Bridge trisections of knotted surfaces in 4-manifolds

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We prove that every smoothly embedded surface in a 4-manifold can be isotoped to be in bridge position with respect to a given trisection of the ambient 4-manifold; that is, after isotopy, the surface meets components of the trisection in trivial disks or arcs. Such a decomposition, which we call a generalized bridge trisection, extends the authors' definition of bridge trisections for surfaces in S^4. Using this construction, we give diagrammatic representations called shadow diagrams for knotted surfaces in 4-manifolds. We also provide a low-complexity classification for these structures and describe several examples, including the important case of complex curves inside $\mathbb{CP}^2$. Using these examples, we prove that there exist exotic 4-manifolds with $(g, 0)$—trisections for certain values of g. We conclude by sketching a conjectural uniqueness result that would provide a complete diagrammatic calculus for studying knotted surfaces through their shadow diagrams.

trisection | knotted surface | bridge trisection | 4-manifold | complex curve

Every knot in S^3 can be cut into two trivial tangles (collections of unknotted arcs) in a classical decomposition known as a *bridge splitting*. This structure provides a convenient measure of complexity, the number of unknotted arcs in each collection, and the smallest number of such arcs in any bridge splitting of a given knot K is the widely studied *bridge number* of K. It is well-known that the idea of a bridge splitting can be extended to other spaces: Every 3-manifold Y admits a Heegaard splitting, a decomposition of Y into two simple pieces called handlebodies, and given a knot $K \subset Y$, there is an isotopy of K after which it meets each handlebody in a collection of unknotted arcs.

In dimension four, decompositions analogous to Heegaard splittings cut spaces into not two but three components. Gay and Kirby proved that every smooth, closed, connected, orientable 4-manifold (henceforth, *4-manifold*) X admits a trisection, splitting X into three simple 4-dimensional pieces (4-dimensional 1-handlebodies) that meet pairwise in 3D handlebodies and have as their common intersection a closed surface. Similarly, in ref. 1 the authors proved that every smoothly embedded, closed surface (henceforth, *knotted surface*) $\mathcal{K}$ in S^4 admits a bridge trisection, a decomposition of the pair $(S^4, \mathcal{K})$ into three collections of unknotted disks in 4-balls that intersect in trivial tangles in 3-balls, akin to classical bridge splittings in S^3. In this paper, we extend this construction to knotted surfaces in arbitrary 4-manifolds. Given a trisection $\mathcal{T}$ splitting a 4-manifold X into $X_1 \cup X_2 \cup X_3$, we say that a knotted surface $\mathcal{K} \subset X$ is in *bridge position* if $\mathcal{K} \cap X_i$ is a collection of unknotted disks and $\mathcal{K} \cap (X_i \cap X_j)$ is a collection of trivial tangles. Our first result is the following.

Theorem 1. *Let X be a 4-manifold with trisection $\mathcal{T}$. Any knotted surface $\mathcal{K}$ in X can be isotoped into bridge position with respect to $\mathcal{T}$.*

If $\mathcal{K} \subset X$ is in bridge position with respect to a trisection, we call the decomposition $(X, \mathcal{K}) = (X_1, \mathcal{K} \cap X_1) \cup (X_2, \mathcal{K} \cap X_2) \cup (X_3, \mathcal{K} \cap X_3)$ a *generalized bridge trisection*.

Returning to dimension three, we note that it can be fruitful to modify a bridge splitting of a knot K in a 3-manifold Y so that the complexity of the underlying Heegaard splitting increases while the number of unknotted arcs decreases. This process involves a technical operation called *meridional stabilization*. We show that there is an analogous operation, which we also call meridional stabilization, in the context of bridge trisections. As a result, we prove the next theorem. (Precise definitions are included in *Section 1*.) A *2-knot* is a knotted surface homeomorphic to S^2.

Theorem 2. *Let $\mathcal{K}$ be a knotted surface with n connected components in a 4-manifold X. The pair $(X, \mathcal{K})$ admits a $(g, k; b, n)$–generalized bridge trisection satisfying $b = 3n - \chi(\mathcal{K})$. In particular, if $\mathcal{K}$ is a 2-knot in X, then $\mathcal{K}$ can be put in 1-bridge position.*

A generalized bridge trisection of the type guaranteed by *Theorem 2* is called *efficient* with respect to the underlying trisection $\mathcal{T}$, since it is the smallest possible b and n for any surface with the same Euler characteristic.

As a corollary to *Theorem 1*, we explain how these decompositions provide a way to encode a knotted surface combinatorially in a 2D diagram, which we call a *shadow diagram*. We anticipate that this paradigm in the study of knotted surfaces will open a window to structures and connections in this field.

Corollary 3. *Every generalized bridge trisection of a knotted surface $\mathcal{K}$ in a 4-manifold X induces a shadow diagram. Moreover, if $\mathcal{K}$ has n components, then an efficient generalized bridge trisection of $\mathcal{K}$ induces a shadow diagram with $9n - 3\chi(\mathcal{K})$ arcs. In particular, if $\mathcal{K}$ is a 2-knot in X, then $(X, \mathcal{K})$ admits a doubly-pointed trisection diagram.*

A knot that has been decomposed into a pair unknotted arcs admits a representation called a *doubly pointed Heegaard diagram*; a doubly-pointed trisection diagram is a direct adaptation of this structure. See *Section 2* for further details.

In *Section 2*, we give shadow diagrams for various examples of simple surfaces in 4-manifolds. First, we give a classification of those 2-knots that can be put in 1-bridge position with respect to a genus one trisection of the ambient 4-manifold. We also study complex curves in complex 4-manifolds, announcing preliminary results related to ongoing work with Peter Lambert-Cole. In particular, we announce the following result, which shows that complex curves in $\mathbb{CP}^2$ have efficient generalized bridge trisections with respect to the genus one trisection of $\mathbb{CP}^2$.

Significance

A common theme in low-dimensional topology is to split a complicated space into simple pieces and to study how these pieces can be glued back together to recover the total space. For example, a bridge splitting of a knotted loop in standard 3D space R_3 cuts the loop into two collections of unknotted arcs. In dimension four, the interesting knotted objects are surfaces, and in previous work, the authors merged ideas from bridge splitting and trisection theories to define bridge trisections, novel decompositions of knotted surfaces in standard four-dimensional space R_4. In this paper, we define generalized bridge trisections for knotted surfaces in more complicated four-dimensional spaces, offering a different approach to knotted surface theory.

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Theorem 4. *Let $\mathcal{C}_d$ be the complex curve of degree d in $\mathbb{CP}^2$. Then, the pair $(\mathbb{CP}^2, \mathcal{C}_d)$ admits an efficient generalized bridge trisection of genus one.*

This theorem can be used to prove the existence of *efficient exotic trisections*, which in this setting are defined to be $(g,0)$–trisections of 4-manifolds that are homeomorphic but not diffeomorphic to a standard 4-manifold.

Section 3 contains the proofs of the main theorems and corollaries. In *Section 4*, we turn our attention to the question of uniqueness of generalized bridge trisections. To this end, we offer the following conjecture.

Conjecture 5. *Any two generalized bridge trisections for a pair $(X,\mathcal{K})$ that induce isotopic trisections of X can be made isotopic after a sequence of elementary perturbation and unperturbation moves.*

1. Preliminaries

We will work in the smooth category throughout this paper. All 4-manifolds are assumed to be orientable. Let $\nu(\cdot)$ denote an open regular neighborhood in an ambient manifold that should be clear from context. A *knotted surface* $\mathcal{K}$ in a 4-manifold X is a smoothly embedded, closed surface, possibly disconnected and possibly nonorientable, considered up to smooth isotopy in X. We will often refer to handlebodies in dimensions three and four; except where a further distinction is appropriate, we will use the term *handlebody* to refer to $\natural^g(S^1\times D^2)$ and the term *1-handlebody* to refer to $\natural^k(S^1\times B^3)$; by the *genus* of these objects, we mean g and k, respectively.

A *trisection* $\mathcal{T}$ of a closed 4-manifold X, introduced by Gay and Kirby (2), is a decomposition $X = X_1\cup X_2\cup X_3$, where X_i is a 1-handlebody, $H_{ij} = X_i\cap X_j$ is a handlebody for $i\neq j$, and $\Sigma = X_1\cap X_2\cap X_3$ is a closed surface. A trisection is uniquely determined by its *spine*, $H_{12}\cup H_{23}\cup H_{31}$, and the spine of a trisection can be encoded with a *trisection diagram* (α,β,γ), a collection of three cut systems α,β,γ on the surface Σ yielding the three handlebodies H_{31}, H_{12}, H_{23}, respectively. (A "cut system" in a genus g surface Σ is a collection of g pairwise disjoint curves cutting Σ into a planar surface, and attaching 2-handles to Σ along a cut system yields a handlebody.) Sometimes it will be useful to assign a complexity to a trisection $\mathcal{T}$: If g is the genus of the central surface Σ and k_i is the genus of the 1-handlebody X_i, we call $\mathcal{T}$ a $(g;k_1,k_2,k_3)$–trisection. In the case that $k_1=k_2=k_3$, we call $\mathcal{T}$ a (g,k)–trisection (with $k=k_1$).

A collection of properly embedded arcs $\tau=\{\tau_i\}$ in the handlebody H is *trivial* if there is an isotopy carrying τ into ∂H. Equivalently, there is a collection of pairwise disjoint disks $\Delta=\{\Delta_i\}$, called *bridge disks*, such that $\partial\Delta_i$ is the endpoint union of τ_i and an arc τ_i' in ∂H. The arc τ_i' is called a *shadow* of τ_i. We also call a collection of trivial arcs a *trivial tangle*. Let L be a link in a 3-manifold Y. A *bridge splitting* of (Y,L) is a decomposition $(Y,L)=(H_1,\tau_1)\cup_\Sigma(H_2,\tau_2)$, where H_i is a handlebody containing a trivial tangle τ_i and $\Sigma = H_1\cap H_2$. It is well known that every pair (Y,L) admits a bridge splitting.

Moving to dimension four, a collection $\mathcal{D}$ of properly embedded disks in a 1-handlebody V is *trivial* if the disks $\mathcal{D}$ are simultaneously isotopic into ∂V. Let $\mathcal{K}$ be a knotted surface in a closed 4-manifold X.

Definition 6:

A generalized bridge trisection of the pair $(X,\mathcal{K})$ is a decomposition $(X,\mathcal{K}) = (X_1,\mathcal{D}_1)\cup(X_2,\mathcal{D}_2)\cup(X_3,\mathcal{D}_3)$, where $X=X_1\cup X_2\cup X_3$ is a trisection, $\mathcal{D}_i$ is a collection of trivial disks in X_i, and for $i\neq j$, the arcs $\tau_{ij}=\mathcal{D}_i\cap\mathcal{D}_j$ form a trivial tangle in H_{ij}.

In ref. 1, the authors proved that every knotted surface in S^4 admits a generalized bridge trisection in which the underlying trisection of S^4 is the standard genus zero trisection. We will refer to such a decomposition simply as a *bridge trisection*. The present article extends this theorem to a given trisection of an arbitrary 4-manifold.

Definition 7:

If $\mathcal{T}$ is a trisection of X given by $X=X_1\cup X_2\cup X_3$, and $\mathcal{K}$ *is* a knotted surface in X such that $(X,\mathcal{K}) = (X_1,\mathcal{K}\cap X_1)\cup(X_2,\mathcal{K}\cap X_2)\cup(X_3,\mathcal{K}\cap X_3)$ is a generalized bridge trisection, we say that $\mathcal{K}$ is in bridge position with respect to $\mathcal{T}$.

The union $(H_{12},\tau_{12})\cup(H_{23},\tau_{23})\cup(H_{31},\tau_{31})$ is called the *spine* of a generalized bridge trisection. As is the case with trisections, bridge trisections are uniquely determined by their spines. Fortunately, the same is true for generalized bridge trisections. To prove this fact, we need the following lemma.

Lemma 8. *Let V be a 1-handlebody, and let L be an unlink contained in ∂V. Up to an isotopy fixing L, the unlink L bounds a unique collection of trivial disks in V.*

Proof: It is well-known that the statement is true when V is a 4-ball (3). Suppose that $\mathcal{D}$ and $\mathcal{D}'$ are two collections of trivial disks properly embedded in V such that $\partial\mathcal{D}=\partial\mathcal{D}'=L$. Let $\mathcal{B}$ denote a collection of properly embedded 3-balls in V cutting V into a 4-ball, and let $\mathcal{S}=\partial\mathcal{B}$. Since L is an unlink, we may isotope $\mathcal{S}$ in ∂V (along with $\mathcal{B}$ in V) so that $L\cap\mathcal{S}=\emptyset$. Since $\mathcal{D}$ is a collection of boundary parallel disks in V, there exists a set of disks $\mathcal{D}_*\subset\partial V$ isotopic to $\mathcal{D}$ via an isotopy fixing L. Choose $\mathcal{D}_*$ so that the number of components of $\mathcal{D}_*\cap\mathcal{S}$ is minimal among all such sets of disks.

We claim that $\mathcal{D}_*\cap\mathcal{S}=\emptyset$. First, we observe that every embedded 2-sphere $S\subset\partial V$ bounds a properly embedded 3-ball in V: If S does not bound a 3-ball in ∂V, then either S is an essential separating sphere, splitting ∂V into two components, each of which is a connected sum of copies of $S^1\times S^2$, or S is an essential nonseparating sphere, and there is an $S^1\times S^2$ summand of ∂V in which S is isotopic to $\{\mathrm{pt}\}\times S^2$. In either case, we can cap off ∂V with 4-dimensional 3-handles and a 4-handle to obtain a 1-handlebody in which S bounds a 3-ball. However, this capping-off process is unique (4), and thus S bounds a 3-ball in V as well.

To prove the claim, suppose by way of contradiction that $\mathcal{D}_*\cap\mathcal{S}\neq\emptyset$, and choose a curve c of $\mathcal{D}_*\cap\mathcal{S}$ that is innermost in a sphere component of $\mathcal{S}$, so c bounds a disk E in this component such that $\mathrm{int}(E)\cap\mathcal{D}_*=\emptyset$. Note that c also bounds a subdisk D of a component of $\mathcal{D}_*$, so $S=E\cup D$ is a 2-sphere embedded in ∂V. By the above argument, S bounds a 3-ball B that is properly embedded in V; thus, $\mathrm{int}(B)\cap\mathcal{D}_*=\emptyset$. It follows that there is an isotopy of $\mathcal{D}_*$ through B in V that pushes D onto E. If $\mathcal{D}'_*$ is the set of disks obtained from $\mathcal{D}_*$ by removing D and gluing on a copy of E (pushed slightly off of $\mathcal{S}$), then $\mathcal{D}'_*$ is isotopic to $\mathcal{D}_*$, with $|{\mathcal{D}'}_*\cap\mathcal{S}|<|\mathcal{D}_*\cap\mathcal{S}|$, a contradiction.

It follows that $\mathcal{D}_*\cap\mathcal{S}=\emptyset$, and we conclude that after isotopy $\mathcal{D}$ is contained in the 4-ball $W=V\setminus\nu(\mathcal{B})$. Similarly, we can assume that after isotopy $\mathcal{D}'$ is contained in W. It now follows from ref. 3 that $\mathcal{D}$ and $\mathcal{D}'$ are isotopic, as desired. □

Corollary 9. *A generalized bridge trisection is uniquely determined by its spine.*

We also observe that we can compute the Euler characteristic of a surface $\mathcal{K}$ from the parameters of a generalized bridge trisection. If $\mathcal{K}\subset X$ is in bridge position with respect to a trisection $\mathcal{T}$ of X, we will set the convention that $c_i=|\mathcal{K}\cap X_i|$ and $b=|\mathcal{K}\cap H_{ij}|=|\mathcal{K}\cap\Sigma|/2$. The next lemma follows from a standard argument in ref. 1.

Lemma 10. *Suppose that $\mathcal{K}\subset X$ is in bridge position with respect to a trisection $\mathcal{T}$. Then*

$$\chi(\mathcal{K}) = c_1+c_2+c_3-b.$$

As with trisections, we may wish to assign a complexity to generalized bridge trisections. The most specific designation has eight parameters. If $\mathcal{T}$ is a generalized bridge trisection, and the underlying trisection has complexity $(g;k_1,k_2,k_3)$, we say that the complexity of the generalized bridge trisection is $(g;k_1,k_2,k_3;b;c_1,c_2,c_3)$. In the case that $k=k_1=k_2=k_3$ and

$c = c_1 = c_2 = c_3$, we say that $\mathcal{T}$ is *balanced* and denote its complexity by (g, k, b, c). Even more generally, a (g, b)-generalized bridge trisection refers to a generalized bridge trisection with a genus g central surface that meets $\mathcal{K}$ in $2b$ points. In *Section 2*, we classify all $(1,1)$-generalized bridge trisections. If the underlying trisection is the genus zero trisection of S^4, as in ref. 1, we call $\mathcal{T}$ a $(b; c_1, c_2, c_3)$-bridge trisection or a (b, c)-bridge trisection in the balanced case.

2. Examples

Before including proofs in the next section, we present several examples of generalized bridge trisections and shadow diagrams of knotted surfaces in 4-manifolds.

A. Shadow Diagrams Just as a trisection diagram determines the spine of a trisection, a type of diagram called a *triplane diagram* determines the spine of a bridge trisection, as shown in ref. 1. Unfortunately, triplane diagrams do not naturally extend from bridge trisections to generalized bridge trisections. Instead, we use a structure called a "*shadow diagram*." Let τ be a trivial tangle in a handlebody H. A *curve-and-arc system* (α, a) determining (H, τ) is a collection of pairwise disjoint simple closed curves α and arcs a in $\Sigma = \partial H$ such that α determines H and a is a collection of shadow arcs for τ. Note that curves in α and arcs in a can be chosen to be disjoint by standard cut-and-paste arguments using compressing disks for H and bridge disks for τ. A *shadow diagram* for a generalized bridge trisection $\mathcal{T}$ is a triple $((\alpha, a), (\beta, b), (\gamma, c))$ of curve-and-arc systems determining the spine $(H_{31}, \tau_{31}) \cup (H_{12}, \tau_{12}) \cup (H_{23}, \tau_{23})$ of $\mathcal{T}$. Since every trivial tangle in a handlebody can be defined by a curve-and-arc system, it is clear that *Corollary 3* follows immediately from *Theorem 1*.

B. The 1-Bridge Trisections One family which deserves special consideration is the collection of 1-bridge trisections, i.e., $(g, 1)$-generalized bridge trisections. If $\mathcal{K}$ has a such a splitting, then it intersects each sector X_i of the underlying trisection in a single disk and each handlebody in a single arc. In this case, we deduce from *Lemma 10* that $\mathcal{K}$ is a 2-knot, and the generalized bridge trisection is efficient. Shadow diagrams for generalized bridge trisections of this type are particularly simple: A *doubly-pointed trisection diagram* is a shadow diagram in which each curve-and-arc system contains exactly one arc. In this case, drawing the arc in the diagram is redundant, since there is a unique way (up to admissible slides of the arcs and curves) to connect the two points in the complement of any one of the sets of curves.

In Fig. 1, we depict several doubly pointed diagrams for low-complexity examples. First, we give diagrams for the two simplest complex curves in $\mathbb{CP}^2$, namely, the line $\mathbb{CP}^1$ and the quadric $\mathcal{C}_2$. Next, we give diagrams for an S^2–fiber in $S^2 \times S^2$ and the sphere $\mathcal{C}_{(1,1)}$ in $S^2 \widetilde{\times} S^2$ representing $(1,1) \in \mathbb{Z} \oplus \mathbb{Z} \cong H_2(S^2 \widetilde{\times} S^2)$. (See ref. 5 for formal definitions.) We postpone the justification for these diagrams until *Section 3*, in which we develop the machinery to make such justification possible.

By results in refs. 1, 6, and 7, any surface with a $(0, b)$–generalized bridge trisection (i.e., a b-bridge trisection) for $b < 4$ is unknotted in S^4. In *Section 3*, we will prove the following classification result.

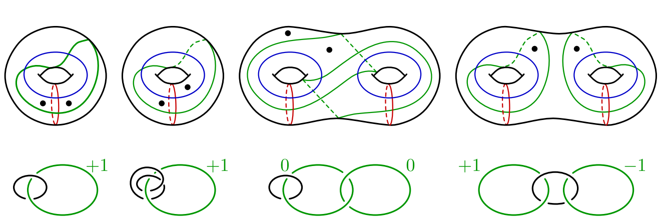

Fig. 1. Some doubly-pointed trisection diagrams. From left to right: $(\mathbb{CP}^2, \mathbb{CP}^1)$, $(\mathbb{CP}^2, \mathcal{C}_2)$, $(S^2 \times S^2, S^2 \times \{*\})$, and $(S^2 \widetilde{\times} S^2, \mathcal{C}_{(1,1)})$.

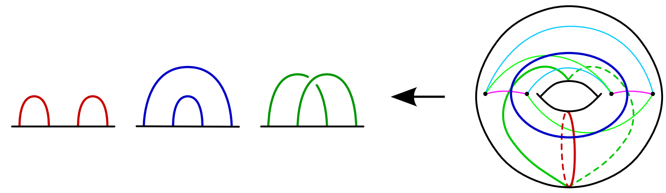

Fig. 2. The branched double covering projection relating the standard cross-cap $(S^4, \mathbb{P}_+)$ and its cover $(\mathbb{CP}^2, \mathbb{RP}^2)$.

Proposition 11. *There are exactly two nontrivial* $(1,1)$*–knots (up to change of orientation and mirroring):* $(\mathbb{CP}^2, \mathbb{CP}^1)$ *and* $(\mathbb{CP}^2, \mathcal{C}_2)$.

On the other hand, there are many 2-knots admitting $(3,1)$-generalized bridge trisections: Perform three meridional stabilizations (defined in *Section 3*) on any $(4,2)$-bridge trisection, of which there are infinitely many (1). We offer the following as worthwhile problems.

Problem 12. *Classify 2-knots admitting* $(2,1)$*-generalized bridge trisections and projective planes admitting* $(1,2)$*-generalized bridge trisections.*

With regard to *Problem 12*, Fig. 2 shows a $(2,1)$–shadow diagram for the standard projective (real) plane $(\mathbb{CP}^2, \mathbb{RP}^2)$ that is the lift of the standard cross-cap in S^4 with normal Euler number -2 under the branched double covering. More generally, consider a surface knot (or link) $(X, \mathcal{K})$, and let $X_n(\mathcal{K})$ denote the n-fold cover of X, branched along $\mathcal{K}$. Let $\widetilde{\mathcal{K}}_n$ denote the lift of $\mathcal{K}$ under this covering.

Proposition 13. *If* $(X, \mathcal{K})$ *admits a* $(g; k_1, k_2, k_3; b; c_1, c_2, c_3)$*–generalized bridge trisection, then* $(X_n(\mathcal{K}), \widetilde{\mathcal{K}})$ *admits a* $(g'; k_1', k_2', k_3'; b, c_1, c_2, c_3)$*–generalized bridge trisection, where* $g' = ng + (n-1)(b-1)$ *and* $k_i' = nk_i + (n-1)(c_i - 1)$.

Proof: It is a standard exercise to show that the n-fold cover of a genus g handlebody branched along a collection of b trivial arcs is a handlebody of genus $g' = ng + (n-1)(b-1)$, with the lift of the original b trivial arcs being a collection of b trivial arcs upstairs. From this, the rest of the proposition follows, once we observe that the trivial disk system $(\natural^k(S^1 \times B^3), \mathcal{D})$ is simply the trivial tangle product $(\natural^k(S^1 \times D^2), \tau) \times I$, and that the branched covering respects this product structure. Thus, each piece of the trisection lifts to a standard piece, so the cover is trisected. □

C. Complex Curves in $\mathbb{CP}^2$. In this subsection, we summarize results that have been obtained in collaboration with Peter Lambert-Cole regarding generalized bridge trisections of complex curves in complex 4-manifolds of low trisection genus (e.g., $\mathbb{CP}^2$, $S^2 \times S^2$, and $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$). Let $\mathcal{C}_d$ denote the complex curve of degree d in $\mathbb{CP}^2$. Note that $\mathcal{C}_d$ is a closed surface of genus $(d-1)(d-2)/2$.

Theorem 4. *The pair* $(\mathbb{CP}^2, \mathcal{C}_d)$ *admits a* $(1, 1; (d-1)(d-2) + 1, 1)$*-generalized bridge trisection.*

In other words, complex curves in $\mathbb{CP}^2$ admit efficient generalized bridge trisections with respect to the genus one trisection of $\mathbb{CP}^2$; each such curve can be decomposed as the union of three disks ($c = 1$). See Fig. 3. Let $X_{n,d}$ denote the 4-manifold obtained as the n-fold cover of $\mathbb{CP}^2$, branched along $\mathcal{C}_d$, which exists whenever n divides d. When $n = d$, we have that $X_{d,d}$ is the degree d hypersurface in $\mathbb{CP}^3$. The next corollary follows from *Theorem 4* and *Proposition 13*.

Corollary 14. $X_{n,d}$ *admits an efficient* $(g, 0)$*-trisection where* $g = n + (n-1)(d-1)(d-2)$.

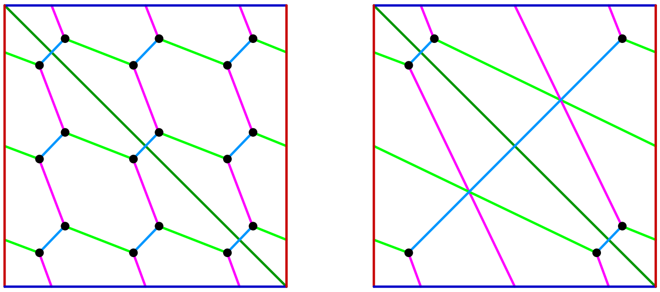

Fig. 3. Two shadow diagrams for $\mathcal{C}_3$ in $\mathbb{CP}^2$. The diagram on the left is due to Peter Lambert-Cole, and the diagram on the right is efficient.

Note that $Z_{p,q,r} = p\mathbb{CP}^2 \# q\overline{\mathbb{CP}^2} \# rS^2 \times S^2$ admits as $(g, 0)$-trisection where $g = p + q + 2r$. It had been speculated that an extension of the main theorems of refs. 6 and 7 would show that every manifold admitting a $(g, 0)$-trisection is diffeomorphic to $Z_{p,q,r}$; however, *Corollary 14* gives many interesting counterexamples to this suspicion.

For example, if d is odd and at least five, then $X_{d,d}$ is homeomorphic, but not diffeomorphic, to $Z_{p,q,0}$ for certain $p, q \geq 0$ (8). Thus, we see that there are pairs of exotic manifolds that are not distinguished by their trisection invariants. We note that Baykur and Saeki have previously given examples of inefficient exotic trisections (9).

3. Proofs

In the first part of this section, we will prove a sequence of lemmas which, taken together, imply *Theorem 1*. In the second part, we prove *Proposition 11*, classifying $(1, 1)$-generalized bridge trisections. In the third part, we introduce the notion of meridional stabilization and prove *Theorem 2*.

A. The Existence of Generalized Bridge Splittings. Here, we discuss the interaction between handle decompositions and trisections of closed 4-manifolds. We will not rigorously define handle decompositions but direct the interested reader to ref. 5.

Suppose $\mathcal{H}$ is a handle decomposition of a 4-manifold X with a single 0-handle and a single 4-handle. Corresponding to $\mathcal{H}$, there is a Morse function $h : X \to \mathbb{R}$, equipped with a gradient-like vector field that induces the handle decomposition $\mathcal{H}$. We will suppose that each Morse function is equipped with a gradient-like vector field (which we will neglect to mention henceforth). After an isotopy, we may assume that every critical point of index i occurs in the level $h^{-1}(i)$. Such a Morse function is called *self-indexing*. For any subset $S \subset \mathbb{R}$, let Y_S denote $X \cap h^{-1}(S)$. Let Z be a compact submanifold of $Y_{\{t\}}$ for some t, and let $[r, s]$ be an interval containing t. We will let $Z_{[r,s]}$ denote the subset of X obtained by pushing Z along the flow of h during time $[r, s]$. (For example, if $t \in (r, s)$, this will involve pushing Z up *and* down the flow.) In particular, if this set does not contain a critical point of h, then $Z_{[r,s]}$ is diffeomorphic to $Z \times [r, s]$. We let $Z_{\{t'\}}$ denote $Z_{[r,s]} \cap h^{-1}(t')$.

Now, let $\mathcal{H}$ be a handle decomposition of X with n_i i-handles for $i = 1, 2, 3$, and let T be the attaching link for the 2-handles, so that T is an n_2-component framed link contained in $\#^{n_1}(S^1 \times S^2)$ with Dehn surgery yielding $\#^{n_3}(S^1 \times S^2)$. In addition, let $h : X \to \mathbb{R}$ be a self-indexing Morse function inducing $\mathcal{H}$. We suppose without loss of generality that T is contained in $Y_{\{3/2\}} = \#^{n_1}(S^1 \times S^2)$, and we let Σ be a genus g Heegaard surface cutting $Y_{\{3/2\}}$ into handlebodies H and H', where a core of the handlebody H contains T.

The following lemma is essentially lemma 14 of ref. 2, in which it is proved in slightly different terms.

Lemma 15. *Let X be a 4-manifold with self-indexing Morse function h, and, using the notation above, consider the sets*

$$X_1 = Y_{[0,3/2]} \cup H'_{[3/2,2]}, \; X_2 = H_{[3/2,5/2]}, \; X_3 = H'_{[2,5/2]} \cup Y_{[5/2,4]}.$$

The decomposition $X = X_1 \cup X_2 \cup X_3$ is a $(g; n_1, g - n_2, n_3)$-trisection with central surface $\Sigma_{\{2\}}$.

Given a self-indexing Morse function h and surface Σ as above, we will let $\mathcal{T}(h, \Sigma)$ denote the trisection described by *Lemma 15*. The next lemma also comes from ref. 2; it is a restatement of lemma 13 from that work.

Lemma 16. *Given a trisection $\mathcal{T}$ of X, there is a self-indexing Morse function h and surface $\Sigma \subset Y_{\{3/2\}}$ such that $\mathcal{T} = \mathcal{T}(h, \Sigma)$.*

Now we turn our focus to knotted surfaces in 4-manifolds. Suppose that $\mathcal{K}$ is a knotted surface in X. A *Morse function of the pair* $h : (X, \mathcal{K}) \to \mathbb{R}$ is a Morse function $h : X \to \mathbb{R}$ with the property that the restriction $h_{\mathcal{K}}$ is also Morse. Note that for any $\mathcal{K} \subset X$, a Morse function $h : X \to \mathbb{R}$ becomes a Morse function of the pair $(X, \mathcal{K})$ after a slight perturbation of $\mathcal{K}$ in X. Expanding upon the previous notation, for $S \subset \mathbb{R}$, we let L_S denote $\mathcal{K} \cap h^{-1}(S)$. Let J be a compact submanifold of $L_{\{t\}}$ for some t, and let $[r, s]$ be an interval containing t. We will let $J_{[r,s]}$ denote the subset of $\mathcal{K}$ obtained by pushing J along the flow of $h_{\mathcal{K}}$ during time $[s, r]$. As above, if this set does not contain a critical point of $h_{\mathcal{K}}$, then $J_{[r,s]}$ is diffeomorphic to $J \times [r, s]$.

Saddle points of $h_{\mathcal{K}}$ can be described as cobordisms between links obtained by resolving bands: Given a link L in a 3-manifold Y, a *band* is an embedded rectangle $R = I \times I$ such that $R \cap L = \partial I \times I$. We *resolve the band* R to get a new link by removing the arcs $\partial I \times I$ from L and replacing them with the arcs $I \times \partial I$. Note that every band R can be represented by a framed arc $\eta = I \times \{1/2\}$, so η meets L only in its endpoints. Let h be a Morse function of the pair $(X, \mathcal{K})$, suppose that all critical points of h and $h_{\mathcal{K}}$ occur at distinct levels, and let $x \in \mathcal{K}$ be a saddle point contained in the level $h^{-1}(t)$. Then, there is a framed arc η with endpoints in the link $L_{\{t-\epsilon\}}$ with the property that the link $L_{\{t+\epsilon\}}$ is obtained from $L_{\{t-\epsilon\}}$ by resolving the band corresponding to η. We will use this fact in the proof of the next lemma, which is related to the notion of a normal form for a 2-knot in S^4 (10, 11).

Lemma 17. *Suppose X is a 4-manifold equipped with a handle decomposition $\mathcal{H}$, and $\mathcal{K}$ is a surface embedded in X. After an isotopy of $\mathcal{K}$, there exists a Morse function of the pair $(X, \mathcal{K})$ such that h is a self-indexing Morse function inducing the handle decomposition $\mathcal{H}$, and index i critical points of $h_{\mathcal{K}}$ occur in the level $Y_{\{i+1\}}$.*

Proof: Let Γ_1 be an embedded wedge of circles containing the cores of the 1-handles, so that $\nu(\Gamma_1)$ is the union of the 0-handle and the 1-handles of $\mathcal{H}$. Similarly, let Γ_3 be an embedded wedge of circles such that $\nu(\Gamma_3)$ is the union of the 3-handles and 4-handle. After isotopy $\mathcal{K}$ meets Γ_1 and Γ_3 transversely; hence $\mathcal{K} \cap \Gamma_1 = \mathcal{K} \cap \Gamma_3 = \emptyset$, and thus we can initially choose a self-indexing Morse function $h : X \to \mathbb{R}$ so that $\nu(\Gamma_1) = Y_{[0,1+\epsilon)}$, $\nu(\Gamma_3) = Y_{(3-\epsilon,4]}$, and $\mathcal{K} \subset Y_{(1+\epsilon,3-\epsilon)}$. For each minimum point of $h_{\mathcal{K}}$, choose a descending arc avoiding $\mathcal{K}$ and the critical points of h and drag the minimum downward within a neighborhood of this arc until it is contained in $Y_{\{1\}}$. Similarly, there is an isotopy of $\mathcal{K}$ after which all maxima are contained in $Y_{\{3\}}$.

It only remains to show that after isotopy, all saddles of $h_{\mathcal{K}}$ are contained in $Y_{\{2\}}$. Let T be the attaching link for the 2-handles of $\mathcal{H}$, considered as a link in $Y_{\{2\}}$. For each saddle point x_i in level $t_i < 2$, let η_i be the framed arc with endpoints in $L_{\{t_i\}}$, where $1 \leq i \leq n$, so that $L_{t_i+\epsilon}$ is obtained from $L_{t_i-\epsilon}$ by resolving the band induced by η_i. Certainly, η_1 is disjoint from $L_{t_1-\epsilon}$ except at its endpoints. A priori, η_2 may intersect the band induced by η_1, but after a small isotopy, we may assume that η_2 avoids η_1 and thus we can push η_2 into $Y_{\{t_1\}}$. Continuing this process, we may push all arcs η_i into $Y_{\{t_1\}}$, and generically, the graph $L_{t_1-\epsilon} \cup \{\eta_i\}$ is disjoint from T, so the entire apparatus can be pushed into $Y_{\{2\}}$. A parallel argument shows that the framed arcs coming from saddles occurring between $t = 2$ and $t = 3$ can be pushed down into $Y_{\{2\}}$, as desired. ☐

We call a Morse function $h:(X,\mathcal{K})\to\mathbb{R}$ that satisfies the conditions in *Lemma 17* a *self-indexing Morse function of the pair* $(X,\mathcal{K})$. Given such a function, we can push the framed arcs $\{\eta_i\}$ corresponding to the saddles of $h_\mathcal{K}$ into the level $Y_{\{3/2\}}$, where the endpoints of $\{\eta_i\}$ are contained in $L_{\{3/2\}}$ and resolving $L_{\{3/2\}}$ along the bands given by $\{\eta_i\}$ yields the link $L_{\{5/2\}}$. A *banded link diagram* for $\mathcal{K}$ consists of the union of $L_{\{3/2\}}$ with the bands given by $\{\eta_i\}$, contained in $Y_{\{3/2\}}$, along with the framed attaching link for the 2-handles in X, denoted by $T\subset Y_{\{3/2\}}$. As such, a banded link diagram completely determines the knotted surface $\mathcal{K}\subset X$. Let $\mathcal{H}$ be the handle decomposition of X determined by h. As above, let Σ be a Heegaard surface cutting $Y_{\{3/2\}}$ into handlebodies H and H', where a core of H contains T.

Let $\Gamma=L_{\{3/2\}}\cup\{\eta_i\}$ in $Y_{\{3/2\}}$. We will show that Γ may be isotoped to be in a relatively nice position with respect to the surface Σ, from which it will follow that there is an isotopy of $\mathcal{K}$ to be in a relatively nice position with respect to the trisection $\mathcal{T}(h,\Sigma)$. An arc $\eta\subset\partial H$ is *dual* to a trivial arc $\tau_i\subset H$ if there is a shadow τ_i' for τ_i that meets η in one endpoint. Finally, a collection of pairwise disjoint arcs $\{\eta_i\}\subset\partial H$ is said to be *dual* to a trivial tangle $\{\tau_i\}$ if there is a collection of shadows $\{\tau'_i\}$ that meet $\{\eta_i\}$ only in their endpoints and such that each component of $\{\eta_i\}\cup\{\tau'_i\}$ is simply connected (in other words, this collection contains only arcs, not loops).

We say that Γ is in *bridge position* with respect to Σ if the link $L_{\{3/2\}}$ is in bridge positions and in addition, $\{\eta_i\}\subset\Sigma$ with framing given by the surface framing and the arcs $\{\eta_i\}$ are dual to the trivial arcs $L_{\{3/2\}}\cap H$. To clarify, the arc $\eta_i\subset\Sigma$ has framing given by the surface framing exactly when the band induced by η_i meets Σ in the single arc η_i. Next, we show that such structures exist, after which we describe how they induce generalized bridge trisections of $(X,\mathcal{K})$.

Lemma 18. *Given a knotted surface $\mathcal{K}\subset X$ and a self-indexing Morse function h of the pair $(X,\mathcal{K})$, let Σ, H, H', T, and Γ be as defined above. There exists an isotopy of Γ in $Y_{\{3/2\}}$ after which Γ is in bridge position with respect to Σ.*

Proof: This decomposition is similar to the notion of a *banded bridge splitting* from ref. 1, where the detailed arguments in theorem 1.3 do not make use of the fact that Σ is sphere and thus transfer directly to this setting. We give a brief outline of the proof, but refer the reader to ref. 1 for further details.

Consider cores $C\subset H$ and $C'\subset H'$, which may be chosen so that $T\subset C$ and both C and C' are disjoint from Γ. Note that $Y_{\{3/2\}}\setminus(C\cup C')$ is diffeomorphic to $\Sigma\times(-1,1)$ and thus there is a natural projection from $Y_{\{3/2\}}\setminus(C\cup C')$ onto $\Sigma=\Sigma\times\{0\}$. By equipping this projection with crossing information, we may view it as an isotopy of Γ within $Y_{\{3/2\}}\setminus(C\cup C')$. First, if the arcs $\{\eta_i\}$ project to arcs that cross themselves or each other, we may stretch $L_{\{3/2\}}$ and shrink $\{\eta_i\}$ so these crossings are slid to $L_{\{3/2\}}$, after which the projection of the collection $\{\eta_i\}$ is embedded in Σ. (see figure 10 of ref. 1). It may be possible that some surface framing of some arc η_i disagrees with its given framing; in this case, an isotopy of $L_{\{3/2\}}$ allows η_i to be pushed off of and back onto Σ with the desired framing, as in figure 11 of ref. 1. Thus, we may assume condition (2) of the definition of bridge position of Γ is satisfied.

Now, we push the projection $L_{\{3/2\}}$ off of Σ so that $L_{\{3/2\}}$ is in bridge position, fulfilling condition (1) of the definition of bridge position. At this point, it may not be the case that the arcs $\{\eta_i\}$ are dual to $L_{\{3/2\}}\cap H$; however, this requirement may be achieved by perturbing $L_{\{3/2\}}$ near the endpoints of the arcs $\{\eta_i\}$ in Σ, as in figure 12 of ref. 1. $\square$

Lemma 19. *Suppose that $\mathcal{K}$ is a knotted surface in X, with self-indexing Morse function h of the pair $(X,\mathcal{K})$ and Σ, H, H', T, and Γ as defined above. Suppose further that Γ is in bridge position with respect to Σ, push the arcs $\{\eta_i\}$ slightly into the interior of H. Let X_1, X_2, and X_3 be defined as in Lemma 15, and define $\mathcal{D}_i=\mathcal{K}\cap X_i$. Then*

$$(X,\mathcal{K})=(X_1,\mathcal{D}_1)\cup(X_2,\mathcal{D}_2)\cup(X_3,\mathcal{D}_3)$$

is a generalized bridge trisection of $(X,\mathcal{K})$.

Proof: By *Lemma 15*, the underlying decomposition $X=X_1\cup X_2\cup X_3$ is a trisection, and thus we must show that $\mathcal{D}_i$ is a trivial disk system in X_i and $\mathcal{D}_i\cap\mathcal{D}_j$ is a trivial tangle in the handlebody $X_i\cap X_j$.

Let $\tau=L_{\{3/2\}}\cap H$ and $\tau'=L_{\{3/2\}}\cap H'$, so that each of τ and τ' is a trivial tangle in H and H', respectively. We note that by construction, $\mathcal{D}_1=L_{[1,3/2]}\cup\tau'_{[3/2,2]}$, $\mathcal{D}_2=\tau_{[3/2,5/2]}$, and $\mathcal{D}_3=L_{[5/2,3]}\cup\tau'_{[2,5/2]}$. Thus, there is a Morse function of the pair $(X_1,\mathcal{D}_1)$ that contains only minimal, so that $\mathcal{D}_1$ is a collection of trivial disks in X_1. Similarly, $(X_3,\mathcal{D}_3)$ contains only maxima, so that $\mathcal{D}_3\subset X_3$ is a collection of trivial disks as well. We also note that $\mathcal{D}_1\cap\mathcal{D}_3=\tau'_{\{2\}}$, a collection of trivial arcs in $X_1\cap X_3$, and $\mathcal{D}_1\cap\mathcal{D}_2=\tau_{\{3/2\}}\cup(\partial\tau)_{[3/2,2]}$, a collection of trivial arcs in $X_1\cap X_2$.

It only remains to show that $\mathcal{D}_2$ is a collection of trivial disks in X_2, and $\mathcal{D}_2\cap\mathcal{D}_3$ is a collection of trivial arcs in $X_2\cap X_3$. However, this follows immediately from lemma 3.1 of ref. 1; although the proof of lemma 3.1 is carried out in the context of the standard trisection of S^4, it can be applied verbatim here. $\square$

Proof of *Theorem 1*: By *Lemma 16*, there exists a self-indexing Morse function $h:X\to\mathbb{R}$ and Heegaard surface $\Sigma\subset Y_{\{3/2\}}$ such that $\mathcal{T}=\mathcal{T}(h,\Sigma)$. Applying *Lemma 17*, we have that there is an isotopy of $\mathcal{K}$ after which $h:(X,\mathcal{K})\to\mathbb{R}$ is a self-indexing Morse function of the pair. Moreover, by *Lemma 18*, there is a further isotopy of $\mathcal{K}$ after which the graph Γ induced by the saddle points of $h_\mathcal{K}$ is in bridge position with respect to Σ. Finally, the decomposition defined in *Lemma 19* is a generalized bridge trisection of $(X,\mathcal{K})$, completing the proof. $\square$

We note that as in lemma 3.3 and remark 3.4 from ref. 1, this process is reversible; in other words, every bridge trisection of $(X,\mathcal{K})$ can be used to extract a handle decomposition of $\mathcal{K}$ within X. The proof of lemma 3.3 applies directly in this case, and when we combine it with *Lemma 16* above, we have the following:

Proposition 20. *If $\mathcal{T}$ is a $(g;k_1,k_2,k_3;b;c_1,c_2,c_3)$-generalized bridge trisection of $(X,\mathcal{K})$, then there is a Morse function h of the pair $(X,\mathcal{K})$ such that h has k_1 index one critical points, $g-k_2$ index two critical points, and k_3 index three critical points; and $h_\mathcal{K}$ has c_1 minima, $b-c_2$ saddles, and c_3 maxima.*

We can now justify the diagrams in Fig. 1. By *Proposition 20*, a 1-bridge trisection will give rise to a banded link diagram without bands corresponding to a Morse function h of the pair $(X,\mathcal{K})$ such that $h_\mathcal{K}$ has a single minimum and maximum. From the shadow diagrams in Fig. 1, we extract banded link diagrams, shown directly beneath each shadow diagram. In each case, the black curve in Fig. 1 bounds a disk (the minimum of $h_\mathcal{K}$) in the 4-dimensional 0-handle and a disk (the maximum of $h_\mathcal{K}$) in the union of the 2-handles with the 4-handle. For example, in the first and third figure, we see that the 2-knot is the union of a trivial disk in the 0-handle, together with a cocore of a 2-handle. The second figure is a well-known description of the quadric. See subsection above. The fourth figure can be obtained by connected summing the first figure with its mirror.

B. Classification of (1, 1)-Generalized Bridge Trisections. In this subsection, we prove *Proposition 11*, classifying $(1,1)$-generalized bridge trisections.

Proof of *Proposition 11*: Suppose that $(X,\mathcal{K})$ admits a $(1,1)$-generalized bridge trisection $\mathcal{T}$. Then $c_1=c_2=c_3=1$, $\chi(\mathcal{K})=2$, and $\mathcal{K}$ is a 2-sphere. In addition, by *Proposition 20*, there is a self-indexing Morse function h on $(X,\mathcal{K})$ so that $h_\mathcal{K}$ has one

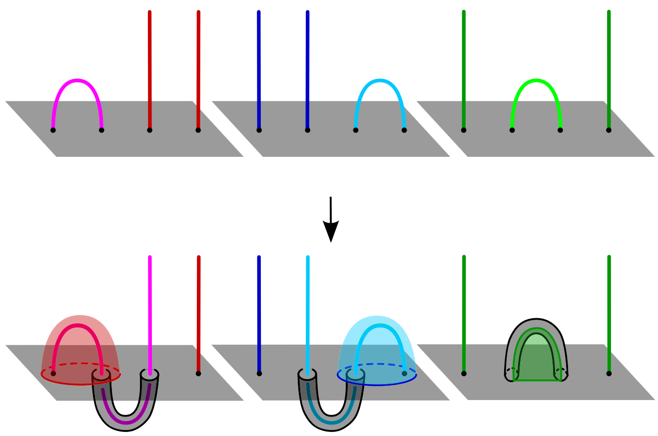

Fig. 4. A sample meridional 1-stabilization along τ' (light green, top right). Meridional stabilization increases the genus of the central surface by one, and a new compressing curve is shown for each handlebody in the bottom half of the figure.

minimum, one maximum, and no saddles. If h has no index two critical points, then $(X, \mathcal{K})$ is the double of a trivial disk in a 4-ball or 1-handlebody; thus, $\mathcal{K}$ is unknotted. If any one $k_i = 1$, then after permuting indices, we may assume that the induced h has no index two critical points. Thus, the only remaining case is $k_1 = k_2 = k_3 = 0$, and so $X = \mathbb{CP}^2$ or $\overline{\mathbb{CP}}^2$.

We will only consider the case $X = \mathbb{CP}^2$; parallel arguments apply by reversing orientations. Let h be a self-indexing Morse function for $\mathcal{T}$, so that $Y_{\{3/2\}}$ is diffeomorphic to S^3, $L_{\{3/2\}}$ is an unknot we call C, and T is a $(+1)$-framed unknot disjoint from C in $Y_{\{3/2\}}$. In addition, attaching a 2-handle to T yields another copy of S^3, in which C remains unknotted. In other words, C is an unknot in S^3 that is still unknotted after $(+1)$-Dehn surgery on T. There are three obvious links $C \cup T$ that satisfy these requirements: a two-component unlink, a Hopf link, and the torus link $T(2,2)$. The first of these three corresponds to the unknotted 2-sphere. The next two correspond to $\mathbb{CP}^1$ and $\mathcal{C}_2$, respectively. We claim no other links $C \cup T$ of this type exist.

Consider T as a (nontrivial) knot in the solid torus $S^3 \setminus \nu(C)$. Since C remains unknotted after $(+1)$-surgery on T, it follows that T is a knot in a solid torus with a solid torus surgery. Let ω denote the linking number of C and T, so that ω is also the winding number of T in $S^3 \setminus \nu(C)$. By ref. 12, one of the following holds: $\omega = 1$ and $T \cup C$ is the Hopf link, $\omega = 2$ and $T \cup C$ is the torus link $T(2,2)$, or $\omega \geq 3$ and the slope of the surgery on T is at least four. The third case contradicts the assumption that the surgery slope is one, completing the proof. $\square$

Remark 21: A similar argument invoking (13) can be used to show that the only nontrivial 2-knots in S^4, $\mathbb{CP}^2$, $\overline{\mathbb{CP}}^2$, or $S^1 \times S^3$ admitting a $(2,1)$-generalized bridge trisection are $\mathbb{CP}^1$ and $\mathcal{C}_2$, as above.

C. Meridional Stabilization. Consider a link L in a 3-manifold Y, equipped with a (g,b)-bridge splitting $(Y, L) = (H_1, \tau_1) \cup (H_2, \tau_2)$, where $b \geq 2$. Fix a trivial arc $\tau' \in \tau_2$, and let $H_1' = H_1 \cup \overline{\nu(\tau')}$ and $H_2' = H_2 \setminus \nu(\tau')$. In addition, let $\tau_i' = L \cap H_i'$, so that $\tau_1' = \tau_1 \cup \tau'$ and $\tau_2' = \tau_2 \setminus \tau'$. Then the decomposition $(Y, L) = (H_1', \tau_1') \cup (H_2', \tau_2')$ is a $(g+1, b-1)$-bridge splitting which is called a *meridional stabilization* of the given (g,b)-splitting. (See ref. 14, for example.)

In this subsection, we will extend meridional stabilization to a similar construction involving generalized bridge trisections to prove *Theorem 2*. Let $\mathcal{T}$ be a generalized bridge trisection for a connected knotted surface $\mathcal{K} \subset X$ with complexity $(g; k_1, k_2, k_3; b; c_1, c_2, c_3)$, and assume that $c_1 \geq 2$. Since $\mathcal{K}$ is connected, there exists an arc $\tau' \in \tau_{23}$ with the property that the two endpoints of τ' lie in different components of $\mathcal{D}_1$. Define $(X_1', \mathcal{D}_1') = \left(X_1 \cup \overline{\nu(\tau')}, \mathcal{D}_1 \cup \left(\overline{\nu(\tau')} \cap \mathcal{K}\right)\right)$ and $(X_j', \mathcal{D}_j') = (X_j \setminus \nu(\tau'), \mathcal{D}_j \setminus \nu(\tau'))$ for $j = 2, 3$, and let $\mathcal{T}'$ be the decomposition

$$(X, \mathcal{K}) = (X_1', \mathcal{D}_1') \cup (X_2', \mathcal{D}_2') \cup (X_3', \mathcal{D}_3').$$

We say that the decomposition $\mathcal{T}'$ is obtained from $\mathcal{T}$ via *meridional 1-stabilization along* τ'. We define meridional i*-stabilization* similarly for $i = 2$ or 3. Observe that the assumption that $\mathcal{K}$ is connected is slightly stronger than necessary; the existence of the arc $\tau' \in \tau_{jk}$ connecting two disks in $\mathcal{D}_i$ is necessary and sufficient. Notably, $\mathcal{T}'$ is a generalized bridge splitting for $(X, \mathcal{K})$, which we verify in the next lemma. Fig. 4 shows the local picture of a meridional 1–stabilization.

Lemma 22. *The decomposition* $\mathcal{T}'$ *of* $(X, \mathcal{K})$ *is a generalized bridge trisection of complexity* $(g+1; k_1+1, k_2, k_3; b-1; c_1-1, c_2, c_3)$.

Proof: Since $\tau' \subset \partial X_j$ for $j = 2$ and 3, we have that $(X_j', \mathcal{D}_j') \cong (X_j, \mathcal{D}_j)$. Let $X' = \overline{\nu(\tau')} \cap (X_2 \cup X_3)$ and $\mathcal{D}' = \overline{\nu(\tau')} \cap (\mathcal{D}_2 \cup \mathcal{D}_3)$. Then X' is a topological 4-ball intersecting X_1 in two 3-balls in ∂X_1; i.e., X' is a 1-handle. It follows that X_1' is obtained from X_1 by the attaching a 1-handle, so $X_1' \cong \natural^{k_1+1}(S^1 \times B^3)$. Similarly, $\mathcal{D}'$ is a band connecting disks D_1 and D_2 in $\mathcal{D}_1$. Since these disks are trivial, we can assume without loss of generality that D_1 and D_2 have been isotoped to lie in ∂X_1, and since $\mathcal{D}'$ is boundary parallel inside X', the disk $D' = D_1 \cup \mathcal{D}' \cup D_2$ is boundary parallel in X_1'. It follows that $\mathcal{D}_1' = \mathcal{D}_1 \setminus (D_1 \cup D_2) \cup D'$ is a trivial $(c_1 - 1)$–disk system.

It remains to verify that the 3D components of the new construction are trivial tangles in handlebodies. Observe that for $\{j,k\} = \{2,3\}$, the decomposition $(\partial X_j, \partial \mathcal{D}_j) = (H_{1j}', \tau_{1j}') \cup (H_{jk}', \tau_{jk}')$ is a 3D meridional stabilization of $(\partial X_j, \partial \mathcal{D}_j) = (H_{1j}, \tau_{1j}) \cup (H_{jk}, \tau_{jk})$. Thus, τ_{ij}' is a trivial $(b-1)$-strand tangle in the genus $g+1$ handlebody H_{ij}', as desired. $\square$

We can now prove *Theorem 2*, which implies *Corollary 3* as an immediate consequence.

Proof of *Theorem 2*: Start with a generalized bridge trisection of $(X, \mathcal{K})$. If there is a spanning arc τ' of the type that is necessary and sufficient for a meridional stabilization, then we perform the stabilization. Thus, we assume there are no such spanning arcs. If $\mathcal{D}_i$ contains c_i disks, then since there are no τ'–type arcs in τ_{jk} for i, j, k distinct, it follows that the c_i disks belong to distinct connected components of $\mathcal{K}$. Thus, $c_i = n$, and $\chi(\mathcal{K}) = c_1 + c_2 + c_3 - b = 3n - b$, so that $b = 3n - \chi(\mathcal{K})$. $\square$

4. Uniqueness of Generalized Bridge Trisections

In general, the types of splittings discussed in this article are not unique up to isotopy, but a guiding principle is that two splittings for a fixed space become isotopic after some number of generic operations, such as the meridional stabilization operation defined above. For example, any two Heegaard splittings for a fixed 3-manifold Y become isotopic after some number of stabilization operations (15, 16), and any two bridge splittings for $K \subset Y$ with a fixed underlying Heegaard splitting become

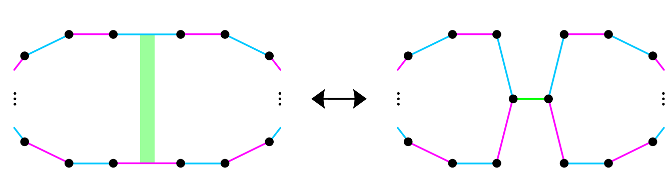

Fig. 5. An illustration (at the level of the shadow diagram) of an elementary 1-perturbation of a generalized bridge splitting.

isotopic after some number of perturbation operations (17, 18). In dimension four, *stabilization* for a trisection $\mathcal{T}$ of a 4-manifold X can be viewed as taking the connected sum of $\mathcal{T}$ and the standard genus three trisection of S^4, and Gay and Kirby proved that any pair of trisections for X become isotopic after some number of trisections (2). The purpose of this section is to define *perturbations* for generalized bridge trisections and lay out steps toward a proof of a corresponding uniqueness theorem in this setting.

Let L be an n-component unlink in $Y=\#^k(S^1\times S^2)$. The *standard* bridge splitting of L is defined to be the connected sum of the standard genus k Heegaard splitting of Y with the standard (classical) n–bridge splitting of L (the connected sum of n copies of the 1-bridge splitting of the unknot). The first ingredient we will need to define perturbation is the following proposition, which uses a result in ref. 19 and follows from a proof identical to that of proposition 2.3 in ref. 1.

Proposition 23. *Every bridge splitting of an unlink L in $\#^k(S^1\times S^2)$ is isotopic to some number of perturbations and stabilizations performed on the standard bridge splitting*.

Consider a bridge trisection $\mathcal{T}$ for a knotted surface $\mathcal{K}\subset X$, with components notated as above. *Proposition 23* implies the key fact that $\mathcal{K}$ admits a shadow diagram $((\alpha,a),(\beta,b),(\gamma,c))$ such that a pair of collections of arcs, say a and b for convenience, do not meet in their interiors, and in addition, the union $a\cup b$ cuts out a collection of embedded disks $\mathcal{D}_*$ from the central surface Σ. Choose a single component D_* of these disks together with an embedded arc δ_* in D_* which connects an arc $a'\in a$ to an arc $b'\in b$. Note that $\mathcal{D}_*$ is a trivialization of the disks $\mathcal{D}_1\subset X_1$ bounded by $\tau_{31}\cup\tau_{12}$ in $\partial X_1=H_{31}\cup H_{12}$, so that we may consider δ_* and D_* to be embedded in the surface $\mathcal{K}$. In addition, there is an isotopy of $\mathcal{D}_*$ in ∂X_1 pushing the shadows $a\cup b$ onto arcs in $\tau_{31}\cup\tau_{12}$, making D_* transverse to Σ and carrying δ_* to an embedded arc in ∂X_1 that meets the central surface Σ in one point.

Let Δ be a rectangular neighborhood of δ_* in D_*, and consider the isotopy of $\mathcal{K}$, supported in Δ, which pushes $\delta_*\subset\mathcal{K}$ away from X_1 in the direction normal to ∂X_1. Let $\mathcal{K}'$ be the resulting embedding, which is isotopic to $\mathcal{K}$. The next lemma follows from the proof of lemma 6.1 in ref. 1.

Lemma 24. *The embedding $\mathcal{K}'$ is in $(b+1)$-bridge position with respect to the trisection $X=X_1\cup X_2\cup X_3$, and if $c_i'=|\mathcal{K}'\cap X_i|$, then $c_1'=c_1+1$, $c_2'=c_2$, and $c_3'=c_3$.*

We call the resulting bridge trisection an *elementary perturbation* of $\mathcal{T}$, and if $\mathcal{T}'$ is the result of some number of elementary perturbations performed on $\mathcal{T}$, we call $\mathcal{T}'$ a *perturbation* of $\mathcal{T}$. Work in ref. 1 also makes clear how to perturb via a shadow diagram. View the rectangle Δ as being contained in Σ, and parameterize it as $\Delta=\delta_*\times I$. Now, crush Δ to a single arc $c'=*\times I$ that meets δ_* transversely once. Considering the arc c' as a shadow arc for the third tangle, the result is a shadow diagram for the elementary perturbation of $\mathcal{T}$. See Fig. 5.

In ref. 1, the authors prove that any two bridge trisections for a knotted surface $(S^4,\mathcal{K})$ are related by a sequence of perturbations and unperturbations. In the setting of generalized bridge trisections, we have the following conjecture.

Conjecture 25. *Any two generalized bridge trisections for $(X,\mathcal{K})$ with the same underlying trisection for X become isotopic after a finite sequence of perturbations and unperturbations.*

The proof of the analogous result for bridge trisections in ref. 1 requires a result of Swenton (20) and Kearton-Kurlin (21) that states that every one-parameter family of Morse functions of the pair $h_t:(S^4,\mathcal{K})\to\mathbb{R}$ such that $h_t:S^4\to\mathbb{R}$ is the standard height function can be made suitably generic. Unfortunately, a more general result does not yet exist for arbitrary pairs $(X,\mathcal{K})$; however, we remark that Conjecture 25 would follow from such a result together with an adaptation of the proof in ref. 1.

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