



Characterizing Dehn surgeries on links via trisections

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We summarize and expand known connections between the study of Dehn surgery on links and the study of trisections of closed, smooth 4-manifolds. In particular, we propose a program in which trisections could be used to disprove the generalized property R conjecture, including a process that converts the potential counterexamples of Gompf, Scharlemann, and Thompson into genus four trisections of the standard 4-sphere that are unlikely to be standard. We also give an analog of the Casson–Gordon rectangle condition for trisections that obstructs reducibility of a given trisection.

Dehn surgery | trisections | 4-manifolds | Generalized Property R Conjecture | Heegaard splitting

The theory of Dehn surgery on knots has been thoroughly developed over the past 40 y. In general, this research has focused on two major questions: First, which manifolds can be obtained by surgery on a knot in a given manifold Y ? Second, given a pair of manifolds Y and Y' , for which knots $K \subset Y$, does there exist a surgery to Y' ? These two questions have contributed to the growth of powerful tools in low-dimensional topology, such as sutured manifold theory, the notion of thin position, and Heegaard Floer homology. For example, over the last 15 y, the Heegaard Floer homology theories of Ozsváth and Szabó have dramatically deepened our collective understanding of Dehn surgeries on knots (for instance, ref. 1).

If we replace the word “knot” with “link” in the preceding paragraph, the situation changes significantly; for example, the classical Lickorish–Wallace theorem asserts that every 3-manifold Y can be obtained by surgery on a link in S^3 (2, 3). For the second general question, concerning which links in a given 3-manifold Y yield a surgery to another given 3-manifold Y' , we observe the following basic fact: Two framed links that are handleslide equivalent surger to the same 3-manifold (4). Thus, surgery classification of links is necessarily considered up to handleslide equivalence, and tools which rely on the topology of a knot exterior $S^3 \setminus \nu(K)$ are not nearly as useful, since handleslides can significantly alter this topology.

The purpose of this paper is to make clear the significant role of the trisection theory of smooth 4-manifolds in the classification of Dehn surgeries on links, including a program that suggests trisections may be used to disprove the generalized property R conjecture (GPRC), Kirby problem 1.82 (5). The GPRC asserts that every n -component link in S^3 with a Dehn surgery to $\#^n(S^1 \times S^2)$ is handleslide equivalent to the n -component zero-framed unlink. We call a link L with such a surgery an R-link. The related stable GPRC asserts that if L is an R-link, then the disjoint union of L and an unlink is handleslide equivalent to an unlink. The GPRC is known to be true when $n = 1$ (6), and the stable GPRC is known to be true in the following special case.

Theorem 1 (7). *If $L \subset S^3$ is an n -component R-link with tunnel number n , then L satisfies the stable GPRC.*

Any n -component R-link L can be used to construct a closed 4-manifold X_L , where X_L has a handle decomposition with a single 0-handle, no 1-handles, n 2-handles, n 3-handles, and a single 4-handle. An elementary argument reveals that X_L is a homotopy 4-sphere, and if L is handleslide equivalent to an unlink, then X_L is the standard S^4 . Thus, both the GPRC and stable GPRC imply the smooth 4D Poincaré conjecture (S4PC)

for geometrically simply connected 4-manifolds (those that can be built without 1-handles). Yet these conjectures are substantially stronger than this instance of the S4PC, since the GPRC implies that not only that is X_L standard, but also that the handle decomposition can be standardized without adding any canceling pairs of handles. (The stable version allows the addition of canceling 2-handle/3-handle pairs, but not canceling 1-handle/2-handle pairs.) Although experts seem divided about the veracity of the S4PC, it is widely believed that the GPRC is false, with the most prominent possible counterexamples appearing in a paper of Gompf, Scharlemann, and Thompson (8), building on work of Akbulut and Kirby (9).

A new tool that has been useful in this context is a trisection of a 4-manifold, introduced by Gay and Kirby (10). A trisection is a decomposition of a 4-manifold X into three simple pieces, a 4-dimensional version of a 3D Heegaard splitting. Elegantly connecting the two theories, Gay and Kirby (10) proved that every smooth 4-manifold admits a trisection, and every pair of trisections for a given 4-manifold has a common stabilization, mirroring the Reidemeister–Singer theorem (11, 12) in dimension three. Unlike Heegaard splittings, however, the stabilization operation of Gay and Kirby can be broken into three separate operations, called unbalanced stabilizations of types 1, 2, and 3 (7). A trisection is said to be standard if it is an unbalanced stabilization of the genus zero trisection of S^4 , and thus every trisection of S^4 becomes standard after some number of stabilizations. Just as trisections were pivotal in the *Proof of Theorem 1* above, we have also used trisections to obtain the following Dehn surgery classification result.

Theorem 2 (13). *If $L \subset S^3$ is a two-component link with tunnel number one with an integral surgery to S^3 , then L is handleslide equivalent to a 0-framed Hopf link or a ± 1 -framed unlink.*

In the present article, we exhibit a program to disprove the GPRC in three steps, of which we complete the first two. The initial step translates the GPRC and the related stable GPRC into statements about trisections of the 4-sphere. In 3. *R-Links and Stabilizations* we prove the following, postponing rigorous definitions for now.

Significance

Dehn surgery is the process in which one cuts out a neighborhood of a knot or a link in 3D space and reglues this neighborhood in a different way to obtain a new 3D space. By viewing this operation as occurring smoothly over a period, there is a natural interpretation of spaces “before” and “after” a Dehn surgery as level sets of a generic function taking a 4D space to the real line. As such, 4D spaces are deeply linked to Dehn surgeries. This paper explores the ways in which trisections of 4-manifolds can be used to answer interesting questions about such surgeries.

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Theorem 3. Suppose L is an R-link and Σ is any admissible surface for L .

- i) If L satisfies the GPRC, then $\mathcal{T}(L, \Sigma)$ is 2-standard.
- ii) The link L satisfies the stable GPRC if and only if $\mathcal{T}(L, \Sigma)$ is 2, 3-standard.

The second step is contained in 4. *Trisecting the Gompf–Scharlemann–Thompson Examples* in which we convert the proposed counterexamples of Gompf–Scharlemann–Thompson into trisections (with explicit diagrams). The final, incomplete step in this program is to prove that the trisections constructed in the second step are not 2-standard, which, together with *Theorem 3*, would imply that the GPRC is false. To accomplish step iii, we must develop machinery to verify that a trisection is non-standard. To this end, in 5. *A Rectangle Condition for Trisection Diagrams* we introduce an analog of the Casson–Gordon rectangle condition (14) for trisection diagrams, giving a sufficient condition for a trisection diagram to correspond to an irreducible (nonstandard) trisection.

We encourage the reader to view this article in full color, as a gray-scale rendering of the figures leads to a loss of information.

1. Trisections

All manifolds are connected and orientable, unless otherwise stated. We let $\nu(\cdot)$ refer to an open regular neighborhood in an ambient manifold that should be clear from context. The tunnel number of a link $L \subset Y$ is the cardinality of the smallest collection of arcs a with the property that $Y \setminus \nu(L \cup a)$ is a handlebody. In this case, $\partial\nu(L \cup a)$ is a Heegaard surface cutting $Y \setminus \nu(L)$ into a handlebody and a compression body. A framed link refers to a link with an integer framing on each component.

Let L be a framed link in a 3-manifold Y , and let a be a framed arc connecting two distinct components of L ; call them L_1 and L_2 . The framings of L_1 , L_2 , and a induce an embedded surface $S \subset Y$, homeomorphic to a pair of pants, such that $L_1 \cup L_2 \cup a$ is a core of S . Note that S has three boundary components, two of which are isotopic to L_1 and L_2 . Let L_3 denote the third boundary component, with framing induced by S . If L' is the framed link $(L \setminus L_1) \cup L_3$, we say that L' is obtained from L by a handleslide of L_1 over L_2 along a .

If two links are related by a finite sequence of handleslides, we say they are handleslide equivalent. It is well known that Dehn surgeries on handleslide-equivalent framed links yield homeomorphic 3-manifolds (4). Recall that an R-link is an n -component link in S^3 with a Dehn surgery to the manifold $\#^n(S^1 \times S^2)$, which we henceforth denote by Y_n . Let U_n denote the n -component zero-framed unlink in S^3 . If an R-link L is handleslide equivalent to U_n , we say that L has property R. If the split union $L \sqcup U_r$ is handleslide equivalent to U_{n+r} for some integer r , we say that L has stable property R. With these definitions the GRPC and stable GPRC can be formulated as follows.

(Stable) GPRC. Every R-link has (stable) property R.

In this section, we explore the relationship between R-links and trisections of the smooth 4-manifolds that can be constructed from these links.

Let X be a smooth, orientable, closed 4-manifold. A $(g; k_1, k_2, k_3)$ -trisection $\mathcal{T}$ of X is a decomposition $X = X_1 \cup X_2 \cup X_3$ such that

- i) each X_i is a 4D 1-handlebody, $\natural^{k_i}(S^1 \times B^3)$;
- ii) if $i \neq j$, then $H_{ij} = X_i \cap X_j$ is a 3D handlebody, $\natural^g(S^1 \times D^2)$; and
- iii) the common intersection $\Sigma = X_1 \cap X_2 \cap X_3$ is a closed genus g surface.

The surface Σ is called the trisection surface or central surface, and the parameter g is called the genus of the trisection. The trisection $\mathcal{T}$ is called balanced if $k_1 = k_2 = k_3 = k$, in which case it is called a (g, k) -trisection; otherwise, it is called unbalanced.

We call the union $H_{12} \cup H_{23} \cup H_{31}$ the spine of the trisection. In addition, we observe that $\partial X_i = Y_{k_i} = H_{ij} \cup_{\Sigma} H_{ik}$ is a genus g Heegaard splitting. Because there is a unique way to cap off Y_{k_i} with $\natural^{k_i}(S^1 \times B^3)$ (15, 16), every trisection is uniquely determined by its spine.

Like Heegaard splittings, trisections can be encoded with diagrams. A cut system for a genus g surface Σ is a collection of g pairwise disjoint simple closed curves that cut Σ into a $2g$ -punctured sphere. A cut system δ is said to define a handlebody H_δ if each curve in δ bounds a disk in H_δ . A triple (α, β, γ) of cut systems is called a $(g; k_1, k_2, k_3)$ -trisection diagram for $\mathcal{T}$ if α, β , and γ define the components H_α, H_β , and H_γ of the spine of $\mathcal{T}$. We set the conventions that $H_\alpha = X_3 \cap X_1$, $H_\beta = X_1 \cap X_2$, and $H_\gamma = X_2 \cap X_3$, which the careful reader may note differ slightly from conventions in ref. 7. With our conventions, (α, β) , (β, γ) , and (γ, α) are Heegaard diagrams for Y_{k_1}, Y_{k_2} , and Y_{k_3} , respectively. In ref. 10, Gay and Kirby proved that every smooth 4-manifold admits a trisection, and trisection diagrams, modulo handleslides within the three collections of curves, are in one-to-one correspondence with trisections.

Given trisections $\mathcal{T}$ and $\mathcal{T}'$ for 4-manifolds X and X' , we can obtain a trisection for $X \# X'$ by removing a neighborhood of a point in each trisection surface and gluing pairs of components of $\mathcal{T}$ and $\mathcal{T}'$ along the boundary of this neighborhood. The resulting trisection is uniquely determined in this manner; we denote it by $\mathcal{T} \# \mathcal{T}'$. A trisection $\mathcal{T}$ is called reducible if $\mathcal{T} = \mathcal{T}' \# \mathcal{T}''$, where neither $\mathcal{T}'$ nor $\mathcal{T}''$ is the genus zero trisection; otherwise, it is called irreducible. Equivalently, $\mathcal{T}$ is reducible if there exists an essential separating curve δ in Σ that bounds compressing disks in H_α, H_β , and H_γ . Such a curve δ represents the intersection of a decomposing 3-sphere with the trisection surface.

In dimension three, stabilization of a Heegaard surface may be viewed as taking the connected sum with the genus one splitting of S^3 , and a similar structure exists for trisections. Let S_i denote the unique genus one trisection of S^4 satisfying $k_i = 1$. Diagrams for these three trisections are shown in Fig. 1. A trisection $\mathcal{T}$ is called i -stabilized if $\mathcal{T} = \mathcal{T}' \# S_i$ and is simply called stabilized if it is i -stabilized for some $i = 1, 2, 3$. Two trisections $\mathcal{T}'$ and $\mathcal{T}''$ are called stably equivalent if there is a trisection $\mathcal{T}$ that is a stabilization of both $\mathcal{T}'$ and $\mathcal{T}''$. Gay and Kirby (10) proved that any two trisections of a fixed 4-manifold are stably equivalent.

We say that a trisection $\mathcal{T}$ of S^4 is standard if $\mathcal{T}$ can be expressed as the connected sum of genus one trisections S_i .

2. Admissible Surfaces

Here we turn our attention to R-links and Dehn surgeries, before connecting these surgeries to the trisections described above. Recall that Y_k denotes $\#^k(S^1 \times S^2)$, and an R-link L is a framed n -component link in S^3 such that Dehn surgery on L yields Y_n . As mentioned above, every R-link L describes a closed 4-manifold X_L with a handle decomposition with a single 0-handle, zero 1-handles, n 2-handles, n 3-handles, and a single 4-handle. Thus, X_L is a homotopy S^4 . An admissible Heegaard surface Σ for L is a Heegaard surface cutting S^3 into two handlebodies H and H' such that a core of H contains L . As such, $M = H \setminus \nu(L)$ is a compression body and Σ may be viewed as a Heegaard surface for the link exterior $E(L) = S^3 \setminus \nu(L)$. Let H_L be the handlebody that results from Dehn filling M (or performing

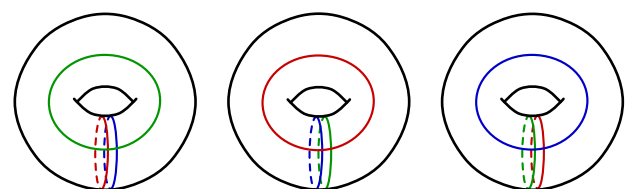


Fig. 1. The three genus one trisections diagrams for S^4 .

Dehn surgery on L in H) along the framing of the link L . An admissible pair consists of an R-link together with an admissible Heegaard surface.

For completeness, we also allow the empty link, $L = \emptyset$, where L has an empty Dehn filling yielding S^3 , giving rise to a handle decomposition of $S^4 = X_\emptyset$ with only a single 0- and 4-handle. An admissible surface Σ for the empty link is a (standard) genus g Heegaard surface for S^3 . A genus g Heegaard diagram (α, β) for Y_k is called standard if $\alpha \cap \beta$ contains k curves, and the remaining $g - k$ curves occur in pairs that intersect once and are disjoint from other pairs. A trisection diagram is called standard if each pair is a standard Heegaard diagram. Note that a standard trisection of S^4 has a standard diagram, since each of its summands S_i has such a diagram.

Lemma 4. *Let L be an n -component R-link. Every admissible pair (L, Σ) gives rise to a trisection $\mathcal{T}(L, \Sigma)$ of X_L with spine $H' \cup H \cup H_L$. If $g(\Sigma) = g$, then $\mathcal{T}(L, \Sigma)$ is a $(g; 0, g - n, n)$ -trisection. Moreover, there is a trisection diagram (α, β, γ) for $\mathcal{T}(L, \Sigma)$ such that*

- i) $H_\alpha = H'$, $H_\beta = H$, and $H_\gamma = H_L$;
- ii) L is a sublink of γ , where γ is viewed as a link framed by Σ in $S^3 = H_\alpha \cup H_\beta$; and
- iii) (β, γ) is a standard diagram for Y_{g-n} , where $\beta \cap \gamma = \gamma \setminus L$.

Proof: This is proved (in slightly different formats) for $L \neq \emptyset$ in both refs. 7 and 10. If $L = \emptyset$, then it follows easily that $S^4 = X_\emptyset$ has a handle decomposition without 1-, 2-, or 3-handles, $H = H_L$, and $H' \cup H \cup H_L$ is the spine for the $(g; 0, g, 0)$ -trisection $\mathcal{T}(L, \Sigma)$ of S^4 . In this case, there is a diagram such that $\beta = \gamma$, the standard genus g diagram for Y_g .

Note that the conventions $H_\alpha = H'$, $H_\beta = H$, and $H_\gamma = H_L$, in conjunction with our earlier conventions, identify the 0-handle with X_1 , the trace of the Dehn surgery on H_β along L with X_2 , and the union of the 3-handles and the 4-handle with X_3 .

Lemma 5 connects R-links, standard trisections, and the stable GPRC.

Lemma 5. *Suppose L is an n -component R-link with admissible genus g surface Σ , and $\mathcal{T}(L, \Sigma)$ is a standard trisection of S^4 . Then L has stable property R.*

Proof: By Lemma 4, the trisection $\mathcal{T}(L, \Sigma)$ has a diagram (α, β, γ) such that (β, γ) is the standard Heegaard diagram for Y_{g-n} . Viewing γ as a g -component link in $S^3 = H_\alpha \cup H_\beta$, we have that $(g - n)$ curves in γ bound disks in H_β , while the remaining n curves are isotopic to L [and are disjoint from the $(g - n)$ disks]. Thus, as a link in S^3 , we have $\gamma = L \sqcup U_{g-n}$.

In addition, the trisection $\mathcal{T}(L, \Sigma)$ is a standard $(g; 0, g - n, n)$ -trisection of S^4 by hypothesis. As such, it must be a connected sum of $g - n$ copies of S_2 and n copies of S_3 , and it has a standard diagram, $(\alpha', \beta', \gamma')$, where $g - n$ curves in γ' are also curves in β' , and the remaining n curves are also curves in α' . Thus, in $S^3 = H_{\alpha'} \cup H_{\beta'}$, the curves γ' compose a g -component unlink, with surface framing equal to the zero framing on each component. Since (α, β, γ) and $(\alpha', \beta', \gamma')$ are trisection diagrams for the same trisection, we have that γ is handleslide equivalent to γ' via slides contained in Σ . Therefore, γ and γ' are handleslide-equivalent links in S^3 . We conclude that L has stable property R, as desired. $\square$

As an aside, we note that Theorem 1 can be obtained quickly using Lemma 5 and the classification of $(g; 0, 1, g - 1)$ -trisections from ref. 7.

3. R-Links and Stabilizations

To prove Theorem 3, we develop the connection between R-links, their induced trisections, and the three types of stabilizations. First, we must introduce several additional definitions. Let (L_1, Σ_1) and (L_2, Σ_2) be two admissible pairs and define the

operation $*$ by

$$(L_1, \Sigma_1) * (L_2, \Sigma_2) = (L_1 \sqcup L_2, \Sigma_1 \# \Sigma_2),$$

where the connected sum is taken so that $L_1 \sqcup L_2$ is not separated by the surface $\Sigma_1 \# \Sigma_2$.

Lemma 6. *If (L_1, Σ_1) and (L_2, Σ_2) are admissible pairs, then $(L, \Sigma) = (L_1, \Sigma_1) * (L_2, \Sigma_2)$ is an admissible pair, and $\mathcal{T}(L, \Sigma) = \mathcal{T}(L_1, \Sigma_1) \# \mathcal{T}(L_2, \Sigma_2)$.*

Proof: It is clear that the framed link $L_1 \sqcup L_2$ has the appropriate surgery, so L is an R-link. Suppose Σ_i bounds a handlebody H_i with core C_i containing L_i . Then there is a core C for $H_1 \natural H_2$ such that $L_1 \sqcup L_2 \subset C_1 \sqcup C_2 \subset C$, and thus $\Sigma_1 \# \Sigma_2$ is admissible as well. For the second claim, note that the separating curve δ arising from the connected sum $\Sigma = \Sigma_1 \# \Sigma_2$ is a reducing curve for $\mathcal{T}(L, \Sigma)$, splitting it into the trisections $\mathcal{T}(L_1, \Sigma_1)$ and $\mathcal{T}(L_2, \Sigma_2)$. $\square$

Let U be a 0-framed unknot in S^3 , and let Σ_U be the genus one splitting of S^3 such that one of the solid tori bounded by Σ_U contains U as a core. In addition, let $\Sigma_\emptyset$ be the genus one Heegaard surface for S^3 , to be paired with the empty link. Note that (U, Σ_U) and $(\emptyset, \Sigma_\emptyset)$ are admissible pairs.

Lemma 7. *The pairs $(\emptyset, \Sigma_\emptyset)$ and (U, Σ_U) yield the following trisections:*

- i) $\mathcal{T}(\emptyset, \Sigma_\emptyset) = S_2$.
- ii) $\mathcal{T}(U, \Sigma_U) = S_3$.

Proof: Note that each trisection in question has genus one. The associated trisections $\mathcal{T}(\emptyset, \Sigma_\emptyset)$ and $\mathcal{T}(U, \Sigma_U)$ are $(1; 0, 1, 0)$ - and $(1; 0, 0, 1)$ -trisections, respectively, and thus they must be S_2 and S_3 . $\square$

By combining Lemmas 6 and 7, we obtain the following.

Corollary 8. *Suppose (L, Σ) is an admissible pair, with $\mathcal{T} = \mathcal{T}(L, \Sigma)$:*

- i) $\mathcal{T}((L, \Sigma) * (\emptyset, \Sigma_\emptyset))$ is the 2-stabilization of $\mathcal{T}$.
- ii) $\mathcal{T}((L, \Sigma) * (U, \Sigma_U))$ is the 3-stabilization of $\mathcal{T}$.

In addition, if Σ_+ is the stabilization of Σ (as a Heegaard surface for Y_k), then $(L, \Sigma_+) = (L, \Sigma) * (\emptyset, \Sigma_\emptyset)$.

Remark 9. *Notably absent from Lemma 7 and Corollary 8 is any reference to 1-stabilization. By generalizing the definition of an admissible pair, we can accommodate 1-stabilization in this context; however, 1-stabilizing a trisection $\mathcal{T}(L, \Sigma)$ that arises from an R-link L corresponds to adding a canceling 1-handle/2-handle pair to the induced handle decomposition of X_L . This addition takes us away from the setting of R-links, so we have chosen not to adopt this greater generality here.*

We say that two trisections $\mathcal{T}_1$ and $\mathcal{T}_2$ of a 4-manifold X are 2-equivalent if there is a trisection $\mathcal{T}$ that is the result of 2-stabilizations performed on both $\mathcal{T}_1$ and $\mathcal{T}_2$.

Lemma 10. *If Σ_1 and Σ_2 are two distinct admissible surfaces for an R-link L , then the trisections $\mathcal{T}(L, \Sigma_1)$ and $\mathcal{T}(L, \Sigma_2)$ are 2-equivalent.*

Proof: Since both Σ_1 and Σ_2 are Heegaard surfaces for $E(L)$, they have a common stabilization Σ by the Reidemeister–Singer theorem (11, 12). By Lemma 6, the surface Σ is admissible, and by Corollary 8, $\mathcal{T}(L, \Sigma)$ can be obtained by 2-stabilizations of $\mathcal{T}(L, \Sigma_i)$.

Observe that 2-equivalence is an equivalence relation. Since Lemma 10 implies that every trisection $\mathcal{T}(L, \Sigma)$ coming from a fixed R-link L belongs to the same 2-equivalence class, it follows that L has a well-defined 2-equivalence class, namely, the 2-equivalence class of $\mathcal{T}(L, \Sigma)$ for any admissible surface Σ . If

two R-links L_1 and L_2 give rise to 2-equivalent trisections, we say that L_1 and L_2 are 2-equivalent.

Suppose that L is an n -component R-link with admissible surface Σ , cutting S^3 into $H \cup H'$, and L is isotopic into a core $C \subset H$ as above. As such, there is a collection of n compressing disks $\mathcal{D}$ with the property that each disk meets a unique component of L once and misses the other components. We call $\mathcal{D}$ a set of dualizing disks. Note that if (α, β, γ) is the trisection diagram for $\mathcal{T}(L, \Sigma)$ guaranteed by Lemma 4, then the n disks bounded by the n curves in β that are not in γ are a set of dualizing disks for L .

Lemma 11. *If R-links L_1 and L_2 are related by a handleslide, then L_1 and L_2 are 2-equivalent.*

Proof: If L_i is an n -component link, then L_1 and L_2 have $n - 1$ components in common and differ by a single component, $L'_1 \subset L_1$ and $L'_2 \subset L_2$, where a slide of L'_1 over another component L' of L_1 along a framed arc a yields L'_2 . Consider $\Gamma = L_1 \cup a$, an embedded graph with $n - 1$ components, and let Σ be a Heegaard surface cutting S^3 into $H \cup H'$, where Γ is contained in a core of H . Then L_1 is also contained in a core of H , and Σ is admissible (with respect to L_1). Let $\mathcal{D}_1$ be a set of dualizing disks for L_1 , which by construction may be chosen so that the arc a avoids all of the disks $\mathcal{D}_1$ (Fig. 2).

There is an isotopy taking Γ into Σ , preserving the intersections of L_i with the dualizing disks $\mathcal{D}_1$, so that the framing of Γ agrees with its surface framing in Σ . As such, we can perform the handleslide of L'_1 over L' along a within the surface Σ , so that the resulting link L_2 is also contained in Σ , with framing given by the surface framing. Let $D'_1 \in \mathcal{D}_1$ be the disk that meets L'_1 once, and let $D' \in \mathcal{D}_1$ be the disk that meets L' once. There is an arc a' , isotopic in Σ to an arc in Γ , that connects D'_1 to D' (Fig. 2). Let D'_2 be the compressing disk obtained by banding D'_1 to D' along a' . Then $\mathcal{D}_2 = (\mathcal{D}_1 \setminus D'_1) \cup D'_2$ is a set of dualizing disks for L_2 . Thus, by pushing L_2 back into H , we see that Σ is an admissible surface for L_2 .

Following Lemma 4, let $H_i \cup H'_i \cup H_{L_i}$ be a spine for $\mathcal{T}(L_i, \Sigma)$. By construction, $H_1 = H_2$ and $H'_1 = H'_2$. Finally, since H_i is Dehn surgery on L_i in H_i , and L_1 and L_2 are related by a single handleslide, we have $H_{L_1} = H_{L_2}$. It follows that $\mathcal{T}(L_1, \Sigma) = \mathcal{T}(L_2, \Sigma)$, and we conclude that L_1 and L_2 are 2-equivalent. $\square$

Recall that a standard trisection of S^4 is the connected sum of copies of S_1 , S_2 , and S_3 and U_n is the zero-framed, n -component unlink, so $X_{U_n} = S^4$.

Lemma 12. *Let Σ be any admissible surface for U_n ; then $\mathcal{T}(U_n, \Sigma)$ is standard.*

Proof: We induct on (n, g) with the dictionary ordering. If $n = 1$, then $E(U_1)$ is a solid torus. If $g = 1$, then $\Sigma = \Sigma_U$, so that $\mathcal{T}(U_1, \Sigma_U) = S_3$ by Lemma 7. If $n = 1$ and $g > 1$, then Σ is stabilized (17, 18), which means that $\mathcal{T}(U_1, \Sigma)$ is 2-stabilized by Corollary 8, and, as such, $\mathcal{T}(U_1, \Sigma)$ is standard by induction.

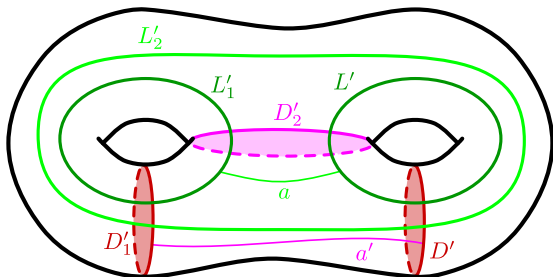


Fig. 2. The disks and arcs used in the Proof of Lemma 11, in which the pairs (L'_1, D'_1) and (L', D') are replaced with (L'_2, D'_1) and (L', D'_2) .

In general, note that the Heegaard genus of an n -component unlink is n ; thus $g \geq n$ for all possible pairs (n, g) . For $n > 1$, we have that $E(U_n)$ is reducible, and so Haken's lemma (19) implies that Σ is reducible, splitting into the connected sum of genus g_1 and g_2 surfaces Σ_1 and Σ_2 , where Σ_i is a Heegaard surface for $E(U_{n_i})$. Then $\mathcal{T}(U_n, \Sigma) = \mathcal{T}(U_{n_1}, \Sigma_1) \# \mathcal{T}(U_{n_2}, \Sigma_2)$, where $(n_i, g_i) < (n, g)$. Since both summands are standard trisections by induction, it follows that $\mathcal{T}(U_n, \Sigma)$ is also standard, completing the Proof. $\square$

A trisection $\mathcal{T}$ is said to be 2-standard if it becomes standard after some number of 2-stabilizations. Similarly, $\mathcal{T}$ is 2, 3-standard if it becomes standard after some number of 2- and 3-stabilizations.

Proof of Theorem 3: Suppose L has property R. By Lemma 11, L and U_n are 2-equivalent links. Thus, $\mathcal{T}(L, \Sigma)$ is 2-equivalent to some trisection coming from U_n , but all trisections induced by U_n are standard by Lemma 12, and thus $\mathcal{T}(L, \Sigma)$ becomes standard after a finite sequence of 2-stabilizations.

If L has stable property R, then $L \sqcup U_n$ has property R for some n , and thus $\mathcal{T}((L, \Sigma) * (U, \Sigma_U) * \dots * (U, \Sigma_U))$ is 2-standard by the above arguments. By Lemma 7 and Corollary 8,

$$\mathcal{T}((L, \Sigma) * (U, \Sigma_U) * \dots * (U, \Sigma_U)) = \mathcal{T}(L, \Sigma) \# S_3 \# \dots \# S_3;$$

hence $\mathcal{T}(L, \Sigma)$ is 2, 3-standard.

Finally, if the trisection $\mathcal{T}(L, \Sigma)$ is 2, 3-standard, then there exist integers s and t such that the connected sum of $\mathcal{T}(L, \Sigma)$ with s copies of S_2 and t copies of S_3 is standard. Let (L_*, Σ_*) be the admissible pair given by

$$(L_*, \Sigma_*) = (L, \Sigma) * \underbrace{(\emptyset, \Sigma_\emptyset) * \dots * (\emptyset, \Sigma_\emptyset)}_s * \underbrace{(U, \Sigma_U) * \dots * (U, \Sigma_U)}_t.$$

By assumption, $\mathcal{T}(L_*, \Sigma_*)$ is standard, so by Lemma 5, the link L_* has stable property R. But by definition of $*$, we have $L_* = L \sqcup U_t$, and thus L also has stable property R, completing the Proof. $\square$

4. Trisecting the Gompf–Scharlemann–Thompson Examples

To use Theorem 3 to disprove the GPRC or the stable GPRC, we must convert the possible counterexamples to these theorems into trisections. In this section, we find admissible surfaces related to the examples proposed by Gompf, Scharlemann, and Thompson (8). We call this family the Gompf–Scharlemann–Thompson (GST) links. First, we outline that construction, and then we define the GST links and discuss how they fit into the broader picture. To proceed, we need several new definitions.

Let K be a knot in S^3 . We say that K is ribbon if K bounds an immersed disk in S^3 whose double points are ribbon singularities. It is well known that every ribbon disk can be viewed as a properly embedded disk in the standard 4-ball B^4 , where $K \subset S^3 = \partial B^4$. Suppose B is any homotopy 4-ball. The knot K is called homotopy ribbon in B if there exists a properly embedded disk $D \subset B$ such that $K = \partial D \subset S^3 = \partial B$ and the inclusion map $(S^3, K) \hookrightarrow (B, D)$ induces a surjection $\pi_1(S^3 \setminus K) \rightarrow \pi_1(B \setminus D)$. Every ribbon knot is homotopy ribbon.

Let $K \subset S^3$ with F a Seifert surface for K . The knot K is fibered with fiber F if its exterior $E(K)$ is homeomorphic to the mapping torus of a homeomorphism $\varphi: F \rightarrow F$ such that $\varphi_{\partial F} = \text{id}$, called the monodromy of K . Let $\hat{Y}$ denote the 3-manifold obtained by 0-surgery on K in S^3 . Then $\hat{Y}$ can be constructed by capping off each copy of F with a disk in the fibration of $E(K)$ to get a closed surface $\hat{F}$, so that $\hat{Y}$ is the mapping torus of $\hat{\varphi}: \hat{F} \rightarrow \hat{F}$. We call $\hat{\varphi}$ the closed monodromy of

K . Finally, we say that φ extends over a handlebody H if there is a homeomorphism $\Phi: H \rightarrow H$ such that $\Phi_{\partial H} = \hat{\varphi}$.

Casson and Gordon (20) proved a remarkable theorem connecting homotopy-ribbon knots to handlebody extensions.

Theorem 13. *Let $K \subset S^3$ be a fibered knot with fiber F and monodromy φ . Then K is homotopy ribbon in a homotopy 4-ball B if and only if the monodromy φ extends over a handlebody H .*

As above, let $K \subset S^3$ be a fibered ribbon knot with fiber F , so that Theorem 13 implies that the monodromy φ of K extends over a handlebody H . Let $L \subset F$ be a link in S^3 such that L is the boundary of a cut system defining H . We call L a Casson–Gordon derivative of K .

Proposition 14. *Suppose K is a fibered ribbon knot with genus g fiber F and Casson–Gordon derivative L . Then both L and $K \cup L$ are R -links. Moreover, L has a genus $2g$ admissible surface, and thus the 4-manifold X_L admits a $(2g; 0, g, g)$ -trisection.*

In the remainder of this section, we spell out the details for the simplest case, $(p, q) = (3, 2)$. Let Q denote the square knot $T(3, 2) \# -T(3, 2)$, let F denote its genus two fiber surface, and let φ denote the monodromy of $E(Q)$. In ref. 21, Scharlemann depicted an elegant way to think about the monodromy φ : We may draw F as a topological annulus A , such that

- an open disk D has been removed from A ,
- each component of ∂A is split into six edges and six vertices, and
- opposite inside edges of ∂A are identified, and opposite outside edges of ∂A are identified, so that the quotient space is homeomorphic to F .

With respect to A , the monodromy φ is a $1/6$ th clockwise rotation of A , followed by an isotopy of D returning it to its original position. As above, let $\hat{Y}$ be the closed 3-manifold obtained by 0-surgery on Q , and let $\hat{\varphi}$ denote the closed monodromy of Q . Note that $\hat{\varphi}$ is an honest $1/6$ th rotation of the annulus in Fig. 3, since, in this case, the puncture has been filled in by the Dehn surgery. Details can be found in refs. 8 and 21, where Lemma 15 is proved.

Lemma 15. *For every rational number p/q with q odd, there is a family $V_{p/q}, V'_{p/q}, V''_{p/q}$ of curves contained in $\hat{F}$ that are permuted by $\hat{\varphi}$.*

Proof: We may subdivide A into six rectangular regions as shown in Fig. 3. It is proved in ref. 21 that $\hat{F}$ is a 3-fold branched cover of a 2-sphere S with four branch points. By naturally identifying S with a 4-punctured sphere constructed by gluing two unit squares along their edges, there is a unique isotopy class of curve $c_{p/q}$ with slope p/q in S . Let $\rho: F \rightarrow S$ denote the cover-

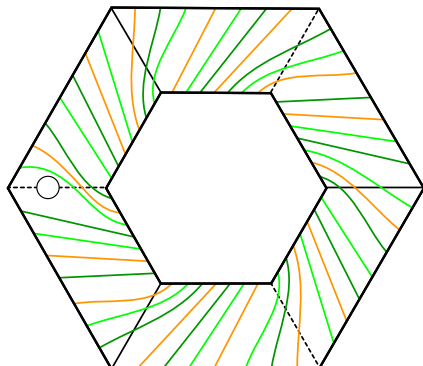


Fig. 3. The curves $V_{3/7}, V'_{3/7}$, and $V''_{3/7}$ on the genus two fiber F for the square knot.

ing map. Scharlemann proves that $\rho^{-1}(c_{p,q}) = V_{p/q}, V'_{p/q}, V''_{p,q}$, and these curves are permuted by $\hat{\varphi}$. $\square$

We note that any 2-component sublink of $V_{p/q} \cup V'_{p/q} \cup V''_{p,q}$ is a Casson–Gordon derivative for Q corresponding to some handlebody extension of φ . Fig. 3 shows the three lifts, $V_{3/7}, V'_{3/7}$, and $V''_{3/7}$, of the rational curve $3/7$ to the fiber F of the square knot. Observe that $\hat{\varphi}^6$ is the identity map, and $\hat{\varphi}^3$ maps $V_{p/q}$ to itself but with reversed orientation.

Finally, we can define the GST links. Lemma 16 is also from ref. 21.

Lemma 16. *The GST link L_n is handleslide equivalent to $Q \cup V_{n/2n+1}$. The R -link L_n has property R for $n = 0, 1, 2$ and is not known to have property R for $n \geq 3$.*

For ease of notation, let $V_n = V_{n/2n+1}$ and $V'_n = V'_{n/2n+1}$, so that $L_n = Q \cup V_n$. Two links L and L' are said to be stably handleslide equivalent or just stably equivalent if there are integers n and n' so that $L \sqcup U_n$ is handleslide equivalent to $L' \sqcup U_{n'}$. While we can find admissible surfaces for L_n , there is a simpler construction for a family of links L'_n stably equivalent to L_n for each n , and we note a link L has stable property R if and only if every link stably equivalent to L has stable property R .

Lemma 17. *The link $L_n = Q \cup V_n$ is stably equivalent to $L'_n = V_n \cup V'_n$.*

Proof: We show that both links are stably equivalent to $Q \cup V_n \cup V'_n$. Since $\hat{\varphi}(V_n) = V'_n$, we have that V'_n is isotopic to V_n in $\hat{Y}$. Carrying this isotopy into S^3 , we see that after some number of handleslides of V'_n over Q , the resulting curve C' is isotopic to V_n . Now C' can be slid over V_n to produce a split unknot U_1 , and $Q \cup V_n \cup V'_n$ is handleslide equivalent to $L_n \sqcup U_1$. On the other hand, V_n and V'_n are homologically independent in the genus two surface F . Thus, there is a sequence of slides of Q over V_n and V'_n in F converting Q to a split unknot, so $Q \cup V_n \cup V'_n$ is handleslide equivalent to $L'_n \sqcup U_1$ as well. $\square$

Next, we define an admissible surface for L'_n . Consider a collar neighborhood $F \times I$ of F , and let $N \subset S^3$ denote the embedded 3-manifold obtained by crushing $\partial F \times I$ to a single curve. Letting $\Sigma = \partial N$, we see that Σ is two copies of F , call them F_0 and F_1 , glued along the curve Q .

Lemma 18. *Consider L'_n embedded in F_0 , and push L'_n slightly into N . Then Σ is an admissible surface for L'_n .*

Proof: First, $F \times I$ is a genus four handlebody, as is N , since N is obtained by crushing the vertical boundary of $F \times I$. Moreover, since the exterior $E(Q)$ is fibered with fiber F , we may view this fibering as an open-book decomposition of S^3 with binding Q , and thus $S^3 \setminus \bar{N}$ is homeomorphic to N , so that Σ is a Heegaard surface for S^3 .

It remains to be seen that there is a core of N containing L'_n , but it suffices to show that there is a pair D_n and D'_n of dualizing disks for L'_n in N . Note that for any properly embedded arc $a \subset F_0$, there is a compressing disk $D(a)$ for N obtained by crushing the vertical boundary of the disk, $a \times I \subset F \times I$. Let a_0 and a'_0 be disjoint arcs embedded in F_0 such that a_0 meets V_n once and avoids V'_n , and a'_0 meets V'_n once and avoids V_n . Then $D(a_0)$ and $D(a'_0)$ are dualizing disks for L'_n , completing the Proof. $\square$

Lemma 18 does more than simply prove Σ is admissible; it provides the key ingredients we need to construct a diagram for $\mathcal{T}(L'_n, \Sigma)$: Let a_1 and a'_1 denote parallel copies of a_0 and a'_0 , respectively, in F_1 , so that $\partial D(a_0) = a_0 \cup a_1$ and $\partial D(a'_0) = a'_0 \cup a'_1$. By Lemma 4, there is a genus four trisection diagram (α, β, γ) for $\mathcal{T}(L'_n, \Sigma)$ so that

$$\beta_1 = \partial D(a_0) \quad \beta_2 = \partial D(a'_0) \quad \gamma_1 = V_n \quad \gamma_2 = V'_n.$$

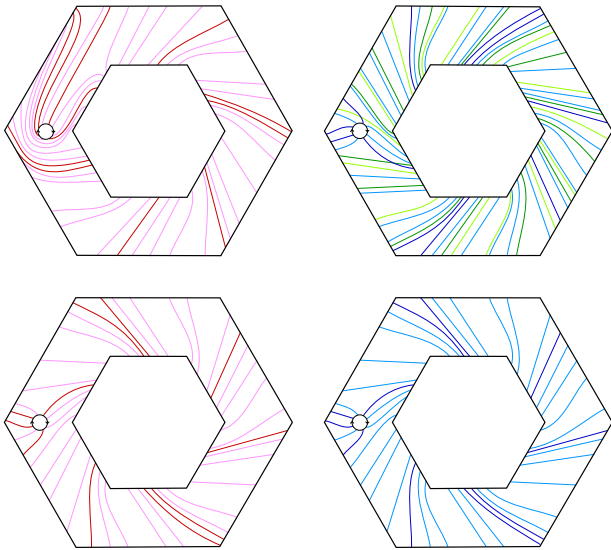


Fig. 4. A trisection diagram for $\mathcal{T}(L'_3, \Sigma)$. *Top row* shows two copies of F_0 , along with arcs: $\varphi(a_0)$ and $\varphi(a'_0)$ (red), $\varphi(b_0)$ and $\varphi(b'_0)$ (pink), b_0 and b'_0 (dark blue), b_1 and b'_1 (light blue), and V_3 (dark green) and V'_3 (light green). *Bottom row* shows two copies of F_1 , along with arcs: a_1 and a'_1 (red and dark blue) and b_1 and b'_1 (pink and light blue). The surfaces in the *Top row* are identified with those in the *Bottom row* along the oriented puncture. Thus, each column describes the closed genus four surface Σ . *Left column* encodes a 4-tuple of curves on this surface, namely, α . *Right column* encodes the 4-tuple β (shades of blue), as well as the two curves γ_1 and γ_2 . The trisection diagram for $\mathcal{T}(L'_3, \Sigma)$ is obtained by overlaying the two columns. (Note that $\gamma_3 = \beta_3$ and $\gamma_4 = \beta_4$.)

Noting that (β, γ) defines a genus four splitting of Y_2 , it follows that any curve disjoint from $\beta_1 \cup \beta_2 \cup \gamma_1 \cup \gamma_2$ that bounds a disk in either of H_β or H_γ also bounds in the other handlebody. Let b_0 and b'_0 denote nonisotopic disjoint arcs in F_0 that are disjoint from $a_0 \cup a'_0 \cup L_n$. Then $b_0 \cup b_1$ and $b'_0 \cup b'_1$ bound disks in N ; thus, letting

$$\beta_3 = \gamma_3 = b_0 \cup b_1 \quad \beta_4 = \gamma_4 = b'_0 \cup b'_1,$$

we have that (β, γ) is a standard diagram, corresponding to two of the cut systems in a diagram for $\mathcal{T}(L_n, \Sigma)$. To find the curves in α , let $N' = \overline{S^3 \setminus N}$, and observe that N' also has the structure of $F \times I$ crushed along its vertical boundary, and $\partial N' = \partial N = F_0 \cup F_1$.

One way to reconstruct S^3 from N and N' , both of which are homeomorphic to crushed products $F \times I$, is to initially glue $F_1 \subset \partial N$ to $F_1 \subset \partial N'$. The result of this initial gluing is again homeomorphic to a crushed product $F \times I$. The second gluing then incorporates the monodromy, so that $F_0 \subset N'$ is glued to $F_0 \subset N$ via φ . The result of this gluing is that if a_1 is an arc in $F_1 \subset N'$ and $D'(a_1)$ is the corresponding product disk in N' , then $\partial D'(a_1) = a_1 \cup \varphi(a_0)$, where a_0 is a parallel copy of a_1 in F_0 (using the product structure of N).

Thus, to find curves in α , we can choose any four arcs in F_1 cutting the surface into a planar component and construct their product disks. However, if we wish to find a diagram with relatively little complication with respect to the β and γ curves we have already chosen, it makes sense to choose those four arcs to be a_1, a'_1, b_1 , and b'_1 . Thus,

$$\begin{aligned} \alpha_1 &= a_1 \cup \varphi(a_0) & \alpha_3 &= b_1 \cup \varphi(b_0) \\ \alpha_2 &= a'_1 \cup \varphi(a'_0) & \alpha_4 &= b'_1 \cup \varphi(b'_0). \end{aligned}$$

We have proved the following.

Proposition 19. *The triple (α, β, γ) forms a $(4; 0, 2, 2)$ -trisection diagram for $\mathcal{T}(L_n, \Sigma)$.*

The diagram (α, β, γ) is depicted in Fig. 4. A generalization of this construction allows us to replace Q with any knot of the form $T(p, q) \# -T(p, q)$.

5. A Rectangle Condition for Trisection Diagrams

In this section, we introduce a tool for potential future use. This tool is an adaptation to the setting of trisection diagrams of the rectangle condition for Heegaard diagrams, which was introduced by Casson and Gordon (14) (also ref. 22). A collection of $3g - 3$ pairwise disjoint and nonisotopic curves in a genus g surface Σ is called a pants decomposition, as the curves cut Σ into $2g - 2$ thrice-punctured spheres, or pairs of pants. A pants decomposition defines a handlebody in the same way a cut system does, although a cut system is a minimal collection of curves defining a handlebody, whereas a pants decomposition necessarily contains superfluous curves. An extended Heegaard diagram is a pair of pants decompositions (α^+, β^+) determining a Heegaard splitting $H_{\alpha^+} \cup H_{\beta^+}$. An extended trisection diagram is a triple of pants decompositions $(\alpha^+, \beta^+, \gamma^+)$ determining the spine $H_{\alpha^+} \cup H_{\beta^+} \cup H_{\gamma^+}$ of a trisection.

Suppose that α^+ and β^+ are pants decompositions of Σ , and let P_{α^+} be a component of $\Sigma \setminus \nu(\alpha^+)$ and P_{β^+} be a component of $\Sigma \setminus \nu(\beta^+)$. Let a_1, a_2 , and a_3 denote the boundary components of P_{α^+} and b_1, b_2 , and b_3 denote the boundary components of P_{β^+} . We say that the pair $(P_{\alpha^+}, P_{\beta^+})$ is saturated if for all $i, j, k, l \in \{1, 2, 3\}$, $i \neq j, k \neq l$, the intersection $P_{\alpha^+} \cap P_{\beta^+}$ contains a rectangle $R_{i,j,k,l}$ with boundary arcs contained in a_i, b_k, a_j , and b_l (Fig. 5, *Left*). We say that that pair of pants P_{α^+} is saturated with respect to β^+ if for every component P_{β^+} of $\Sigma \setminus \nu(\beta^+)$, the pair $(P_{\alpha^+}, P_{\beta^+})$ is saturated.

An extended Heegaard diagram (α^+, β^+) satisfies the Casson–Gordon rectangle condition if for every component P_{α^+} of $\Sigma \setminus \nu(\alpha^+)$, we have that P_{α^+} is saturated with respect to β^+ . Casson and Gordon (14) proved the following.

Theorem 20. *Suppose that an extended Heegaard diagram (α^+, β^+) satisfies the rectangle condition. Then the induced Heegaard splitting $H_{\alpha^+} \cup H_{\beta^+}$ is irreducible.*

Now, let $(\alpha^+, \beta^+, \gamma^+)$ be an extended trisection diagram. We say that $(\alpha^+, \beta^+, \gamma^+)$ satisfies the rectangle condition if for every component P_{α^+} of $\Sigma \setminus \nu(\alpha^+)$, we have that either P_{α^+} is saturated with respect to β^+ or P_{α^+} is saturated with respect to γ^+ .

Remark 21. *Note that since (α^+, β^+) and (α^+, γ^+) are extended Heegaard diagrams for the standard manifolds Y_{k_1} and Y_{k_3} , it is not possible for either pair to satisfy the rectangle condition of Casson and Gordon (14). In other words, it is not possible that every component P_{α^+} of α^+ be saturated with respect to, say, β^+ .*

Proposition 22. *Suppose that an extended trisection diagram $(\alpha^+, \beta^+, \gamma^+)$ satisfies the rectangle condition. Then the induced trisection $\mathcal{T}$ with spine $H_{\alpha^+} \cup H_{\beta^+} \cup H_{\gamma^+}$ is irreducible.*

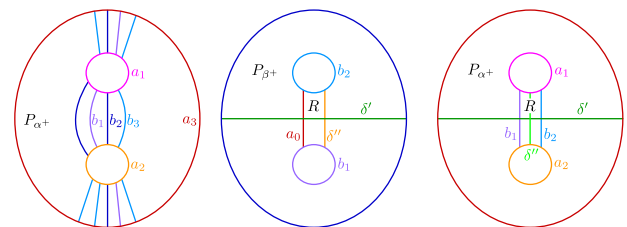


Fig. 5. (*Left*) A pair of pants P_{α^+} that is saturated with respect to a second pair of pants P_{β^+} . (*Center*) A depiction of the contradiction incurred under the assumption that $\delta \in \alpha^+$ but $\delta \notin \beta^+$. (*Right*) A depiction of the contradiction incurred under the assumption that $\delta \notin \alpha^+ \cup \beta^+ \cup \gamma^+$.

Proof: Suppose by way of contradiction that $\mathcal{T}$ is reducible. Then there exists a curve $\delta \subset \Sigma = \partial H_{\alpha+}$ that bounds disks $D_1 \subset H_{\alpha+}$, $D_2 \subset H_{\beta+}$, and $D_3 \subset H_{\gamma+}$. Let $D_{\alpha+}$ denote the set of $3g-3$ disks in $H_{\alpha+}$ bounded by the curves α^+ , and define $D_{\beta+}$ and $D_{\gamma+}$ similarly. There are several cases to consider. First, suppose that $\delta \in \alpha^+$, so that $D_1 \in D_{\alpha+}$, and let $P_{\alpha+}$ be a component of $\Sigma \setminus \nu(\alpha^+)$ that contains δ as a boundary component. Suppose without loss of generality that $P_{\alpha+}$ is saturated with respect to β^+ . Then, for any curve $b \in \beta^+$, we have that b is the boundary of a component $P_{\beta+}$ of $\Sigma \setminus \nu(\beta^+)$, where $P_{\alpha+} \cap P_{\beta+}$ contains a rectangle with boundary arcs in δ and b . It follows that δ meets every curve $b \in \beta^+$, so $\delta \notin \beta^+$.

Suppose that D_2 and $D_{\beta+}$ have been isotoped to intersect minimally, so that these disks meet in arcs by a standard argument. There must be an outermost arc of intersection in D_2 , which bounds a subdisk of D_2 with an arc $\delta' \subset \delta$, and δ' is a wave (an arc with both endpoints on the same boundary curve) contained in a single component $P_{\beta+}$ of $\Sigma \setminus \nu(\beta^+)$. Let b_1 and b_2 be the boundary components of $P_{\beta+}$ disjoint from δ' . Since $P_{\alpha+}$ is saturated with respect to β^+ , there is a rectangle $R \subset P_{\alpha+} \cap P_{\beta+}$ with boundary arcs contained in b_1 , δ , b_2 , and some other curve in $\partial P_{\alpha+}$ (Fig. 5, Center). Let δ'' be the arc component of ∂R contained in δ . Since the wave δ' separates b_1 from b_2 in $P_{\beta+}$, it follows that $\delta' \cap \delta'' \neq \emptyset$, a contradiction.

In the second case, suppose that δ is a curve in β^+ . Note that the Heegaard splitting determined by (α^+, γ^+) is reducible, and thus by the contrapositive of the Casson–Gordon rectangle condition, there must be some pair of pants $P_{\alpha+}$ of $\Sigma \setminus \nu(\alpha^+)$ such that $P_{\alpha+}$ is not saturated with respect to γ^+ , so that $P_{\alpha+}$ is saturated with respect to β^+ . Let $P_{\beta+}$ be a component of $\Sigma \setminus \nu(\beta^+)$ that contains δ as a boundary component. By the above argument, $\delta \notin \alpha^+$, and if we intersect D_1 with $D_{\alpha+}$, we can run an argument parallel to the one above to show that δ has a

self-intersection, a contradiction. A similar argument shows that $\delta \notin \gamma^+$.

Finally, suppose that δ is not contained in any of α^+ , β^+ , or γ^+ . By intersecting the disks D_1 and $D_{\alpha+}$, we see that there is a wave $\delta' \subset \delta$ contained in some pants component $P_{\alpha+}$ of $\Sigma \setminus \nu(\alpha^+)$. Suppose without loss of generality that $P_{\alpha+}$ is saturated with respect to β^+ . By intersecting D_2 with $D_{\beta+}$, we see that there is a wave $\delta'' \subset \delta$ contained in some pants component $P_{\beta+}$ of $\Sigma \setminus \nu(\beta^+)$. Let a_1 and a_2 be the components of $\partial P_{\alpha+}$ that avoid δ' , and let b_1 and b_2 be the components of $\partial P_{\beta+}$ that avoid δ'' . By the rectangle condition, $P_{\alpha+} \cap P_{\beta+}$ contains a rectangle R whose boundary is made of arcs in a_1 , b_1 , a_2 , and b_2 . As such, $\delta' \cap R$ contains an arc connecting b_1 to b_2 , while $\delta'' \cap R$ contains an arc connecting a_1 to a_2 , but this implies that $\delta' \cap \delta'' \neq \emptyset$, a contradiction. We conclude that no such curve δ exists. $\square$

Of course, at this time, the rectangle condition is a tool without an application, which elicits the following question.

Question 23. *Is there an extended trisection diagram that satisfies the rectangle condition?*

Note that while it is easy to find three pants decompositions that satisfy the rectangle condition, the difficulty lies in finding three such pants decompositions which also determine a trisection; in pairs, they must be extended Heegaard diagrams for the 3-manifolds Y_{k_i} .

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