpha \psi(x)\,dx = 0$ for $|\alpha| \le K$ we have

$$\int_{\mathbb{R}^n} x^\alpha F_r(x)\,dx = 0 \quad \text{for } |\alpha| \le K,\ r \in \mathbb{Z}.$$

We next decompose each function F_r into atoms. To this end we need a Whitney type cover for Ω_r, given in the following

Lemma 3. *Suppose Ω is an open proper subset of $\mathbb{R}^n$ and let $\rho(x) := \operatorname{dist}(x, \Omega^c)$. Then there exists a constant $K > 0$, depending only on n, and a sequence of points $\{\xi_j\}_{j\in\mathbb{N}}$ in Ω with the following properties, where $\rho_j := \operatorname{dist}(\xi_j, \Omega^c)$:*

(a) $\Omega = \cup_{j\in\mathbb{N}} B(\xi_j, \rho_j/2)$.

(b) $\{B(\xi_j, \rho_j/5)\}$ *are disjoint.*

(c) *If* $B\big(\xi_j, \frac{3\rho_j}{4}\big) \cap B\big(\xi_\nu, \frac{3\rho_\nu}{4}\big) \ne \emptyset$, *then* $7^{-1}\rho_\nu \le \rho_j \le 7\rho_\nu$.

(d) *For every* $j \in \mathbb{N}$ *there are at most* K *balls* $B\big(\xi_\nu, \frac{3\rho_\nu}{4}\big)$ *intersecting* $B\big(\xi_j, \frac{3\rho_j}{4}\big)$.

Variants of this simple lemma are well known and frequently used. To prove it one simply selects $\{B(\xi_j, \rho(\xi_j)/5)\}_{j\in\mathbb{N}}$ to be a maximal disjoint subset of $\{B(x, \rho(x)/5)\}_{x\in\Omega}$ and then properties (a)–(d) follow readily, see [5, pp. 15–16].

We apply Lemma 3 to each set $\Omega_r \neq \emptyset$, $r \in \mathbb{Z}$. Fix $r \in \mathbb{Z}$ and assume $\Omega_r \neq \emptyset$. Denote by $B_j := B(\xi_j, \rho_j/2)$, $j = 1, 2, \dots$, the balls given by Lemma 3, applied to Ω_r, with the additional assumption that these balls are ordered so that $\rho_1 \geq \rho_2 \geq \cdots$. We will adhere to the notation from Lemma 3. We will also use the more compact notation $\mathcal{B}_r := \{B_j\}_{j\in\mathbb{N}}$ for the set of balls covering Ω_r.

For each ball $B \in \mathcal{B}_r$ and $k \geq s_r$ we define

$$E_{rk}^B := E_{rk} \cap \left\{x : \operatorname{dist}(x, B) < 2^{-k+1}\right\} \quad \text{if} \quad B \cap E_{rk} \neq \emptyset \tag{23}$$

and set $E_{rk}^B := \emptyset$ if $B \cap E_{rk} = \emptyset$.

We also define, for $\ell = 1, 2, \dots$,

$$R_{rk}^{B_\ell} := E_{rk}^{B_\ell} \setminus \cup_{\nu>\ell} E_{rk}^{B_\nu} \quad \text{and} \tag{24}$$

$$F_{B_\ell}(x) := \sum_{k\geq s_r} \int_{R_{rk}^{B_\ell}} \psi_k(x-y)\tilde{\psi}_k * f(y)\, dy. \tag{25}$$

Lemma 4. *For every $\ell \geq 1$ the function F_{B_ℓ} is well defined, more precisely, the series in* (25) *converges point-wise and in distributional sense. Furthermore,*

$$\operatorname{supp} F_{B_\ell} \subset 7B_\ell, \tag{26}$$

$$\int_{\mathbb{R}^n} x^\alpha F_{B_\ell}(x)\, dx = 0 \quad \textit{for all } \alpha \textit{ with } |\alpha| \leq n(p^{-1} - 1), \tag{27}$$

and

$$\|F_{B_\ell}\|_\infty \leq c_\sharp 2^r, \tag{28}$$

where the constant $c_\sharp$ is independent of r, ℓ.

In addition, for any $k \geq s_r$

$$E_{rk} = \cup_{\ell\geq 1} R_{rk}^{B_\ell} \quad \textit{and} \quad R_{rk}^{B_\ell} \cap R_{rk}^{B_m} = \emptyset, \quad \ell \neq m. \tag{29}$$

Hence

$$F_r = \sum_{B\in\mathcal{B}_r} F_B \qquad (\textit{convergence in } \mathcal{S}'). \tag{30}$$

Proof. Fix $\ell \geq 1$. Observe that using Lemma 3 we have

$$B_\ell \subset \left\{x \in \mathbb{R}^n : \operatorname{dist}(x, \Omega_r^c) < 2\rho_\ell\right\}$$

and hence $E_{rk}^{B_\ell} := \emptyset$ if $2^{-k+1} \geq 2\rho_\ell$. Define $k_0 := \min\{k : 2^{-k} < \rho_\ell\}$. Hence $\rho_\ell/2 \leq 2^{-k_0} < \rho_\ell$. Consequently,

$$F_{B_\ell}(x) := \sum_{k\geq k_0} \int_{R_{rk}^{B_\ell}} \psi_k(x-y)\tilde{\psi}_k * f(y)\, dy.$$

Therefore, $\operatorname{supp} F_{B_\ell} \subset B\big(\xi_\ell, (7/2)\rho_\ell\big) = 7B_\ell$, which confirms (26).

To prove (28) we will use the following

Lemma 5. *For an arbitrary set $S \subset \mathbb{R}^n$ let*

$$S_k := \{x \in \mathbb{R}^n : \operatorname{dist}(x, S) < 2^{-k+1}\}$$

and set

$$F_S(x) := \sum_{k \ge \kappa_0} \int_{E_{rk} \cap S_k} \psi_k(x-y) \tilde{\psi}_k * f(y)\, dy \tag{31}$$

for some $\kappa_0 \ge s_r$. Then $\|F_S\|_\infty \le c\, 2^r$, where $c > 0$ is a constant independent of S and κ_0. Moreover, the above series converges in $\mathcal{S}'$.

Proof. From (8) it follows that $F_S(x) = 0$ if $\operatorname{dist}(x, S) \ge 3 \times 2^{-\kappa_0}$

Let $x \in S$. Evidently, $B(x, 2^{-k}) \subset S_k$ for every k and hence

$$\begin{aligned} F_S(x) &= \sum_{k \ge \kappa_0} \int_{E_{rk} \cap B(x, 2^{-k})} \psi_k(x-y) \tilde{\psi}_k * f(y)\, dy \\ &= \sum_{k \ge \kappa_0} \int_{E_{rk}} \psi_k(x-y) \tilde{\psi}_k * f(y)\, dy = F_{r,\kappa_0}(x). \end{aligned}$$

On account of Lemma 2 (b) we obtain $|F_S(x)| = |F_{r,\kappa_0}(x)| \le c\, 2^r$.

Consider the case when $x \in S_\ell \setminus S_{\ell+1}$ for some $\ell \ge \kappa_0$. Then $B(x, 2^{-k}) \subset S_k$ if $\kappa_0 \le k \le \ell - 1$ and $B(x, 2^{-k}) \cap S_k = \emptyset$ if $k \ge \ell + 2$. Therefore,

$$\begin{aligned} F_S(x) &= \sum_{k=\kappa_0}^{\ell-1} \int_{E_{rk}} \psi_k(x-y) \tilde{\psi}_k * f(y)\, dy + \sum_{k=\ell}^{\ell+1} \int_{E_{rk} \cap S_k} \psi_k(x-y) \tilde{\psi}_k * f(y)\, dy \\ &= F_{r,\kappa_0,\ell-1}(x) + \sum_{k=\ell}^{\ell+1} \int_{E_{rk} \cap S_k} \psi_k(x-y) \tilde{\psi}_k * f(y)\, dy, \end{aligned}$$

where we used the notation from (7). By Lemma 2 (b) and (9) it follows that $|F_S(x)| \le c\, 2^r$.

We finally consider the case when $2^{-\kappa_0+1} \le \operatorname{dist}(x, S) < 3 \times 2^{-\kappa_0}$. Then we have $F_S(x) = \int_{E_{r\kappa_0} \cap S_{\kappa_0}} \psi_{\kappa_0}(x-y) \tilde{\psi}_{\kappa_0} * f(y)\, dy$ and the estimate $|F_S(x)| \le c\, 2^r$ is immediate from (9).

The convergence in $\mathcal{S}'$ in (31) is established as in the proof of Lemma 2. □

Fix $\ell \ge 1$ and let $\{B_j : j \in \mathcal{J}\}$ be the set of all balls $B_j = B(\xi_j, \rho_j/2)$ such that $j > \ell$ and

$$B\Big(\xi_j, \frac{3\rho_j}{4}\Big) \cap B\Big(\xi_\ell, \frac{3\rho_\ell}{4}\Big) \ne \emptyset.$$

By Lemma 3 it follows that $\#\mathcal{J} \le K$ and $7^{-1}\rho_\ell \le \rho_j \le 7\rho_\ell$ for $j \in \mathcal{J}$. Define

$$k_1 := \min\big\{k : 2^{-k+1} < 4^{-1} \min\{\rho_j : j \in \mathcal{J} \cup \{\ell\}\}\big\}. \tag{32}$$

From this definition and $2^{-k_0} < \rho_\ell$ we infer

$$2^{-k_1+1} \ge 8^{-1} \min_{j\in\mathcal{J}\cup\{\ell\}} \rho_j > 8^{-2}\rho_\ell > 8^{-2}\, 2^{-k_0} \implies k_1 \le k_0+7. \tag{33}$$

Clearly, from (32)

$$\{x : \mathrm{dist}(x, B_j) < 2^{-k+1}\} \subset B(\xi_j, 3\rho_j/4), \qquad \forall k \ge k_1,\ \forall j \in \mathcal{J}\cup\{\ell\}.$$

Denote $S := \cup_{j\in\mathcal{J}} B_j$ and $\tilde S := \cup_{j\in\mathcal{J}} B_j \cup B_\ell = S \cup B_\ell$. As in Lemma 5 we set

$S_k := \{x \in \mathbb{R}^n : \mathrm{dist}(x, S) < 2^{-k+1}\}$ and $\tilde S_k := \{x \in \mathbb{R}^n : \mathrm{dist}(x, \tilde S) < 2^{-k+1}\}$.

It readily follows from the definition of k_1 in (32) that

$$R_{rk}^{B_\ell} := E_{rk}^{B_\ell} \setminus \cup_{\nu>\ell} E_{rk}^{B_\nu} = (E_{rk}\cap \tilde S_k) \setminus (E_{rk}\cap S_k) \quad \text{for} \quad k \ge k_1. \tag{34}$$

Denote

$$F_S(x) := \sum_{k\ge k_1} \int_{E_{rk}\cap S_k} \psi_k(x-y)\tilde\psi_k * f(y)\, dy, \quad \text{and}$$

$$F_{\tilde S}(x) := \sum_{k\ge k_1} \int_{E_{rk}\cap \tilde S_k} \psi_k(x-y)\tilde\psi_k * f(y)\, dy.$$

From (34) and the fact that $S \subset \tilde S$ it follows that

$$F_{B_\ell}(x) = F_{\tilde S}(x) - F_S(x) + \sum_{k_0\le k<k_1} \int_{R_{rk}^{B_\ell}} \psi_k(x-y)\tilde\psi_k * f(y)\, dy.$$

By Lemma 5 we get $\|F_S\|_\infty \le c\,2^r$ and $\|F_{\tilde S}\|_\infty \le c\,2^r$. On the other hand from (33) we have $k_1 - k_0 \le 7$. We estimate each of the (at most 7) integrals above using (9) to conclude that $\|F_{B_\ell}\|_\infty \le c\,2^r$.

We deal with the convergence in (25) and (30) as in the proof of Lemma 2.

Clearly, (27) follows from the fact that $\int_{\mathbb{R}^n} x^\alpha \psi(x)\, dx = 0$ for all α with $|\alpha| \le K$.

Finally, from Lemma 3 we have $\Omega_r \subset \cup_{j\in\mathbb{N}} B_\ell$ and then (29) is immediate from (23) and (24). □

We are now prepared to complete the proof of Theorem 1. For every ball $B \in \mathcal{B}_r$, $r \in \mathbb{Z}$, provided $\Omega_r \ne \emptyset$, we define $B^\star := 7B$,

$$a_B(x) := {c_\sharp}^{-1}|B^\star|^{-1/p}2^{-r}F_B(x) \quad \text{and} \quad \lambda_B := c_\sharp |B^\star|^{1/p}2^r,$$

where $c_\sharp > 0$ is the constant from (28). By (26) $\mathrm{supp}\, a_B \subset B^\star$ and by (28)

$$\|a_B\|_\infty \le {c_\sharp}^{-1}|B^\star|^{-1/p}2^{-r}\|F_B\|_\infty \le |B^\star|^{-1/p}.$$

Furthermore, from (27) it follows that $\int_{\mathbb{R}^n} x^\alpha a_B(x)\, dx = 0$ if $|\alpha| \le n(p^{-1}-1)$. Therefore, each a_B is an atom for H^p.

We set $\mathcal{B}_r := \emptyset$ if $\Omega_r = \emptyset$. Now, using the above, (22), and Lemma 4 we get

$$f = \sum_{r\in\mathbb{Z}} F_r = \sum_{r\in\mathbb{Z}} \sum_{B\in\mathcal{B}_r} F_B = \sum_{r\in\mathbb{Z}} \sum_{B\in\mathcal{B}_r} \lambda_B a_B,$$

where the convergence is in $\mathcal{S}'$, and

$$\sum_{r\in\mathbb{Z}} \sum_{B\in\mathcal{B}_r} |\lambda_B|^p \le c \sum_{r\in\mathbb{Z}} 2^{pr} \sum_{B\in\mathcal{B}_r} |B| = c \sum_{r\in\mathbb{Z}} 2^{pr} |\Omega_r| \le c\|f\|_{H^p}^p,$$

which is the claimed atomic decomposition of $f \in H^p$. Above we used that $|B^\star| = |7B| = 7^n|B|$. The proof of Theorem 1 is complete. $\square$

Remark. The proof of Theorem 1 can be considerably simplified and shortened if one seeks to establish atomic decomposition of the H^p spaces in terms of q-atoms with $p < q < \infty$ rather than ∞-atoms as in Theorem 1, i.e. atoms satisfying $\|a\|_{L^q} \le |B|^{1/q-1/p}$ with $q < \infty$ rather than $\|a\|_{L^\infty} \le |B|^{-1/p}$. We will not elaborate on this here.

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