Topology and Nesting of the Zero Set Components of Monochromatic Random Waves

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Abstract

This paper is dedicated to the study of the topologies and nesting configurations of the components of the zero set of monochromatic random waves. We prove that the probability of observing any diffeomorphism type and any nesting arrangement among the zero set components is strictly positive for waves of large enough frequencies. Our results are a consequence of building Laplace eigenfunctions in euclidean space whose zero sets have a component with prescribed topological type or an arrangement of components with prescribed nesting configuration. © 2018 Wiley Periodicals, Inc.

1 Introduction

For $n \geq 1$ let $E_1(\mathbb{R}^n)$ denote the linear space of entire (real-valued) eigenfunctions f of the Laplacian Δ whose eigenvalue is 1,

$$\Delta f + f = 0. \tag{1.1}$$

The zero set of f is the set

$$V(f) = \{x \in \mathbb{R}^n : f(x) = 0\}.$$

The zero set decomposes into a collection of connected components, which we denote by $\mathcal{C}(f)$. Our interest is in the topology of $V(f)$ and of the members of $\mathcal{C}(f)$. Let $H(n-1)$ denote the (countable and discrete) set of diffeomorphism classes of compact, connected, smooth $(n-1)$-dimensional manifolds that can be embedded in $\mathbb{R}^n$. The compact components c in $\mathcal{C}(f)$ give rise to elements $t(c)$ in $H(n-1)$ (here we are assuming that f is generic with respect to a Gaussian measure so that $V(f)$ is smooth; see Section 2). The connected components of $\mathbb{R}^n \setminus V(f)$ are the nodal domains of f, and our interest is in their nesting properties, again for generic f. To each compact $c \in \mathcal{C}(f)$ we associate a finite connected rooted tree as follows. By the Jordan-Brouwer separation theorem [11], each component $c \in \mathcal{C}(f)$ has an exterior and interior. We choose the interior to be the compact end. The nodal domains of f, which are in the interior of c, are taken to be the

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vertices of a graph. Two vertices share an edge if the respective nodal domains have a common boundary component (unique if there is one). This gives a finite, connected rooted tree denoted $e(c)$, the root being the domain adjacent to c (see Figure 4.1). Let $\mathcal{T}$ be the collection (countable and discrete) of finite connected rooted trees. Our main results are that any topological type and any rooted tree can be realized by elements of $E_1(\mathbb{R}^n)$.

THEOREM 1.1. *Given $t \in H(n-1)$ there exists $f \in E_1(\mathbb{R}^n)$ and $c \in \mathcal{C}(f)$ for which $t(c) = t$.*

THEOREM 1.2. *Given $T \in \mathcal{T}$ there exists $f \in E_1(\mathbb{R}^n)$ and $c \in \mathcal{C}(f)$ for which $e(c) = T$.*

Theorems 1.1 and 1.2 are of basic interest in the understanding of the possible shapes of nodal sets and domains of eigenfunctions in $\mathbb{R}^n$ (it applies equally well to any eigenfunction with eigenvalue $\lambda^2 > 0$ instead of 1). Our main purpose, however, is to apply it to derive a basic property of the universal monochromatic measures μ_C and μ_X whose existence was proved in [16]. We proceed to introduce these measures.

Let $(\mathbb{S}^n, g)$ be the n-sphere endowed with a smooth Riemannian metric g. Our results apply equally well with $\mathbb{S}^n$ replaced by any compact smooth manifold M; we restrict to $\mathbb{S}^n$ as it allows for a very clean formulation. Consider an orthonormal basis $\{\varphi_j\}_{j=1}^{\infty}$ for $L^2(\mathbb{S}^n, g)$ consisting of real-valued eigenfunctions, $\Delta_g \phi_j = -\lambda_j^2 \phi_j$. A monochromatic random wave on $(\mathbb{S}^n, g)$ is the Gaussian random field $f = f_{\eta,\lambda}$,

$$f := D_{\eta,\lambda}^{-1/2} \sum_{\lambda_j \in [\lambda, \lambda+\eta_\lambda]} a_j \phi_j, \tag{1.2}$$

where the a_j's are real-valued i.i.d. standard Gaussians, $a_j \sim N(0,1)_{\mathbb{R}}$, $\eta_\lambda = \eta(\lambda)$ is a nonnegative function satisfying $\eta(\lambda) = o(\lambda)$ as $\lambda \to \infty$, and $D_{\eta,\lambda} = \#\{j : \lambda_j \in [\lambda, \lambda + \eta]\}$. When choosing $\eta \equiv 0$ the λ's we consider in forming the $f_{\eta,\lambda}$'s are the square roots of the Laplace eigenvalues. To a monochromatic random wave we associate its (compact) nodal set $V(f)$ and a corresponding finite set of nodal domains. The connected components of $V(f)$ are denoted by $\mathcal{C}(f)$, and each $c \in \mathcal{C}(f)$ yields a $t(c) \in H(n-1)$. Each $c \in \mathcal{C}(f)$ also gives a tree end $e(c)$ in T that is chosen to be the smaller of the two rooted trees determined by the inside and outside of $c \subset \mathbb{S}^n$. The topology of $V(f)$ is described completely by the probability measure $\mu_{\mathcal{C}(f)}$ on $H(n-1)$ given by

$$\mu_{C(f)} := \frac{1}{|\mathcal{C}(f)|} \sum_{c \in \mathcal{C}(f)} \delta_{t(c)},$$

where δ_t is a point mass at $t \in H(n-1)$. Similarly, the distribution of nested ends of nodal domains of f is described by the measure $\mu_{X(f)}$ on T given by

$$\mu_{X(f)} := \frac{1}{|\mathcal{C}(f)|} \sum_{c \in \mathcal{C}(f)} \delta_{e(c)},$$

with δ_e being the point mass at $e \in \mathcal{T}$.

The main theorem in [16] asserts that there exist probability measures $\mu_{\mathcal{C}}$ and μ_X on $H(n-1)$ and $\mathcal{T}$, respectively, to which $\mu_{\mathcal{C}(f)}$ and $\mu_{X(f)}$ approach as $\lambda \to \infty$, for almost all $f = f_{\eta,\lambda}$, provided that for every $x_0 \in \mathbb{S}^n$

$$\sup_{u,v \in B(0,r_\lambda)} \left| \partial_u^k \partial_v^j \left[\operatorname{Cov}\left(f_{\eta,\lambda}^{x_0}(u), f_{\eta,\lambda}^{x_0}(v) \right) - \operatorname{Cov}\left(f_\infty^{x_0}(u), f_\infty^{x_0}(v) \right) \right] \right| = o(1) \tag{1.3}$$

as $\lambda \to \infty$. Here, $r_\lambda = o(\lambda)$, $f_{\eta,\lambda}^{x_0} : T_{x_0}\mathbb{S}^n \to \mathbb{R}$ is the localized wave on $T_{x_0}\mathbb{S}^n$ defined as $f_{\eta,\lambda}^{x_0}(u) = f_{\eta,\lambda}(\exp_{x_0}(\frac{u}{\lambda}))$, and $f_\infty^{x_0}$ is the Gaussian random field on $T_{x_0}\mathbb{S}^n$ characterized by the covariance kernel $\operatorname{Cov}(f_\infty^{x_0}(u), f_\infty^{x_0}(v)) = \int_{S_{x_0}\mathbb{S}^n} e^{i\langle u-v,w\rangle_{g_{x_0}}}\, dw$ (see Section 2). The probability measures $\mu_{\mathcal{C}}$ and μ_X are universal in that they only depend on the dimension n of M.

Monochromatic random waves on the n-sphere equipped with the round metric are known as random spherical harmonics whenever $\eta \equiv 0$. It is a consequence of the Mehler-Heine [12] asymptotics that they satisfy condition (1.3) for all $x_0 \in \mathbb{S}^n$. Also, on any $(\mathbb{S}^n, g)$ the fields $f_{\eta,\lambda}$ with $\eta \to \infty$ satisfy condition (1.3) for all $x_0 \in \mathbb{S}^n$. Finally, monochromatic random waves $f_{\eta,\lambda}$ on $(\mathbb{S}^n, g)$ with $\eta \equiv c$, for some $c > 0$, satisfy condition (1.3) for every $x_0 \in \mathbb{S}^n$ satisfying that the set of geodesic loops that close at x_0 has measure 0 (see [4]). On general manifolds one can define monochromatic random waves just as in $(\mathbb{S}^n, g)$. Monochromatic random waves with $\eta \equiv 0$ on the n-torus are known as arithmetic random waves. They satisfy condition (1.3) for all $x_0 \in \mathbb{T}^n$ if $n \geq 5$, and on $\mathbb{T}^n$ with $2 \leq n \leq 4$ provided we work with a density one subsequence of λ's [8]. On general (M, g) monochromatic random waves with $\eta \equiv c$ for some $c > 0$ satisfy condition (1.3) for every $x_0 \in M$ satisfying that the set of geodesic loops that close at x_0 has measure 0 (see [4]). Examples of such manifolds are surfaces without conjugate points and manifolds whose sectional curvature is negative everywhere.

Our main application of Theorems 1.1 and 1.2 is the following result.

THEOREM 1.3. *Let* $(\mathbb{S}^n, g)$ *be the* n*-sphere equipped with a smooth Riemannian metric. Let* $\mu_{\mathcal{C}}$ *and* μ_X *be the limit measures (introduced in* [16]*) arising from monochromatic random waves on* $(\mathbb{S}^n, g)$ *for which condition* (1.3) *is satisfied for every* $x_0 \in \mathbb{S}^n$.

(i) *The support of* $\mu_{\mathcal{C}}$ *is* $H(n-1)$. *That is, every atom of* $H(n-1)$ *is positively charged by* $\mu_{\mathcal{C}}$.
(ii) *The support of* μ_X *is all of* $\mathcal{T}$. *That is, every atom of* $\mathcal{T}$ *is positively charged by* μ_X.

Remark 1.4. Theorem 1.3 asserts that every topological type that can occur will do so with a positive probability for the universal distribution of topological types of random monochromatic waves in [16]. The reduction from Theorems 1.1 and 1.2 to Theorem 1.3 is abstract and is based on the "soft" techniques in [14, 16] (see also Section 2). In particular, it offers us no lower bounds for these probabilities. Developing such lower bounds is an interesting problem. The same applies to the tree ends.

Remark 1.5. Theorem 1.3 holds for monochromatic random waves on general compact, smooth Riemannian manifolds (M, g) without boundary. Part (i) actually holds without modification. The reason that we state the result on the round sphere $\mathbb{S}^n$ is that, by the Jordan-Brouwer separation theorem [11], on $\mathbb{S}^n$ every component of the zero set separates $\mathbb{S}^n$ into two distinct components. This gives that the nesting graph for the zero sets is a rooted tree. On general (M, g) this is not necessarily true, so there is no global way to define a tree that describes the nesting configuration of the zero set in all of M for all $c \in C(f)$. However, according to [15] almost all c's localize to small coordinate patches and hence our arguments apply.

We end the introduction with an outline of the paper. Theorem 1.1 for $n = 3$ (which is the first interesting case) is proved in [16] by deformation of the eigenfunction

$$u(x, y, z) = \sin(\pi x)\sin(\pi y) + \sin(\pi x)\sin(\pi z) + \sin(\pi y)\sin(\pi z). \tag{1.4}$$

The proof exploits that the space $H(2)$ is simply the set of orientable compact surfaces that are determined by their genus. So in engineering a component of a deformation of f to have a given genus it is clear what to aim for in terms of how the singularities (all are conic) of $f = 0$ resolve.

For $n \geq 4$, little is known about the space $H(n-1)$ and we proceed in Section 3 quite differently. We apply Whitney's approximation theorem to realize t as an embedded real analytic submanifold of $\mathbb{R}^n$. Then, following some techniques in [7] we find suitable approximations of $f \in E_1(\mathbb{R}^n)$ and whose zero set contains a diffeomorphic copy of t. The construction of f hinges on the Lax-Malgrange theorem and Thom's isotopy theorem. Regarding Theorem 1.2, the case $n = 2$ is resolved in [16] using a deformation of $\sin(\pi x)\sin(\pi y)$ and a combinatorial chessboard-type argument. In higher dimensions, for example $n = 3$, we proceed in Section 4 by deforming

$$u(x, y, z) = \sin(\pi x)\sin(\pi y)\sin(\pi z). \tag{1.5}$$

This f has enough complexity (as compared to the u in (1.4)) to produce all elements in $\mathcal{T}$ after deformation. However, it is much more difficult to study. Unlike (1.4) or $\sin(\pi x)\sin(\pi y)$, the zero set $u^{-1}(0)$ in (1.5) has point and 1-dimensional edge singularities. The analysis of its resolution under deformation requires a lot of care, especially as far as engineering elements of $\mathcal{T}$. The payoff as we noted is that it is rich enough to prove Theorem 1.2.

In Section 2 we review some of the theory of monochromatic Gaussian fields and their representations. Section 3 is devoted to the proof of Theorem 1.1. Section 4 is devoted to the proof of Theorem 1.2. The latter begins with an interpolation theorem of Mergelyan type for elements in $E_1(\mathbb{R}^n)$. We use that to engineer deformations of (1.5) that achieve the desired tree end, this being the most delicate aspect of the paper.

2 Monochromatic Gaussian Waves

Our interest is in the monochromatic Gaussian field on $\mathbb{R}^n$, which is a special case of the band-limited Gaussian fields considered in [16], and which is fundamental in the proof of [16, theorem 1.1]. For $0 \le \alpha \le 1$, define the annulus $A_\alpha = \{\xi \in \mathbb{R}^n : \alpha \le |\xi| \le 1\}$, and let σ_α be the Haar measure on A_α normalized so that $\sigma_\alpha(A_\alpha) = 1$. Using that the transformation $\xi \mapsto -\xi$ preserves A_α, we choose a real-valued orthonormal basis $\{\phi_j\}_{j=1}^\infty$ of $L^2(A_\alpha, \sigma_\alpha)$ satisfying

$$\phi_j(-\xi) = (-1)^{\eta_j}\phi_j(\xi), \quad \eta_j \in \{0,1\}. \tag{2.1}$$

The band-limited Gaussian field $H_{n,\alpha}$ is defined to be the random real-valued functions f on $\mathbb{R}^n$ given by

$$f(x) = \sum_{j=1}^\infty b_j i^{\eta_j} \widehat{\phi_j}(x) \tag{2.2}$$

where

$$\widehat{\phi_j}(x) = \int_{\mathbb{R}^n} \phi_j(\xi) e^{-i\langle x,\xi\rangle} d\sigma_\alpha(\xi), \tag{2.3}$$

and the b_j's are identically distributed, independent, real-valued, standard Gaussian variables. We note that the field $H_{n,\alpha}$ does not depend on the choice of the orthonormal basis $\{\phi_j\}$.

The distributional identity $\sum_{j=1}^\infty \phi_j(\xi)\phi_j(\eta) = \delta(\xi-\eta)$ on A_α together with (2.1) lead to the explicit expression for the covariance function:

$$\mathrm{Cov}(x,y) := \mathbb{E}_{H_{n,\alpha}}(f(x)f(y)) = \int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} d\sigma_\alpha(\xi). \tag{2.4}$$

From (2.4), or directly from (2.2), it follows that almost all f's in $H_{n,\alpha}$ are analytic in x [2]. For the monochromatic case $\alpha = 1$ we have

$$\mathrm{Cov}(x,y) = \frac{1}{(2\pi)^{n/2}} \frac{J_\nu(|x-y|)}{|x-y|^\nu}, \tag{2.5}$$

where to ease notation we have set

$$\nu := \frac{n-2}{2}.$$

Here there is also a natural choice of a basis for $L^2(\mathbb{S}^{n-1}, d\sigma) = L^2(A_1, \mu_1)$ given by spherical harmonics. Let $\{Y_m^\ell\}_{m=1}^{d_\ell}$ be a real-valued basis for the space of spherical harmonics $\mathcal{E}_\ell(\mathbb{S}^{n-1})$ of eigenvalue $\ell(\ell+n-2)$, where $d_\ell = \dim \mathcal{E}_\ell(\mathbb{S}^{n-1})$. We compute the Fourier transforms for the elements of this basis.

PROPOSITION 2.1. *For every* $\ell \geq 0$ *and* $m = 1, 2, \ldots, d_\ell$, *we have*

$$\widehat{Y_m^\ell|_{\mathbb{S}^{n-1}}}(x) = (2\pi)^{\frac{n}{2}}\, i^\ell\, Y_m^\ell\Big(\frac{x}{|x|}\Big)\frac{J_{\ell+\nu}(|x|)}{|x|^\nu}. \tag{2.6}$$

PROOF. We give a proof using the theory of point pair invariants [17], which places such calculations in a general and conceptual setting. The sphere $\mathbb{S}^{n-1}$, with its round metric, is a rank 1 symmetric space, and $\langle \dot{x}, \dot{y}\rangle$ for $\dot{x}, \dot{y} \in \mathbb{S}^{n-1}$ is a point pair invariant (here $\langle\,,\rangle$ is the standard inner product on $\mathbb{R}^n$ restricted to $\mathbb{S}^{n-1}$). Hence, by the theory of these pairs we know that for every function $h : \mathbb{R} \to \mathbb{C}$ we have

$$\int_{\mathbb{S}^{n-1}} h(\langle \dot{x}, \dot{y}\rangle)\, Y(\dot{y})\, d\sigma(\dot{y}) = \lambda_h(\ell) Y(\dot{x}), \tag{2.7}$$

where Y is any spherical harmonic of degree ℓ and $\lambda_h(\ell)$ is the spherical transform. The latter can be computed explicitly using the zonal spherical function of degree ℓ. Fix any $\dot{x} \in \mathbb{S}^{n-1}$ and let $Z_{\dot{x}}^\ell$ be the unique spherical harmonic of degree ℓ that is rotationally invariant by motions of $\mathbb{S}^{n-1}$ fixing $\dot{x}$ and so that $Z_{\dot{x}}^\ell(\dot{x}) = 1$. Then,

$$\lambda_h(\ell) = \int_{\mathbb{S}^{n-1}} h(\langle \dot{x}, \dot{y}\rangle) Z_{\dot{x}}^\ell(\dot{y})\, d\sigma(\dot{y}). \tag{2.8}$$

The function $Z_{\dot{x}}^\ell(\dot{y})$ may be expressed in terms of the Gegenbauer polynomials [9, (8.930)] as

$$Z_{\dot{x}}^\ell(\dot{y}) = \frac{C_\ell^\nu\big(\langle \dot{x}, \dot{y}\rangle\big)}{C_\ell^\nu(1)}. \tag{2.9}$$

Now, for $x \in \mathbb{R}^n$,

$$\widehat{Y_m^\ell|_{\mathbb{S}^{n-1}}}(x) = \int_{\mathbb{S}^{n-1}} h_x\big(\big\langle \tfrac{x}{|x|}, \dot{y}\big\rangle\big)\, Y_m^\ell(\dot{y}) d\sigma(\dot{y}),$$

where we have set $h_x(t) = e^{-i|x|t}$. Hence, by (2.7) we have

$$\widehat{Y_m^\ell|_{\mathbb{S}^{n-1}}}(x) = \lambda_{h_x}(\ell)\, Y_m^\ell\big(\tfrac{x}{|x|}\big),$$

with

$$\begin{aligned}\lambda_{h_x}(\ell) &= \int_{\mathbb{S}^{n-1}} e^{-i|x|\left\langle \frac{x}{|x|}, \dot{y}\right\rangle} Z_{\dot{x}}^\ell(\dot{y}) d\sigma(\dot{y}) \\ &= \frac{\mathrm{vol}(\mathbb{S}^{n-2})}{C_\ell^\nu(1)} \int_{-1}^1 e^{-it|x|}\, C_\ell^\nu(t)(1-t^2)^{\nu-\frac{1}{2}}\, dt.\end{aligned} \tag{2.10}$$

The last term in (2.10) can be computed using [9, (7.321)]. This gives

$$\lambda_{h_x}(\ell) = (2\pi)^{\frac{n}{2}}\, i^\ell\, \frac{J_{\ell+\nu}(|x|)}{|x|^\nu},$$

as desired. $\square$

COROLLARY 2.2. *The monochromatic Gaussian ensemble $H_{n,1}$ is given by random f's of the form*

$$f(x) = (2\pi)^{\frac{n}{2}} \sum_{\ell=0}^{\infty} \sum_{m=1}^{d_\ell} b_{\ell,m}\, Y_m^\ell\left(\frac{x}{|x|}\right) \frac{J_{\ell+\nu}(|x|)}{|x|^\nu},$$

where the $b_{\ell,m}$'s are i.i.d. standard Gaussian variables.

The functions

$$x \mapsto Y_m^\ell\left(\frac{x}{|x|}\right) \frac{J_{\ell+\nu}(|x|)}{|x|^\nu}, \quad x \mapsto e^{i\langle x,\xi\rangle},$$

with $|\xi| = 1$, and those in (2.2) for which the series converges rapidly (e.g., for almost all f in $H_{n,1}$), all satisfy (1.1), that is, $f \in E_1(\mathbb{R}^n)$. In addition, consider the subspaces P_1 and T_1 of $E_1(\mathbb{R}^n)$ defined by

$$P_1 := \operatorname{span}\left\{x \mapsto Y_m^\ell\left(\frac{x}{|x|}\right) \frac{J_{\ell+\nu}(|x|)}{|x|^\nu} : \ell \geq 0,\ m = 1, 2, \dots, d_\ell\right\},$$
$$T_1 := \operatorname{span}\left\{x \mapsto \frac{e^{i\langle x,\xi\rangle} + e^{-i\langle x,\xi\rangle}}{2},\ x \mapsto \frac{e^{i\langle x,\xi\rangle} - e^{-i\langle x,\xi\rangle}}{2i} : |\xi| = 1\right\}.$$

PROPOSITION 2.3. *Let $f \in E_1(\mathbb{R}^n)$ and let $K \subset \mathbb{R}^n$ be a compact set. Then, for any $t \geq 0$ and $\varepsilon > 0$ there are $g \in P_1$ and $h \in T_1$ such that*

$$\|f - g\|_{C^t(K)} < \varepsilon \quad \text{and} \quad \|f - h\|_{C^t(K)} < \varepsilon.$$

That is, we can approximate f on compact subsets in the C^t-topology by elements of P_1 and T_1, respectively.

PROOF. Let $f \in E_1$. Since f is analytic we can expand it in a rapidly convergent series in the Y_m^ℓ's. That is,

$$f(x) = \sum_{\ell=0}^{\infty} \sum_{m=1}^{d_\ell} a_{m,\ell}(|x|) Y_m^\ell\left(\frac{x}{|x|}\right).$$

Moreover, for $r > 0$,

$$\int_{\mathbb{S}^{n-1}} |f(r\dot{x})|^2\, d\sigma(\dot{x}) = \sum_{\ell=0}^{\infty} \sum_{m=1}^{d_\ell} |a_{m,\ell}(r)|^2. \tag{2.11}$$

In polar coordinates, $(r, \theta) \in (0, +\infty) \times \mathbb{S}^{n-1}$, the Laplace operator in $\mathbb{R}^n$ is given by

$$\Delta = \partial_r^2 + \frac{n-1}{r}\partial_r + \frac{1}{r^2}\Delta_{\mathbb{S}^{n-1}},$$

and hence for each ℓ, m we have that

$$r^2 a''_{m,\ell}(r) + (n-1) r a'_{m,\ell}(r) + (r^2 - \ell(\ell + n - 2)) a_{m,\ell}(r) = 0, \tag{2.12}$$

where ℓ is some positive integer. There are two linearly independent solutions to (2.12). One is $r^{-\nu} J_{\ell+\nu}(r)$ and the other blows up as $r \to 0$. Since the left-hand side of (2.11) is finite as $r \to 0$, it follows that the $a_{m,\ell}$'s cannot pick up any component of the blowing-up solution. That is, for $r \geq 0$

$$a_{m,\ell}(r) = c_{\ell,m} \frac{J_{\ell+\nu}(r)}{r^\nu}$$

for some $c_{m,\ell} \in \mathbb{R}$. Hence,

$$f(x) = \sum_{\ell=0}^{\infty} \sum_{m=1}^{d_\ell} c_{\ell,m}\, Y_m^\ell\left(\frac{x}{|x|}\right) \frac{J_{\ell+\nu}(|x|)}{|x|^\nu}. \tag{2.13}$$

Furthermore, this series converges absolutely and uniformly on compact subsets, as do its derivatives. Thus, f can be approximated by members of P_1 as claimed by simply truncating the series in (2.13).

To deduce the same for T_1 it suffices to approximate each fixed

$$Y_m^\ell\left(\frac{x}{|x|}\right) \frac{J_{\ell+\nu}(|x|)}{|x|^\nu}.$$

To this end let $\xi_1, -\xi_1, \xi_2, -\xi_2, \ldots, \xi_N, -\xi_N$ be a sequence of points in $\mathbb{S}^{n-1}$ that become equidistributed with respect to $d\sigma$ as $N \to \infty$. Then, as $N \to \infty$,

$$\frac{1}{2N} \sum_{j=1}^{N} \Big(e^{-i\langle x,\xi_j\rangle} Y_m^\ell(\xi_j) + (-1)^\ell e^{i\langle x,\xi_j\rangle} Y_m^\ell(\xi_j) \Big) \longrightarrow \int_{\mathbb{S}^{n-1}} e^{-i\langle x,\xi\rangle} Y_m^\ell(\xi)\, d\sigma(\xi). \tag{2.14}$$

The proof follows since

$$(2\pi)^{\frac{n}{2}} i^\ell Y_m^\ell\left(\frac{x}{|x|}\right) \frac{J_{\ell+\nu}(|x|)}{|x|^\nu} = \int_{\mathbb{S}^{n-1}} e^{-i\langle x,\xi\rangle} Y_m^\ell(\xi)\, d\sigma(\xi).$$

Indeed, the convergence in (2.14) is uniform over compact subsets in x. □

Remark 2.4. For $\Omega \subset \mathbb{R}^n$ open, let $E_1(\Omega)$ denote the eigenfunctions on Ω satisfying $\Delta f(x) + f(x) = 0$ for $x \in \Omega$. Any function g on Ω that is a limit (uniform over compact subsets of Ω) of members of E_1 must be in $E_1(\Omega)$. While the converse is not true in general, note that if $\Omega = B$ is a ball in $\mathbb{R}^n$, then the proof of Proposition 2.3 shows that the uniform limits of members of E_1 (or P_1 or T_1) on compact subsets in B is precisely $E_1(B)$.

With these equivalent means of approximating functions by suitable members of $H_{n,1}$, and particularly $E_1(\mathbb{R}^n)$, we are ready to prove Theorems 1.1 and 1.2. Indeed, as shown in [16] the extension of condition (ρ4) of [15, theorem 1] suffices. Namely, for $c \in H(n-1)$ it is enough to find an $f \in T_1$ with $f^{-1}(0)$ containing c as one of its components for Theorem 1.1, and for $T \in \mathcal{T}$ it suffices to find an $f \in T_1$ such that $e(c) = T$ for some component c of $f^{-1}(0)$.

3 Topology of the Zero Set Components

In this section we prove Theorem 1.1. By the discussion above it follows that given a representative c of a class $t(c) \in H(n-1)$, it suffices to find $f \in E_1(\mathbb{R}^n)$ for which $\mathcal{C}(f)$ contains a diffeomorphic copy of c.

PROOF OF THEOREM 1.1. To begin the proof we claim that we may assume that c is real analytic. Indeed, if we start with $\widetilde{c}$ smooth, of the desired topological type, we may construct a tubular neighborhood $V_{\widetilde{c}}$ of $\widetilde{c}$ and a smooth function

$$H_{\widetilde{c}} : V_{\widetilde{c}} \to \mathbb{R} \quad \text{with } \widetilde{c} = H_{\widetilde{c}}^{-1}(0).$$

Note that without loss of generality we may assume that $\inf_{x\in V_{\widetilde{c}}} \|\nabla H_{\widetilde{c}}(x)\| > 0$. Fix any $\epsilon > 0$. We apply Thom's isotopy theorem [1, theorem 20.2] to obtain the existence of a constant $\delta_{\widetilde{c}} > 0$ so that for any function F with $\|F - H_{\widetilde{c}}\|_{C^1(V_{\widetilde{c}})} < \delta_{\widetilde{c}}$ there exists $\Psi_F : \mathbb{R}^n \to \mathbb{R}^n$ diffeomorphism with

$$\Psi_F(\widetilde{c}) = F^{-1}(0) \cap V_{\widetilde{c}}.$$

To construct a suitable F we use Whitney's approximation theorem [19, lemma 6], which yields the existence of a real analytic approximation $F : V_{\widetilde{c}} \to \mathbb{R}^{m_{\widetilde{c}}}$ of $H_{\widetilde{c}}$ that satisfies $\|F - H_{\widetilde{c}}\|_{C^1(V_{\widetilde{c}})} < \delta_{\widetilde{c}}$. It follows that $\widetilde{c}$ is diffeomorphic to $c := \Psi_F(\widetilde{c})$ and c is real analytic as desired.

By the Jordan-Brouwer separation theorem [11], the hypersurface c separates $\mathbb{R}^n$ into two connected components. We write A_c for the corresponding bounded component of $\mathbb{R}^n \setminus c$. Let λ^2 be the first Dirichlet eigenvalue for the domain A_c and let h_λ be the corresponding eigenfunction:

$$\begin{cases} (\Delta + \lambda^2) h_\lambda(x) = 0, & x \in \overline{A_c}, \\ h_\lambda(x) = 0, & x \in c. \end{cases}$$

Consider the rescaled function

$$h(x) := h_\lambda(x/\lambda),$$

defined on the rescaled domain $\lambda A_c := \{x \in \mathbb{R}^n : x/\lambda \in A_c\}$. Since $(\Delta+1)h = 0$ in $\overline{\lambda A_c}$ and $\partial(\lambda A_c)$ is real analytic, h may be extended to some open set $B_c \subset \mathbb{R}^n$ with $\overline{\lambda A_c} \subset B_c$ so that

$$\begin{cases} (\Delta + 1) h(x) = 0, & x \in B_c, \\ h(x) = 0, & x \in \lambda c, \end{cases}$$

where λc is the rescaled hypersurface $\lambda c := \{x \in \mathbb{R}^n : x/\lambda \in c\}$. Note that since h_λ is the first Dirichlet eigenfunction, then we know that there exists a tubular neighborhood $V_{\lambda c}$ of λc on which $\inf_{x\in V_{\lambda c}} \|\nabla h(x)\| > 0$ (see lemma 3.1 in [3]). Without loss of generality assume that $V_{\lambda c} \subset B_c$.

We apply Thom's isotopy theorem [1, theorem 20.2] to obtain the existence of a constant $\delta > 0$ so that for any function f with $\|f - h\|_{C^1(V_{\lambda c})} < \delta$ there exists

$\Psi_f : \mathbb{R}^n \to \mathbb{R}^n$ diffeomorphism so that

$$\Psi_f(\lambda c) = f^{-1}(0) \cap V_{\lambda c}.$$

Since $\mathbb{R}^n \setminus B_c$ has no compact components, Lax-Malgrange's theorem [13, theorem 3.10.7] yields the existence of a global solution $f : \mathbb{R}^n \to \mathbb{R}$ to the elliptic equation $(\Delta + 1) f = 0$ in $\mathbb{R}^n$ with

$$\|f - h\|_{C^1(B_c)} < \delta.$$

We have then constructed a solution to $(\Delta+1) f = 0$ in $\mathbb{R}^n$, i.e., $f \in E_1$, for which $f^{-1}(0)$ contains a diffeomorphic copy of c (namely, $\Psi_f(\lambda c)$). This concludes the proof of the theorem. □

We note that finding a solution to $(\Delta + 1) f = 0$ for which $C(f)$ contains a diffeomorphic copy of c is related to the work [7] of A. Enciso and D. Peralta-Salas. In [7] the authors seek to find solutions to the problem $(\Delta - q) f = 0$ in $\mathbb{R}^n$ so that $C(f)$ contains a diffeomorphic copy of c, where q is a nonnegative, real analytic potential and c is a (possibly infinite) collection of compact or unbounded "tentacled" hypersurfaces. The construction of the solution f that we presented shares ideas with [7]. Since our setting and goals are simpler than theirs, the construction of f is much shorter and straightforward.

4 Nesting of Nodal Domains

The proof of Theorem 1.2 consists of perturbing the zero set of the eigenfunction $u_0(x_1, \dots, x_n) = \sin(\pi x_1) \cdots \sin(\pi x_n)$ so that the zero set of the perturbed function will have the desired nesting. The nodal domains of u_0 build an n-dimensional chessboard made out of unit cubes. By adding a small perturbation to u_0 the changes of topology in $u_0^{-1}(0)$ can only occur along the singularities of $u_0^{-1}(0)$. Therefore, we will build an eigenfunction f satisfying $-\Delta f = f$ by prescribing it along the singularities $L = \bigcup_{a,b \in \mathbb{Z}} \cup_{i,j=1, i \neq j}^{n} \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i = a,\ x_j = b\}$ of the zero set of u_0. We then construct a new eigenfunction $u_\varepsilon = u_0 + \varepsilon f$ that will have the desired nesting among a subset of its nodal domains. The idea is to prescribe f on the singularities of the zero set of u_0 in such a way that two adjacent cubes of the same sign will either glue or disconnect along the singularity. The following theorem shows that one can always find a solution f to $-\Delta f = f$ with prescribed values on a set of measure zero (such as L). We prove this result following the first step of Carleson's proof [5] of Mergelyan's classical theorem about analytic functions.

THEOREM 4.1. *Let $K \subset \mathbb{R}^n$ be a compact set with Lebesgue measure* 0 *and so that $\mathbb{R}^n \setminus K$ is connected. Then, for every $\delta > 0$ and $h \in C_c^2(\mathbb{R}^n)$ there exists $f : \mathbb{R}^n \to \mathbb{R}$ satisfying*

$$-\Delta f = f \quad \textit{and} \quad \sup_K \{|f - h| + \|\nabla f - \nabla h\|\} \le \delta.$$

Remark 4.2. In the statement of the theorem the function $h \in C_c^2(\mathbb{R}^n)$ can be replaced by $h \in C_c^1(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is any open set with $K \subset \Omega$. This is because $C_c^2(\mathbb{R}^n)$ is dense in $C_c^1(\Omega)$ in the C^1-topology.

PROOF. Consider the sets

$$\mathcal{A} = \{(\phi, \partial_{x_1}\phi, \dots, \partial_{x_n}\phi) : \phi \in \ker(\Delta + 1)\},$$
$$\mathcal{B} = \{(\phi, \partial_{x_1}\phi, \dots, \partial_{x_n}\phi) : \phi \in C_c^2(\mathbb{R}^n)\},$$

and write $\mathcal{A}_K, \mathcal{B}_K$ for the restrictions of $\mathcal{A}, \mathcal{B}$ to K. Both $\mathcal{A}_K$ and $\mathcal{B}_K$ are subsets of the Banach space $\bigoplus_{k=0}^n C(K)$, and clearly $\mathcal{A}_K \subset \overline{\mathcal{B}_K}^{\|\cdot\|_{C^0}}$. It follows that the claim in the theorem is equivalent to proving that

$$\mathcal{B}_K \subset \overline{\mathcal{A}_K}^{\|\cdot\|_{C^0}}. \tag{4.1}$$

To prove (4.1), note that a distribution D in the dual space $(\bigoplus_{k=0}^n C(K))^*$ can be identified with an $(n+1)$-tuple of measures $(\nu_0, \nu_1, \dots, \nu_n)$ with $\nu_j \in (C(K))^*$ for each $j = 0, 1, \dots, n$. That is, for each $(\psi_0, \psi_1, \dots, \psi_n) \in \bigoplus_{j=0}^n C(K)$,

$$D(\psi_0, \psi_1, \dots, \psi_n) = \sum_{j=0}^n \int_K \psi_j \, d\nu_j. \tag{4.2}$$

Since $\overline{\mathcal{A}_K}^{\|\cdot\|_{C^0}} = (\mathcal{A_K}^\perp)^\perp$, proving (4.1) is equivalent to showing that for each $D \in (\bigoplus_{k=0}^n C(K))^*$ satisfying $D(\Phi) = 0$ for all $\Phi \in \mathcal{A}_K$, one has that $D(\Phi) = 0$ for all $\Phi \in \mathcal{B}_K$. Using that each $D \in (\bigoplus_{k=0}^n C(K))^*$ is supported in K, we have reduced our problem to showing that

$$\begin{aligned}&\text{If } D \in (\textstyle\bigoplus_{k=0}^n C(K))^* \text{ satisfies } D(\Psi) = 0 \ \forall \Psi \in \mathcal{A},\\ &\text{then } D(\Phi) = 0 \ \forall \Phi \in \mathcal{B}.\end{aligned} \tag{4.3}$$

We proceed to prove the claim in (4.3). Fix $D \in (\bigoplus_{k=0}^n C(K))^*$ so that it satisfies the assumption in (4.3). Given $\phi \in C_c^2(\mathbb{R}^n)$ we need to prove that $D(\phi, \partial_{y_1}\phi, \dots, \partial_{y_n}\phi) = 0$. Consider the fundamental solution

$$N(x, y) := \frac{1}{n(n-2)\omega_n} \frac{1}{|x-y|^{n-2}},$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. Note that there exists $C > 0$ so that

$$\left|\frac{\partial N}{\partial y_j}(x, y)\right| < \frac{C}{|x-y|^{n-1}}$$

for all $j = 0, 1, \dots, n$. Therefore, for y fixed, $N(x, y)$ and $\frac{\partial N}{\partial y_j}(x, y)$ are locally integrable in $\mathbb{R}^n$. In particular, $N(x, y)|d\nu_0(y)|dx$ and $\frac{\partial N}{\partial y_j}(x, y)|d\nu_j(y)|dx$ are integrable on the product $K \times \mathbb{R}^n$, where the ν_j's are as in (4.2). Also, note that

$$\phi(y) = \int_{\mathbb{R}^n} (\Delta + 1)\phi(x) N(x, y) dx \quad \text{and}$$
$$\frac{\partial \phi}{\partial y_j}(y) = \int_{\mathbb{R}^n} (\Delta + 1)\phi(x) \frac{\partial N}{\partial y_j}(x, y) dx.$$

By these observations, and since K has measure zero, we may apply Fubini to get

$$\begin{aligned}
&D(\phi, \partial_{y_1}\phi, \dots, \partial_{y_n}\phi) \\
&\quad = \int_K \phi(y) d\nu_0(y) + \sum_{j=1}^n \int_K \frac{\partial \phi}{\partial y_j}(y) d\nu_j(y) \\
&\quad = \int_K \int_{\mathbb{R}^n \setminus K} (\Delta + 1)\phi(x) N(x, y) dx \, d\nu_0(y) \\
&\qquad + \sum_{j=1}^n \int_K \int_{\mathbb{R}^n \setminus K} (\Delta + 1)\phi(x) \frac{\partial N}{\partial y_j}(x, y) dx \, d\nu_j(y) \\
&\quad = \int_{\mathbb{R}^n \setminus K} \int_K (\Delta + 1)\phi(x) N(x, y) dx \, d\nu_0(y) \\
&\qquad + \sum_{j=1}^n \int_{\mathbb{R}^n \setminus K} \int_K (\Delta + 1)\phi(x) \frac{\partial N}{\partial y_j}(x, y) dx \, d\nu_j(y) \\
&\quad = \int_{\mathbb{R}^n \setminus K} (\Delta + 1)\phi(x) F(x) dx,
\end{aligned}$$

where

$$F(x) := \int_K N(x, y) d\nu_0(y) + \sum_{j=1}^n \int_K \frac{\partial N}{\partial y_j}(x, y) d\nu_j(y).$$

The claim that $D(\phi, \partial_{y_1}\phi, \dots, \partial_{y_n}\phi) = 0$ follows from the fact that $F(x) = 0$ for $x \in \mathbb{R}^3 \setminus K$. To see this, let $R > 0$ be large enough so that $K \subset B(0, R)$. Then, for $x \in \mathbb{R}^n \setminus B(0, R)$, the map $\psi^x(y) := N(x, y)$ is in $\ker(\Delta+1)|_{B(0,R)}$. Applying Proposition 2.3, we know that there exists a sequence $\{\psi_\ell^x\}_\ell \subset \ker(\Delta + 1)$ for which

$$\left\| \psi_\ell^x - \psi^x \right\|_{C^1(B(0,R))} \overset{\ell \to \infty}{\longrightarrow} 0.$$

Hence, by the assumption in (4.3), for each $x \in \mathbb{R}^n \setminus B(0, R)$

$$\begin{aligned}
0 &= D(\psi^x, \partial_{y_1}\psi^x, \dots, \partial_{y_n}\psi^x) \\
&= \int_K N(x, y) d\nu_0(y) + \sum_{j=1}^n \int_K \frac{\partial N}{\partial y_j}(x, y) d\nu_j(y) = F(x).
\end{aligned} \tag{4.4}$$

Now, the integral defining $F(x)$ converges absolutely for $x \in \mathbb{R}^n \setminus K$ and defines an analytic function of x in this set. Since $F(x)$ vanishes for $x \in \mathbb{R}^n \setminus B(0, R)$ and

$\mathbb{R}^n \setminus K$ is connected, it follows that

$$F(x) = 0 \quad \text{for all } x \in \mathbb{R}^n \setminus K,$$

as claimed. □

4.1 Construction of the Rough Domains

We will give a detailed proof Theorem 1.2 in $\mathbb{R}^3$ since in this setting it is easier to visualize how the argument works. In Section 4.6 we explain the modifications one needs to carry in order for the same argument to hold in $\mathbb{R}^n$.

Let $u_0 : \mathbb{R}^3 \to \mathbb{R}$ be defined as

$$u_0(x, y, z) = \sin(\pi x) \sin(\pi y) \sin(\pi z).$$

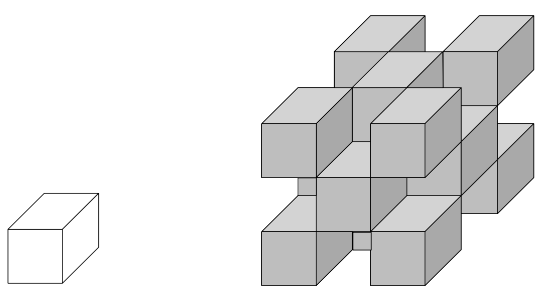
Its nodal domains consist of a collection of cubes whose vertices lie on the grid $\mathbb{Z}^3$. Throughout this note the cubes are considered to be closed sets, so faces and vertices are included. We say that a cube is positive (resp., negative) if u_0 is positive (resp., negative) when restricted to it. We define the collection $\mathcal{B}^+$ of all sets Ω that are built as a finite union of cubes with the following two properties:

- $\mathbb{R}^3 \setminus \Omega$ is connected.
- All the cubes in Ω that have a face in $\partial\Omega$ are positive.

We define $\mathcal{B}^-$ in the same way only that the faces in $\partial\Omega$ should belong to negative cubes.

In what follows, we define the "engulf" and "join" operations on the set of cubes that will allow us to create any nesting configuration. These operations are inspired by those introduced in [16] but do not coincide with them despite having the same names.

Engulf operation. Let $C \in \mathcal{B}^+$. We define the "engulf" operation as follows. We define $E(C)$ to be the set obtained by adding to C all the negative cubes that touch C, even if they share only one point with C. By construction $E(C) \in \mathcal{B}^-$. If $C \in \mathcal{B}^-$, the set $E(C)$ is defined in the same form only that one adds positive cubes to C. In this case $E(C) \in \mathcal{B}^+$.



C $E(C)$

Join operation. Given $C \in \mathcal{B}^+ \cup \mathcal{B}^-$ we distinguish two vertices using the lexicographic order. Namely, for any set of vertices $\Gamma \subset \mathbb{Z}^3$, for $i \in \{1,2,3\}$ we set

$$A_i^{\min} = \{(x_1^*, x_2^*, x_3^*) \in \Gamma : x_i^* = \min\{x_i : (x_1, x_2, x_3) \in \Gamma\}\} \subset \mathbb{Z}^3.$$

In the same way we define $A_i^{\max}$, replacing the minimum function above by the maximum one. For $C \in \mathcal{B}^+ \cup \mathcal{B}^-$, let $\Gamma_C = C \cap \mathbb{Z}^3$ be the set of vertices of cubes in C. We then set

$$v_+(C) = A_1^{\max}(A_2^{\max}(A_3^{\max}(\Gamma_C))) \quad \text{and} \quad v_-(C) = A_1^{\min}(A_2^{\min}(A_3^{\min}(\Gamma_C))).$$

Given the vertex $v_+(C)$ we define the edge $e_+(C)$ to be the edge in ∂C that has vertex $v_+(C)$ and is parallel to the x-axis. The edge $e_-(C)$ is defined in the same way.

We may now define the "join" operation. Given $C_1 \in \mathcal{B}^+$ and $C_2 \in \mathcal{B}^+$ we define $J(C_1, C_2) \in \mathcal{B}^+$ as follows. Let $\widetilde{C}_2$ be the translated copy of C_2 for which $e_+(C_1)$ coincides with $e_-(\widetilde{C}_2)$. We "join" C_1 and C_2 as

$$J(C_1, C_2) = C_1 \cup \widetilde{C}_2.$$

In addition, for a single set C we define $J(C) = C$, and if there are multiple sets $C_1, \dots, C_n$ we define

$$J(C_1, \dots, C_n) = J\big(C_1, J\big(C_2, J(C_3, \dots J(C_{n-1}, C_n))\big)\big).$$

DEFINITION 4.3 (Rooted trees). Let $T_\infty := \bigcup_{k=0}^{\infty} \mathbb{N}^k$. A *rooted tree* is characterized as a finite set of nodes $T \subset T_\infty$ satisfying

- $\varnothing \in T$,
- $(k_1, \dots, k_{\ell+1}) \in T \implies (k_1, \dots, k_\ell) \in T$,
- $(k_1, \dots, k_\ell, j) \in T \implies (k_1, \dots, k_\ell, i) \in T$ for all $i \le j$.

To condense notation, if $v \in T$ is a node with N children, we denote the children by $(v, 1), \dots, (v, N)$.

Given a tree T we associate to each node $v \in T$ a structure $C_v \subset \mathbb{R}^3$ defined as follows. If the node $v \in T$ is a leaf, then C_v is a cube with a prescribed sign. For the rest of the nodes we set

$$C_v = J\big(E(C_{(v,1)}), \dots, E(C_{(v,N)})\big),$$

where N is the number of children of the node v. See Figure 4.2 for an illustration of this operation. It is convenient to identify the original structures $E(C_{(v,j)})$ with the translated ones $\widetilde{E}(C_{(v,j)})$ that are used to build C_v. After this identification,

$$C_v := \bigcup_{j=1}^{N} E(C_{(v,j)}).$$

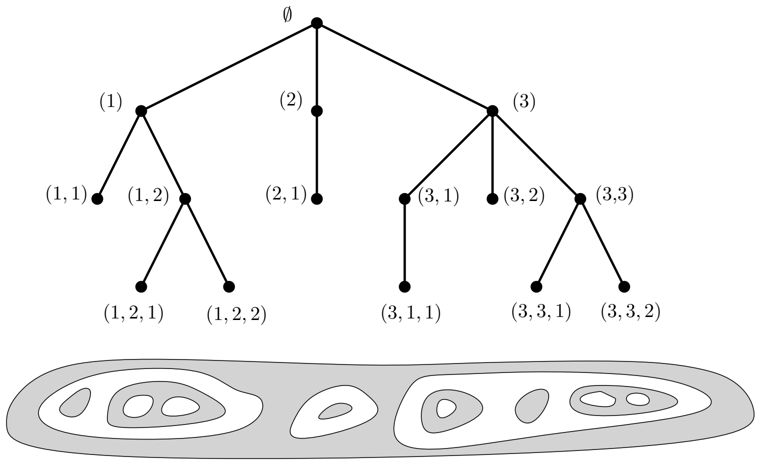


FIGURE 4.1. Example of a tree and a transversal cut of the corresponding nesting of nodal domains. All the domains in figures below are labeled after this example.

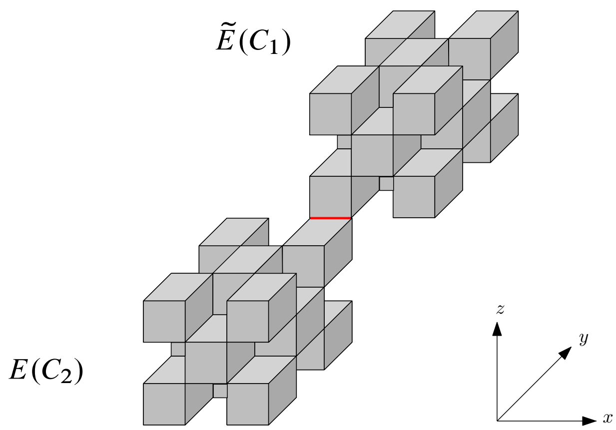


FIGURE 4.2. This picture shows $J(E(C_1), E(C_2))$. The edge $e_+(E(C_2) = e_-(\widetilde{E}(C_1)$ is depicted in red.

4.2 Building the Perturbation

Let $v \in T$ be a node with N children. We define the set of edges connected to C_v on which the perturbation will be defined.

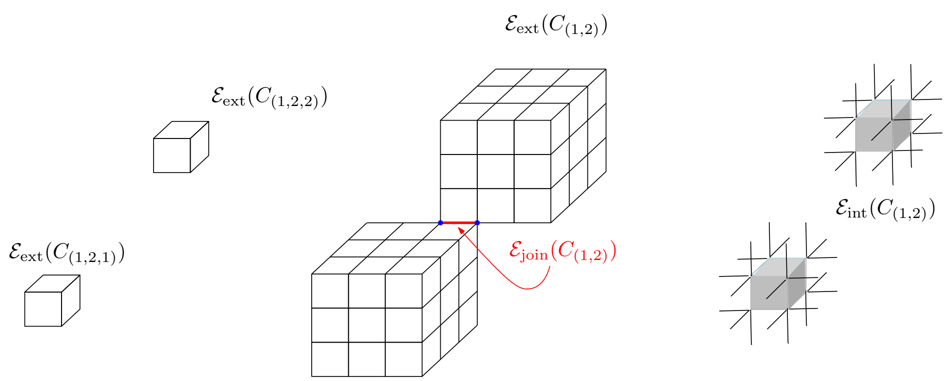
- We let $\mathcal{E}_{\text{join}}(C_v)$ be the set of edges in ∂C_v through which the structures $\{E(C_{(v,j)})\}_{j=1}^N$ are joined. We will take these edges to be open. That is, the edges in $\mathcal{E}_{\text{join}}(C_v)$ do not include their vertices.

- We let $\mathcal{E}_{\mathrm{ext}}(C_v)$ be the set of edges in $\mathcal{S}_{\mathrm{ext}}(C_v)$ that are not in $\mathcal{E}_{\mathrm{join}}(C_v)$. Here $\mathcal{S}_{\mathrm{ext}}(C_v)$ is the surface

$$\mathcal{S}_{\mathrm{ext}}(C_v) := \big\{x \in \mathbb{R}^3 : d_{\max}\big(x, \textstyle\bigcup_{j=1}^N C_{v,j}\big) = 1\big\}. \tag{4.5}$$

  If v is a leaf, we set $\mathcal{S}_{\mathrm{ext}}(C_v) = \partial C_v$. All the edges in $\mathcal{E}_{\mathrm{ext}}(C_v)$ are taken to be closed (so they include the vertices).
- We let $\mathcal{E}_{\mathrm{int}}(C_v)$ be the set of edges that connect $\mathcal{S}_{\mathrm{ext}}(C_v)$ with $\mathcal{S}_{\mathrm{ext}}(C_{v,j})$ for some $j \in \{1, 2, \ldots, N\}$. If v is a leaf, then we set $\mathcal{E}_{\mathrm{int}}(C_v) = \varnothing$.

Remark 4.4. Note that if $v \in T$ and $C_v \in \mathcal{B}^-$, then $E(C_v) \setminus C_v$ is the set of positive cubes that are in the bounded component of $\mathcal{S}_{\mathrm{ext}}(C_v)$ and touch $\mathcal{S}_{\mathrm{ext}}(C_v)$. Also, if a negative cube in $\mathbb{R}^3 \setminus C_v$ is touching C_v, then it does so through an edge in $\mathcal{E}_{\mathrm{ext}}(C_v)$.



Remark 4.5. Given a node v with children $\{(v, j)\}_{j=1}^N$, let $\mathcal{G}(C_{(v,j)})$ be the set of edges in $\{x \in \mathbb{R}^3 : d(x, C_{(v,j)}) = 1\}$. It is clear that for each $j = 1, 2, \ldots, N$ the set $\mathcal{G}(C_{(v,j)})$ is connected. Also, $\mathcal{E}_{\mathrm{ext}}(C_v) = \bigcup_{j=1}^N \mathcal{G}(C_{(v,j)}) \setminus \mathcal{E}_{\mathrm{join}}(C_v)$. Since the edges in $\mathcal{E}_{\mathrm{join}}(C_v)$ are open, the structures $\mathcal{E}_{\mathrm{ext}}(C_v)$ are connected.

We proceed to define a perturbation $h : K \to \mathbb{R}$, where

$$K = \bigcup_{v \in T} \mathcal{E}_{\mathrm{ext}}(C_v) \cup \mathcal{E}_{\mathrm{int}}(C_v) \cup \mathcal{E}_{\mathrm{join}}(C_v).$$

We note that by construction K is formed by all the edges in $C_\varnothing$. Also, it is important to note that if two adjacent cubes have the same sign, then they share an edge in K. The function h is defined by the rules A, B, and C below.

(A) *Perturbation on* $\mathcal{E}_{\mathrm{ext}}(C_v)$. Let $v \in T$ and assume $C_v \in \mathcal{B}^-$. We define h on every edge of $\mathcal{E}_{\mathrm{ext}}(C_v)$ to be 1. If $C_v \in \mathcal{B}^+$, we define h on every edge of $\mathcal{E}_{\mathrm{ext}}(C_v)$ to be -1.

Rule A is meant to separate C_v from all the exterior cubes of the same sign that surround it. Note that for all $v \in T$ we have $\mathcal{E}_{\mathrm{ext}}(C_v) \cap \mathcal{E}_{\mathrm{ext}}(C_{(v,j)}) = \varnothing$, where (v, j) is any of the children of v, so Rule A is well-defined.

(B) *Perturbation on* $\mathcal{E}_{\text{int}}(C_v)$. Let e be an edge in $\mathcal{E}_{\text{int}}(C_v)$. Then, we already know that h is 1 on one vertex and -1 on the other vertex. We extend h smoothly to the entire edge e, at which it has a unique zero at the midpoint of e, and so that the absolute value of the derivative of h is ≥ 1. We also ask for the derivative of h to be 0 at the vertices. For example, if the edge is $\{(a,b,z) : z \in [0,1]\}$ where $a,b,c \in \mathbb{Z}$, we could take $h(a,b,z) = \cos(\pi z)$.

Rule B is enforced to ensure that no holes are added between edges that join a structure C_v with any of its children structures $C_{(v,j)}$.

Next, assume $C_v \in B^-$. Note that for any edge e in $\mathcal{E}_{\text{join}}(C_v)$ we have that the function h takes the value 1 at their vertices, since those vertices belong to edges in $\mathcal{E}_{\text{ext}}(C_v)$ and the function h is defined to be 1 on $\mathcal{E}_{\text{ext}}(C_v)$. We have the same picture if $C_v \in \mathcal{B}^+$, only that h takes the value -1 on the vertices of all the joining edges. We therefore extend h to be defined on e as follows.

(C) *Perturbation on* $\mathcal{E}_{\text{join}}(C_v)$. Let $v \in T$ and assume $C_v \in \mathcal{B}^-$. Given an edge in $\mathcal{E}_{\text{join}}(C_v)$ we already know that h takes the value 1 at the vertices of the edge. We extend h smoothly to the entire edge so that it takes the value -1 at the midpoint of the edge, and so that it only has two roots at which the absolute value of the derivative of h is ≥ 1. We further ask h to have zero derivative at the endpoints of the edge. For example, if the edge is $\{(a,b,z) : z \in [c, c+1]\}$ where $a,b,c \in \mathbb{Z}$, we could take $h(a,b,z) = \cos(2\pi z)$. In the case in which $C_v \in \mathcal{B}^+$ we need h to take the value $+1$ at the midpoint of the edge.

Rule C is meant to glue the structures $\{E(C_{(v,j)})\}_{j=1}^N$ through the middle point of the edges that join them, without generating new holes.

Remark 4.6. By construction the function h is smooth in the interior of each edge. Furthermore, since we ask the derivative of h to vanish at the vertices in K, the function h can be extended to a function $h \in C^1(\Omega)$ where $\Omega \subset \mathbb{R}^3$ is an open neighborhood of K.

DEFINITION 4.7. Given a tree T, let $h \in C^1(\Omega)$ be defined following Rules A, B, and C and Remark 4.6, where $\Omega \subset \mathbb{R}^3$ is an open neighborhood of K. Since K is compact and $\mathbb{R}^3 \setminus K$ is connected, Theorem 4.1 gives the existence of $f : \mathbb{R}^3 \to \mathbb{R}$ that satisfies

$$-\Delta f = f \quad \text{and} \quad \sup_K \{|f - h| + \|\nabla f - \nabla h\|\} \leq \tfrac{1}{100}.$$

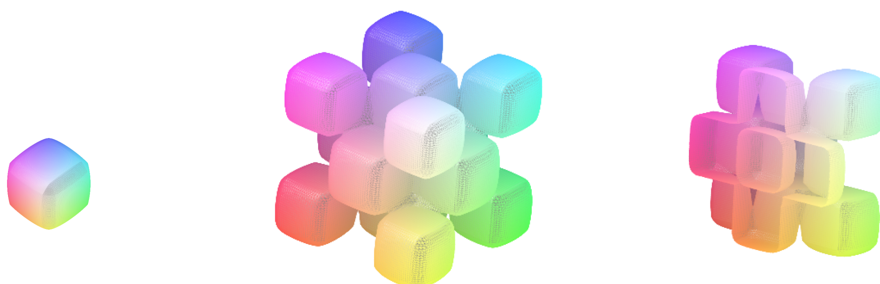
For $\varepsilon > 0$ small

$$u_\varepsilon := u_0 + \varepsilon f.$$

We will show in Lemma 4.9 that the perturbation was built so that the nodal domain of u_ε corresponding to $v \in T$ is constituted by the deformed cubes in $\bigcup_{j=1}^N E(C_{(v,j)}) \setminus C_{(v,j)}$ after the perturbation is performed.

We illustrate how Rules A, B, and C work in the following examples. In what follows we shall use repeatedly that the singularities of the zero set of u_0 are on the edges and vertices of the cubes. Therefore, the changes of topology in the zero set can only occur after perturbing the function u_0 along the edges and vertices of the cubes.

Example 1. As an example of how Rules A and B work, we explain how to create a domain that contains another nodal domain inside of it. The tree corresponding to this picture is given by two nodes, 1 and $(1,1)$, that are joined by an edge. We start with a positive cube $C_{(1,1)} \in \mathcal{B}^+$ and work with its engulfment $C_1 = E(C_{(1,1)}) \in \mathcal{B}^-$. All the edges of $C_{(1,1)}$ belong to $\mathcal{E}_{\text{ext}}(C_{(1,1)})$. Therefore, the function u_ε takes the value $-\varepsilon$ on $\mathcal{E}_{\text{ext}}(C_{(1,1)})$. Also, all the positive cubes that touch $C_{(1,1)}$ do so through an edge in $\mathcal{E}_{\text{ext}}(C_{(1,1)})$. It follows that all the positive cubes surrounding $C_{(1,1)}$ are disconnected from $C_{(1,1)}$ after the perturbation is performed. The cube $C_{(1,1)}$ then becomes a positive nodal domain $\Omega_{(1,1)}$ of u_ε that is contractible to a point.

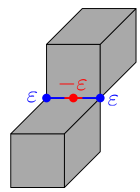


$\Omega_{(1,1)}$ Ω_1 transversal cut of Ω_1

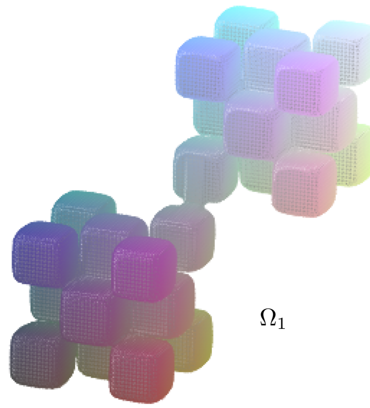
Next, note that all the negative cubes that touch $C_{(1,1)}$ (i.e., cubes in $E(C_{(1,1)}) \setminus C_{(1,1)}$) do so through a face whose edges are in $\mathcal{E}_{\text{ext}}(C_{(1,1)})$, or through a vertex that also belongs to one of the edges in $\mathcal{E}_{\text{ext}}(C_{(1,1)})$. Therefore, all the negative cubes are glued together after the perturbation is performed and belong to a nodal domain Ω_1 that contains the connected set $\mathcal{E}_{\text{ext}}(C_{(1,1)})$.

Up until this point we have seen that Ω_1 contains the perturbation of the cubes in $E(C_{(1,1)}) \setminus C_{(1,1)}$. We claim that no other cubes are added to Ω_1. Indeed, all the negative cubes that touch the boundary of $E(C_{(1,1)}) = C_1$ do so through edges in $\mathcal{E}_{\text{ext}}(C_1)$. Then, since u_ε takes the value ε on $\mathcal{E}_{\text{ext}}(C_1)$, all the surrounding negative cubes are disconnected from $E(C_{(1,1)})$ after we apply the perturbation. Since along the edges connecting $\partial C_{(1,1)}$ with ∂C_1 the function u_ε has only one sign change (it goes from $-\varepsilon$ to ε) it is clear that Ω_1 can be retracted to $\partial\Omega_{(1,1)}$.

Example 2. Here we explain how Rule C works. Suppose we want to create a nodal domain that contains two disjoint nodal domains inside of it. The tree corresponding to this picture is given by three nodes, 1, $(1,1)$, and $(1,2)$. The node 1 is joined by an edge to $(1,1)$ and by another edge to $(1,2)$. Assume that $C_{(1,1)}$ and $C_{(1,2)}$ belong to $\mathcal{B}^+$. Then, $C_1 = E(C_{(1,1)}) \cup E(C_{(1,2)}) \in \mathcal{B}^-$. When each of the structures $E(C_{(1,1)})$ or $E(C_{(1,1)})$ are perturbed, we get a copy of the negative nodal domain in Example 1. Since in C_1 the structures $E(C_{(1,1)})$ and $E(C_{(1,1)})$ are joined by an edge, the two copies of Ω_1 will also be glued. The reason for this is that the function u_ε takes the value $-\varepsilon$ in the middle point of the edge joining $E(C_{(1,1)})$ and $E(C_{(1,1)})$. Therefore, a small negative tube connects both structures.



joining cubes



Ω_1

4.3 Local Behavior of the Zero Set

In this section we explain what our perturbation does to the zero set of u_0 at a local level. Given a tree T and $\varepsilon > 0$, let

$$u_\varepsilon = u_0 + \varepsilon f$$

be defined as in Definition 4.7. Using that f is a continuous function and that we are working on a compact region of $\mathbb{R}^n$ (we call it D), it is easy to see that there exists a $\delta_0 > 0$, so that if $\mathcal{T}_\delta$ is the δ-tubular neighborhood of K, then u_ε has no zeros in $\mathcal{T}_\delta^c \cap C_\varnothing$ as long as $\delta \le \delta_0$ and

$$\varepsilon = c_1 \delta^2,$$

where c_1 is some positive constant that depends only on $\|f\|_{C^0(D)}$. This follows after noticing that $|u_0|$ takes the value 1 at the center of each cube and decreases radially until it takes the value 0 on the boundary of the cube.

The construction of the tubular neighborhood $\mathcal{T}_\delta$ yields that in order to understand the behavior of the zero set of u_ε, we may restrict ourselves to study it inside $\mathcal{T}_\delta$ for $\delta \le \delta_0$. We proceed to study the zero set of u_ε in a δ-tubular neighborhood of each edge in K. Assume, without loss of generality, that the edge is the set of points $\{(0,0,z) : z \in [0,1]\}$.

Vertices. At the vertex $(0,0,0)$ the function h takes the value 1 or -1. Assume $h(0,0,0) = -1$ (the study when the value is 1 is identical). In this case, we claim that the zero set of $u_\varepsilon(x,y,z)$ near the vertex is diffeomorphic to that of the function $\ell_\varepsilon(x,y,z) := u_0(x,y,z) - \varepsilon$ provided δ (and hence $\varepsilon = \varepsilon(\delta)$) is small enough. To see this, for $\eta > 0$ set V_η to be one of the connected components of $u_\varepsilon^{-1}(B(0,\eta))$ intersected with $\mathcal{T}_\delta$.

We apply the version of Thom's isotopy theorem given in [7, theorem 3.1], which asserts that for every smooth function ℓ satisfying

$$\|u_\varepsilon - \ell\|_{C^1(V_\eta)} \leq \min\{\eta/4,\ 1,\ \inf_{V_\eta} \|\nabla u_\varepsilon\|\}, \tag{4.6}$$

there exists a diffeomorphism $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ making

$$\Phi(u_\varepsilon^{-1}(0) \cap V_\eta) = \ell^{-1}(0) \cap V_\eta.$$

We observe that the statement of [7, theorem 3.1] gives the existence of an $\alpha > 0$ so that the diffeomorphism can be built provided $\|\ell_\varepsilon - u\|_{C^1(V_\eta)} \leq \alpha$. However, it can be tracked from the proof that α can be chosen to be as in the right-hand side of (4.6).

Applying [7, theorem 3.1] to the function ℓ_ε, we obtain what we claim provided we can verify (4.6). First, note that $\|u_\varepsilon - \ell_\varepsilon\|_{C^1(V_\eta)} = \|\varepsilon(f-1)\|_{C^1(V_\eta)}$. It is then easy to check that

$$\|u_\varepsilon - \ell_\varepsilon\|_{C^1(V_\eta)} \leq c_2\varepsilon \tag{4.7}$$

for some $c_2 > 0$ depending only on $\|\nabla f\|_{C^0(D)}$. Next, we find a lower bound for the gradient of u_ε when restricted to the zero set $u_\varepsilon^{-1}(0)$. Note that for $(x,y,z) \in \mathcal{T}_\delta \cap u_\varepsilon^{-1}(0)$ we have

$$\begin{aligned} &\|\nabla u_\varepsilon(x,y,z)\| \\ &\quad = \varepsilon\big\|-\pi f(x,y,z)\big(\cot(\pi x), \cot(\pi y), \cot(\pi z)\big) + \nabla f(x,y,z)\big\| \\ &\quad \geq \varepsilon\left(\sqrt{\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}} - \|\nabla f(x,y,z)\|\right) + O(\varepsilon\delta) \\ &\quad \geq \varepsilon\left(\frac{1}{\delta} - \|\nabla f(x,y,z)\|\right) + O(\varepsilon\delta). \end{aligned} \tag{4.8}$$

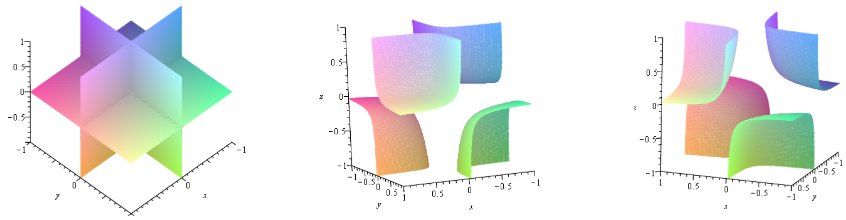
On the other hand, since $\|\nabla\langle \operatorname{Hess} u_\varepsilon(x,y,z), (x,y,z)\rangle\| = O(\eta)$ for all $(x,y,z) \in V_\eta$, we conclude

$$\inf_{V_\eta} \|\nabla u_\varepsilon\| > \varepsilon\left(\frac{1}{\delta} - \|\nabla f(x,y,z)\|\right) + O(\varepsilon\delta) + O(\eta) \tag{4.9}$$

whenever δ is small enough.

By using the bounds in (4.7) and (4.9), it is immediate to check that (4.6) holds provided we choose $\eta = c_3\varepsilon$ for a constant $c_3 > 0$ depending only on f and for δ small enough.

In the following image the first figure shows the zero set of u_0 near 0. The other two figures are of the zero set of $\ell_\varepsilon(x,y,z)$.

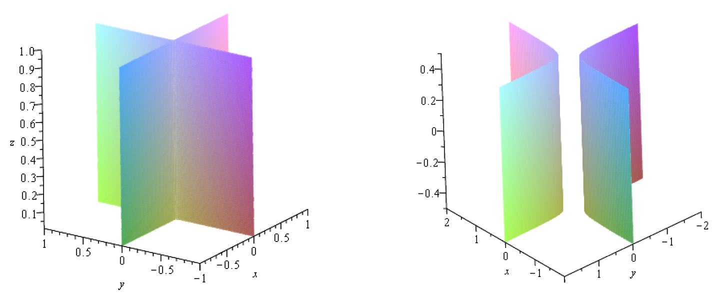


This shows that at each vertex where h takes the value -1 the negative cubes that touch the vertex are glued together while the positive ones are disconnected.

Edges. Having dealt with the vertices we move to describe the zero set of the perturbation near a point inside the edge.

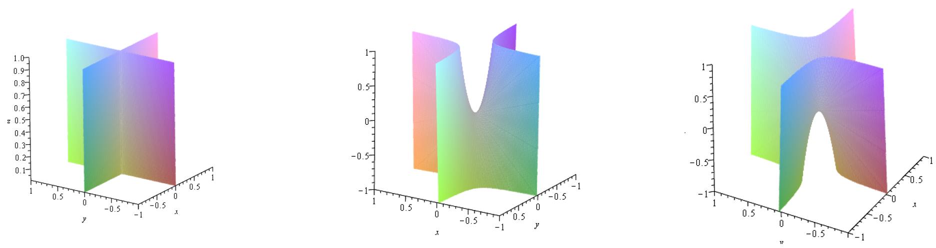
There are three cases. In the first case (case A) the perturbation h is strictly positive (approximately ε) or strictly negative (approximately $-\varepsilon$) along the edge. In the second case (case B) the perturbation f is strictly positive (approximately ε) at one vertex and strictly negative (approximately $-\varepsilon$) at the other vertex. In the third case (case C), the edge is joining two adjacent structures so the perturbation f takes the same sign at the vertices (it is approximately ε) and the opposite sign (it is approximately ε) at the midpoint of the edge having only two zeros along the edge.

In case A the zero set of $u_\varepsilon(x, y, z)$ near the edge is diffeomorphic to the zero set of the map $\ell_\varepsilon(x, y, z) := u_0(x, y, z) - \varepsilon$. The proof of this claim is the same as the one given near the vertices, so we omit it. In the picture below the first figure shows the zero set of u_0 near the edge while the second figure shows the zero set of ℓ_ε.



This shows that two cubes of the same sign, say negative, that are connected through an edge are going to be either glued if the perturbation takes the value -1 along the edge or disconnected if the perturbation takes the value $+1$ along the edge.

In case B, it is clear that the only interesting new behavior occurs near the points on the edge at which the function f vanishes. Since $\|h - f\|_{C^1(\Omega)} < \frac{1}{100}$ and $h(0, 0, b) = 0$, there is only one point at which f vanishes; say the point is $(0, 0, b)$. Note that f was built so that $(0, 0, b)$ is the only zero of f along the edge. We claim that the zero set of u_ε near $(0, 0, b)$ is diffeomorphic to the zero set of the map $\ell_\varepsilon(x, y, z) := u_0(x, y, z) - \varepsilon f(0, 0, z)$. The proof of this claim is similar to the one given near the vertices, so we omit it. The only relevant difference is that in order to bound $\|\nabla u_\varepsilon\|$ from below, one uses that $\|\nabla u_\varepsilon(x, y, z)\| \geq \|\varepsilon \nabla f(x, y, z)\| - \|\nabla u_0(x, y, z)\|$ and that $\|\nabla u_0(x, y, z)\| = O(\beta)$ in a ball of radius β centered at $(0, 0, b)$, while $\|\nabla f(0, 0, b)\| > 1 - \frac{1}{100}$. Of course, if one is away from the value $z = b$, then the analysis is the same as that of case A. The first figure in the picture below shows the zero set of u_0 along the edge, while the second figure shows the zero set of ℓ_ε when $f(0, 0, z) = \cos(\pi z)$.

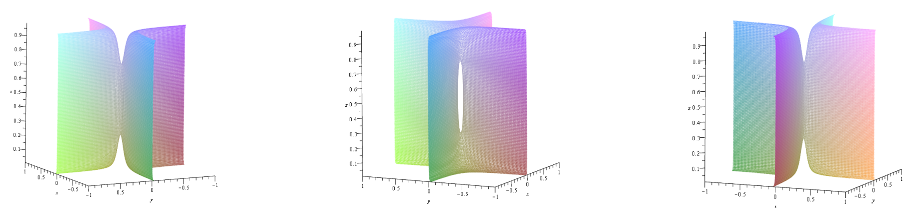


This shows that two consecutive cubes sharing an edge along which the perturbation changes sign will be glued on one half of the edge and disconnected along the other half.

In case C, the zero set of u_ε is diffeomorphic to that of

$$\ell_\varepsilon(x, y, z) = u_0(x, y, z) + \varepsilon f(0, 0, z)$$

where f satisfies $\|h - f\|_{C^1(\Omega)} < \frac{1}{100}$ and $h(0, 0, 0) = h(0, 0, 1) = 1$ and $h(0, 0, \frac{1}{2}) = -1$. The zero set of ℓ_ε when $f(0, 0, z) = \cos(2\pi z)$ is plotted in the figure below.



This shows that two cubes that are joining two consecutive structures will be glued through the midpoint while being disconnected at the vertices.

4.4 Definition of the Nodal Domains

Given a tree T and $\varepsilon > 0$ we continue to work with

$$u_\varepsilon = u_0 + \varepsilon f,$$

as defined in Definition 4.7. Fix $v \in T$, and suppose it has N children. Assume without loss of generality that $C_v \in \mathcal{B}^+$. For every $j \in \{1, 2, \ldots, n\}$ the perturbed function u_ε takes the value ε on $\mathcal{E}_{\text{ext}}(C_{(v,j)})$, and $\mathcal{E}_{\text{ext}}(C_{(v,j)})$ is connected. It follows that for each $j \in \{1, 2, \ldots, N\}$ there exists a positive nodal domain $\mathcal{N}_{(v,j)}$

of u_ε that contains $\mathcal{E}_{\text{ext}}(C_{(v,j)})$. We define the set $\Omega_v = \Omega_v(\varepsilon)$ as

$$\Omega_v := \bigcup_{j=1}^{N} \mathcal{N}_{(v,j)}. \tag{4.10}$$

Throughout this section we use the description of the local behavior of $u_\varepsilon^{-1}(0)$ that we gave in Section 4.3. In the following lemma we prove that Ω_v is a nodal domain of u_ε.

LEMMA 4.8. *Let T be a tree and for each $\varepsilon > 0$ let u_ε be the perturbation defined in* (4.7)*. Then, for each $\varepsilon > 0$ and $v \in T$, the set $\Omega_v = \Omega_v(\varepsilon)$ defined in* (4.10) *is a nodal domain of u_ε.*

PROOF. Let $v \in T$ and suppose v has N children. Assume without loss of generality that $C_v \in \mathcal{B}^-$. By definition, $\Omega_v = \bigcup_{j=1}^{N} \mathcal{N}_{(v,j)}$ where $\mathcal{N}_{(v,j)}$ is the nodal domain of u_ε that contains $\mathcal{E}_{\text{ext}}(C_{(v,j)})$. To prove that Ω_v is itself a nodal domain, we shall show that $\mathcal{N}_{(v,j)} = \mathcal{N}_{(v,j+1)}$ for all $j \in \{1, 2, \dots, N-1\}$.

Fix $j \in \{1, 2, \dots, N-1\}$. The structures $E(C_{(v,j)})$ and $E(C_{(v,j+1)})$ are joined through an edge e_j in $\mathcal{E}_{\text{join}}(C_v)$. If we name the middle point of e_j as m_j, then by rule C we have $u_\varepsilon(m_j) = \varepsilon f(m_j) < 0$.

The edge e_j is shared by a cube $c_j \in E(C_{(v,j)})$ and a cube $c_{j+1} \in E(C_{(v,j+1)})$. Note that every cube in $E(C_{(v,j)})$ has at least one vertex that belongs to an edge in $\mathcal{E}_{\text{ext}}(C_{(v,j)})$ (same with $E(C_{(v,j+1)})$). Let p_j be a vertex of c_j that belongs to an edge in $\mathcal{E}_{\text{ext}}(C_{(v,j)})$. In the same way we choose q_j to be a vertex in c_{j+1} that belongs to an edge in $\mathcal{E}_{\text{ext}}(C_{(v,j+1)})$. In particular, by rule A we have that $u_\varepsilon(p_j) < 0$ and $u_\varepsilon(q_j) < 0$.

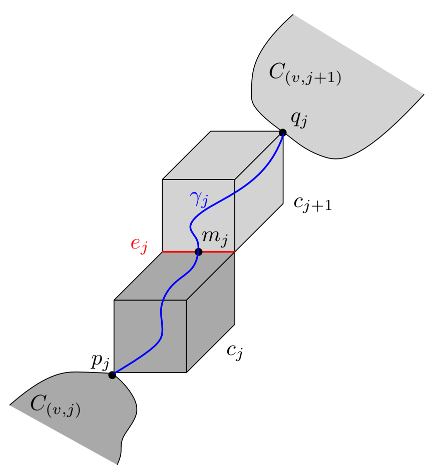


FIGURE 4.3.

Since c_j and c_{j+1} are negative cubes, there exists a curve $\gamma_j \subset u_\varepsilon^{-1}((-\infty,0))$ that joins p_j with q_j while passing through the middle point m_j. See Figure 4.2 for an illustration.

Finally, since $p_j \in \mathcal{E}_{\text{ext}}(C_{(v,j)}) \subset \mathcal{N}_{(v,j)}$, $q_j \in \mathcal{E}_{\text{ext}}(C_{(v,j+1)}) \subset \mathcal{N}_{(v,j+1)}$, and γ_j is a connected subset of $u_\varepsilon^{-1}((-\infty,0))$, we must have that $\mathcal{N}_{(v,j)} = \mathcal{N}_{(v,j+1)}$, as claimed. $\square$

In the following lemma we describe the set of cubes that end up building a nodal domain after the perturbation is performed.

LEMMA 4.9. *Let T be a tree and for each $\varepsilon > 0$ let u_ε be the perturbation defined in* (4.7). *For each $v \in T$ with N children we have*

$$\lim_{\varepsilon\to 0} \Omega_v(\varepsilon) = \bigcup_{j=1}^{N} E(C_{(v,j)}) \setminus C_{(v,j)}.$$

PROOF. First, we show that all the cubes in $\bigcup_{j=1}^{N} E(C_{(v,j)}) \setminus C_{(v,j)}$ glue together to form part of Ω_v after the perturbation is performed. Assume, without loss of generality, that $C_v \in \mathcal{B}^+$. Then, $C_{(v,j)} \in \mathcal{B}^-$ for every child (v,j) of v. All the cubes in $\bigcup_{j=1}^{N} E(C_{(v,j)}) \setminus C_{(v,j)}$ have an edge in $\mathcal{E}_{\text{ext}}(C_{(v,j)})$. Since such cubes are positive and u_ε takes the value ε on $\mathcal{E}_{\text{ext}}(C_{(v,j)})$, it follows that the cubes become part of the nodal domain that contains $\mathcal{E}_{\text{ext}}(C_{(v,j)})$. 1That is, all the cubes in $\bigcup_{j=1}^{N} E(C_{(v,j)}) \setminus C_{(v,j)}$ become part of Ω_v after the perturbation is added to u_0.

Second, we show that no cubes, other than those in $\bigcup_{j=1}^{N} E(C_{(v,j)}) \setminus C_{(v,j)}$, will glue together to form part of Ω_v. Indeed, any other positive cube in $\mathbb{R}^3 \setminus \bigcup_{j=1}^{N} E(C_{(v,j)})$ touching $\partial(\bigcup_{j=1}^{N} E(C_{(v,j)}))$ does so through an edge in $\mathcal{E}_{\text{ext}}(C_v)$. Since the function u_ε takes the value $-\varepsilon$ on $\mathcal{E}_{\text{ext}}(C_v)$, those cubes will disconnect from $\bigcup_{j=1}^{N} E(C_{(v,j)})$ after we perturb. On the other hand, any positive cube in $\bigcup_{j=1}^{N} C_{(v,j)} \in \mathcal{B}^-$ touches $\bigcup_{j=1}^{N} E(C_{(v,j)})$ through edges in $\bigcup_{i=1}^{N_j} \mathcal{E}_{\text{ext}}(C_{(v,j,i)})$ where N_j is the number of children of (v,j). Since f takes the value $-\varepsilon$ on $\bigcup_{i=1}^{N_j} \mathcal{E}_{\text{ext}}(C_{(v,j,i)})$, the cubes in $\bigcup_{j=1}^{N} C_{(v,j)}$ will also disconnect from

$$\bigcup_{j=1}^{N} E(C_{(v,j)}) \setminus C_{(v,j)}. \qquad \square$$

It is convenient to define the partial collections of nested domains. Given a tree T , a perturbation u_ε, and $v \in T$, we define the collection $\Omega_v^* = \Omega_v^*(\varepsilon)$ of all nodal domains that are descendants of Ω_v as follows. If v is a leaf then $\Omega_v^* = \Omega_v$. If v is not a leaf and has N children, we set

$$\Omega_v^* := \overline{\Omega_v} \cup \bigcup_{j=1}^{N} \Omega_{(v,j)}^*.$$

Remark 4.10. A direct consequence of Lemma 4.9 is the following. Let T be a tree, and for each $\varepsilon > 0$ let u_ε be the perturbation defined in (4.7). For each $v \in T$,

$$\lim_{\varepsilon \to 0} \Omega_v^*(\varepsilon) = C_v.$$

4.5 Proof of Theorem 1.2

We will use throughout this section that we know how the zero set behaves at a local scale (as described in Section 4.3). Let T be a tree, and for each $\varepsilon > 0$ let u_ε be the perturbation defined in (4.7). We shall prove that there is a subset of the nodal domains of u_ε that are nested as prescribed by T. Since for every $v \in T$ the set Ω_v is a nodal domain of u_ε, the theorem would follow if we had that for all $v \in T$:

(i) $\Omega^*_{(v,j)} \subset \mathrm{int}(\Omega^*_v)$ for every (v, j) child of v.
(ii) $\Omega^*_{(v,j)} \cap \Omega^*_{(v,k)} = \varnothing$ for all $j \neq k$.
(iii) $\mathbb{R}^3 \setminus \Omega^*_v$ has no bounded component.

Statements (i), (ii), and (iii) imply that $\mathbb{R}^3 \setminus \Omega_v$ has $N + 1$ components. One component is unbounded, and each of the other N components is filled by $\Omega^*_{(v,j)}$ for some j. We prove statements (i), (ii), and (iii) by induction. The statements are obvious for the leaves of the tree.

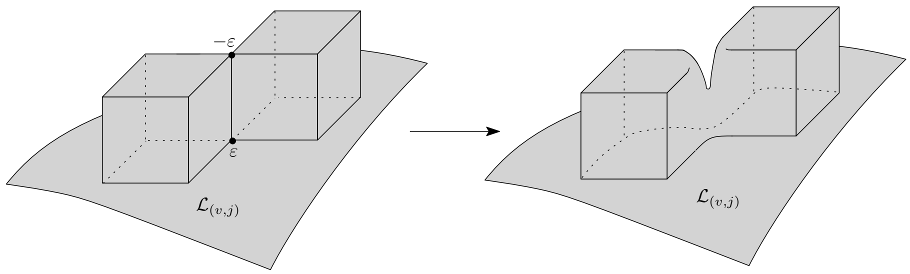
Remark 4.11. The proof of claim (iii) actually shows that Ω_v can be retracted to the arc connected set $\bigcup_{j=1}^N \Omega^*_{(v,j)} \cup \bigcup_{j=1}^{N-1} \gamma_j$ where $\gamma_j \subset \Omega_v$ is the curve introduced in Lemma 4.8 connecting $\mathcal{E}_{\text{ext}}(C_{(v,j)})$ with $\mathcal{E}_{\text{ext}}(C_{(v,j+1)})$ that passes through the midpoint of the edge joining $E(C_{(v,j)})$ with $E(C_{(v,j+1)})$.

Proof of Claim (i). Since $\Omega^*_v = \overline{\Omega_v} \cup \bigcup_{j=1}^N \Omega^*_{(v,j)}$, we shall show that there exists an open neighborhood $\mathcal{U}_{(v,j)}$ of $\Omega^*_{(v,j)}$ so that $\mathcal{U}_{(v,j)} \subset \Omega^*_v$.

Assume without loss of generality that $C_v \in \mathcal{B}^+$. Then, for every child (v, j), all the faces in $\partial C_{(v,j)}$ belong to cubes in $C_{(v,j)}$ that are negative. Also, all the other negative cubes in $\mathbb{R}^3 \setminus C_{(v,j)}$ that touch $\partial C_{(v,j)}$ do so through an edge in $\mathcal{E}_{\text{ext}}(C_{(v,j)})$. Since the function u_ε takes the value ε on $\mathcal{E}_{\text{ext}}(C_{(v,j)})$, all the negative cubes in $C_{(v,j)}$ are disconnected from those in $\mathbb{R}^3 \setminus C_{(v,j)}$ after the perturbation is performed. While all the negative cubes touching $C_{(v,j)}$ are disconnected, an open positive layer $\mathcal{L}_{(v,j)}$ that surrounds $\Omega^*_{(v,j)}$ is created. The layer $\mathcal{L}_{(v,j)}$ contains the grid $\mathcal{E}_{\text{ext}}(C_{(v,j)})$ and so it is contained inside Ω_v. The result follows from setting $\mathcal{U}_{(v,j)} := \mathcal{L}_{(v,j)} \cup \Omega^*_{(v,j)}$.

Proof of Claim (ii). This is a consequence of how we proved the statement (i) since both $\Omega^*_{(v,j)}$ and $\Omega^*_{(v,k)}$ are surrounded by a positive layer inside Ω^*_v.

Proof of Claim (iii). Note that $\lim_{\varepsilon \to 0} \bigcup_{j=1}^N \Omega^*_{(v,j)}(\varepsilon) = \bigcup_{j=1}^N C_{(v,j)}$ and that by the induction assumption $\mathbb{R}^3 \setminus \bigcup_{j=1}^N \Omega^*_{(v,j)}$ has no bounded components. On the other hand, we also have that $\lim_{\varepsilon \to 0} \Omega_v(\varepsilon) = \bigcup_{j=1}^N E(C_{(v,j)}) \setminus C_{(v,j)}$.



This shows that, in order to prove that $\mathbb{R}^3 \setminus \Omega_v^*$ has no bounded components, we should show that the cubes in $\bigcup_{j=1}^N E(C_{(v,j)}) \setminus C_{(v,j)}$ glue together to those in $\bigcup_{j=1}^N C_{(v,j)}$ leaving no holes. Note that all the cubes in $\bigcup_{j=1}^N E(C_{(v,j)}) \setminus C_{(v,j)}$ are attached to the mesh $\bigcup_{j=1}^N \mathcal{E}_{\mathrm{ext}}(C_{(v,j)})$ through some faces or vertices.

Assume without loss of generality that $C_v \in \mathcal{B}^+$. For each $j \in \{1, 2, \dots, N\}$ the layer $\mathcal{L}_{(v,j)}$ is contained in Ω_v and all the cubes in $E(C_v) \setminus C_v$ are glued to the layer through an entire face or vertex. The topology of Ω_v will depend exclusively on how the cubes in $E(C_{(v,j)}) \setminus C_{(v,j)}$ join with or disconnect from each other along the edges that start at $\mathcal{E}_{\mathrm{ext}}(C_{(v,j)})$ and end at a distance 1 from $\mathcal{E}_{\mathrm{ext}}(C_{(v,j)})$. The function u_ε takes the value ε on $\mathcal{E}_{\mathrm{ext}}(C_{(v,j)})$. Also, note that if a pair of positive cubes in the unbounded component of $\mathbb{R}^3 \setminus \mathcal{L}_{(v,j)}$ share an edge e that starts at $\mathcal{E}_{\mathrm{ext}}(C_{(v,j)})$ and ends at a distance 1 from it, then the end vertex belongs to $\mathcal{E}_{\mathrm{ext}}(C_v)$, and the function u_ε takes the value $-\varepsilon$ at this point. Since the function u_ε has only one root on e, we have that no holes are added to Ω_v when applying the perturbation to those two cubes. For cubes in the bounded component that share an edge, one argues similarly and uses the value of u_ε on $\bigcup_{i=1}^{N_j} \mathcal{E}_{\mathrm{ext}}(C_{v,j,i})$ where N_j is the number of children of (v, j).

To finish, we note that two consecutive structures $E(C_{(v,j)})$ and $E(C_{(v,j+1)})$ are joined through an edge separating two cubes as shown in Figure 4.3. The function u_ε is negative (approximately equal to $-\varepsilon$) at the vertices of the edge and is positive at the middle point (approximately equal to $+\varepsilon$). Since along the edge u_ε was prescribed to have only two roots, no holes are introduced when joining the structures.

4.6 Higher Dimensions

The argument in higher dimensions is analogous to the one in dimension 3. We briefly discuss the modifications that need to be made in this setting. Let

$$u_0(x_1, \dots, x_n) = \sin(\pi x_1) \cdots \sin(\pi x_n).$$

We will work with cubes in $\mathbb{R}^n$ that we identify with a point $c \in \mathbb{Z}^n$. That is, the cube corresponding to $c = (c_1, \dots, c_n) \in \mathbb{Z}^n$ is given by $c = \{x \in \mathbb{R}^n : x_k \in [c_k, c_k + 1]\}$. As before, we say that a cube is positive (resp., negative) if u_0 is positive (resp., negative) when restricted to it. The collection of faces of the cube

c is $\bigcup_{1\le i\le n}\cup_{x_i\in\{c_i,c_i+1\}}\{x\in\mathbb{R}^n:\ x_k\in[c_k,c_k+1]\ \forall k\neq i\}$. The collection of edges is

$$\bigcup_{1\le i,j\le n}\ \bigcup_{\substack{a_i\in\{c_i,c_i+1\}\\ a_j\in\{c_j,c_j+1\}}} H_c(a_i,a_j)$$

where each edge is described as the set

$$H_c(a_i,a_j)=\{x\in\mathbb{R}^n : x_i=a_i,\ x_j=a_j,\ x_k\in[c_k,c_k+1]\ \forall k\neq i,j\}.$$

We note that if two cubes of the same sign are adjacent, then they are connected through an edge or a subset of it. In analogy with the $\mathbb{R}^3$ case, we define the collection $\mathcal{B}^+$ of all sets Ω that are built as a finite union of cubes with the following two properties:

- $\mathbb{R}^n\setminus\Omega$ is connected.
- If c is a cube in $\mathcal{B}^+$ with a face in ∂B^+, then c must be a positive cube.

We define $\mathcal{B}^-$ in the same way only that the cubes with faces in $\partial\Omega$ should be negative cubes.

Engulf operation. Let $C\in\mathcal{B}^+$. We define $E(C)$ to be the set obtained by adding to C all the negative cubes that touch C, even if they share only one point with C. By construction $E(C)\in\mathcal{B}^-$. If $C\in\mathcal{B}^-$, the set $E(C)$ is defined in the same form only that one adds positive cubes to C. In this case $E(C)\in\mathcal{B}^+$.

Join operation. Given $C\in\mathcal{B}^+\cup\mathcal{B}^-$ we distinguish two vertices using the lexicographic order. For $C\in\mathcal{B}^+\cup\mathcal{B}^-$, let $\Gamma_C=C\cap\mathbb{Z}^n$ be the set of its vertices. We let $v_+(C)$ be the largest vertex in Γ_C and $v_-(C)$ be the smallest vertex in Γ_C. Given the vertex $v_+(C)$ we define the edge $e_+(C)$ to be the edge in ∂C that contains the vertex $v_+(C)$ and is parallel to the hyperplane defined by the $x_1,\dots,x_{n-2}$ coordinates. The edge $e_-(C)$ is defined in the same way.

Given $C_1\in\mathcal{B}^+$ and $C_2\in\mathcal{B}^+$ we define $J(C_1,C_2)\in\mathcal{B}^+$ as follows. Let $\widetilde{C}_2$ be the translated copy of C_2 for which $e_+(C_1)$ coincides with $e_-(\widetilde{C}_2)$. We "join" C_1 and C_2 as $J(C_1,C_2)=C_1\cup\widetilde{C}_2$.

In addition, for a single set C we define $J(C)=C$, and if there are multiple sets $C_1,\dots,C_n$ we define $J(C_1,\dots,C_n)=J(C_1,J(C_2,J(C_3,\dots J(C_{n-1},C_n))))$.

Definition of the rough nested domains. Given a tree T we associate to each node $v\in T$ a structure $C_v\subset\mathbb{R}^n$ defined as follows. If the node $v\in T$ is a leaf, then C_v is a cube of the adequate sign. For the rest of the nodes we set $C_v=J(E(C_{(v,1)}),\dots,E(C_{(v,N)}))$, where N is the number of children of the node v. We continue to identify the original structures $E(C_{(v,j)})$ with the translated ones $\widetilde{E}(C_{(v,j)})$ that are used to build C_v. After this identification,

$$C_9=\cup_{j=1}^N E(C_{(v,j)}).$$

Building the perturbation. Let $v \in T$ be a node with N children. We define the sets of edges $\mathcal{E}_{\text{ext}}(C_v)$, $\mathcal{E}_{\text{int}}(C_v)$, and $\mathcal{E}_{\text{join}}(C_v)$ in exactly the same way as we did in $\mathbb{R}^3$ (see Section 4.2). We proceed to define a perturbation $h : K \to \mathbb{R}$, where

$$K = \bigcup_{v \in T} \mathcal{E}_{\text{ext}}(C_v) \cup \mathcal{E}_{\text{int}}(C_v) \cup \mathcal{E}_{\text{join}}(C_v).$$

The function h is defined by the rules A, B, and C below. Let $\chi : [0, \infty] \to [-1, 1]$ be a smooth increasing function satisfying

$$\chi(0) = -1, \quad \chi(1/2) = 0, \quad \text{and} \quad \chi(t) = 1 \quad \text{for } t \geq 1.$$

We also demand

$$\chi'(0) = 0 \quad \text{and} \quad \chi'(1/2) \geq 1. \tag{4.11}$$

(A) *Perturbation on* $\mathcal{E}_{\text{ext}}(C_v)$. Let $v \in T$ and assume $C_v \in \mathcal{B}^-$. We define h on every edge of $\mathcal{E}_{\text{ext}}(C_v)$ to be 1. If $C_v \in \mathcal{B}^+$, we define h on every edge of $\mathcal{E}_{\text{ext}}(C_v)$ to be -1.

(B) *Perturbation on* $\mathcal{E}_{\text{int}}(C_v)$. Let $H_c(a_i, a_j)$ be an edge that touches both $\mathcal{E}_{\text{ext}}(C_v)$ and $\mathcal{E}_{\text{ext}}(C_{(v,\ell)})$ for some of the child structures $C_{(v,\ell)}$ of C_v. Assume $C_v \in \mathcal{B}^-$. Then we know that we must have $h|_{\mathcal{E}_{\text{ext}}(C_v)} = 1$ and $h|_{\mathcal{E}_{\text{ext}}(C_{(v,\ell)})} = -1$. Let $x_{i_1}, \dots, x_{i_k}$ be the set of directions in $H_c(a_i, a_j)$ that connect $\mathcal{E}_{\text{ext}}(C_v)$ and $\mathcal{E}_{\text{ext}}(C_{(v,\ell)})$. We let

$$h|_{H_c(a_i,a_j)} : H_c(a_i, a_j) \to [-1, 1]$$

be defined as

$$h(x_1, \dots, x_n) = \chi\left(\sqrt{\sum_{m=1}^{k} (x_{i_m} - c_{i_m})^2}\right).$$

With this definition, since whenever $x \in \mathcal{E}_{\text{ext}}(C_{(v,\ell)})$ we have $x_{i_m} = c_{i_m}$ for all $m = 1, 2, \dots, k$, we get $h(x) = \chi(0) = -1$. Also, whenever $x \in \mathcal{E}_{\text{ext}}(C_v)$ we have that there exists a coordinate x_{i_m} for which $x_{i_m} = c_{i_m} + 1$. Then, $\sum_m (x_{i_m} - c_{i_m})^2 \geq 1$ and so $h(x) = 1$. Note that h vanishes on the sphere $\mathcal{S} = \{x \in \mathbb{R}^n : \sum_{m=1}^{k} (x_{i_m} - c_{i_m})^2 = \frac{1}{4}\}$ and that $\|\nabla h\| \geq 1$ on $\mathcal{S}$ because of (4.11). If $C_v \in \mathcal{B}^+$, simply multiply χ by -1.

(C) *Perturbation on* $\mathcal{E}_{\text{join}}(C_v)$. Let $v \in T$ and assume $C_v \in \mathcal{B}^-$. We set

$$h(x_1, \dots, x_n) = \chi\left(2\sqrt{\sum_{k=1}^{n-2} \left(x_{i_k} - \frac{2c_{i_k} + 1}{2}\right)^2}\right),$$

where i_k ranges over the indices $\{1, 2, \dots, n\} \setminus \{i, j\}$. With this definition, whenever x is at the center of the edge $H_c(a_i, a_j)$ we have $h(x) = \chi(0) =$

-1. Also, if $x \in \partial H_c(a_i, a_j)$ we have $\big(x_k - \frac{2c_k+1}{2}\big)^2 = \frac{1}{4}$ for some k, and so $h(x) = 1$. Also note that h vanishes on a sphere of radius $\frac{1}{4}$ centered at the midpoint of $H_c(a_i, a_j)$ and that the gradient of h does not vanish on the sphere because of (4.11). If $C_v \in \mathcal{B}^+$, simply multiply χ by -1.

Remark 4.12. By construction the function h is smooth in the interior of each edge. Furthermore, since according to (4.11) we have $\chi'(0) = 0$ and $\chi'(1) = 0$, the gradient of h vanishes on the boundaries of the edges in K. Therefore, the function h can be extended to a function $h \in C^1(\Omega)$ where $\Omega \subset \mathbb{R}^n$ is an open neighborhood of K.

Given a tree T, let $h \in C^1(\Omega)$ be defined following Rules A, B, and C and Remark 4.12, where $\Omega \subset \mathbb{R}^n$ is an open neighborhood of K. Since K is compact and $\mathbb{R}^n \setminus K$ is connected, Theorem 4.1 gives the existence of $f : \mathbb{R}^n \to \mathbb{R}$ that satisfies

$$-\Delta f = f \quad \text{and} \quad \sup_K \{|f - h| + \|\nabla f - \nabla h\|\} \le \tfrac{1}{100}.$$

For $\varepsilon > 0$ small, set

$$u_\varepsilon := u_0 + \varepsilon f.$$

The definitions in Rules A, B, and C are analogues to those in dimension 3. For example, when working in dimension 3 on the edge $e = \{(0,0,z) : z \in [0,1]\}$, we could have set

$$h(0,0,z) = \chi(z) \quad \text{if } e \in \mathcal{E}_{\text{int}}(C_v) \text{ with } C_v \in \mathcal{B}^-$$

and

$$h(0,0,z) = \chi(2|z - 1/2|)) \quad \text{if } e \in \mathcal{E}_{\text{join}}(C_v) \text{ with } C_v \in \mathcal{B}^-.$$

Note that all the edges in $C_\varnothing$ are edges in K. Also, it is important to note that if two adjacent cubes have the same sign, then they share a subset of an edge in K.

If two adjacent cubes are connected through a subset of $\mathcal{E}_{\text{ext}}(C_v)$, then the cubes will be either glued or separated along that subset. This is because the function f is built to be strictly positive (approximately ε) or strictly negative (approximately $-\varepsilon$) along the entire edge.

If two adjacent cubes share an edge through which two structures are being joined, then they will be glued to each other near the midpoint of the edge. This is because f is built so that it has the same sign as the cubes in an open neighborhood of the midpoint of the joining edge.

If two adjacent cubes in C_v of the same sign share a subset of an edge in $H_c(a_i, a_j) \in \mathcal{E}_{\text{int}}(C_v)$, then with the same notation as in Rule B, there exists a subset of directions $\{x_{i_{m_1}}, \ldots, x_{i_{m_s}}\} \subset \{x_{i_1}, \ldots, x_{i_k}\}$ so that the set $\mathcal{R} = \{x \in H_c(a_i, a_j) : x_{i_{m_t}} \in [c_{i_{m_t}}, c_{i_{m_t}} + 1] \ \forall t = 1, 2, \ldots, s\}$ is shared by the cubes. By construction, the cubes will be glued through the portion $\mathcal{R}_1$ of $\mathcal{R}$ that joins $(c_{i_{m_1}}, \ldots, c_{i_{m_s}})$ with the point $(z_1, \ldots, z_s)$ near the midpoint $(c_{i_{m_1}} + \frac{1}{2}, \ldots, c_{i_{m_s}} + \frac{1}{2})$, while being disconnected through the portion $\mathcal{R}_2$ of $\mathcal{R}$ that joins the point $(z_1, \dots, z_s)$ with $(c_{i_{m_1}}+1, \dots, c_{i_{m_s}}+1)$. This is because f is prescribed to have the same sign as the cubes along $\mathcal{R}_1$, while taking the opposite sign of the cubes along $\mathcal{R}_2$.

Let $C_v \in B^-$, with $C_v = \bigcup_{j=1}^N E(C_{(v,\ell)})$. Using a similar argument to the one given in $\mathbb{R}^3$ one obtains that all the cubes in $\mathcal{E}_{\text{ext}}(C_{(v,\ell)}) \setminus C_{(v,\ell)}$ will glue together to form a negative nodal domain Ω_v of u_ε. We sketch the argument in what follows. All the negative cubes in $\mathbb{R}^n \setminus C_v$ that touch C_v do so through an edge in $\mathcal{E}_{\text{ext}}(C_v)$ since they will be at distance 1 from the children structures $\{C_{(v,\ell)}\}_\ell$. Since the perturbation f takes a strictly positive value (approximately $+\varepsilon$) along any edge in $\mathcal{E}_{\text{ext}}(C_v)$, the negative cubes in $\mathbb{R}^n \setminus C_v$ will be separated from those in C_v. Simultaneously, for each ℓ, all the cubes in $E(C_{(v,\ell)}) \setminus C_{(v,\ell)}$ are glued to each other since they are negative cubes that touch $\mathcal{E}_{\text{ext}}(C_{(v,\ell)})$, and $\mathcal{E}_{\text{ext}}(C_{(v,\ell)})$ is a connected set on which the perturbation f takes a strictly negative value (approximately $-\varepsilon$). This gives that $\mathcal{E}_{\text{ext}}(C_{(v,\ell)})$ belongs to a negative nodal domain of u_ε, and that the negative cubes in $E(C_{(v,\ell)}) \setminus C_{(v,\ell)}$ are glued to the nodal domain after the perturbation is performed.

Furthermore, two consecutive structures $E(C_{(v,\ell)})$ and $E(C_{(v,\ell+1)})$ are joined through an edge in $\mathcal{E}_{\text{int}}(C_v)$. This edge, which joins a negative cube in $E(C_{(v,\ell)})$ and a negative cube in $E(C_{(v,\ell+1)})$, has its boundary inside $\mathcal{E}_{\text{ext}}(C_{(v,\ell)})$. Since f is strictly positive (approximately $+\varepsilon$) on $\mathcal{E}_{\text{ext}}(C_{(v,\ell)})$, we know that the parts of the two cubes that are close to the boundary will be disconnected. However, since the perturbation was built so that f is strictly negative (approximately $-\varepsilon$) at the midpoint of the edge, both negative cubes are glued to each other. In fact, one can build a curve γ_ℓ contained inside the nodal domain that joins $\mathcal{E}_{\text{ext}}(C_{(v,\ell)})$ with $\mathcal{E}_{\text{ext}}(C_{(v,\ell+1)})$. It then follows that all the cubes in $\bigcup_{j=1}^N E(C_{(v,\ell)}) \setminus C_{(v,\ell)}$ are glued to each other after the perturbation is performed, and they will form the nodal domain Ω_v of u_ε containing $\bigcup_{\ell=1}^n \mathcal{E}_{\text{ext}}(C_{(v,\ell)})$. One can carry the same stability arguments we presented in Section 4.3 to obtain that at a local level there are no unexpected new nodal domains. For this to hold, as in the $\mathbb{R}^3$ case, the argument hinges on the fact that in the places where both u_0 and f vanish, the gradient of f is not zero (as explained at the end of each rule). Finally, Rule B is there to ensure that the topology of each nodal domain is controlled in the sense that when the cubes in $\mathcal{E}_{\text{ext}}(C_{(v,\ell)}) \setminus C_{(v,\ell)}$ glue to each other they do so without creating unexpected handles. Indeed, the cubes in $\mathcal{E}_{\text{ext}}(C_{(v,\ell)}) \setminus C_{(v,\ell)}$ can be retracted to the set $\bigcup_{\ell=1}^N \Omega^*_{(v,\ell)} \cup \bigcup_{\ell=1}^{N-1} \gamma_\ell$ where $\Omega^*_{(v,\ell)} := \overline{\Omega_{v,\ell}} \cup \bigcup_{\ell=1}^{N_\ell} \Omega^*_{(v,\ell,j)}$ and $\{(v,\ell,j) : j = 1, 2, \dots, N_\ell\}$ are the children of (v,ℓ).

The argument we just sketched also shows that the nodal domains Ω_v with $v \in T$ are nested as prescribed by the tree T. Indeed, claims (i), (ii), and (iii) in the proof of Theorem 1.2 are proved in $\mathbb{R}^n$ in exactly the same way we carried the argument in $\mathbb{R}^3$.

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