On singularity formation in a Hele-Shaw model

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ABSTRACT. We discuss a lubrication approximation model of the interface between two immiscible fluids in a Hele-Shaw cell, derived in [**CDG+93**] and widely studied since. The model consists of a single one dimensional evolution equation for the thickness $2h = 2h(x,t)$ of a thin neck of fluid,

$$\partial_t h + \partial_x (h\, \partial_x^3 h) = 0\,,$$

for $x \in (-1,1)$ and $t \geq 0$. The boundary conditions fix the neck height and the pressure jump:

$$h(\pm 1, t) = 1, \qquad \partial_x^2 h(\pm 1, t) = P > 0.$$

We prove that starting from smooth and positive h, as long as $h(x,t) > 0$, for $x \in [-1,1],\ t \in [0,T]$, no singularity can arise in the solution up to time T. As a consequence, we prove for any $P > 2$ and any smooth and positive initial datum that the solution pinches off in either finite or infinite time, *i.e.*, $\inf_{[-1,1]\times[0,T_*)} h = 0$, for some $T_* \in (0,\infty]$. These facts have been long anticipated on the basis of numerical and theoretical studies.
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1. Introduction

In the Hele-Shaw problem, two immiscible viscous fluids are placed in a narrow gap between two plates. Neglecting variations transversal to the plates, the problem is modeled by two dimensional incompressible and irrotational hydrodynamical equations. In the presence of surface tension, boundary conditions connect the mean curvature of the interface separating the two fluids to the pressure jump. The fluids form characteristic patterns [**ST58**]. The zero surface tension limit has been associated in the physical literature to Laplacian growth [**KMWZ04**], integrable systems [**MWWZ00**], and to diffusion-limited aggregation [**WS81, Vic84, Hal00**]. A dimension reduction, using lubrication approximation, leads to degenerate fourth order parabolic equations in one space dimension. The original derivations are related to wetting, thin films, and the triple junction between two fluids and a solid substrate (see [**DG85, SH88, ODB97, BEI+09**] and [**ED74, Gre78, Hoc81**]). Some of the mathematical papers related to the spreading of thin films and bubbles are [**BF90, BP96, BP98, GO03, GKO08, BW02, Knu15, KM15, GIM17**].

Our focus in this paper is on singularity formation. In this context, a one dimensional model for topology change in a Hele-Shaw set-up was discussed in [**CDG+93**]. The equation describes the evolution of the thickness h of a thin neck of fluid. The paper [**CDG+93**] derives the evolution equation of h using lubrication approximation, describes its variational dissipative structure and its steady states, and discusses the possibility of reaching zero thickness in finite or infinite time. This singularity formation was investigated theoretically and numerically in quite a number of studies. In [**DGKZ93**] a first numerical evidence of finite time pinch off was obtained. Systematic expansions and numerical results for a wider range of problems indicated finite time pinch off and velocity singularities in [**GPS93**]. A family of equations was considered in [**BBDK94**], numerical results supporting selfsimilar behavior were obtained, and finite or infinite time pinch off was asserted. In [**ED94**] numerical studies and physical arguments compared lubrication approximation equations to careful experiments of drop formation ([**CR80, CM80, PSS90**]). In [**CBEN99**] experiment and scaling near equal viscosities are accompanied by studies of the dependence of the breaking rate and shape of the drop on the viscosity ratio. A comprehensive survey of selfsimilar behaviors is given in [**EF08**], including a discussion of the pinch off scenarios presented on the basis of numerical evidence in [**ABB96**].

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In spite of the remarkable success of the dramatically reduced model obtained by lubrication approximation (see (1.1)–(1.2) below) to quantitatively describe experimental reality, as evidenced by numerical studies and theoretical investigations, the finite time pinch off has yet to be rigorously proved. In this paper, we prove an old conjecture of one of us, recorded in [**ED94**], that as long as $h > 0$ no singularity can arise from smooth and positive initial data (see Theorem 1.1 below). We also prove that indeed, as suggested in [**CDG$^+$93**] and in [**BBDK94**], global in time behavior leads to pinch off, just as finite time singularities do (see Theorem 1.7 below). To the best of our knowledge, this is the first rigorous proof for the emergence of a pinching singularity in the one dimensional Hele-Shaw model of [**CDG$^+$93**].

The equation we study ([**CDG$^+$93**])

$$\partial_t h(x,t) + \partial_x(h\,\partial_x^3 h)(x,t) = 0, \quad (x,t) \in (-1,1)\times(0,\infty), \tag{1.1}$$

is supplemented with boundary conditions

$$\begin{aligned} h(\pm 1, t) &= 1, \quad t > 0, \\ \partial_x^2 h(\pm 1, t) &= P, \quad t > 0. \end{aligned} \tag{1.2}$$

Here, $P > 0$ is the pressure of the less viscous fluid and $h \geq 0$ is half of the width of the thin neck. The equation has a steady solution h_P, given by (1.8) below, which is unique in a class of relatively smooth solutions (see Proposition A.2). This steady solution has a neck singularity if $P > 2$ (a segment where it is identically zero). The main result of the paper is to prove convergence to this solution in finite or infinite time. In order to do so we start by obtaining a strong enough local existence result. We exploit further the structure of the equation to pass to limit of infinite time, and prove that the limits have to be formed from pieces of parabolas and straight lines where they do not vanish. Then we prove that the only possible valid limit there is h_P.

We denote $I = (-1,1)$ and for any $T \in (0,\infty]$, we define

$$X(T) = \left\{ f \in L^\infty\big([0,T]; H^3(I)\big) : \partial_x^3 f \in L^2\big([0,T]; H^2(I)\big) \right\}$$

endowed with its natural norm. When T is finite, by interpolation $X(T)$ is equivalent to the space

$$L^\infty\big([0,T]; H^3(I)\big) \cap L^2\big([0,T]; H^5(I)\big).$$

THEOREM 1.1 (**Local existence of strong solutions and continuation criterion**). *Let $h_0 \in H^3(I)$ satisfy the boundary conditions (1.2) with $P > 0$, and assume $h_{0,m} := \inf_I h_0 > 0$. There exists a positive finite time T, depending only on P, $\|h_0\|_{H^3(I)}$ and $h_{0,m}$, such that problem (1.1)-(1.2) with initial data h_0 has a unique solution $h \in X(T)$ with $\inf_{I\times[0,T]} h > 0$.*

Moreover, there exists an increasing function $\mathcal{F} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ depending only on P such that

$$\|h\|_{X(T)} \leq \mathcal{F}\big(\frac{1}{\inf_{I\times[0,T]} h}, \|h_0\|_{H^3}\big) \tag{1.3}$$

Therefore, h blows up in the sense that it leaves the space $X(T)$ at a finite time T^* if and only if

$$\inf_{x\in I} h(t,x) \searrow 0 \text{ as } t \nearrow T^*. \tag{1.4}$$

Furthermore, if we denote

$$D(h(t)) = \int_I h\,|\partial_x^3 h|^2(x,t)dx \tag{1.5}$$

then

$$\int_0^T D(h(t))dt \leq C(\|h_0\|_{H^3(I)} + 1) \tag{1.6}$$

for some $C > 0$ depending only on P, and

$$D(h(t)) = D(h(0)) + \int_0^t \Big(\int_I \partial_t h |\partial_x^3 h|^2(x,s)dxds - 2\int_I |\partial_x\partial_t h|^2(x,s)dx \Big) ds \tag{1.7}$$

for a.e. $t \in [0,T]$.

REMARK 1.2. We observe that the right-hand side of (1.3) does not explicitly depend on T. This fact is used in the proof of Theorem 1.7 because it permits unique continuation of the solution uniformly in time as long as h is bounded below.

The problem (1.1)-(1.2) has the energy

$$E(h(t)) = \frac{1}{2}\int_I |\partial_x h(x,t)|^2 dx + P\int_I h(x,t)dx$$

which dissipates according to

$$\frac{d}{dt}E(h(t)) = -D(h(t)) = -\int_I h(x,t)|\partial_x^3 h(x,t)|^2 dx \leq 0$$

(see the proof of (2.11) below).

Define the steady solution h_P by

$$h_P(x) = \frac{P}{2}(x^2-1)+1, \qquad P \in (0,2], \tag{1.8a}$$

$$h_P(x) = \begin{cases} \frac{P}{2}(|x|-x_P)^2, & x_P \leq |x| \leq 1, \\ 0, & |x| < x_P, \end{cases} \qquad P > 2, \tag{1.8b}$$

where $x_P = 1-\sqrt{\frac{2}{P}}$ for $P > 2$. The energy dissipation rate $D(h)$ vanishes for $h = h_P$. When $P \in (0,2]$, h_P is a smooth, nonnegative solution of (1.1)-(1.2). When $P > 2$, $h_P \in W^{2,\infty}(I)$ and has a jump of its second derivative at $\pm x_P$. In the second case, h_P is a weak solution in the sense of the following definition.

DEFINITION 1.3 (**Weak solution**). *We say that a nonnegative function $h \in L^2([0,T];H^2(I))$ is a weak solution of* (1.1)-(1.2) *on* $[0,T]$ *if there exists* $\delta > 0$ *such that for a.e.* $t \in [0,T]$, $h(t) \in C^2([-1,-1+\delta]) \cap C^2([1-\delta,1])$, $h(t)$ *verifies the boundary conditions* (1.2), *and*

$$\int_0^T\int_I h\partial_t\varphi dxdt - \int_0^T\int_I \big(h\partial_x^2 h - \frac{1}{2}|\partial_x h|^2\big)\partial_x^2\varphi dxdt = 0 \tag{1.9}$$

for all $\varphi \in C_0^\infty(I\times(0,T))$.

The preceding definition is based on the identity

$$\partial_x(h\partial_x^3 h) = \partial_x^2\big(h\partial_x^2 h - \frac{1}{2}|\partial_x h|^2\big). \tag{1.10}$$

REMARK 1.4 (**Global weak solutions**). We prove in Theorem A.1 of the appendix that for any nonnegative H^1 data that is smooth near ± 1 and satisfies the boundary condition (1.2), there exists a global weak solution to (1.1)-(1.2). Related results for different boundary conditions can be found in [**BF90, BP96, BP98**].

The next proposition implies that h_P has the least energy among all weak solutions.

PROPOSITION 1.5 (**Energy minimizer**). *For any nonegative function $h \in H^1(I)$ taking value 1 at ± 1 we have $E(h) \geq E(h_P)$. Moreover, $E(h) = E(h_P)$ if and only if $h = h_P$.*

In order to prove the finite or infinite time pinch off, we show that a sequence of functions with bounded energy E and vanishing energy dissipation rate D converges weakly to the energy minimizer h_p.

THEOREM 1.6 (**Relaxation to energy minimizer**). *Let (h_n) be sequence of nonnegative $H^3(I)$ functions satisfying* (1.2). *Assume that (h_n) is uniformly bounded in $H^1(I)$ and $D(h_n) \to 0$. Then we have $h_n \rightharpoonup h_P$ in $H^1(I)$ and $h_n \to h_P$ in $H^3_{loc}(\{x : h_P(x) > 0\})$. When $P \in (0,2)$, $h_n \to h_P$ in $H^3(I)$.*

As a corollary of Theorems 1.1 and 1.6 we have the main result of this paper:

THEOREM 1.7 (**Stability for $P<2$ and pinch off for $P>2$**).
Part 1. *If $P \in (0,2)$, then h_P is asymptotically stable in $H^1(I)$. More precisely, there exist $\delta, c, C > 0$ depending only on P such that the following holds. If $h_0 \in H^3(I)$ satisfies $\inf_I h_0 > 0$ and $\|h_0 - h_P\|_{H^1} \leq \delta$ then $h \in X(T)$ for any $T > 0$, $\inf_{I\times\mathbb{R}^+} h \geq c$ and*

$$\|h(t) - h_P\|_{H^1(I)} \leq C\|h_0 - h_P\|_{H^1(I)} \exp(-ct) \quad \forall t > 0.$$

Moreover, $h(t) \to h_P$ in $H^3(I)$ as $t \to \infty$.

Part 2. *If $P \geq 2$, then starting from any positive $h_0 \in H^3(I)$, the solution h of (1.1)-(1.2), constructed in Theorem 1.1, pinches off at either finite or infinite time. In the latter case, by Theorem 1.6, $h(t_n) \rightharpoonup h_P$ in $H^1(I)$ and $h(t_n) \to h_P$ in $H^3_{loc}(\{x : h_P(x) > 0\})$ for some $t_n \to \infty$.*

REMARK 1.8. When $P > 2$, if h is global in X, the bound (1.3) blows up since h is pinched at infinite time. In particular, the bound for h in $L^\infty([0,T]; H^3(I))$ blows up as $T \to \infty$. Nevertheless, along an unbounded sequence of times, h converges to h_P in $H^3_{loc}(\{x : h_P(x) > 0\})$.

REMARK 1.9. Assume that h is a positive smooth solution of (1.1)-(1.2) on $[0, T^*)$, $T^* \in (0,\infty)$, and that $\min_{x\in I} h(x, T^*) = 0$. Let $x_m(t)$ be a position of the minimum of h in x at time t and denote $h_m(t) = h(x_m(t), t)$. Since $(\partial_x h)(x_m(t), t) = 0$, it is easy to see that

$$\frac{d}{dt} \ln h_m(t) = -(\partial_x^4 h)(x_m(t), t) \quad \forall t \in [0, T^*).$$

This implies

$$\int_0^{T^*} (\partial_x^4 h)(x_m(t), t)dt = \infty.$$

We also remark that in the derivation of model (1.1) (see [**CDG**$^+$**93**]), the speed of the flow is given by $v = \partial_x^3 h$, and hence

$$\int_0^{T^*} (\partial_x v)(x_m(t), t)dt = \infty.$$

This is one kind of singularity occurring when h touches 0 in finite time.

Throughout this paper, $\mathcal{F}(\cdot, ..., \cdot)$ denotes nonnegative functions which are increasing in each argument. $\mathcal{F}$ may change from line to line unless it is enumerated.

2. A linear problem

Let T be a positive real number and let g be a positive function satisfying

$$g \in L^\infty([0,T]; H^2(I)), \quad \partial_t g \in L^1([0,T]; L^\infty(I)). \tag{2.1}$$

We study in this section the linear problem

$$\begin{cases} \partial_t h(x,t) + \partial_x (g \partial_x^3 h)(x,t) = 0, & (x,t) \in I \times (0,T), \\ h(\pm 1, t) = 1, \quad \partial_x^2 h(\pm 1, t) = P, & t > 0, \\ h(x,t) = h_0(x), & t = 0. \end{cases} \tag{2.2}$$

We prove the following well-posedness result.

THEOREM 2.1 (**Strong solution for the linear problem**). *For every $h_0 \in H^3(I)$ satisfying the boundary conditions (1.2), there exists a unique solution $h \in X(T)$ to problem (2.2). Moreover, denoting*

$$c_T = \inf_{(x,t)\in I\times[0,T]} g(x,t) > 0,$$

then h obeys the bounds

$$\|h\|_{X(T)} \le \mathcal{F}\big(\frac{1}{c_T}, \|g\|_{L^\infty([0,T];H^2(I))}, \|\partial_t g\|_{L^1([0,T];L^\infty(I))}, \|h_0\|_{H^3}\big), \tag{2.3}$$

$$\|h\|_{L^\infty([0,T];H^1(I))} \le C(1+\|h_0\|_{H^1(I)}), \tag{2.4}$$

$$\int_0^T\!\!\int_I g\,|\partial_x^3 h|^2\,dxdt \le C\big(1+\|h_0\|^2_{H^1(I)}\big). \tag{2.5}$$

Here, $\mathcal{F}$ and C depend only on P. Furthermore, denoting $w = g\,\partial_x^3 h$ we have that

$$\int_I \frac{w^2(x,t)}{g(x,t)}dx = \int_I \frac{w^2(x,0)}{g(x,0)}dx + \int_0^t\!\!\int_I \frac{\partial_t g}{g^2}w^2(x,s)dxds - 2\int_0^t\!\!\int_I |\partial_x^2 w|^2(x,s)dxds, \tag{2.6}$$

$$\left\|\frac{w}{\sqrt{g}}(\cdot,t)\right\|_{L^2(I)} \le \left\|\frac{w}{\sqrt{g}}(\cdot,0)\right\|_{L^2(I)} + \frac{1}{2}\int_0^t \left\|\frac{\partial_t g}{g^{\frac{3}{2}}}(\cdot,s)\right\|_{L^2(I)} \|w(\cdot,s)\|_{L^\infty(I)}ds, \tag{2.7}$$

and

$$\int_0^t \|\partial_x^2 w(\cdot,s)\|^2_{L^2}ds \le \frac{1}{2}\left\|\frac{w}{\sqrt{g}}(\cdot,0)\right\|^2_{L^2(I)} + \frac{1}{2}\int_0^t \left\|\frac{\partial_t g}{g^{\frac{3}{2}}}(\cdot,s)\right\|_{L^2(I)} \left\|\frac{w}{\sqrt{g}}(\cdot,s)\right\|_{L^2(I)} \|w(\cdot,s)\|_{L^\infty(I)}ds \tag{2.8}$$

hold for a.e. $t \in [0,T]$.

The remainder of this section contains the proof of Theorem 2.1. Let (g^n) a sequence of $C^\infty([0,T]\times\overline{I})$ functions such that $g^n(x,t) \ge c_T/2$ and

$$g^n \to g \in L^\infty([0,T];H^2(I)), \quad \partial_t g^n \to \partial_t g \in L^1([0,T];L^\infty(I)). \tag{2.9}$$

Let h_0^n be a sequence of $C^\infty(\overline{I})$ functions satisfying (1.2) and converging to h_0 in $H^3(I)$. By the classical parabolic theory (see Theorem 6.2 [**LM72**]), there exists for each n a unique solution $h^n \in C^\infty(\overline{I})$ to the problem (2.2) with g replaced by g^n and h_0 replaced by h_0^n. We prove a closed a priori estimate for h^n in $X(T)$, a contraction estimate in $H^1(I)$, and then pass to the limit $n \to \infty$ to obtain the existence and uniqueness of a $h \in X(T)$ solving (2.2). To this end, we set

$$u^n = h^n - \frac{P}{2}(x^2-1) - 1.$$

Then,

$$\partial_t u^n = -\partial_x(g^n\,\partial_x^3 u^n) \quad \text{on } [0,T], \quad u^n|_{t=0} = h_0^n - \frac{P}{2}(x^2-1) - 1, \tag{2.10}$$

and

$$u^n(\pm 1,\cdot) = 0, \quad u^n_{xx}(\pm 1,\cdot) = 0.$$

Throughout sections 2.1, 2.2 and 2.3 we write $u^n = u$, $h^n = h$, $h_0^n = h_0$ and $g^n = g$ to simplify notation.

2.1. H^1 energy. We first claim that h satisfies

$$\frac{d}{dt}\int_I \big(\frac{1}{2}|\partial_x h|^2 + Ph\big) = -\int_I g|\partial_x^3 h|^2 \le 0. \tag{2.11}$$

Indeed, we have

$$\begin{aligned}\frac{d}{dt}\int_I \frac{1}{2}|\partial_x h|^2 &= \int_I \partial_t\partial_x h\partial_x h = \partial_t h\partial_x h|_{-1}^1 - \int_I \partial_t h\partial_x^2 h \\ &= \int_I \partial_x(g\partial_x^3 h)\partial_x^2 h = -\int_I g|\partial_x^3 h|^2 + Pg\partial_x^3 h|_{-1}^1,\end{aligned}$$

and

$$\frac{d}{dt}\int_I Ph = -P\int_I \partial_x(g\partial_x^3 h) = -Pg\partial_x^3 h|_{-1}^1,$$

where we use the fact that $\partial_t h(\pm1,\cdot) = 0$ (because $h(\pm1,\cdot) = 1$). This proves (2.11).

Next, multiplying (2.10) by $-\partial_x^2 u$, then integrating by parts we get

$$-\int_I \partial_t u\partial_x^2 u = \int_I \partial_x(g\partial_x^3 u)\partial_x^2 u = g\partial_x^3 u\partial_x^2 u|_{-1}^1 - \int_I g|\partial_x^3 u| = -\int_I g|\partial_x^3 u|^2.$$

But

$$-\int_I \partial_t u\partial_x^2 u = -\partial_t u\partial_x u|_{-1}^1 + \int_I \partial_t\partial_x u\partial_x u = \frac{1}{2}\frac{d}{dt}\int_I |\partial_x u|^2$$

noticing that $\partial_t u(\pm1,\cdot) = 0$ (because $u(\pm1,\cdot) = 0$). Denoting

$$E_1 = \|\partial_x u\|_{L^2(I)}, \quad D_1 = \|\sqrt{g}\partial_x^3 u\|_{L^2(I)},$$

we obtain

$$\frac{1}{2}\frac{d}{dt}E_1^2 + D_1^2 = 0, \tag{2.12}$$

and hence

$$\frac{1}{2}E_1(T)^2 + \|D_1\|^2_{L^2([0,T])} = \frac{1}{2}E_1(0)^2. \tag{2.13}$$

In particular, (2.13) and the definition of u gives

$$\|\partial_x u\|_{L^2(I)} \le \|\partial_x u(0)\|_{L^2(I)} \le \|\partial_x h_0\|_{L^2(I)} + P. \tag{2.14}$$

Since $u(\pm1,t) = 0$, the Poincaré inequality also gives

$$\|u\|_{L^2(I)} \le C\|\partial_x u\|_{L^2(I)} \le C(1 + \|h_0\|_{H^1(I)}) \tag{2.15}$$

which implies together with (2.14) and the definition of u that

$$\|h\|_{L^\infty([0,T];H^1(I))} \le C(1 + \|h_0\|_{H^1(I)}). \tag{2.16}$$

where C only depends on P.

Moreover, by (2.13) we obtain

$$\int_0^T\int_I g\,|\partial_x^3 h|^2 dxdt \le C\big(1 + \|h_0\|^2_{H^1(I)}\big), \tag{2.17}$$

and by the positivity of g,

$$\|\partial_x^3 h\|_{L^2([0,T];L^2(I))} \le \frac{C}{\sqrt{c_T}}\big(1 + \|h_0\|_{H^1(I)}\big) \tag{2.18}$$

where c_T is as in the statement of the theorem.

2.2. *H^2* **energy.** We multiply (2.10) by $\partial_x^4 u$ and integrate. On one hand,

$$\begin{aligned}\int_I \partial_t u\partial_x^4 u &= \partial_t u\partial_x^3 u|_{-1}^1 - \int_I \partial_t\partial_x u\partial_x^3 u = -\int_I \partial_t\partial_x u\partial_x^3 u\\ &= -\partial_t\partial_x u\partial_x^2 u|_{-1}^1 + \int_I \partial_t\partial_x^2 u\partial_x^2 u = \frac{1}{2}\frac{d}{dt}\int_I |\partial_x^2 u|^2.\end{aligned}$$

On the other hand,

$$-\int_I \partial_x(g\partial_x^3 u)\partial_x^4 u = -\int_I g|\partial_x^4 u|^2 - \int_I \partial_x g\partial_x^3 u\partial_x^4 u.$$

Denoting

$$E_2 = \|\partial_x^2 u\|_{L^2(I)}, \quad D_2 = \|\sqrt{g}\partial_x^4 u\|_{L^2(I)},$$

it follows that

$$\frac{1}{2}\frac{d}{dt}E_2^2 + D_2^2 = -\int_I \partial_x g \partial_x^3 u \partial_x^4 u \le \frac{1}{c_T}\|\partial_x g\|_{L^\infty(I\times[0,T])} D_1 D_2 \le \frac{1}{2c_T^2}\|\partial_x g\|^2_{L^\infty(I\times[0,T])} D_1^2 + \frac{1}{2}D_2^2.$$

In view of (2.13), this yields

$$\begin{aligned} E_2^2(T) + \int_0^T D_2^2 dt &\le E_2^2(0) + \frac{1}{c_0^2}\|\partial_x g\|^2_{L^\infty(I\times[0,T])}\int_0^T D_1^2 dt \\ &\le E_2^2(0) + \frac{1}{2c_0^2}\|\partial_x g\|^2_{L^\infty(I\times[0,T])} E_1^2(0), \end{aligned}$$

and consequently,

$$\|\partial_x^2 h\|_{L^\infty([0,T];L^2(I))} + \sqrt{c_T}\|\partial_x^4 h\|_{L^2([0,T];L^2(I))} \le C(1+\|h_0\|_{H^2}) + \frac{1}{c_T}\|\partial_x g\|_{L^\infty(I\times[0,T])}(\|h_0\|_{H^1(I)} + C). \tag{2.19}$$

This, together with (2.16) implies

$$\|h\|_{L^\infty([0,T];H^2(I))} + \sqrt{c_T}\|\partial_x^4 h\|_{L^2([0,T];L^2(I))} \le C(1+\|h_0\|_{H^2}) + \frac{1}{c_T}\|\partial_x g\|_{L^\infty(I\times[0,T])}(\|h_0\|_{H^1(I)} + C). \tag{2.20}$$

2.3. H^3 energy. A direct L^2 estimate for $\partial_x^3 u$ would make high order boundary terms appear (up to order 5) which are not given by the boundary conditions. Instead, we exploit further the structure of the equation. Setting $w = g\,\partial_x^3 h$, we have $\partial_t h = -\partial_x w$, and thus $\partial_x w(\pm 1) = \partial_x^3 w(\pm 1) = 0$ in view of (1.2). From the identity

$$\partial_t w = \partial_t g \partial_x^3 h + g\partial_x^3 \partial_t h = \frac{\partial_t g}{g} w - g\partial_x^4 w$$

we conclude

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_I \frac{w^2}{g} &= \int_I \partial_t w \frac{w}{g} - \frac{1}{2}\int_I \frac{\partial_t g}{g^2} w^2 = \int_I \frac{\partial_t g}{g^2} w^2 - \int_I w\partial_x^4 w - \frac{1}{2}\int_I \frac{\partial_t g}{g^2} w^2 \\ &= \frac{1}{2}\int_I \frac{\partial_t g}{g^2} w^2 - \int_I w\partial_x^4 w. \end{aligned} \tag{2.21}$$

Integrating by parts twice and using the boundary conditions for w gives

$$\int_I w\partial_x^4 w = \int_I |\partial_x^2 w|^2,$$

which yields

$$\int_I \frac{w^2(x,t)}{g(x,t)}dx = \int_I \frac{w^2(x,0)}{g(x,0)}dx + \int_0^t\int_I \frac{\partial_t g}{g^2} w^2(x,s)dxds - 2\int_0^t\int_I |\partial_x^2 w|^2(x,s)dxds, \tag{2.22}$$

$$\frac{1}{2}\frac{d}{dt}\|\frac{w}{\sqrt{g}}\|^2_{L^2} + \|\partial_x^2 w\|^2_{L^2} \le \frac{1}{2}\|\frac{\partial_t g}{g^{\frac{3}{2}}}\|_{L^2}\|\frac{w}{\sqrt{g}}\|_{L^2}\|w\|_{L^\infty} \tag{2.23}$$

and

$$\frac{1}{2}\frac{d}{dt}\|\frac{w}{\sqrt{g}}\|^2_{L^2} + \|\partial_x^2 w\|^2_{L^2} \le \frac{1}{2}\|\frac{\partial_t g}{g}\|_{L^\infty}\|\frac{w}{\sqrt{g}}\|^2_{L^2}. \tag{2.24}$$

By (2.24) and Grönwall's lemma,

$$\|\frac{w}{\sqrt{g}}\|_{L^\infty([0,T];L^2)} + \|\partial_x^2 w\|_{L^2([0,T];L^2)} \le \|\frac{w_0}{\sqrt{g}_0}\|_{L^2}\exp\big(2\int_0^T \|\frac{\partial_t g}{g}\|_{L^\infty}ds\big). \tag{2.25}$$

Moreover, since

$$\partial_x^2 w = \partial_x^2 g\partial_x^3 h + 2\partial_x g\partial_x^4 h + g\partial_x^5 h$$

and

$$\|\partial_x^3 h\|_{L^\infty(I)} \le C\|\partial_x^4 h\|_{L^2(I)}, \tag{2.26}$$

which follows from Poincaré-Wirtinger's inequality and the fact that

$$\int_I \partial_x^3 h dx = \partial_x^2 h(1) - \partial_x^2 h(-1) = P - P = 0,$$

we get

$$\begin{aligned} \|g\partial_x^5 h\|_{L^2} &\le \|\partial_x^2 g\|_{L^2}\|\partial_x^3 h\|_{L^\infty} + 2\|\partial_x g\|_{L^\infty}\|\partial_x^4 h\|_{L^2} + \|\partial_x^2 w\|_{L^2} \\ &\le C\|g\|_{H^2}\|\partial_x^4 h\|_{L^2} + \|\partial_x^2 w\|_{L^2}. \end{aligned} \tag{2.27}$$

In view of (2.20), (2.25), (2.27), and the lower bound $g \ge c_T$, we thus obtain

$$\|\partial_x^3 h\|_{L^\infty([0,T];L^2)} + \|\partial_x^5 h\|_{L^2([0,T];L^2)} \le \mathcal{F}\big(\frac{1}{c_T}, \|g\|_{L^\infty([0,T];H^2)}, \|\partial_t g\|_{L^1([0,T];L^\infty)}, \|h_0\|_{H^3}\big). \tag{2.28}$$

2.4. Proof of Theorem 2.1. A combination of (2.20), (2.18) and (2.28) leads to

$$\begin{aligned} \|h^n\|_{X(T)} &\le \mathcal{F}\big(\frac{1}{c_T}, \|g^n\|_{L^\infty([0,T];H^2(I))}, \|\partial_t g^n\|_{L^1([0,T];L^\infty)}, \|h_0^n\|_{H^3}\big) \\ &\le \mathcal{F}\big(\frac{1}{c_T}, \|g\|_{L^\infty([0,T];H^2(I))}, \|\partial_t g\|_{L^1([0,T];L^\infty)}, \|h_0\|_{H^3}\big). \end{aligned} \tag{2.29}$$

Recall that $\partial_t h^n = -\partial_x w^n$ and $\partial_x w^n(\pm 1) = 0$. It then follows from Poincaré's inequality and (2.25) that

$$\|\partial_t h^n\|_{L^2([0,T];H^1)} \le C\|w^n\|_{L^2([0,T];H^2)} \le \mathcal{F}\big(\frac{1}{c_T}, \|\partial_t g\|_{L^1([0,T];L^\infty)}, \|h_0\|_{H^3}\big). \tag{2.30}$$

By virtue of Aubin-Lions's lemma applied with the triple $H^3(I) \subset C^2(\overline{I}) \subset H^1(I)$, there exists $h \in X(T)$ such that

$$h^n \rightharpoonup h \quad \text{in } L^2([0,T];H^5(I))), \tag{2.31}$$

$$h^n \rightharpoonup * h \quad \text{in } L^\infty([0,T];H^3(I)), \tag{2.32}$$

$$h^n \to h \quad \text{in } C([0,T];C^2(\overline{I})). \tag{2.33}$$

For any test function $\phi \in C_0^\infty(I \times (0,T))$,

$$\int_0^T \int_I h^n \partial_t \phi dx dt + \int_0^T \int_I g^n \partial_x^3 h^n \partial_x \phi dx dt = 0.$$

The convergences (2.31) and (2.9) ensure that (h,g) satisfies the same weak formulation. Then because $h \in L^2([0,T];H^4(I))$ and $g \in L^\infty([0,T];H^2(I))$, we actually have $\partial_t h + \partial_x(g\partial_x^3 h) = 0$ in $L^2([0,T];H^1)$. Next, (2.33) implies that $h(0) = h_0$ and the boundary conditions $\partial_x h(\pm 1, t) = 1$, $\partial_x^2 h(\pm 1, t) = P$ are observed for any $t \in [0,T]$. The bounds (2.3), (2.4) and (2.5) on h are inherited from the corresponding bounds (2.29), (2.16) and (2.17) on h^n. Letting $n \to \infty$ in (2.22) yields (2.6). Finally, integrating (2.23) and letting $n \to \infty$ we obtain (2.7) and (2.8).

The uniqueness of solutions follows from the energy inequality. Let h_1, h_2 be two solutions of (2.2) with the same initial condition h_0. The difference $k = h_1 - h_2$ solves

$$\begin{cases} \partial_t k(x,t) + \partial_x(g\partial_x^3 k)(x,t) = 0, & (x,t) \in I \times (0,T), \\ k(\pm 1, t) = \partial_x^2 k(\pm 1, t) = 0, & t > 0, \\ k(x,t) = 0, & t = 0. \end{cases} \tag{2.34}$$

Similarly to the H^1 energy estimate for u above, we multiply the first equation in (2.34) by $-\partial_x^2 k$ and integrate by parts to get

$$\frac{1}{2}\frac{d}{dt}\|\partial_x k\|_{L^2(I)}^2 = -\int_I g|\partial_x^3 k|^2 \le 0,$$

consequently $\partial_x k = 0$. Since $k(\pm 1) = 0$ we conclude that $k = 0$, concluding the proof of uniqueness.

3. A nondegenrerate problem

Fixing a small positive real number ε, we prove in this section the global well-posedness of the following nondegenerate nonlinear parabolic problem

$$\begin{cases} \partial_t h(x,t) + \partial_x(\sqrt{h^2+\varepsilon^2}\partial_x^3 h)(x,t) = 0, & (x,t) \in I \times (0,\infty), \\ h(\pm 1,t) = 1, \partial_x^2 h(\pm 1,t) = P, & t > 0, \\ h(x,t) = h_0(x), & t = 0. \end{cases} \tag{3.1}$$

THEOREM 3.1 (**Strong solution for the nondegenerate nonlinear problem**). *For every $h_0 \in H^3$ satisfying the boundary conditions (1.2), and for every $T > 0$, there exists a unique solution $h \in X(T)$ to problem (3.1). Moreover, h obeys the bounds*

$$\|h\|_{X(T)} \le \mathcal{F}\big(\frac{1}{\inf_{I\times[0,T]}|h| + \varepsilon}, \|h_0\|_{H^3}\big), \tag{3.2}$$

$$\|h\|_{L^\infty([0,T];H^1(I))} \le C(1 + \|h_0\|_{H^1(I)}) \tag{3.3}$$

with $\mathcal{F}$ and C depending only on P. Furthermore, (2.5), (2.6), (2.7) and (2.8) hold with $g = \sqrt{h^2+\varepsilon^2}$.

3.1. Uniqueness. If h_1 and h_2 are two solutions of (3.1), we set $k = h_1 - h_2$ and $g_j = \sqrt{h_j^2 + \varepsilon^2}$, $j = 1, 2$. Observe that k solves

$$\begin{cases} \partial_t k(x,t) + \partial_x(g_1\partial_x^3 k)(x,t) + \partial_x((g_1-g_2)\partial_x^3 h_2)(x,t) = 0, & (x,t) \in I \times (0,\infty), \\ k(\pm 1,t) = \partial_x^2 k(\pm 1,t) = 0, & t > 0, \\ k(x,t) = 0, & t = 0. \end{cases} \tag{3.4}$$

Multiplying the first equation in (3.4) by $-\partial_x^2 k$ and integrating by parts (note that $\partial_t h_j \in L^2([0,T]; H_0^1(I))$) we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\partial_x k\|_{L^2(I)}^2 &= -\int_I g_1|\partial_x^3 k|^2 - \int_I (g_1-g_2)\partial_x^3 h_2 \partial_x^3 k \\ &\le -\varepsilon\int_I |\partial_x^3 k|^2 - \int_I (g_1-g_2)\partial_x^3 h_2\partial_x^3 k. \end{aligned}$$

It is readily seen that

$$|g_1(x) - g_2(x)| \le \|h_1 - h_2\|_{L^\infty(I)} \le C\|k\|_{H^1(I)}$$

which implies

$$\left|\int_I (g_1-g_2)\partial_x^3 h_2 \partial_x^3 k\right| \le C\|k\|_{H^1(I)}\|\partial_x^3 h_2\|_{L^2(I)}\|\partial_x^3 k\|_{L^2(I)}.$$

Since $k(\pm 1, \cdot) = 0$, Poincaré's inequality gives $\|k\|_{H^1(I)} \le C\|\partial_x k\|_{L^2(I)}$. This combined with a Young inequality leads to

$$\frac{1}{2}\frac{d}{dt}\|\partial_x k\|_{L^2(I)}^2 \le C_\varepsilon \|\partial_x k\|_{L^2(I)}^2\|\partial_x^3 h_2\|_{L^2(I)}^2.$$

Because $\|\partial_x^3 h_2\|_{L^2(I)} \in L^2([0,T])$ for any $T > 0$ we conclude by Grönwall's lemma that $\partial_x k = 0$ and thus $k = 0$.

3.2. Local existence. The existence of a local-in-time solution is obtained by Picard's iterations. We set $h^0(x,t) = h_0(x)$ for all $t > 0$ and define recursively h^{n+1}, $n \geq 0$, to be the solution of the problem

$$\begin{cases} \partial_t h^{n+1}(x,t) + \partial_x(g^n \partial_x^3 h^{n+1})(x,t) = 0, & (x,t) \in I \times (0,\infty), \\ g^n = \sqrt{|h^n|^2 + \varepsilon^2}, & \\ h(\pm 1, t) = 1, \quad \partial_x^2 h(\pm 1, t) = P, & t > 0, \\ h(x,t) = h_0(x), & t = 0. \end{cases} \tag{3.5}$$

Applying recursively Theorem 2.1 we find that $h^n \in X(T)$ for any $T > 0$. We now prove by induction that there exist $T_0, C_0 > 0$,

$$T_0 = T_0(\frac{1}{\varepsilon}, \|h_0\|_{H^3}), \quad C_0 = C_0(\frac{1}{\varepsilon}, \|h_0\|_{H^3}),$$

such that for any $n \geq 0$,

$$\|h^n\|_{X(T_0)} + \|\partial_t h^n\|_{L^1([0,T_0];L^\infty)} \leq C_0. \tag{3.6}$$

In view of the identities

$$\partial_t g^n = \frac{h^n \partial_t h^n}{\sqrt{|h^n|^2 + \varepsilon^2}}, \quad \partial_x g^n = \frac{h^n \partial_x h^n}{\sqrt{|h^n|^2 + \varepsilon^2}}, \quad \partial_x^2 g^n = \frac{|\partial_x h^n|^2 + h^n \partial_x^2 h^n}{\sqrt{|h^n|^2 + \varepsilon^2}} - \frac{|h^n|^2 |\partial_x h^n|^2}{(|h^n|^2 + \varepsilon^2)^{\frac{3}{2}}}, \tag{3.7}$$

we find

$$\|\partial_t g^n\|_{L^1([0,T];L^\infty)} \leq \|\partial_t h^n\|_{L^1([0,T];L^\infty)} \tag{3.8}$$

and

$$\|g^n\|_{L^\infty([0,T];H^2)} \leq \mathcal{F}_1(\frac{1}{\varepsilon}, \|h^n\|_{L^\infty([0,T];H^2)}) \tag{3.9}$$

This together with (2.3) yields

$$\|h^{n+1}\|_{X(T)} \leq \mathcal{F}_2(T, \frac{1}{\varepsilon}, \|h^n\|_{L^\infty([0,T];H^2)}, \|\partial_t h^n\|_{L^1([0,T];L^\infty)}, \|h_0\|_{H^3}). \tag{3.10}$$

Thus

$$\|h^{n+1}\|_{L^2([0,T];H^5)} \leq \mathcal{F}_2(T, \frac{1}{\varepsilon}, \|h^n\|_{L^\infty([0,T];H^2)}, \|\partial_t h^n\|_{L^1([0,T];L^\infty)}, \|h_0\|_{H^3}) \tag{3.11}$$

possibly with another $\mathcal{F}_2$. From the equation for h^{n+1} we deduce that

$$\begin{aligned} \|\partial_t h^{n+1}\|_{L^1([0,T];L^\infty)} &\leq \sqrt{T} \|\partial_t h^{n+1}\|_{L^2([0,T];L^\infty)} \\ &\leq C\sqrt{T} \|g^n\|_{L^\infty([0,T];H^2)} \|h^{n+1}\|_{L^2([0,T];H^5)} \\ &\leq \sqrt{T} \mathcal{F}_3(T, \frac{1}{\varepsilon}, \|h^n\|_{L^\infty([0,T];H^2)}, \|\partial_t h^n\|_{L^1([0,T];L^\infty)}, \|h_0\|_{H^3}). \end{aligned} \tag{3.12}$$

Thus (3.6) holds for $n = 0, 1$ with arbitrary $T_0 \in (0,1)$ and

$$C_0 > \max\Big\{\|h_0\|_{H^3}, \mathcal{F}_2(1, \frac{1}{\varepsilon}, \|h_0\|_{H^3}, 0, \|h_0\|_{H^3}), \mathcal{F}_3(1, \frac{1}{\varepsilon}, \|h_0\|_{H^3}, 0, \|h_0\|_{H^3})\Big\} =: M. \tag{3.13}$$

Assume (3.6) for $0, 1, ...n$ with $n \geq 1$ we now prove it for $n+1$. A direct induction based on (3.10) would amplify the bound for h^{n+1}, and thus additional considerations are needed.

LEMMA 3.2. *There exist $\delta \in (0,1)$ and $\mathcal{F}_5$, $\mathcal{F}_6$ such that for all $T \leq 1$ and $n \geq 1$,*

$$\|h^{n+1}\|_{X(T)} \leq \mathcal{F}_5\Big(\frac{1}{\varepsilon}, T^\delta \mathcal{F}_6\big(\|h^{n-1}\|_{L^\infty([0,T];H^2(I))}, \|\partial_t h^{n-1}\|_{L^2([0,T];H^1)}\big), \|h_0\|_{H^3(I)}\Big). \tag{3.14}$$

PROOF. We first note that $u^n := h^n - \frac{P}{2}x^2$ solves

$$\begin{cases} \partial_t u^n(x,t) + \partial_x(g^{n-1} \partial_x^3 u^n)(x,t) = 0, & (x,t) \in I \times (0,\infty), \\ g^{n-1} = \sqrt{|h^{n-1}|^2 + \varepsilon^2}, & \\ u^n(\pm 1, t) = \partial_x^2 u^n(\pm 1, t) = 0, & t > 0, \\ u^n(x,t) = u_0^n(x) := h_0(x) - \frac{P}{2}x^2, & t = 0. \end{cases} \tag{3.15}$$

Then as in section 2.2, we multiply the first equation in (3.15) by $\partial_x^4 u^n$ and integrate by parts to obtain

$$\frac{1}{2}\frac{d}{dt}\|\partial_x^2 u^n\|_{L^2(I)}^2 = -\int_I g^{n-1}|\partial_x^4 u^n|^2 - \int_I \partial_x g^{n-1}\partial_x^3 u^n \partial_x^4 u^n.$$

Let us note that $\partial_t\partial_x^2 u \in L^2([0,T];H^{-1}(I))$ and $\partial_x^2 u \in L^2([0,T];H_0^1(I))$. Employing the Gagliardo-Nirenberg inequality

$$\|\partial_x^3 f\|_{L^2(I)} \le C\|\partial_x^4 f\|_{L^2(I)}^{\alpha}\|f\|_{L^2(I)}^{1-\alpha} + C\|f\|_{L^2(I)}, \quad \alpha = \frac{3}{4},$$

we bound

$$\begin{aligned}\left|\int_I \partial_x g^{n-1}\partial_x^3 u^n \partial_x^4 u^n\right| &\le \|\partial_x g^{n-1}\|_{L^\infty(I)}\|\partial_x^3 u^n\|_{L^2(I)}\|\partial_x^4 u^n\|_{L^2(I)}\\ &\le C\|g^{n-1}\|_{H^2(I)}\|\partial_x^4 u^n\|_{L^2(I)}^{1+\alpha}\|u^n\|_{L^2(I)}^{1-\alpha}\\ &\quad + C\|g^{n-1}\|_{H^2(I)}\|\partial_x^4 u^n\|_{L^2(I)}\|u^n\|_{L^2(I)}.\end{aligned}$$

Consequently

$$\begin{aligned}&\|\partial_x^2 u^n\|_{L^\infty([0,T];L^2(I))}^2 + \|\sqrt{g^{n-1}}\partial_x^4 u^n\|_{L^2([0,T];L^2(I))}^2\\ &\le \|\partial_x^2 u^n(0)\|_{L^2(I)}^2 + C\|g^{n-1}\|_{L^\infty([0,T];H^2(I))}\|u^n\|_{L^\infty([0,T];L^2(I))}^{1-\alpha}\int_0^T \|\partial_x^4 u^n\|_{L^2(I)}^{1+\alpha}\\ &\quad + C\|g^{n-1}\|_{L^\infty([0,T];H^2(I))}\|u^n\|_{L^\infty([0,T];L^2(I))}\int_0^T \|\partial_x^4 u^n\|_{L^2(I)}.\end{aligned}$$

Appealing to Hölder's inequality we can gain small factors of powers of T:

$$\int_0^T \|\partial_x^4 u^n\|_{L^2(I)}^{1+\alpha} \le T^{\frac{1-\alpha}{2}}\|\partial_x^4 u^n\|_{L^2([0,T];L^2(I))}^{\alpha+1}, \quad \int_0^T \|\partial_x^4 u^n\|_{L^2(I)} \le T^{\frac{1}{2}}\|\partial_x^4 u^n\|_{L^2([0,T];L^2(I))}.$$

Invoking (3.9) and (3.10) with n replaced by $n-1$ leads to

$$\begin{aligned}&\|\partial_x^2 u^n\|_{L^\infty([0,T];L^2(I))} + \|\sqrt{g^{n-1}}\partial_x^4 u^n\|_{L^2([0,T];L^2(I))}\\ &\le T^\beta \mathcal{F}_3(\frac{1}{\varepsilon}, \|h^{n-1}\|_{L^\infty([0,T];H^2(I))}, \|\partial_t h^{n-1}\|_{L^1([0,T];L^\infty)}, \|h_0\|_{H^3}) + \|u_0\|_{H^2(I)}\end{aligned}$$

for some $\beta \in (0,1)$ and for all $T \le 1$, $n \ge 1$. We thus obtain by virtue of (2.4),

$$\|h^n\|_{L^\infty([0,T];H^2)} \le T^\beta \mathcal{F}_4\big(\frac{1}{\varepsilon}, \|h^{n-1}\|_{L^\infty([0,T];H^2)}, \|\partial_t h^{n-1}\|_{L^1([0,T];L^\infty)}, \|h_0\|_{H^3}\big) + C\|h_0\|_{H^2} + C.$$

Substituting this and (3.12) (with n replaced by $n-1$) in (3.10) yields

$$\begin{aligned}&\|h^{n+1}\|_{X(T)}\\ &\le \mathcal{F}_2\Big(T, \frac{1}{\varepsilon}, T^\beta \mathcal{F}_4\big(\frac{1}{\varepsilon}, \|h^{n-1}\|_{L^\infty([0,T];H^2)}, \|\partial_t h^{n-1}\|_{L^2([0,T];H^1)}, \|h_0\|_{H^3}\big) + C\|h_0\|_{H^2} + C,\\ &\quad \sqrt{T}\mathcal{F}_3\big(T, \frac{1}{\varepsilon}, \|h^{n-1}\|_{L^\infty([0,T];H^2)}, \|\partial_t h^{n-1}\|_{L^1([0,T];L^\infty)}, \|h_0\|_{H^3}\big), \|h_0\|_{H^3}\Big)\\ &\le \mathcal{F}_5\Big(\frac{1}{\varepsilon}, T^\gamma \mathcal{F}_6\big(\|h^{n-1}\|_{L^\infty([0,T];H^2)}, \|\partial_t h^{n-1}\|_{L^2([0,T];H^1)}\big), \|h_0\|_{H^3}\Big)\end{aligned}$$

for some $\gamma \in (0,1)$, for all $T \le 1$ and $n \ge 1$. $\square$

Now we choose

$$C_0 > \max\Big\{M, \mathcal{F}_5(\frac{1}{\varepsilon}, 1, \|h_0\|_{H^3})\Big\}$$

and $T_0 \in (0,1)$ satisfying

$$T_0^\gamma \mathcal{F}_6(C_0, C_0) \le 1, \quad \sqrt{T_0}\mathcal{F}_3(\frac{1}{\varepsilon}, C_0, C_0, \|h_0\|_{H^3}) \le C_0$$

then owing to (3.12), (3.14) and the induction hypothesis,

$$\|h^{n+1}\|_{X(T_0)} + \|\partial_t h^{n+1}\|_{L^1([0,T_0];L^\infty)} \le C_0$$

which completes the proof of the uniform bounds (3.6). In fact, using the first equation in (3.5) and the uniniform boundedness of h^n in $X(T_0)$ we deduce that $\partial_t h^n$ is uniformly bounded in $L^2([0,T_0]; H^1(I))$. Passing to the limit $n \to \infty$ with the use of Aubin-Lions's lemma, we obtain a solution $h \in X(T_0)$ of (3.1). Moreover, $T_0 \in (0,1)$ depends only on $\|h_0\|_X$ and ε, and the bound

$$\|h\|_{X(T_0)} \le C_0 \le \mathcal{F}(\frac{1}{\varepsilon}, \|h_0\|_{H^3})$$

holds. Finally, (2.5), (2.6), (2.7) and (2.8) hold with $g = \sqrt{h^2 + \varepsilon^2}$ by applying Theorem 2.1 to (3.5) then letting $n \to \infty$.

3.3. Global existence. We now iterate the above procedure over time intervals $\mathcal{T}_m$ of length less than 1 and glue the solutions together to obtain a maximal solution h defined on $[0, T^*)$ with $T^* \in (0, \infty]$.

PROPOSITION 3.3. *For any $T < T^*$, h obeys the bound*

$$\|h\|_{X(T)} \le \mathcal{F}\big(\frac{1}{h_m(T) + \varepsilon}, \|h_0\|_{H^3}\big), \quad h_m(T) := \inf_{I \times [0,T]} |h|. \tag{3.16}$$

PROOF. We revisit the energy estimates leading to Theorem 2.1 but with g replaced by h. First, the inequality (2.11) holds,

$$\frac{d}{dt}\int_I (\frac{1}{2}|\partial_x h|^2 + Ph) = -\int_I g|\partial_x^3 h|^2 \le 0.$$

Letting $u = h - \frac{P}{2}(x^2 - 1) - 1$ and $g = \sqrt{h^2 + \varepsilon^2}$, as in sections 2.1 and 2.2 we have that

$$\frac{1}{2}\frac{d}{dt}E_1^2 + D_1^2 \le 0 \tag{3.17}$$

and

$$\frac{1}{2}\frac{d}{dt}E_2^2 + D_2^2 = -\int_I \partial_x g \partial_x^3 u \partial_x^4 u$$

hold, where

$$E_1 = \|\partial_x u\|_{L^2(I)}, \quad D_1 = \|\sqrt{g}\partial_x^3 u\|_{L^2}, \quad E_2 = \|\partial_x^2 u\|_{L^2(I)}, \quad D_2 = \|\sqrt{g}\partial_x^4 u\|_{L^2}.$$

In particular, we deduce as for (2.16) that

$$\|h\|_{L^\infty([0,T];H^1(I))} \le C(1 + \|h_0\|_{H^1(I)}). \tag{3.18}$$

Writing $\partial_x g = \partial_x h \frac{h}{g} = (\partial_x u + Px)\frac{h}{g}$ and noting that $|h| \le g$ we bound

$$\begin{aligned}
\left|\int_I \partial_x g \partial_x^3 u \partial_x^4 u dx\right| &\le \int_I \left|\partial_x u \partial_x^3 u \partial_x^4 u\right| dx + P\int_I \left|x \partial_x^3 u \partial_x^4 u\right| dx \\
&\le \frac{1}{h_m(T)+\varepsilon}\|\partial_x u\|_{L^\infty(I)} D_1 D_2 + \frac{P}{h_m(T)+\varepsilon} D_1 D_2 \\
&\le \frac{1}{h_m(T)+\varepsilon}\|\partial_x u\|_{H^1(I)} D_1 D_2 + \frac{P}{h_m(T)+\varepsilon} D_1 D_2 \\
&\le \frac{C}{h_m(T)+\varepsilon} E_2 D_1 D_2 + \frac{P}{h_m(T)+\varepsilon} D_1 D_2, \\
&\le \frac{1}{2} D_2^2 + \frac{C}{h_m^2(T)+\varepsilon^2} E_2^2 D_1^2 + \frac{C}{h_m^2(T)+\varepsilon^2} D_1^2
\end{aligned}$$

where the bound

$$\|\partial_x u\|_{H^1(I)} \le C\|\partial_x^2 u\|_{L^2(I)},$$

which follows from Poincaré-Wirtinger's inequality together with the fact that $\int_I \partial_x u = 0$, was used. Thus

$$\frac{1}{2}\frac{d}{dt}E_2^2 + \frac{1}{2}D_2^2 \le \frac{C}{h_m^2(T)+\varepsilon^2}E_2^2 D_1^2 + \frac{C}{h_m^2(T)+\varepsilon^2}D_1^2$$

which combined with (3.17) yields

$$\frac{1}{2}\frac{d}{dt}E^2 + \frac{1}{2}D_2^2 \le \frac{C}{h_m^2(T)+\varepsilon^2}E_2^2 D_1^2 \le \frac{C}{h_m^2(T)+\varepsilon^2}E^2 D_1^2$$

with $E^2 = \frac{C}{h_m^2(T)+\varepsilon^2}E_1^2 + E_2^2$. Then by the Grönwall lemma,

$$\begin{aligned}
\|E_2\|_{L^\infty([0,T])} \le \|E\|_{L^\infty([0,T])} &\le E(0)\exp\left(\frac{C}{h_m^2(T)+\varepsilon^2}\|D_1\|^2_{L^2([0,T])}\right) \\
&\le E(0)\exp\left(\frac{C}{h_m^2(T)+\varepsilon^2}E_1^2(0)\right).
\end{aligned} \tag{3.19}$$

It follows that

$$\begin{aligned}
\|D_2\|_{L^2([0,T])} &\le \frac{C}{h_m(T)+\varepsilon}\|E_2\|_{L^\infty([0,T])}\|D_1\|_{L^2([0,T])} \\
&\le \frac{C}{h_m(T)+\varepsilon}E(0)\exp\left(\frac{C}{h_m^2(T)+\varepsilon^2}E_1^2(0)\right)E_1(0).
\end{aligned} \tag{3.20}$$

A combination of (3.18), (3.17), (3.19) and (3.20) leads to

$$\|h\|_{L^\infty([0,T];H^2(I))} + \|\partial_x^3 h\|_{L^2([0,T];L^2(I))} + \|\partial_x^4 h\|_{L^2([0,T];L^2(I))} \le \mathcal{F}\left(\frac{1}{h_m(T)+\varepsilon}, \|h_0\|_{H^2}\right). \tag{3.21}$$

We now turn to the H^3 estimate. As proved in section 3.2, (2.7) and (2.8) (with $g = \sqrt{h^3+\varepsilon^2}$) hold on each iterative time interval $\mathcal{T}_m$, and thus hold on $[0,T]$ by gluing them together. In other words, we have for *a.e.* $t \in [0,T]$ that

$$\left\|\frac{w}{\sqrt{g}}(\cdot,t)\right\|_{L^2(I)} \le \left\|\frac{w}{\sqrt{g}}(\cdot,0)\right\|_{L^2(I)} + \frac{1}{2}\int_0^t \left\|\frac{\partial_t g}{g^{\frac{3}{2}}}(\cdot,s)\right\|_{L^2(I)}\|w(\cdot,s)\|_{L^\infty} ds \tag{3.22}$$

and

$$\int_0^t \|\partial_x^2 w(\cdot,s)\|^2_{L^2} ds \le \frac{1}{2}\left\|\frac{w}{\sqrt{g}}(\cdot,0)\right\|^2_{L^2(I)} + \frac{1}{2}\int_0^t \left\|\frac{\partial_t g}{g^{\frac{3}{2}}}(\cdot,s)\right\|_{L^2(I)}\left\|\frac{w}{\sqrt{g}}(\cdot,s)\right\|_{L^2(I)}\|w(\cdot,s)\|_{L^\infty} ds. \tag{3.23}$$

But by (2.26) it is readily seen that

$$\|w\|_{L^\infty} \le C\|g\|_{L^\infty}\|\partial_x^3 h\|_{L^\infty} \le C(\|h\|_{L^\infty}+\varepsilon)\|\partial_x^4 h\|_{L^2}$$

and

$$\|\partial_t g\|_{L^2} \le \|\partial_t h\|_{L^2} \le C\|h\|_{H^2}(\|\partial_x^3 h\|_{L^2} + \|\partial_x^4 h\|_{L^2}). \tag{3.24}$$

Consequently

$$\|\frac{w}{\sqrt{g}}\|_{L^\infty([0,T];L^2)} \le \|\frac{w_0}{\sqrt{g}_0}\|_{L^2} + \frac{C}{(h_m(T)+\varepsilon)^{\frac{3}{2}}}A$$

with

$$\begin{aligned} A &= \|w\|_{L^\infty([0,T];L^\infty)}\|\partial_t g\|_{L^2([0,T];L^2)} \\ &\le C(\|h\|_{H^1}+\varepsilon)\|h\|_{L^\infty([0,T];H^2)}\Big(\|\partial_x^3 h\|_{L^2([0,T];L^2)}\|\partial_x^4 h\|_{L^2([0,T];L^2)} + \|\partial_x^4 h\|^2_{L^2([0,T];L^2)}\Big) \\ &\le \mathcal{F}\big(\frac{1}{h_m(T)+\varepsilon}, \|h_0\|_{H^2}\big) \end{aligned}$$

in view of (3.21), and

$$\begin{aligned} \|\partial_x^2 w\|^2_{L^2([0,T];L^2)} &\le \frac{1}{2}\|\frac{w_0}{\sqrt{g}_0}\|^2_{L^2} + \frac{C}{(h_m(T)+\varepsilon)^{\frac{3}{2}}}\|\frac{w}{\sqrt{g}}\|_{L^\infty([0,T];L^2)}\|\partial_t g\|_{L^\infty([0,T];L^2)}\|w\|_{L^\infty([0,T];L^\infty)} \\ &\le \frac{1}{2}\|\frac{w_0}{\sqrt{g}_0}\|^2_{L^2} + \frac{C}{(h_m(T)+\varepsilon)^{\frac{3}{2}}}\Big(\|\frac{w_0}{\sqrt{g}_0}\|_{L^2} + \frac{C}{(h_m(T)+\varepsilon)^{\frac{3}{2}}}A\Big)A \\ &\le \mathcal{F}\big(\frac{1}{h_m(T)+\varepsilon}, \|h_0\|_{H^2}\big). \end{aligned}$$

Appealing to (2.27) with $g = h$ we deduce that

$$\|\partial_x^3 h\|_{L^\infty([0,T];L^2)} + \|\partial_x^5 h\|_{L^\infty([0,T];L^2)} \le \mathcal{F}\big(\frac{1}{h_m(T)+\varepsilon}, \|h_0\|_{H^3}\big) \tag{3.25}$$

from which (3.16) follows. □

Now (3.16) implies the global bound

$$\|h\|_{X(T)} \le \mathcal{F}\big(\frac{1}{\varepsilon}, \|h_0\|_{H^3}\big)$$

for any $T < T^*$. We thus conclude that $T^* = \infty$. Furthermore, the bounds (3.2) and (3.3) follow from (3.18), (3.25) and (3.21).

4. Proof of Theorem 1.1

Let $h_0 \in H^3$ satisfy the boundary conditions (1.2) and

$$h_{0,m} := \inf_I h_0 > 0.$$

Step 1. (Approximate equations). For each $\varepsilon \in (0,1]$, let h_ε be the solution of the nondegenrate problem

$$\begin{cases} \partial_t h_\varepsilon(x,t) + (\sqrt{h_\varepsilon^2+\varepsilon^2}\,\partial_x^3 h_\varepsilon)_x(x,t) = 0, & (x,t) \in (-1,1)\times(0,\infty), \\ h_\varepsilon(\pm 1,t) = 1, \quad \partial_x^2 h_\varepsilon(\pm 1,t) = P, & t > 0, \\ h_\varepsilon(x,t) = h_0(x), & t = 0. \end{cases} \tag{4.1}$$

According to Theorem 3.1, $h^\varepsilon \in X(T)$ for any $T > 0$ and h_ε obeys the bounds

$$\|h_\varepsilon\|_{X(T)} \le \mathcal{F}\big(\frac{1}{h_{\varepsilon,m}(T)+\varepsilon}, \|h_0\|_{H^3}\big), \tag{4.2}$$

$$\|h_\varepsilon\|_{L^\infty([0,T];H^1(I))} \le C(1 + \|h_0\|_{H^1(I)}) \tag{4.3}$$

with

$$h_{\varepsilon,m}(T) = \inf_{(x,t)\in I\times[0,T]} |h_\varepsilon(x,t)|.$$

Moreover, (2.5) and (2.6) hold with $g=\sqrt{h_\varepsilon^2+\varepsilon^2}$.

Using the equation for h_ε and (4.2) we get

$$\|\partial_t h_\varepsilon\|_{L^2([0,T];H^1)}\le \mathcal{F}\big(\frac{1}{h_{\varepsilon,m}(T)+\varepsilon},\|h_0\|_{H^3}\big) \tag{4.4}$$

for all $T\le 1$. This implies

$$\begin{aligned} h_\varepsilon(x,t)&\ge h_\varepsilon(x,0)-\big|\int_0^t\partial_t h_\varepsilon(x,s)ds\big| \\ &\ge h_{0,m}-\sqrt{T}\|\partial_t h_\varepsilon\|_{L^2([0,T];L^\infty)} \\ &\ge h_{0,m}-\sqrt{T}\mathcal{F}\big(\frac{1}{h_{\varepsilon,m}(T)+\varepsilon},\|h_0\|_{H^3}\big)\quad \forall t\le T\le 1. \end{aligned} \tag{4.5}$$

Step 2. (Bootstrap) Denote

$$d_\varepsilon(T)=\frac{1}{h_{\varepsilon,m}(T)+\varepsilon},\quad T\le 1.$$

We choose C_0 sufficiently large and T_0 sufficiently small so that

$$C_0>\frac{1}{h_{0,m}}, \tag{4.6}$$

$$\sqrt{T_0}\mathcal{F}(C_0,\|h_0\|_{H^3})\le\frac{h_{0,m}}{2}, \tag{4.7}$$

$$C_0>\frac{1}{h_{0,m}-\sqrt{T_0}\mathcal{F}_2(C_0,\|h_0\|_{H^3})}. \tag{4.8}$$

This is possible by taking

$$C_0>\frac{2}{h_{0,m}},\quad \sqrt{T_0}\mathcal{F}_2(C_0,\|h_0\|_{H^3})\le\frac{h_{0,m}}{2}.$$

We claim that

$$d_\varepsilon(T_0)\le C_0\quad\forall\varepsilon>0. \tag{4.9}$$

Indeed, if (4.9) is not true then there exists $\varepsilon_0>0$ such that $d_{\varepsilon_0}(T_0)>C_0$. By (4.6),

$$d_{\varepsilon_0}(0)=\frac{1}{h_{0,m}+\varepsilon}\le\frac{1}{h_{0,m}}<C_0.$$

By the continuity of $d_{\varepsilon_0}(\cdot)$, there exists $T_1\in(0,T_0)$ such that $d_{\varepsilon_0}(T_1)=C_0$. Then (4.7) implies

$$\sqrt{T_1}\mathcal{F}(d_{\varepsilon_0}(T_1),\|h_0\|_{H^3})=\sqrt{T_1}\mathcal{F}(C_0,\|h_0\|_{H^3})\le\sqrt{T_0}\mathcal{F}(C_0,\|h_0\|_{H^3})\le\frac{h_{0,m}}{2}.$$

We deduce from (4.5) that

$$\inf_{I\times[0,T_1]}h_{\varepsilon_0}\ge\frac12 h_{0,m}>0$$

and

$$h_{\varepsilon_0,m}(T_1)\ge h_{0,m}-\sqrt{T_0}\mathcal{F}(C_0,\|h_0\|_{H^3})>0.$$

Hence

$$C_0=d_{\varepsilon_0}(T_1)=\frac{1}{h_{\varepsilon_0,m}(T_1)+\varepsilon_0}\le\frac{1}{h_{0,m}-\mathcal{F}(C_0,\|h_0\|_{H^3})}.$$

This contradicts (4.8), and thus we conclude the claim (4.9). Coming back to (4.5) we find

$$\inf_{I\times[0,T_0]}h_\varepsilon\ge\frac12 h_{0,m}\quad\forall\varepsilon>0.$$

Step 3. (Conclusion of the argument) Inserting (4.9) into (4.2) and (4.4) yields

$$\|h_\varepsilon\|_{X(T_0)} + \|\partial_t h_\varepsilon\|_{L^2([0,T_0];H^1(I))} \le M_0$$

for some M_0 depending only on $\|h_0\|_{H^3(I)}$ and $h_{0,m}$. Set $\varepsilon = \frac{1}{n}$ and rename $h_n = h_\varepsilon$, $d_n = d_\varepsilon$. According to Aubin-Lions's lemma, there exists $h \in X(T_0)$ such that

$$h_n \rightharpoonup h \quad \text{in } L^2([0,T_0];H^5(I)), \tag{4.10}$$

$$h^n \rightharpoonup * \, h \quad \text{in } L^\infty([0,T_0];H^3(I)), \tag{4.11}$$

$$h_n \to h \quad \text{in } C([0,T_0];C^2(\bar{I})). \tag{4.12}$$

Moreover, it is easy to check that h solves the problem (1.1)-(1.2). Letting $\varepsilon \to 0$ in (4.5) we find

$$\inf_{I\times[0,T_0]} h \ge \frac{1}{2} h_{0,m} > 0.$$

Next, it follows from (4.2) and the convergences (4.10), (4.11) that

$$\|h\|_{X(T_0)} \le \liminf_{n\to\infty} \|h_n\|_{X(T_0)} \le \liminf_{n\to\infty} \mathcal{F}\big(\frac{1}{h_{n,m}(T_0)+\frac{1}{n}}, \|h_0\|_{H^3}\big).$$

We can replace $\liminf$ by $\lim$ of a subsequence $n_k \to \infty$. For some $(x_k,t_k) \in I\times[0,T_0]$, $h_{n_k,m}(T_0) = h_{n_k}(x_k,t_k)$. By the compactness of $[-1,1]\times[0,T_0]$, there exists a subsequence $n_{k_j} \to \infty$ such that

$$(x_{k_j},t_{k_j}) \to (x_0,t_0) \in [-1,1]\times[0,T_0], \quad h_{n_{k_j}}(x_{k_j},t_{k_j}) \to h(x_0,t_0) \ge \inf_{I\times[0,T_0]} h$$

where (4.12) was used in the second convergence. Consequently

$$\begin{aligned}\|h\|_{X(T_0)} &\le \mathcal{F}\big(\frac{1}{\lim_{j\to\infty} h_{n_{k_j}}(x_{k_j},t_{k_j}) + \frac{1}{n_{k_j}}}, \|h_0\|_{H^3}\big)\\ &\le \mathcal{F}\big(\frac{1}{\inf_{I\times[0,T_0]} h}, \|h_0\|_{H^3}\big)\end{aligned}$$

where the fact that $\mathcal{F}$ is increasing was used.

In addition, passing to the limit in (2.5) and (2.6) leads to (1.6) and (1.7) repsectively.

Finally, because h is positive on I, it is unique by the same argument as in section 3.1.

5. Proof of Proposition 1.5

Let $h \in H^1(I)$ be a nonnegative function satisfying $h(\pm 1) = 1$. We have

$$\begin{aligned}E(h(t)) =& \frac{1}{2}\int_I |\partial_x h|^2 dx + P\int_I h dx\\ =& \frac{1}{2}\int_I |\partial_x(h-h_P)|^2 dx + \frac{1}{2}\int_I |\partial_x h_P|^2 dx + \int_I \partial_x(h-h_P)\partial_x h_P dx + P\int_I h dx.\end{aligned}$$

Integration by parts in the cross term gives

$$\int_I \partial_x(h-h_P)\partial_x h_P dx = (h-h_P)\partial_x h_P|_{-1}^1 - \int_I (h-h_P)\partial_x^2 h_P dx = -\int_I (h-h_P)\partial_x^2 h_P dx$$

since $h = h_P$ at ± 1.

<u>Case 1:</u> $P \in (0,2]$. In this case $\partial_x^2 h_P = P$, and thus

$$E(h(t)) = \frac{1}{2}\int_I |\partial_x(h-h_P)|^2 dx + \frac{1}{2}\int_I |\partial_x h_P|^2 dx + P\int_I h_P \ge E(h_P).$$

Moreover, $E(h(t)) = E(h_P)$ if and only if $\partial_x(h - h_P) = 0$ which is equivalent to $h = h_P$ by the boundary condition $h(\pm 1) = h_P(\pm 1) = 1$.

Case 2: $P > 2$. Then $\partial_x^2 h_P(x) = P$ if $|x| > x_P$ and $= 0$ if $|x| < x_P$. Thus

$$\begin{aligned} E(h(t)) &= \frac{1}{2}\int_I |\partial_x(h - h_P)|^2 dx + \frac{1}{2}\int_I |\partial_x h_P|^2 dx + P\int_I h - P\int_{x_P<|x|<1} (h - h_P) \\ &= \frac{1}{2}\int_I |\partial_x(h - h_P)|^2 dx + \frac{1}{2}\int_I |\partial_x h_P|^2 dx + P\int_{x_P<|x|<1} h_P + P\int_{-x_P}^{x_P} h \\ &= \frac{1}{2}\int_I |\partial_x(h - h_P)|^2 dx + \frac{1}{2}\int_I |\partial_x h_P|^2 dx + P\int_I h_P + P\int_{-x_P}^{x_P} h \\ &\geq E(h_P). \end{aligned}$$

Moreover, $E(h(t)) = E(h_P)$ if and only if

$$\begin{cases} \partial_x(h - h_P) = 0 \quad \text{on } I, \\ h = 0 \quad \text{on } (-x_P, x_P). \end{cases}$$

Again, owing to the boundary condition $h(\pm 1) = h_P(\pm 1) = 1$, this is equivalent to $h(x, \cdot) = h_P(x)$ for $|x| > x_P$ and $h = 0$ on $(-x_P, x_P)$. In other words, $h = h_P$.

6. Proof of Theorem 1.6

Let h_n be a sequence of nonnegative $H^3(I)$ functions satisfying (1.2). Assume that h_n is uniformly bounded in $H^1(I)$ and $D(h_n) \to 0$. Note that in view of the Gagliardo-Nirenberg inequality

$$\|f\|_{L^2(I)} \leq C\|\partial_x f\|_{L^2}^{\frac{1}{2}}\|f\|_{L^1}^{\frac{1}{2}} + C\|f\|_{L^1(I)},$$

the energy E defines a norm which is equivalent to the $H^1(I)$ norm. Then, by extracting a subsequence, still denoted t_n, we have $h_n \rightharpoonup h_\infty$ in $H^1(I)$. In particular,

$$h_n \to h_\infty \quad \text{in } C(\overline{I}). \tag{6.1}$$

Observe that if at some $x_0 \in \overline{I} = [-1, 1]$, $h_\infty(x_0) > 0$ then for some $\delta > 0$, $h_\infty \geq \frac{2}{3}h_\infty(x_0)$ on $I_{x_0,\delta} := (x_0 - \delta, x_0 + \delta) \cap I$. By (6.1), $h_n \geq \frac{1}{2}h_\infty(x_0)$ on $I_{x_0,\delta}$ for sufficiently large n. By the definition of $D(h)$ we get

$$\int_{I_{x_0,\delta}} |\partial_x^3 h_n(x)|^2 dx \to 0. \tag{6.2}$$

By interpolation, the quantity

$$N_3(u) := \int_{I_{x_0,\delta}} (|u|^2 + |\partial_x^3 u|^2) dx$$

defines a norm which is equivalent to the $H^3(I_{x_0,\delta})$ norm. It follows from (6.1) and (6.2) that $h_n \rightharpoonup h_\infty$ in N_3 and

$$\begin{aligned} N_3(h_\infty) \leq \liminf_{n\to\infty} N_3(h(t_n)) &= \lim_{n\to\infty}\int_{I_{x_0,\delta}} |h_n(x)|^2 dx + \lim_{n\to\infty}\int_{I_{x_0,\delta}} |\partial_x^3 h_n(x)|^2 dx \\ &= \int_{I_{x_0,\delta}} |h_\infty(x)|^2 dx, \end{aligned}$$

hence

$$\int_{I_{x_0,\delta}} |\partial_x^3 h_\infty(x)|^2 dx = 0.$$

We have proved that

LEMMA 6.1. *If $h_\infty(x_0) > 0$, $x_0 \in \overline{I}$, then there exists a neighborhood $I_{x_0,\delta} = (x_0-\delta, x_0+\delta) \cap I$ of x_0 in which h_n, h_∞ are positive, $\partial_x^3 h_\infty = 0$, and $h_n \to h_\infty$ in $H^3(I_{x_0,\delta})$. Consequently, $\partial_x^3 h_\infty = 0$ on $Z = \{x \in I : h_\infty(x) > 0\}$, hence h_∞ is either a parabola or a straight line on each connected component (which are open intervals) of Z.*

The next lemma rules out the possibility that h_n goes down to 0 at a non-zero angle.

LEMMA 6.2. *Let $x_0 \in I$ and $J = (x_0, x_0+\delta) \subset I$. Let $k \in C^2(J)$ be such that $k > 0$ on J and k, $\partial_x k$, $\partial_x^2 k$ are right-continuous at x_0 with $k(x_0^+) = 0$ and $\partial_x k(x_0^+) \neq 0$. Let k_n be a sequence of nonnegative functions in $H^3(I)$ such that $k_n(\pm 1) = c > 0$ and $k_n \to k$ in $C^2(J)$. Then,*

$$\int_I k_n |\partial_x^3 k_n|^2 \not\to 0.$$

The same conclusion holds if J is placed by $(x_0-\delta, x_0) \subset I$ and x_0^+ is replaced by x_0^- in the assumptions on k.

PROOF. Assume by contradiction

$$\int_I k_n |\partial_x^3 k_n|^2 \to 0. \tag{6.3}$$

Then in view of Höder's inequality and the boundedness of k_n in $L^\infty(I)$, we have for any $I' \subset I$ that

$$\left|\int_{I'} k_n \partial_x^3 k_n\right| \le \sqrt{|I'|}\Big(\int_{I'} k_n^2 |\partial_x^3 k_n|^2 dx\Big)^{\frac12} \le \sqrt{|I'|}\sup_n \|k_n\|_{L^\infty(I)}\Big(\int_{I'} k_n |\partial_x^3 k_n|^2 dx\Big)^{\frac12}$$

from which it follows that

$$\int_{I'} k_n \partial_x^3 k_n \to 0 \quad \forall I' \subset I. \tag{6.4}$$

Since

$$k(x_0^+)\partial_x^2 k(x_0^+) - \frac12 (\partial_x k(x_0^+))^2 = -\frac12 (\partial_x k(x_0^+))^2 < 0$$

there exists $\varepsilon \in (0,\delta)$ so small that $\partial_x k(x_0+\varepsilon) \neq 0$ and

$$k(x_0+\varepsilon)\partial_x^2 k(x_0+\varepsilon) - \frac12 (\partial_x k(x_0+\varepsilon))^2 < 0.$$

Here, the assumptions that $k \in C^2(J)$ and $k, \partial_x k, \partial_x^2 k$ are right continuous at x_0 were used. We note that $k_n(x) \ge c > 0$ on $J_1 = (x_0+\varepsilon, x_0+\delta)$ for all n. This combined with (6.3) yields $\int_{J_1} |\partial_x^3 k_n|^2 \to 0$, and thus $k_n \to k$ in $H^3(J_1)$ since we know $k_n \to k$ in $C^0(J_1)$. In particular, $k \in C^2(\overline{J}_1)$ and

$$k_n(x_0+\varepsilon) \to k(x_0+\varepsilon) > 0, \quad \partial_x k_n(x_0+\varepsilon) \to \partial_x k(x_0+\varepsilon) \neq 0, \quad \partial_x^2 k_n(x_0+\varepsilon) \to \partial_x^2 k(x_0+\varepsilon).$$

Let x_n be the global minimum of k_n on $\overline{I}$. We know that $k_n \ge 0$, $k_n(\pm 1) = c > 0$ and $k_n(x_0) \to k(x_0) = 0$, hence $x_n \in I$ for n sufficiently large. Then $\partial_x k(x_n) = 0$ and $\partial_x^2 k_n(x_n) > 0$. Now we compute

$$\begin{aligned}\int_{x_n}^{x_0+\varepsilon} k_n \partial_x^3 k_n &= k_n \partial_x^2 k_n\Big|_{x_n}^{x_0+\varepsilon} - \int_{x_n}^{x_0+\varepsilon} \partial_x k_n \partial_x^2 k_n \\ &= k_n(x_0+\varepsilon)\partial_x^2 k_n(x_0+\varepsilon) - k_n(x_n)\partial_x^2 k_n(x_n) - \frac12(\partial_x k_n(x_0+\epsilon))^2 + \frac12 (\partial_x k_n(x_n))^2 \\ &= k_n(x_0+\varepsilon)\partial_x^2 k_n(x_0+\varepsilon) - k_n(x_n)\partial_x^2 k_n(x_n) - \frac12(\partial_x k_n(x_0+\varepsilon))^2.\end{aligned}$$

Since $k_n(x_n)\partial_x^2 k_n(x_n) \ge 0$, the right-hand side is smaller than or equal to

$$k_n(x_0+\varepsilon)\partial_x^2 k_n(x_0+\varepsilon) - \frac12(\partial_x k_n(x_0+\varepsilon))^2$$

which converges to

$$k(x_0+\varepsilon)\partial_x^2 k(x_0+\varepsilon) - \frac{1}{2}(\partial_x k(x_0+\varepsilon))^2 < 0$$

while the left-hand side converges to 0, according to (6.4). This contradiction concludes the proof. $\square$

We now proceed to show $h_\infty = h_P$. First, $h_\infty(1) = \lim h_n(1) = 1$. By Lemma 6.1, there exists $\delta_0 \in (0,1)$ such that $h_n \to h_\infty$ in $H^3((1-\delta_0, 1))$, $h_\infty > 0$ and $\partial_x^3 h_\infty = 0$ on $(1-\delta_0, 1)$. In particular, $h_n \to h_\infty$ in $C^2([1-\delta_0, 1])$ and $\partial_x^2 h_\infty(1) = \lim \partial_x^2 h_n(1) = P$. Let $J = (1, 1-\delta)$ be the connected component of $Z = \{x \in I : h_\infty(x) > 0\}$ whose closure contains 1. Then h_∞ is a parabola of the form

$$h_\infty(x) = \frac{P}{2}x^2 + ax + b, \quad \frac{P}{2} + a + b = 1 \tag{6.5}$$

on J.

<u>Case 1</u>: $P \in (0,2)$. We claim that $\delta > 1$. Assume by contradiction $\delta \le 1$. Then $h_\infty(x_0) = 0$ with $x_0 := 1-\delta \in [0,1)$. According to Lemma 6.2, $\partial_x h_\infty(x_0) = 0$. This is equivalent to

$$\begin{cases} \Delta := a^2 - 2P(1 - a - \frac{P}{2}) = (a+P)^2 - 2P = 0, \\ x_0 = -\frac{a}{P}, \end{cases}$$

where the first condition is equivalent to $a = a_1 = \sqrt{2P} - P$ or $a = a_2 = -\sqrt{2P} - P$. If $a = a_1$ then $x_0 = -\frac{\sqrt{2P}-P}{P} = 1 - \sqrt{\frac{2}{P}} < 0$. If $a = a_2$ then $x_0 = \frac{\sqrt{2P}+P}{P} > 1$. Both cases being impossible, we conclude that $\delta > 1$. In particular, h assumes the form (6.5) on $[-\varepsilon, 1]$ with some $\varepsilon > 0$.

Similarly, if we start from $x = -1$ we also have that $h_\infty(x) = \frac{P}{2}x^2 + a'x + b'$ for $x \in [-1, \varepsilon']$ for some $\varepsilon' \in (0,1)$ and $a', b' \in \mathbb{R}$. Necessarily $ax + b = a'x + b'$ on $[-\varepsilon, \varepsilon']$, and thus $(a', b') = (a, b)$. In other words, h_∞ assumes the form (6.5) on the whole interval $[-1,1]$. Equalizing $h_\infty(-1) = h_\infty(1) = 1$ leads to $a = 0$. We thus conclude that

$$h(x) = \frac{P}{2}(x^2 - 1) + 1 = h_P \quad \text{on } [-1,1].$$

<u>Case 2:</u> $P \ge 2$. Arguing as in Case 1 we find $\delta \le 1$ and $h_\infty(x_0) = 0$ with

$$x_0 = 1 - \delta = 1 - \sqrt{\frac{2}{P}} = x_P \in [0,1),$$

and $a = \sqrt{2P} - P$.

When $P = 2$, $x_0 = 0$ and $a = 0$. Hence $h_\infty(x) = x^2$ on $[0,1]$. A similar argument also gives $h_\infty(x) = x^2$ on $[-1, 0]$, hence $h_\infty = h_P$.

Consider now the case $P > 2$. Then $x_0 = x_P \in (0,1)$ and

$$h_\infty(x) = \frac{P}{2}x^2 + ax + b = \frac{P}{2}x^2 + (\sqrt{2P} - P)x + 1 - \sqrt{2P} + \frac{P}{2} = \frac{P}{2}(x - x_P)^2$$

on $[x_P, 1]$. We claim that $h_\infty = 0$ on $[0, x_P)$, then by symmetry $h_\infty = h_P$. Assume by contradiction $h_\infty(x_1) > 0$ for some $x_1 \in [0, x_P)$. Let $(a,b) \subset I$ be the connected component of $Z = \{x \in I : h_\infty > 0\}$ that contains x_1. Necessarily $h_\infty(b) = 0$ and $b \le x_P$. By Lemma 6.1, h_∞ is either a parabola or a straight line (a,b). Let us show that both cases are impossible. Indeed, if h_∞ is a straight line on (a,b) then h_∞ hits 0 at $x = b$ (from the left) with an angle, which is impossible according to Lemma 6.2. Assume now that h_∞ is a parabola on (a,b). Since h_∞ must touch down from the left of b at zero angle, the only possibility is that the parabola $\frac{P}{2}x^2 + ax + b$ is positive while its slope is negative on $(-\infty, b)$. Thus $h_\infty(x) = \frac{P}{2}x^2 + ax + b$ on the whole interval $[-1, b]$. But then $h_\infty(-1) = h_\infty(1) = 1$ yields $a = 0$ which contradicts the fact that $a = \sqrt{2P} - P < 0$. Therefore, $h_\infty = h_P$ when $P > 2$.

By Lemma 6.1, $h_n \to h_P$ in $H^3_{loc}(\{x : h_P(x) > 0\})$ for any $P > 0$. Furthermore, when $P \in (0, 2)$, $h_P > 0$ on I and one can take in Lemma 6.1 $I_{x_0,\delta} = I$ for any $x_0 \in I$, hence $h_n \to h_P$ in $H^3(I)$. We have actually proved that any subsequence of (h_n) has a subsequence with desired convergence properties. Because the limit is unique (and is equal to h_P) we conclude that in fact the whole sequence h_n has those properties.

7. Proof of Theorem 1.7

Part 1. Let $P \in (0, 2)$, and let $h_0 \in H^3(I)$ satisfy (1.2) and $\inf_I h_0 > 0$. According to Theorem 1.1, there exist a maximal time of existence $T^* \in (0, \infty]$ and a unique solution $h \in X(T)$ with $\inf_{I\times[0,T]} h > 0$ for any $T < T^*$. Set $u = h - h_P$, then because $\partial_x^3 h_P = 0$ we have

$$\begin{cases} \partial_t u(x,t) + \partial_x(h\partial_x^3 u)(x,t) = 0, & (x,t) \in I \times (0,T^*), \\ u(\pm 1,t) = \partial_x^2 u(\pm 1,t) = 0, & t > 0. \end{cases} \tag{7.1}$$

Multiplying the first equation in (7.1) by $-\partial_x^2 u$ and integrating by parts, we obtain as in section 2.1,

$$\frac{1}{2}\frac{d}{dt}\|\partial_x u(\cdot,t)\|^2_{L^2(I)} = -\int_I h(t,x)|\partial_x^3 u(x,t)|^2 dx, \quad t \in (0,T^*). \tag{7.2}$$

In particular,

$$\|\partial_x u(\cdot,t)\|_{L^2(I)} \le \|\partial_x u(\cdot,0)\|_{L^2(I)}, \quad t \in (0,T^*).$$

Since $u(\pm 1,\cdot) = 0$, Poincaré's inequality together with the embedding $H^1(I) \subset C(I)$ yields

$$\|u(\cdot,t)\|_{L^\infty(I)} \le C_1\|\partial_x u(\cdot,t)\|_{L^2(I)} \le C_1\|\partial_x u(\cdot,0)\|_{L^2(I)}, \quad t \in (0,T^*).$$

Consequently,

$$h(x,t) \ge h_P(x) - C_1\|\partial_x u(\cdot,0)\|_{L^2(I)} \ge \frac{2-P}{2} - C_1\|\partial_x u(\cdot,0)\|_{L^2(I)},$$

and thus

$$h(x,t) \ge \frac{1}{2}\frac{2-P}{2} \tag{7.3}$$

for all $(x,t) \in I \times [0,T^*)$ provided

$$\|\partial_x u(\cdot,0)\|_{L^2(I)} \le \frac{1}{2C_1}\frac{2-P}{2}.$$

Therefore, $T^* = \infty$ according to the blow-up criterion (1.4).

Next, we show that h converges to h_∞ exponentially in $H^1(I)$. Indeed, because $\partial_x^2 u(\pm 1,\cdot) = 0$ and $\int_I \partial_x u dx = u(1) - u(-1) = 0$, Poincaré's inequalities yield

$$\|\partial_x^3 u(x,t)\|_{L^2(I)} \ge C_2\|\partial_x^2 u(x,t)\|_{L^2(I)} \ge C_3\|\partial_x u(x,t)\|_{L^2(I)}$$

which combined with (7.3) and (7.2) leads to

$$\frac{d}{dt}\|\partial_x u(\cdot,t)\|^2_{L^2(I)} \le -C_4\|\partial_x u(\cdot,t)\|^2_{L^2(I)}.$$

By Grönwall's lemma,

$$\|\partial_x u(\cdot,t)\|_{L^2(I)} \le \|\partial_x u(\cdot,0)\|_{L^2(I)} \exp(-C_4 t) \quad \forall t > 0.$$

Finally, note that $u(\pm 1,\cdot) = 0$ we conclude by Poincaré's inequality that

$$\|u(\cdot,t)\|_{H^1(I)} \le C\|u(\cdot,0)\|_{H^1(I)} \exp(-C_4 t) \quad \forall t > 0. \tag{7.4}$$

Let us now turn to prove that $D(h) \in W^{1,1}(\mathbb{R}^+)$. According to (1.6), $D(h) \in L^1(\mathbb{R}^+)$. Thus, by virtue of (1.7), it remains to show that

$$A := \int_I \partial_t h|\partial_x^3 h|^2(x,s)dx - 2\int_I |\partial_x\partial_t h|^2(x,s)dx \in L^1(\mathbb{R}^+).$$

In the rest of this proof, we write $L^pL^q \equiv L^p(\mathbb{R}^+; L^q(I))$. We first note that by (3.24),

$$\|\partial_t h\|_{L^2L^2} \le C\|h\|_{L^\infty H^2}(\|\partial_x^3 h\|_{L^2L^2} + \|\partial_x^4 h\|_{L^2L^2}). \tag{7.5}$$

Consider next $\partial_x\partial_t h = -\partial_x^2 h\partial_x^3 h - 2\partial_x h\partial_x^4 h - h\partial_x^5 h$. It is readily seen that

$$\|\partial_x h\partial_x^4 h\|_{L^2L^2} \le C\|h\|_{L^\infty H^2}\|\partial_x^4 h\|_{L^2L^2}, \quad \|h\partial_x^5 h\|_{L^2L^2} \le C\|h\|_{L^\infty H^1}\|\partial_x^5 h\|_{L^2L^2}.$$

Using (2.26) we bound

$$\|\partial_x^2 h\partial_x^3 h\|_{L^2L^2} \le \|\partial_x^2 h\|_{L^\infty L^2}\|\partial_x^3 h\|_{L^2L^\infty} \le C\|\partial_x^2 h\|_{L^\infty L^2}\|\partial_x^4 h\|_{L^2L^2}.$$

Consequently

$$\|\partial_x\partial_t h\|_{L^2L^2} \le C\|h\|_{L^\infty H^2}\|\partial_x^4 h\|_{L^2L^2} + C\|h\|_{L^\infty H^1}\|\partial_x^5 h\|_{L^2L^2}. \tag{7.6}$$

In view of the lower bound (7.3), it follows from (1.3) that

$$\|h\|_{X(\mathbb{R}^+)} \le \mathcal{F}(\|h_0\|_{H^3}). \tag{7.7}$$

This together with (7.6) yields

$$\int_0^\infty \int_I |\partial_x\partial_t h|^2(x,s)dxds = \|\partial_x\partial_t h\|_{L^2L^2}^2 \le \mathcal{F}(\|h_0\|_{H^3}). \tag{7.8}$$

On the other hand, using (2.26) and Hölder's inequality we get

$$\int_I \partial_t h|\partial_x^3 h|^2 dx \le \|\partial_t h\|_{L^2(I)}\|\partial_x^3 h\|_{L^2(I)}\|\partial_x^3 h\|_{L^\infty(I)} \le C\|\partial_t h\|_{L^2(I)}\|\partial_x^3 h\|_{L^2(I)}\|\partial_x^4 h\|_{L^2(I)},$$

hence

$$\int_0^\infty \left|\int_I \partial_t h|\partial_x^3 h|^2(x,s)dx\right| ds \le C\|\partial_t h\|_{L^2L^2}\|\partial_x^3 h\|_{L^\infty L^2}\|\partial_x^4 h\|_{L^2L^2}.$$

Employing (7.5) and (7.7) we deduce that

$$\int_0^\infty \left|\int_I \partial_t h|\partial_x^3 h|^2(x,s)dx\right| ds \le \mathcal{F}(\|h_0\|_{H^3})$$

which combined with (7.8) concludes that $A \in L^1(\mathbb{R}^+)$. This completes the proof of $D(h) \in W^{1,1}(\mathbb{R}^+)$. According to Corollary 8.9 [**Bre11**] we then have $D(h(t)) \to 0$ as $t \to \infty$, and thus Theorem 1.6 implies that $h(t) \to h_P$ in $H^3(I)$ as $t \to \infty$.

Part 2. Let $P \ge 2$, and let $h_0 \in H^3(I)$ satisfy (1.2) and $\inf_I h_0 > 0$. Suppose that the solution h to (1.1)-(1.2) with initial data h_0 is not pinched at finite time neither at infinite time, then according to Theorem 1.1, h is global, $h \in X(T)$ for any $T > 0$, and

$$\inf_{I\times[0,\infty)} h \ge c_0 \tag{7.9}$$

for some $c_0 > 0$. Set

$$h_\infty(x) = \frac{P}{2}(x^2 - 1) + 1.$$

Observe that h_∞ is a stationary solution of (1.1)-(1.2) and h_∞ vanishes at $\pm\sqrt{1 - \frac{2}{P}}$. As before, $u = h - h_\infty$ satisfies (7.1). By virtue of (7.9), the proof of (7.4) also gives

$$\|u(\cdot,t)\|_{H^1(I)} \le C\|u(\cdot,0)\|_{H^1(I)}\exp(-Ct) \quad \forall t > 0.$$

In particular,

$$\lim_{t\to\infty} \|h(\cdot,t) - h_\infty(\cdot)\|_{C(I)} = 0.$$

Because $h_\infty(\sqrt{1 - \frac{2}{P}}) = 0$, we deduce that $\lim_{t\to\infty} h(\sqrt{1 - \frac{2}{P}}, t) = 0$ which contradicts (7.9).

Assume now that h is global in time. Since $D(h) \in L^1(\mathbb{R}^+)$ there exists $t_n \to \infty$ such that $D(h(t_n)) \to 0$. By virtue of Theorem 1.6, $h(t_n) \rightharpoonup h_P$ in $H^1(I)$ and $h(t_n) \to h_P$ in $H^3_{loc}(\{x : h_P(x) > 0\})$.

Appendix A. Weak solutions

THEOREM A.1 (**Existence of global weak solutions**). *Let $h_0 \in H^1(I)$ be a nonnegative function such that $h_0 \in H^3((-1,-1+\delta_0)) \cap H^3((1-\delta_0,1))$ for some $\delta_0 \in (0,1)$ and h_0 satisfies (1.2). Let T be a positive real number. Then there exists a global weak solution h of (1.1)-(1.2) in the sense of Definition 1.3. More precisely,*

$$h \in C(\overline{I} \times [0,T]) \cap L^\infty([0,T]; H^1(I)) \cap L^2([0,T]); H^2(I)) \cap H^1((0,T); H^{-1}(I))$$

and there exists $\delta \in (0,1)$ independent of T such that

$$h \in L^2\big([0,T]; H^3((-1,-1+\delta)) \cap H^3((1-\delta,1))\big).$$

PROOF. Let $h_0^n \in H^3(I)$ be a sequence of nonnegative functions satisfying (1.2) such that $h_0^n \to h$ in $H^1(I) \cap H^3(J)$. According to Theorem 3.1 there exists for each n a unique solution $h^n \in X([0,T])$, for any $T > 0$, to the problem

$$\begin{cases} \partial_t h^n(x,t) + \partial_x(\sqrt{|h^n|^2 + n^{-2}}\partial_x^3 h^n)(x,t) = 0, & (x,t) \in I \times (0,\infty), \\ h^n(\pm 1, t) = 1, \partial_x^2 h^n(\pm 1,t) = P, & t > 0, \\ h^n(x,t) = h_0^n(x), & t = 0. \end{cases} \tag{A.1}$$

Moreover, there exists $C > 0$ independent of n and T such that

$$\|h^n\|_{L^\infty([0,T];H^1(I))} \leq C\|h_0^n\|_{H^1(I)} \tag{A.2}$$

and

$$\int_0^T \int_I g^n |\partial_x^3 h^n|^2(x,s)dxds \leq C(\|h_0^n\|^2_{H^1(I)} + 1), \quad g^n = \sqrt{|h^n|^2 + n^{-2}}. \tag{A.3}$$

Writing $g^n \partial_x^3 h^n = \partial_x(g^n \partial_x^2 h^n) - \partial_x g^n \partial_x^2 h^n$ we have

$$0 = \partial_t h^n + \partial_x(g^n \partial_x^3 h^n) = \partial_t h^n + \partial_x^2(g^n \partial_x^2 h^n) - \partial_x(\partial_x g^n \partial_x^2 h^n).$$

Then, for any $\varphi \in C_0^\infty(I \times (0,T))$,

$$-\int_0^T \int_I h^n \partial_t \varphi + \int_0^T \int_I g^n \partial_x^2 h^n \partial_x^2 \varphi + \int_0^T \int_I \partial_x g^n \partial_x^2 h^n \partial_x \varphi = 0. \tag{A.4}$$

Because $h^n(\pm 1, \cdot) = 1$ and h^n is uniformly bounded in $L^\infty(\mathbb{R}^+; C^{\frac{1}{2}}(\overline{I}))$ (by virtue of (A.2) and the embedding $H^1(I) \subset C^{\frac{1}{2}}(\overline{I})$), there exists $\delta > 0$ sufficiently small such that

$$h^n(x,t) \geq \frac{1}{2} \quad \forall t \geq 0,\ \forall x \in J_1 := [-1,-1+\delta] \cup [1-\delta, 1] := J_{1,l} \cup J_{1,r}.$$

It then follows from (A.3) that

$$\|\partial_x^3 h^n\|_{L^2(\mathbb{R}^+;L^2(J_1))} \leq C = C(\|h_0\|_{H^1(I)}) \tag{A.5}$$

which combined with (A.2) and interpolation yields

$$\|h^n\|_{L^2([0,T];H^3(J_1))} \leq C = C(\|h_0\|_{H^1(I)}, T), \quad \forall T > 0. \tag{A.6}$$

Let $A > 0$ depend only on $\|h_0\|_{H^1(I)}$ such that $\|h^n\|_{L^\infty(I\times\mathbb{R}^+)} \leq A$ for all n. We define

$$f_n(s) = -\int_s^A \frac{dr}{\sqrt{r^2 + n^{-2}}}, \quad F_n(s) = -\int_s^A f_n(r)dr.$$

Note that $g_n(s) \leq 0$ and $F_n(s) \geq 0$ for any $s \leq A$. Let χ be a nonnegative cut-off function equal to 1 on $I_1 := I \setminus J_1$ and supported on $(-1,1)$. Multiplying the first equation in (A.1) by $f_n(h^n(x,t))\chi(x)$ then integrating by parts we obtain

$$\begin{aligned}\int_I \partial_t h^n f_n(h^n)\chi dx &= -\int_I \partial_x(g^n\partial_x^3 h^n) f_n(h^n)\chi dx\\ &= \int_I g^n \partial_x^3 h^n f_n'(h^n)\partial_x h^n \chi dx + \int_I g^n\partial_x^3 h^n f_n(h^n)\partial_x\chi dx\\ &= \int_I \partial_x^3 h^n \partial_x h^n \chi dx + \int_I g^n \partial_x^3 h^n f_n(h^n)\partial_x \chi dx\\ &= -\int_I |\partial_x^2 h^n|^2\chi dx - \int_I \partial_x^2 h^n \partial_x h^n \partial_x\chi dx + \int_I g^n\partial_x^3 h^n f_n(h^n)\partial_x\chi dx.\end{aligned}$$

Since

$$\int_I \partial_t h^n f_n(h^n)\chi dx = \frac{d}{dt}\int_I F_n(h^n)\chi dx$$

we deduce that

$$\begin{aligned}&\int_I F_n(h^n)(x,T)\chi dx + \int_0^T\int_{I_1} |\partial_x^2 h^n|^2\chi dx ds\\ &\leq \int_I F_n(h^n)(x,0)\chi dx - \int_0^T\int_I \partial_x^2 h^n\partial_x h^n\partial_x\chi dx ds + \int_0^T\int_I g^n\partial_x^3 h^n f_n(h^n)\partial_x\chi dx ds.\end{aligned} \tag{A.7}$$

We split

$$\int_0^T\int_I \partial_x^2 h^n\partial_x h^n\partial_x\chi dx = \int_0^T\int_{I_1}\partial_x^2 h^n\partial_x h^n\partial_x\chi dx + \int_0^T\int_{J_1}\partial_x^2 h^n\partial_x h^n\partial_x\chi dx =: H_1 + H_2.$$

Using Hölder's inequality and (A.2) we get

$$|H_1| \leq C\|\partial_x^2 h^n\|_{L^2([0,T];L^2(I_1))}, \quad C = C(\|h_0\|_{H^1(I)}, T).$$

On the other hand, (A.6) gives

$$|H_2| \leq C = C(\|h_0\|_{H^1(I)}, T).$$

Thus

$$\left|\int_0^T\int_I \partial_x^2 h^n\partial_x h^n\partial_x\chi dx ds\right| \leq C\|\partial_x^2 h^n\|_{L^2([0,T];L^2(I_1))} + C, \quad C = C(\|h_0\|_{H^1(I)}, T). \tag{A.8}$$

Applying Hölder's inequality together with (A.2) and (A.3) we find

$$\left|\int_0^T\int_I g^n\partial_x^3 h^n f_n(h^n)\partial_x\chi dx ds\right| \leq C = C(\|h_0\|_{H^1(I)}, T). \tag{A.9}$$

In addition, it is easy to see that

$$\int_I F_n(h^n)(x,0)\chi dx \leq C = C(\|h_0\|_{H^1(I)}). \tag{A.10}$$

Putting together (A.7), (A.8), (A.9) and (A.10) yields

$$\|\partial_x^2 h^n\|^2_{L^2([0,T];L^2(I_1))} \leq C\|\partial_x^2 h^n\|_{L^2([0,T];L^2(I_1))} + C, \quad C = C(\|h_0\|_{H^1(I)}, T).$$

Consequently, there exists $C = C(\|h_0\|_{H^1(I)}, T)$ such that

$$\|\partial_x^2 h^n\|_{L^2([0,T];L^2(I_1))} \leq C \quad \forall n.$$

This together with (A.6) implies

$$\|\partial_x^2 h^n\|_{L^2([0,T];L^2(I))} \leq C \quad \forall n. \tag{A.11}$$

Let us fix a positive (finite) time T. A combination of (A.2) and (A.3) leads to the uniform boundedness of $g^n\partial_x^3 h^n$ in $L^2([0,T];L^2(I))$, hence the uniform boundedness of $\partial_t h^n$ in $L^2([0,T];H^{-1}(I))$. Using this, (A.2), (A.6), (A.11) and Aubin-Lions's lemma we conclude that up to extracting a subsequence,

$$h^n \rightharpoonup h \text{ in } L^2([0,T];H^2(I)), \quad h^n \to h \text{ in } C(\overline{I}\times[0,T])\cap L^2([0,T];H^1(I))\cap L^2([0,T];C^2(\overline{J_1}))$$

for some

$$h\in C(\overline{I}\times[0,T])\cap L^\infty([0,T];H^1(I))\cap L^2([0,T];H^2(I))\cap L^2([0,T];H^3(J_{1,l})\cap H^3(J_{1,r}))$$

with $\partial_t h\in L^2((0,T);H^{-1}(I))$. In particular, h satisfies the boundary conditions (1.2) for *a.e.* $t\in[0,T]$. We claim that

$$h(x,t)\geq 0 \quad \forall (x,t)\in I\times[0,T].$$

Indeed, coming back to (A.7) we deduce from (A.8), (A.9) and (A.10) that

$$\int_I F_n(h^n(x,t))dx \leq C(\|h_0\|_{H^1(I)},T) \tag{A.12}$$

for all $n\geq 0$ and $t\leq T$. Assume by contradiction $h(x_0,t_0)<0$ for some $(x_0,t_0)\in I\times[0,T]$. Since $h^n\to h$ uniformly on $\overline{I}\times[0,T]$, there exist $\eta>0$ and $n_0\in\mathbf{N}$ such that

$$h_n(x,t_0)<-\eta \quad \text{if } |x-x_0|\leq\delta,\ n\geq n_0.$$

But for such x,

$$F_n(h^n(x,t_0)) = -\int_{h^n(x,t_0)}^A f_n(s)ds \geq -\int_{-\eta}^0 f_n(s)ds \to -\int_{-\eta}^0 f_\infty(s)ds \quad \text{as } n\to\infty$$

by the monotone convergence theorem, here

$$f_\infty(s) := \lim_{n\to\infty} f_n(s) = -\infty$$

for any $s\leq 0$. It follows that

$$\int_I F_n(h^n(x,t_0)) = +\infty$$

which contradicts (A.12), and thus $h\geq 0$.

Then letting $n\to\infty$ in (A.4) leads to

$$-\int_0^T\int_I h\partial_t\varphi + \int_0^T\int_I h\partial_x^2 h\partial_x^2\varphi + \int_0^T\int_I \partial_x h\partial_x^2 h\partial_x\varphi = 0 \quad \forall\varphi\in C_0^\infty(I\times(0,T)). \tag{A.13}$$

Writing $\partial_x h\partial_x^2 h = \frac{1}{2}\partial_x|\partial_x h|^2$ and integrating by parts in the last integral we arrive at

$$-\int_0^T\int_I h\partial_t\varphi + \int_0^T\int_I \big(h\partial_x^2 h - \frac{1}{2}|\partial_x h|^2\big)\partial_x^2\varphi = 0 \quad \forall\varphi\in C_0^\infty(I\times(0,T)). \tag{A.14}$$

In other words, h is a weak solution of (1.1)-(1.2) in the sense of Definition 1.3. □

In general, weak solutions can be non-unique. Nevertheless, the steady weak solution h_P is unique as shown in the next Proposition.

PROPOSITION A.2 (**Uniqueness of** h_P). *For any $P>0$, h_P is the unique even weak steady solution, in the sense of Definition 1.3, to (1.1)-(1.2).*

PROOF. It is easy to check that h_P is an even weak steady solution in the sense of Definition 1.3. Assume now that h is an even weak steady solution, we prove that $h=h_P$. We first notice that the weak formulation (1.9) is equivalent to $\partial_x\partial_x(h\partial_x^2 h-\frac{1}{2}|\partial_x h|^2)=0$ in $\mathscr{D}'(I)$, or again $\partial_x(h\partial_x^2 h-\frac{1}{2}|\partial_x h|^2)=C$ in $\mathscr{D}'(I)$ for some constant C. We claim that $C=0$. Indeed, writing $h\partial_x^2 h=\partial_x(h\partial_x h)-|\partial_x h|^2)$ we get

$$C\int_I\varphi = -\langle h\partial_x^2 h-\frac{1}{2}|\partial_x h|^2,\partial_x\varphi\rangle_{\mathscr{D}'(I),\mathscr{D}(I)} = -(h\partial_x^2 h-\frac{1}{2}|\partial_x h|^2,\partial_x\varphi)_{L^2(I),L^2(I)}$$

for any $\varphi \in \mathscr{D}(I)$. Noting that h is even, we can make the change of variables $x \mapsto -x$ to obtain

$$C \int_I \varphi = \langle h\partial_x^2 h - \frac{1}{2}|\partial_x h|^2, \partial_x \varphi_1 \rangle_{L^2(I),L^2(I)} = -C \int_I \varphi_1$$

with $\varphi_1(\cdot) = \varphi(-\cdot) \in \mathscr{D}(I)$. Since $\int_I \varphi_1 = \int_I \varphi$ for any $\varphi \in \mathscr{D}(I)$ we conclude that $C = 0$ as claimed. We thus have

$$\begin{aligned} 0 &= (h\partial_x^2 h - \frac{1}{2}|\partial_x h|^2, \partial_x \varphi)_{L^2(I),L^2(I)} \\ &= (\partial_x^2 h, \partial_x (h\varphi))_{L^2(I),L^2(I)} - (\partial_x^2 h, \partial_x h \varphi)_{L^2(I),L^2(I)} + \frac{1}{2}(\partial_x |\partial_x h|^2, \varphi)_{L^2(I),L^2(I)} \\ &= -\langle \partial_x^3 h, h\varphi \rangle_{H^{-1}(I),H_0^1(I)} \end{aligned} \tag{A.15}$$

for any $\varphi \in H_0^1(I)$. If $h(x_0) > 0$, $x_0 \in \overline{I}$, there exists a neighborhood I_{x_0} of x_0 in I such that $h \geq \frac{1}{2}h(x_0)$ on I_{x_0}. For any $\psi \in H_0^1(I_{x_0})$, defining

$$\varphi(x) = \begin{cases} \frac{\psi}{h}, & x \in I_{x_0}, \\ 0, & x \in I \setminus I_{x_0} \end{cases}$$

we have $\varphi \in H_0^1(I_{x_0}) \subset H_0^1(I)$ and by (A.15),

$$\langle \partial_x^3 h, \psi \rangle_{H^{-1}(I_{x_0}),H_0^1(I_{x_0})} = 0.$$

This implies $\partial_x^3 h = 0$ in $\mathscr{D}'(I_{x_0})$, and thus $\partial_x^3 h = 0$ in $\mathscr{D}'(\{h > 0\})$. Consequently, on each connected component (which are open intervals) of $\{h > 0\}$, h is either a parabola or a straight line. In addition, h cannot hit 0 at a non-zero angle because $h \in H^2(I)$. We are thus in the same situation as in the proof of Theorem 1.6 which allows us to conclude that $h = h_P$. □

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