Inviscid limit for SQG in bounded domains

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ABSTRACT. We prove that the limit of any weakly convergent sequence of Leray-Hopf solutions of dissipative SQG equations is a weak solution of the inviscid SQG equation in bounded domains.

1. Introduction

The behavior of high Reynolds number fluids is a broad, important and mostly open problem of nonlinear physics and of PDE. Here we consider a model problem, the surface quasi-geostrophic equation, and the limit of its viscous regularizations of certain types. We prove that the inviscid limit is rigid, and no anomalies arise in the limit.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Denote

$$\Lambda = \sqrt{-\Delta}$$

where $-\Delta$ is the Laplacian operator with Dirichlet boundary conditions. The dissipative surface quasi-geostrophic (SQG) equation in Ω is the equation

$$\partial_t \theta^\nu + u^\nu \cdot \nabla \theta^\nu + \nu \Lambda^s \theta^\nu = 0, \quad \nu > 0,\ s \in (0, 2], \tag{1.1}$$

where $\theta^\nu = \theta^\nu(x,t)$, $u^\nu = u^\nu(x,t)$ with $(x,t) \in \Omega \times [0,\infty)$ and with the velocity u^ν given by

$$u^\nu = R_D^\perp \theta^\nu := \nabla^\perp \Lambda^{-1} \theta^\nu, \quad \nabla^\perp = (-\partial_2, \partial_1). \tag{1.2}$$

We refer to the parameter ν as "viscosity". Fractional powers of the Laplacian $-\Delta$ are based on eigenfunction expansions. The inviscid SQG equation has zero viscosity

$$\partial_t \theta + u \cdot \nabla \theta = 0, \quad u = R_D^\perp \theta. \tag{1.3}$$

The dissipative SQG (1.1) has global weak solutions for any L^2 initial data:

THEOREM 1.1. *For any initial data $\theta_0 \in L^2(\Omega)$ there exists a global weak solution θ*

$$\theta \in C_w(0,\infty; L^2(\Omega)) \cap L^2(0,\infty; D(\Lambda^{\frac{s}{2}}))$$

to the dissipative SQG equation (1.1)*. More precisely, θ satisfies the weak formulation*

$$\int_0^\infty \int_\Omega \theta \varphi(x) dx \partial_t \phi(t) dt + \int_0^\infty \int_\Omega u\theta \cdot \nabla \varphi(x) dx \phi(t) dt - \nu \int_0^\infty \int_\Omega \Lambda^{\frac{s}{2}} \theta \Lambda^{\frac{s}{2}} \varphi(x) dx \phi(t) dt = 0 \tag{1.4}$$

for any $\phi \in C_c^\infty((0,\infty))$ and $\varphi \in D(\Lambda^2)$. Moreover, θ obeys the energy inequality

$$\frac{1}{2}\|\theta(\cdot,t)\|^2_{L^2(\Omega)} + \nu \int_0^t \int_\Omega |\Lambda^{\frac{s}{2}} \theta|^2 dx dr \le \frac{1}{2}\|\theta_0\|^2_{L^2(\Omega)} \tag{1.5}$$

and the balance

$$\frac{1}{2}\|\theta(\cdot,t)\|^2_{D(\Lambda^{-\frac{1}{2}})} + \nu \int_0^t \int_\Omega |\Lambda^{\frac{s-1}{2}} \theta|^2 dx dr = \frac{1}{2}\|\theta_0\|^2_{D(\Lambda^{-\frac{1}{2}})} \tag{1.6}$$

for a.e. $t > 0$. In addition, $\theta \in C([0,\infty); D(\Lambda^{-\varepsilon}))$ for any $\varepsilon > 0$ and the initial data θ_0 is attained in $D(\Lambda^{-\varepsilon})$.

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We refer to any weak solutions of (1.1) satisfying the properties (1.4), (1.5), (1.6) as a "Leray-Hopf weak solution".

REMARK 1.2. Theorem 1.1 for critical dissipative SQG $s = 1$ was obtained in [**6**].

REMARK 1.3. Note that $C_c^\infty(\Omega)$ is not dense in $D(\Lambda^2)$ since the $D(\Lambda^2)$ norm is equivalent to the $H^2(\Omega)$ norm and $C_c^\infty(\Omega)$ is dense in $H_0^2(\Omega)$ which is strictly contained in $D(\Lambda^2)$.

The existence of L^2 global weak solutions for inviscid SQG (1.3) was proved in [**8**]. More precisely, (see Theorem 1.1, [**8**]) for any initial data $\theta_0 \in L^2(\Omega)$ there exists a global weak solution $\theta \in C_w(0, \infty; L^2(\Omega))$ satisfying

$$\int_0^\infty \int_\Omega \theta \partial_t \varphi dx dt + \int_0^\infty \int_\Omega u\theta \cdot \nabla \varphi dx dt = 0 \quad \forall \varphi \in C_c^\infty(\Omega \times (0, \infty)), \tag{1.7}$$

and such that the Hamiltonian

$$H(t) := \|\theta(t)\|^2_{D(\Lambda^{-\frac{1}{2}})} \tag{1.8}$$

is constant in time. Moreover, the initial data is attained in $D(\Lambda^{-\varepsilon})$ for any $\varepsilon > 0$.

Our main result in this note establishes the convergence of weak solutions of the dissipative SQG to weak solutions of the inviscid SQG in the inviscid limit $\nu \to 0$.

THEOREM 1.4. *Let $\{\nu_n\}$ be a sequence of viscosities converging to* 0 *and let $\{\theta_0^{\nu_n}\}$ be a bounded sequence in $L^2(\Omega)$. Any weak limit θ in $L^2(0, T; L^2(\Omega))$, $T > 0$, of any subsequence of $\{\theta^{\nu_n}\}$ of Leray-Hopf weak solutions of the dissipative SQG equation* (1.1) *with viscosity ν_n and initial data $\theta_0^{\nu_n}$ is a weak solution of the inviscid SQG equation* (1.3) *on* $[0, T]$. *Moreover, $\theta \in C(0, T; D(\Lambda^{-\varepsilon}))$ for any $\varepsilon > 0$, and the Hamiltonian of θ is constant on* $[0, T]$.

REMARK 1.5. The same result holds true on the torus $\mathbb{T}^2$. The case of the whole space $\mathbb{R}^2$ was treated in [**1**].

REMARK 1.6. With more singular constitutive laws $u = \nabla^\perp \Lambda^{-\alpha}\theta$, $\alpha \in [0, 1)$, L^2 global weak solutions of the inviscid equations were obtained in [**3, 15**]. Theorem 1.4 could be extended to this case. It is also possible to consider L^p initial data in light of the work [**12**].

It is worth noting that in order for a general weak solution θ of the inviscid SQG to conserve the Hamiltonian, the Onsager-type critical condition requires $\theta \in L^3_{t,x}$ (see [**14**] for $\Omega = \mathbb{T}^2$). On the other hand, the vanishing viscosity solutions obtained in Theorem 1.4 conserve the Hamiltonian, even though they are only in $L^\infty_t L^2_x$. In [**4**], a result in the same spirit has been obtained regarding the energy conservation of weak solutions of the Euler equation on the torus $\mathbb{T}^2$.

As a corollary of the proof of Theorem 1.4 we have the following weak rigidity of inviscid SQG in bounded domains:

COROLLARY 1.7. *Any weak limit in $L^2(0, T; L^2(\Omega))$, $T > 0$, of any sequence of weak solutions of the inviscid SQG equation* (1.3) *is a weak solution of* (1.3). *Here, weak solutions of* (1.3) *are interpreted in the sense of* (1.7).

REMARK 1.8. On tori, this result was proved in [**14**]. If the weak limit occurs in $L^\infty(0, T; L^2(\Omega))$ and the sequence of weak solutions conserves the Hamiltonian then so is the limiting weak solution.

The paper is organized as follows. Section 2 is devoted to basic facts about the spectral fractional Laplacian and results on commutator estimate. The proofs of Theorems 1.1 and 1.4 are given respectively in sections 3 and 4. Finally, an auxiliary lemma is given in Appendix A.

2. Fractional Laplacian and commutators

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain with smooth boundary. The Laplacian $-\Delta$ is defined on $D(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$. Let $\{w_j\}_{j=1}^{\infty}$ be an orthonormal basis of $L^2(\Omega)$ comprised of L^2-normalized eigenfunctions w_j of $-\Delta$, i.e.

$$-\Delta w_j = \lambda_j w_j, \quad \int_\Omega w_j^2 dx = 1,$$

with $0 < \lambda_1 < \lambda_2 \leq ... \leq \lambda_j \to \infty$.
The fractional Laplacian is defined using eigenfunction expansions,

$$\Lambda^s f \equiv (-\Delta)^{\frac{s}{2}} f := \sum_{j=1}^{\infty} \lambda_j^{\frac{s}{2}} f_j w_j \quad \text{with } f = \sum_{j=1}^{\infty} f_j w_j, \quad f_j = \int_\Omega f w_j dx$$

for $s \geq 0$ and $f \in D(\Lambda^s)$ where

$$D(\Lambda^s) := \{f \in L^2(\Omega) : \big(\lambda_j^{\frac{s}{2}} f_j\big) \in \ell^2(\mathbb{N})\}.$$

The norm of f in $D(\Lambda^s)$ is defined by

$$\|f\|_{D(\Lambda^s)} := \|(\lambda_j^{\frac{s}{2}} f_j)\|_{\ell^2(\mathbb{N})}.$$

It is also well-known that $D(\Lambda)$ and $H_0^1(\Omega)$ are isometric. In the language of interpolation theory,

$$D(\Lambda^\alpha) = [L^2(\Omega), D(-\Delta)]_{\frac{\alpha}{2}} \quad \forall \alpha \in [0, 2].$$

As mentioned above,

$$H_0^1(\Omega) = D(\Lambda) = [L^2(\Omega), D(-\Delta)]_{\frac{1}{2}},$$

hence

$$D(\Lambda^\alpha) = [L^2(\Omega), H_0^1(\Omega)]_\alpha \quad \forall \alpha \in [0, 1].$$

Consequently, we can identify $D(\Lambda^\alpha)$ with usual Sobolev spaces (see Chapter 1, [**17**]):

$$D(\Lambda^\alpha) = \begin{cases} H_0^1(\Omega) \cap H^\alpha(\Omega) & \text{if } \alpha \in (1, 2], \\ H_0^\alpha(\Omega) & \text{if } \alpha \in (\frac{1}{2}, 1], \\ H_{00}^{\frac{1}{2}}(\Omega) := \{u \in H_0^{\frac{1}{2}}(\Omega) : u/\sqrt{d(x)} \in L^2(\Omega)\} & \text{if } \alpha = \frac{1}{2}, \\ H^\alpha(\Omega) & \text{if } \alpha \in [0, \frac{1}{2}). \end{cases} \tag{2.1}$$

Here and below $d(x)$ denote the distance from x to the boundary $\partial\Omega$.

Next, for $s > 0$ we define

$$\Lambda^{-s} f = \sum_{j=1}^{\infty} \lambda_j^{-\frac{s}{2}} f_j w_j$$

if $f = \sum_{j=1}^{\infty} f_j w_j \in D(\Lambda^{-s})$ where

$$D(\Lambda^{-s}) := \left\{ \sum_{j=1}^{\infty} f_j w_j \in \mathscr{D}'(\Omega) : f_j \in \mathbb{R}, \sum_{j=1}^{\infty} \lambda_j^{-\frac{s}{2}} f_j w_j \in L^2(\Omega) \right\}.$$

The norm of f is then defined by

$$\|f\|_{D(\Lambda^{-s})} := \|\Lambda^{-s} f\|_{L^2(\Omega)} = \big(\sum_{j=1}^{\infty} \lambda_j^{-s} f_j^2\big)^{\frac{1}{2}}.$$

It is easy to check that $D(\Lambda^{-s})$ is the dual of $D(\Lambda^s)$ with respect to the pivot space $L^2(\Omega)$.

LEMMA 2.1 (Lemma 2.1, [**15**]). *The embedding*

$$D(\Lambda^s) \subset H^s(\Omega) \tag{2.2}$$

is continuous for all $s \geq 0$.

LEMMA 2.2. *For* $s, r \in \mathbb{R}$ *with* $s > r$, *the embedding* $D(\Lambda^s) \subset D(\Lambda^r)$ *is compact.*

PROOF. Let $\{u_n\}$ be a bounded sequence in $D(\Lambda^s)$. Then $\{\Lambda^r u_n\}$ is bounded in $D(\Lambda^{s-r})$. Choosing $\delta > 0$ smaller than $\min(s-r, \frac{1}{2})$ we have $D(\Lambda^{s-r}) \subset D(\Lambda^\delta) = H^\delta(\Omega) \subset L^2(\Omega)$ where the first embedding is continuous and the second is compact. Consequently the embedding $D(\Lambda^{s-r}) \subset L^2(\Omega)$ is compact and thus there exist a subsequence n_j and a function $f \in L^2(\Omega)$ such that $\Lambda^r u_{n_j}$ converge to f strongly in $L^2(\Omega)$. Then u_{n_j} converge to $u := \Lambda^{-r} f$ strongly in $D(\Lambda^r)$ and the proof is complete. $\square$

A bound for the commutator between Λ and multiplication by a smooth function was proved in [**6**] using the method of harmonic extension:

THEOREM 2.3 (Theorem 2, [**6**]). *Let* $\chi \in B(\Omega)$ *with* $B(\Omega) = W^{2,d}(\Omega) \cap W^{1,\infty}(\Omega)$ *if* $d \geq 3$, *and* $B(\Omega) = W^{2,p}(\Omega)$ *with* $p > 2$ *if* $d = 2$. *There exists a constant* $C(d, p, \Omega)$ *such that*

$$\|[\Lambda, \chi]\psi\|_{D(\Lambda^{\frac{1}{2}})} \leq C(d, p, \Omega)\|\chi\|_{B(\Omega)}\|\psi\|_{D(\Lambda^{\frac{1}{2}})}.$$

Pointwise estimates for the commutator between fractional Laplacian and differentiation were established in [**8**]:

THEOREM 2.4 (Theorem 2.2, [**8**]). *For any* $p \in [1, \infty]$ *and* $s \in (0, 2)$ *there exists a positive constant* $C(d, s, p, \Omega)$ *such that for all* $\psi \in C_c^\infty(\Omega)$ *we have*

$$|[\Lambda^s, \nabla]\psi(x)| \leq C(d, s, p, \Omega) d(x)^{-s-1-\frac{d}{p}} \|\psi\|_{L^p(\Omega)}$$

holds for all $x \in \Omega$.

This pointwise bound implies the following commutator estimate in Lebesque spaces.

THEOREM 2.5. *Let* $p,\ q \in [1, \infty]$, $s \in (0, 2)$ *and* φ *satisfy*

$$\varphi(\cdot) d(\cdot)^{-s-1-\frac{d}{p}} \in L^q(\Omega).$$

Then the operator $\varphi[\Lambda^s, \nabla]$ *can be uniquely extended from* $C_c^\infty(\Omega)$ *to* $L^p(\Omega)$ *such that there exists a positive constant* $C = C(d, s, p, \Omega)$ *such that*

$$\|\varphi[\Lambda^s, \nabla]\psi\|_{L^q(\Omega)} \leq C\|\varphi(\cdot)d(\cdot)^{-s-1-\frac{d}{p}}\|_{L^q(\Omega)}\|\psi\|_{L^p(\Omega)} \tag{2.3}$$

holds for all $\psi \in L^p(\Omega)$.

(2.3) is remarkable in that the commutator between an operator of order $s \in (0, 2)$ and an operator of order 1 is an operator of order 0.

3. Proof of Theorem 1.1

We use Galarkin approximations. Denote by $\mathbb{P}_m$ the projection in $L^2(\Omega)$ onto the linear span L^2_m of eigenfunctions $\{w_1, ..., w_m\}$, i.e.

$$\mathbb{P}_m f = \sum_{j=1}^{m} f_j w_j \quad \text{for } f = \sum_{j=1}^{\infty} f_j w_j. \tag{3.1}$$

The mth Galerkin approximation of (1.1) is the following ODE system in the finite dimensional space L^2_m:

$$\begin{cases} \dot{\theta}_m + \mathbb{P}_m(u_m \cdot \nabla\theta_m) + \nu\Lambda^s\theta_m = 0 & t > 0, \\ \theta_m = P_m\theta_0 & t = 0, \end{cases} \tag{3.2}$$

with $\theta_m(x,t) = \sum_{j=1}^m \theta_j^{(m)}(t)w_j(x)$ and $u_m = {R_D}^{\perp}\theta_m$ satisfying $\operatorname{div} u_m = 0$. Note that (3.2) is equivalent to

$$\frac{d\theta_l^{(m)}}{dt} + \sum_{j,k=1}^m \gamma_{jkl}^{(m)}\theta_j^{(m)}\theta_k^{(m)} + \nu\lambda_l^s\theta_l^{(m)} = 0, \quad l = 1,2,...,m, \tag{3.3}$$

with

$$\gamma_{jkl}^{(m)} = \lambda_j^{-\frac{1}{2}}\int_\Omega \Big(\nabla^\perp w_j \cdot \nabla w_k\Big) w_l dx.$$

The local existence of θ_m on some time interval $[0,T_m]$ follows from the Cauchy-Lipschitz theorem. On the other hand, the antisymmetry property $\gamma_{jkl}^{(m)} = -\gamma_{jlk}^{(m)}$ yields

$$\frac{1}{2}\|\theta_m(\cdot,t)\|^2_{L^2(\Omega)} + \nu\int_0^t\int_\Omega |\Lambda^{\frac{s}{2}}\theta_m|^2 dxdr = \frac{1}{2}\|\mathbb{P}_m\theta_0\|^2_{L^2(\Omega)} \le \frac{1}{2}\|\theta_0\|^2_{L^2(\Omega)} \tag{3.4}$$

for all $t\in[0,T_m]$. This implies that θ_m is global and (3.4) holds for all positive times. The sequence θ_m is thus uniformly bounded in $L^\infty(0,\infty;L^2(\Omega))\cap L^2(0,\infty;D(\Lambda^{\frac{s}{2}}))$. Upon extracting a subsequence, we have θ_m converge to some θ weakly-* in $L^\infty(0,\infty;L^2(\Omega))$ and weakly in $L^2(0,\infty;D(\Lambda^{\frac{s}{2}}))$. In particular, θ obeys the same energy inequality as in (3.4). On the other hand, if one multiplies (3.3) by $\lambda_l^{-1/2}\theta_l^{(m)}$ and uses the fact that $\gamma_{jkl}^{(m)}\lambda_l^{-1/2} = -\gamma_{lkj}^{(m)}\lambda_j^{-1/2}$, one obtains

$$\frac{1}{2}\|\theta_m(\cdot,t)\|^2_{D(\Lambda^{-\frac{1}{2}})} + \nu\int_0^t\int_\Omega |\Lambda^{\frac{s-1}{2}}\theta_m|^2 dxdr = \frac{1}{2}\|\mathbb{P}_m\theta_0\|^2_{D(\Lambda^{-\frac{1}{2}})}. \tag{3.5}$$

We derive next a uniform bound for $\partial_t\theta_m$. Let $N>0$ be an integer to be determined. For any $\varphi\in D(\Lambda^{2N})$ we integrate by parts to get

$$\begin{aligned}\int_\Omega \partial_t\theta_m\varphi dx &= -\int_\Omega \mathbb{P}_m \operatorname{div}(u_m\theta_m)\varphi dx - \int_\Omega \nu\Lambda^s\theta_m\varphi dx \\ &= \int_\Omega (u_m\theta_m)\cdot\nabla(\mathbb{P}_m\varphi)dx - \int_\Omega \nu\theta_m\Lambda^s\phi dx.\end{aligned}$$

The first term is controlled by

$$\left|\int_\Omega (u_m\theta_m)\cdot\nabla(\mathbb{P}_m\varphi)dx\right| \le \|u_m\theta_m\|_{L^1(\Omega)}\|\nabla\mathbb{P}_m\varphi\|_{L^\infty(\Omega)} \le C\|\mathbb{P}_m\varphi\|_{H^3(\Omega)}.$$

According to Lemma A.1, for N and k satisfying $N > \frac{k}{2}+1$ there exists a positive constant $C_{N,k}$ such that

$$\|\mathbb{P}_m\varphi\|_{H^k(\Omega)} \le C_{N,k}\|\varphi\|_{D(\Lambda^{2N})} \quad \forall m\ge 1,\ \forall\varphi\in D(\Lambda^{2N}). \tag{3.6}$$

With $k=3$ and $N=3$ we have

$$\left|\int_\Omega (u_m\theta_m)\cdot\nabla(\mathbb{P}_m\varphi)dx\right| \le C\|\varphi\|_{D(\Lambda^6)}.$$

On the other hand,

$$\left|\int_\Omega \nu\theta_m\Lambda^s\varphi dx\right| \le C\|\theta_m\|_{L^2(\Omega)}\|\varphi\|_{D(\Lambda^2)}.$$

We have proved that

$$\left|\int_\Omega \partial_t\theta_m\varphi dx\right| \le C\|\varphi\|_{D(\Lambda^6)} \quad \forall\varphi\in D(\Lambda^6).$$

Because $L^2(\Omega) \times D(\Lambda^6) \ni (f,g) \mapsto \int_\Omega fgdx$ extends uniquely to a bilinear from on $D(\Lambda^{-6}) \times D(\Lambda^6)$, we deduce that $\partial_t \theta_m$ are uniformly bounded in $L^\infty(0,\infty; D(\Lambda^{-6}))$. Note that we have used only the uniform regularity $L^\infty(0;\infty; L^2(\Omega))$ of θ_m. We have the embeddings $D(\Lambda^{\frac{s}{2}}) \subset D(\Lambda^{(s-1)/2}) \subset D(\Lambda^{-6})$ where the first one is compact by virtue of Lemma 2.2, and the second is continuous. Fix $T > 0$. Aubin-Lions' lemma (see [**16**]) ensures that for some function f and along some subsequence θ_m converge to f weakly in $L^2(0,T; D(\Lambda^{\frac{s}{2}}))$ and strongly in $L^2(0,T; D(\Lambda^{(s-1)/2}))$. Apriori, both f and the subsequence depend on both T. However, we already know that $\theta_m \to \theta$ weakly in $L^2(0,\infty; D(\Lambda^{\frac{s}{2}}))$. Therefore, $f = \theta$ and the convergences to θ hold for the whole sequence. Similarly, applying Aubin-Lions' lemma with the embeddings $L^2(\Omega) \subset D(\Lambda^{-\varepsilon}) \subset D(\Lambda^{-6})$ for sufficiently small $\varepsilon > 0$ we obtain that $\theta_m \to \theta$ strongly in $C([0,T]; D(\Lambda^{-\varepsilon}))$. Integrating (3.2) against an arbitrary test function of the form $\phi(t)\varphi(x)$ with $\phi \in C_c^\infty((0,T))$, $\varphi \in D(\Lambda^6)$ yields

$$\int_0^T \int_\Omega \theta_m \varphi(x) dx \partial_t \phi(t) dt + \int_0^T \int_\Omega u_m \theta_m \cdot \nabla \mathbb{P}_m \varphi(x) dx \phi(t) dt - \nu \int_0^T \int_\Omega \Lambda^{\frac{s}{2}} \theta_m \Lambda^{\frac{s}{2}} \varphi(x) dx \phi(t) dt = 0.$$

By Lemma A.1,

$$\|(\mathbb{I} - \mathbb{P}_m)\varphi\|_{L^\infty(\Omega)} \leq C \|(\mathbb{I} - \mathbb{P}_m)\varphi\|_{H^3(\Omega)} \to 0 \quad \text{as } m \to \infty.$$

The weak convergence of θ_m in $L^2(0,T; D(\Lambda^{\frac{s}{2}}))$ allows one to pass to the limit in the two linear terms. The strong convergence of θ_m in $L^2(0,T; L^2(\Omega))$ together with the weak convergence of u_m in the same space allows one to pass to the limit in the nonlinear term and conclude that θ satisfies the weak formulation (1.4) with $\varphi \in D(\Lambda^6)$. In fact, $\theta \in L^2(0,\infty; D(\Lambda^{\frac{s}{2}})) \subset L^2(0,\infty; L^p(\Omega))$ for some $p > 2$, hence $u\theta \in L^2(0,\infty; L^q(\Omega))$ for some $q > 1$. In addition, if $\varphi \in D(\Lambda^2)$ then $\nabla \varphi \in L^r$ for all $r < \infty$, and thus the nonlinearity $\int_\Omega u\theta \cdot \nabla \varphi dx$ makes sense. Then because $D(\Lambda^2)$ is dense in $D(\Lambda^6)$, (1.4) holds for $\varphi \in D(\Lambda^2)$.

We now pass to the limit in (3.5). The strong convergence $\theta_m \to \theta$ in $C(0,T; D(\Lambda^{-\varepsilon}))$ gives the convergence of the first term. On the other hand, the strong convergence $\theta_m \to \theta$ in $L^2(0,T; D(\Lambda^{(s-1)/2}))$ yields the convergence of the second term. The right hand side converges to $\frac{1}{2}\|\theta_0\|^2_{D(\Lambda^{-\frac{1}{2}})}$ since $\mathbb{P}_m \theta_0$ converge to θ_0 in $L^2(\Omega)$. We thus obtain (1.6).

Since $\theta_m \to \theta$ in $C([0,T]; D(\Lambda^{-\varepsilon}))$ we deduce that

$$\theta_0 = \lim_{m \to \infty} \mathbb{P}_m \theta_0 = \lim_{m \to \infty} \theta_m|_{t=0} = \theta|_{t=0} \quad \text{in } D(\Lambda^{-\varepsilon}).$$

For a.e. $t \in [0,T]$, $\theta_m(t)$ are uniformly bounded in $L^2(\Omega)$, and thus along some subsequence m_j, a priori depending on t, we have $\theta_{m_j}(t)$ converge weakly to some $f(t)$ in $L^2(\Omega)$. But we know $\theta_m(t) \to \theta(t)$ in $D(\Lambda^{-\varepsilon})$. Thus, $f(t) = \theta(t)$ and $\theta_m(t) \rightharpoonup \theta(t)$ in $L^2(\Omega)$ as a whole sequence for a.e. $t \in [0,T]$. Recall that $\frac{d}{dt}\theta_m$ are uniformly bounded in $L^\infty(0,T; D(\Lambda^{-6}))$. For all $\varphi \in D(\Lambda^6)$ and $t \in [0,T]$ we write

$$\langle \theta_m(t), \varphi \rangle_{L^2(\Omega), L^2(\Omega)} = \langle \theta_m(0), \varphi \rangle_{L^2(\Omega), L^2(\Omega)} + \int_0^t \langle \frac{d}{dt}\theta_m(r), \varphi \rangle_{D(\Lambda^{-6}), D(\Lambda^6)} dr.$$

Because $\frac{d}{dt}\theta_m$ converge to $\frac{d}{dt}\theta$ weakly-* in $L^\infty(0,T; D(\Lambda^{-6}))$, letting $m \to \infty$ yields

$$\langle \theta(t), \varphi \rangle_{L^2(\Omega), L^2(\Omega)} = \langle \theta_0, \varphi \rangle_{L^2(\Omega), L^2(\Omega)} + \int_0^t \langle \frac{d}{dt}\theta(r), \varphi \rangle_{D(\Lambda^{-6}), D(\Lambda^6)} dr$$

for a.e. $t \in [0,T]$. Taking the limit $t \to 0$ gives

$$\lim_{t \to 0} \langle \theta(t), \varphi \rangle_{L^2(\Omega), L^2(\Omega)} = \langle \theta_0, \varphi \rangle_{L^2(\Omega), L^2(\Omega)}$$

for all $\varphi \in D(\Lambda^6)$. Finally, since $D(\Lambda^6)$ is dense in $L^2(\Omega)$ and $\theta \in L^\infty(0,T; L^2(\Omega))$ we conclude that $\theta \in C_w(0,T; L^2(\Omega))$ for all $T > 0$.

4. Proof of Theorem 1.4

First, using approximations and commutator estimates we justify the commutator structure of the SQG nonlinearity derived in [**8**].

LEMMA 4.1. *For all $\psi \in H_0^1(\Omega)$ and $\varphi \in C_c^\infty(\Omega)$ we have*

$$\int_\Omega \Lambda\psi\nabla^\perp\psi \cdot \nabla\varphi dx = \frac{1}{2}\int_\Omega [\Lambda, \nabla^\perp]\psi \cdot \nabla\varphi\psi dx - \frac{1}{2}\int_\Omega \nabla^\perp\psi \cdot [\Lambda, \nabla\varphi]\psi dx. \tag{4.1}$$

Here, the commutator $[\Lambda, \nabla^\perp]\psi \cdot \nabla\varphi$ is understood in the sense of the extended operator defined in Theorem 2.5.

PROOF. Let $\psi_n \in C_c^\infty(\Omega)$ converging to ψ in $H_0^1(\Omega)$. Integrating by parts and using the fact that $\nabla^\perp \cdot \nabla\varphi = 0$ gives

$$\int_\Omega \Lambda\psi_n\nabla^\perp\psi_n \cdot \nabla\varphi dx = -\int_\Omega \psi_n\nabla^\perp\Lambda\psi_n \cdot \nabla\varphi dx,$$

Because ψ_n is smooth and has compact support inside Ω, $\nabla^\perp\psi_n \in D(\Lambda)$, and thus we can commute $\nabla^\perp$ with Λ to obtain

$$\begin{aligned}
&\int_\Omega \Lambda\psi_n\nabla^\perp\psi_n \cdot \nabla\varphi dx \\
&= -\int_\Omega \psi_n[\nabla^\perp, \Lambda]\psi_n \cdot \nabla\varphi dx - \int_\Omega \psi_n\Lambda\nabla^\perp\psi_n \cdot \nabla\varphi dx \\
&= -\int_\Omega \psi_n[\nabla^\perp, \Lambda]\psi_n \cdot \nabla\varphi dx - \int_\Omega \nabla^\perp\psi_n \cdot \Lambda(\psi\nabla\varphi) dx \\
&= -\int_\Omega [\nabla^\perp, \Lambda]\psi_n \cdot \nabla\varphi\psi_n dx - \int_\Omega \nabla^\perp\psi_n \cdot [\Lambda, \nabla\varphi]\psi_n dx - \int_\Omega \nabla^\perp\psi_n \cdot \nabla\varphi\Lambda\psi_n dx.
\end{aligned}$$

Noticing that the last term on the right-hand side is exactly the negative of the left-hand side, we deduce that

$$\int_\Omega \Lambda\psi_n\nabla^\perp\psi_n \cdot \nabla\varphi dx = \frac{1}{2}\int_\Omega [\Lambda, \nabla^\perp]\psi_n \cdot \nabla\varphi\psi_n dx - \frac{1}{2}\int_\Omega \nabla^\perp\psi_n \cdot [\Lambda, \nabla\varphi]\psi_n dx.$$

The commutator estimates in Theorems 2.3 and 2.5 then allow us to pass to the limit in the preceding representation and conclude that (4.1) holds. □

Now let $\nu_n \to 0^+$ and let $\theta_0^{\nu_n}$ be a bounded sequence in $L^2(\Omega)$. For each n let $\theta_n \equiv \theta^{\nu_n}$ be a Leray-Hopf weak solution of (1.1) with viscosity ν_n and initial data $\theta_0^{\nu_n}$. In view of the energy inequality (1.5), θ_n are uniformly bounded in $L^\infty(0, \infty; L^2(\Omega))$ and satisfies

$$\int_0^\infty \int_\Omega \theta_n\varphi(x)dx\partial_t\phi(t)dt + \int_0^\infty \int_\Omega u_n\theta_n \cdot \nabla\varphi(x)dx\phi(t)dt - \nu_n\int_0^\infty \int_\Omega \Lambda^{\frac{s}{2}}\theta_n\Lambda^{\frac{s}{2}}\varphi(x)dx\phi(t)dt = 0 \tag{4.2}$$

for all $\phi \in C_c^\infty((0, \infty))$ and $\varphi \in D(\Lambda^2)$. Fix $T > 0$. Assume that along a subsequence, still labeled by n, θ_n converge to θ weakly in $L^2(0, T; L^2(\Omega))$. We prove that θ is a weak solution of the inviscid SQG equation. We first prove a uniform bound for $\partial_t\theta_n$ provided only the uniform regularity $L^\infty(0, T; L^2(\Omega))$ of θ_n. To this end, let us define for a.e. $t \in [0, T]$ the function $f_n(t) \in H^{-3}(\Omega)$ by

$$\langle f_n(t), \varphi\rangle_{H^{-3}(\Omega), H_0^3(\Omega)} := \int_\Omega (u_n(x,t)\theta_n(x,t) \cdot \nabla\varphi(x) - \nu_n\theta_n(x,t)\Lambda^s\varphi(x))dx$$

for all $\varphi \in H_0^3(\Omega) \subset D(\Lambda^2)$, where $H_0^\mu(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in $H^\mu(\Omega)$ for any $\mu > 0$. Indeed, we have

$$\left|\int_\Omega (u_n(x,t)\theta_n(x,t) \cdot \nabla\varphi(x) - \nu_n\theta_n(x,t)\Lambda^s\varphi(x))dx\right| \le C\big(\|\theta_n(t)\|_{L^2(\Omega)}^2 + 1\big)\|\varphi\|_{H^3(\Omega)}.$$

This shows that f_n are uniformly bounded in $L^\infty(0,T;H^{-3}(\Omega))$. Then for any $\phi\in C_c^\infty((0,T))$, it follows from (4.2) that

$$\int_0^T \theta_n\partial_t\phi dt=-\int_0^T f_n\phi dt$$

in $H^{-3}(\Omega)$. In other words, $\partial_t\theta_n=f_n$ and the desired uniform bound for $\partial_t\theta_n$ follows. Fix $\varepsilon\in(0,\frac{1}{2})$. Aubin-Lions' lemma applied with the embeddings $L^2(\Omega)\subset D(\Lambda^{-\varepsilon})\subset H^{-3}(\Omega)$ then ensures that θ_n converge to θ strongly in $C(0,T;D(\Lambda^{-\varepsilon}))$. Consequently ψ_n converge to $\psi:=\Lambda^{-1}\theta$ strongly in $C(0,T;D(\Lambda^{1-\varepsilon}))$.

Now we take $\phi\in C_c^\infty((0,\infty))$ and $\varphi\in C_c^\infty(\Omega)$. Because of Lemma 4.1, the weak formulation (1.4) gives

$$\begin{aligned}\int_0^T\int_\Omega \theta_n\varphi(x)dx\partial_t\phi(t)dt&+\frac{1}{2}\int_0^T\int_\Omega[\Lambda,\nabla^\perp]\psi_n\cdot\nabla\varphi(x)\psi_n dx\phi(t)dt\\ &-\frac{1}{2}\int_0^T\int_\Omega\nabla^\perp\psi_n\cdot[\Lambda,\nabla\varphi(x)]\psi_n dx\phi(t)dt-\nu_n\int_0^T\int_\Omega\theta_n\Lambda^s\varphi(x)dx\phi(t)dt=0,\end{aligned}$$

where $\psi_n:=\Lambda^{-1}\theta_n$ are uniformly bounded in $L^\infty(0,T;H_0^1(\Omega))$. The weak convergence $\theta_n\rightharpoonup\theta$ in $L^2(0,T;L^2(\Omega))$ readily yields

$$\lim_{n\to\infty}\int_0^T\int_\Omega\theta_n\varphi(x)dx\partial_t\phi(t)dt=\int_0^T\int_\Omega\theta\varphi(x)dx\partial_t\phi(t)dt$$

and

$$\lim_{n\to\infty}\nu_n\int_0^T\int_\Omega\theta_n\Lambda^s\varphi(x)dx\phi(t)dt=0.$$

Next we pass to the limit in the two nonlinear terms. Applying the commutator estimate in Theorem 2.3 we have

$$\begin{aligned}&\left|\int_0^T\int_\Omega\nabla^\perp\psi_n\cdot[\Lambda,\nabla\varphi]\psi_n dx\phi dt-\int_0^T\int_\Omega\nabla^\perp\psi\cdot[\Lambda,\nabla\varphi]\psi dx\phi dt\right|\\ &\le\left|\int_0^T\int_\Omega\nabla^\perp(\psi_n-\psi)\cdot[\Lambda,\nabla\varphi]\psi dx\phi dt\right|+\|\phi\nabla^\perp\psi_n\|_{L^2(0,T;L^2(\Omega))}\|[\Lambda,\nabla\varphi](\psi_n-\psi)\|_{L^2(0,T;L^2(\Omega))}\\ &\le\left|\int_0^T\int_\Omega\nabla^\perp(\psi_n-\psi)\cdot[\Lambda,\nabla\varphi]\psi dx\phi dt\right|+C\|\psi_n-\psi\|_{L^2(0,T;D(\Lambda^{\frac{1}{2}}))}.\end{aligned}$$

The first term converges to 0 due to the weak convergence of ψ_n to ψ in $L^2(0,T;H_0^1(\Omega))$ and the fact that $[\Lambda,\nabla\varphi]\psi\in D(\Lambda^{\frac{1}{2}})\subset L^2(\Omega)$ in view of Theorem 2.3. The second term also converges to 0 due to the strong convergence of ψ_n to ψ in $C(0,T;D(\Lambda^{1-\varepsilon}))$ with $\varepsilon\in(0,\frac{1}{2})$. Finally, we apply the commutator estimate in Theorem 2.5 to obtain

$$\begin{aligned}&\left|\int_0^T\int_\Omega[\Lambda,\nabla^\perp]\psi_n\cdot\nabla\varphi\psi_n dx\phi dt-\int_0^T\int_\Omega[\Lambda,\nabla^\perp]\psi\cdot\nabla\varphi\psi dx\phi dt\right|\\ &\le\|\nabla\varphi[\Lambda,\nabla^\perp](\psi_n-\psi)\|_{L^2(0,T;L^2(\Omega))}\|\phi\psi_n\|_{L^2(0,T;L^2(\Omega))}\\ &\qquad+\|[\Lambda,\nabla^\perp]\psi\cdot\nabla\varphi\|_{L^2(0,T;L^2(\Omega))}\|\phi(\psi_n-\psi)\|_{L^2(0,T;L^2(\Omega))}\\ &\le C\|\psi_n-\psi\|_{L^2(0,T;L^2(\Omega))}\end{aligned}$$

which converges to 0. Putting together the above considerations leads to

$$\int_0^T\int_\Omega\theta\varphi(x)dx\partial_t\phi(t)dt+\int_0^T\int_\Omega u\theta\cdot\nabla\varphi(x)dx\phi(t)dt=0,\quad\forall\phi\in C_c^\infty((0,T)),\ \varphi\in C_c^\infty(\Omega).$$

Therefore, θ is a weak solution of the inviscid SQG equation on $[0,T]$.

Finally, let us show the Hamiltonian conservation of θ. We have the energy balance (1.6) for each θ_n. If $s \le 1$, then the uniform boundedness of θ_n in $L^\infty(0,T;L^2(\Omega))$ implies

$$\lim_{n\to\infty} \nu_n \int_0^t \int_\Omega |\Lambda^{\frac{s-1}{2}}\theta_n|^2 dxdr = 0, \quad t \in [0,T]. \tag{4.3}$$

In addition, $\theta_n \to \theta$ strongly in $C(0,T;D(\Lambda^{-\varepsilon})) \subset C(0,T;D(\Lambda^{-\frac{1}{2}}))$. Letting $\nu = \nu_n \to 0$ in the balance (1.6) we conclude that the Hamiltonian of θ is constant on $[0,T]$. Consider next the case $s \in (1,2]$. Then since $\frac{s-1}{2} \in (0, \frac{s}{2})$ it follows by interpolation that

$$\|\Lambda^{\frac{s-1}{2}}\theta_n\|^2_{L^2(\Omega)} \le \|\theta_n\|^{2(1-\lambda)}_{L^2(\Omega)} \|\Lambda^{\frac{s}{2}}\theta_n\|^{2\lambda}_{L^2(\Omega)} \le C\|\Lambda^{\frac{s}{2}}\theta_n\|^{2\lambda}_{L^2(\Omega)}$$

for some $\lambda \in (0,1)$ depending only on s. Thus, for any $\delta > 0$,

$$\nu_n \int_0^t \|\Lambda^{\frac{s-1}{2}}\theta_n\|^2_{L^2(\Omega)} dt \le Ct\nu_n \delta^{-\frac{\lambda}{1-\lambda}} + C\delta\nu_n \int_0^T \|\Lambda^{\frac{s}{2}}\theta_n\|^2_{L^2(\Omega)} dr, \quad t \in [0,T].$$

Because of (1.5) the energy dissipation quantities $\nu_n \int_0^t \int_\Omega |\Lambda^{\frac{s}{2}}\theta_n|^2 dxdt$, $t \in [0,T]$, are uniformly bounded. Sending $\nu_n \to 0$ and then $\delta \to 0$ yields (4.3) for this case. This completes the proof.

Appendix A. A bound on $\mathbb{P}_m$

Recall the definition (3.1) of $\mathbb{P}_m$. The following lemma is essentially taken from [**8**]. We include the proof for the sake of completeness.

LEMMA A.1. *Let $\Omega \subset \mathbb{R}^d$, $d \ge 2$, be a bounded domain with smooth boundary. For every N and $k \in \mathbf{N}$ satisfying $N > \frac{k}{2} + \frac{d}{2}$ there exists a positive constant $C_{N,k}$ such that*

$$\|\mathbb{P}_m \varphi\|_{H^k(\Omega)} \le C_{N,k} \|\varphi\|_{D(\Lambda^{2N})} \tag{A.1}$$

for all $m \ge 1$ and $\varphi \in D(\Lambda^{2N})$; moreover, we have

$$\lim_{m\to\infty} \|(\mathbb{I} - \mathbb{P}_m)\varphi\|_{H^k(\Omega)} = 0. \tag{A.2}$$

PROOF. As $\varphi \in D(\Lambda^{2N})$, we have $\Delta^\ell \varphi \in H^1_0(\Omega)$ for all $\ell = 0, 1, ...N-1$. This allows repeated integration by parts with w_j using the relation $-\Delta w_j = \lambda_j w_j$. Using Hölder's inequality and the fact that w_j is normalized in L^2, we obtain

$$|\varphi_j| \le \lambda_j^{-N} \|\Delta^N \varphi\|_{L^2}, \quad \varphi_j = \int_\Omega \varphi w_j dx.$$

By elliptic regularity estimates and induction, we have for all $k \in \mathbb{N}$ that

$$\|w_j\|_{H^k(\Omega)} \le C_k \lambda_j^{\frac{k}{2}}.$$

We know from the easy part of Weyl's asymptotic law that $\lambda_j \ge C j^{\frac{2}{d}}$. Consequently, with $N > \frac{k}{2} + \frac{d}{2}$ we deduce that

$$\begin{aligned}
\sum_{j=1}^\infty |\varphi_j| \|w_j\|_{H^k(\Omega)} &\le C_k \|\Delta^N \varphi\|_{L^2} \sum_{j=1}^\infty \lambda_j^{-N+\frac{k}{2}} \\
&\le C_k \|\varphi\|_{D(\Lambda^{2N})} \sum_{j=1}^\infty j^{(-N+\frac{k}{2})\frac{2}{d}} \\
&= C_{N,k} \|\varphi\|_{D(\Lambda^{2N})}
\end{aligned}$$

where $C_{N,k} < \infty$ depends only on N and k. Because

$$(\mathbb{I} - \mathbb{P}_m)\varphi = \sum_{j=m+1}^{\infty} \varphi_j w_j,$$

this proves both (A.1) and (A.2). The proof is complete. $\square$

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References

[1] L. C. Berselli. Vanishing Viscosity Limit and Long-time Behavior for 2D Quasi-geostrophic Equations. *Indiana Univ. Math. J.* 51(4) (2002), 905–930.

[2] T. Buckmaster, S. Shkoller, V. Vicol. Nonuniqueness of weak solutions to the SQG equation. arXiv:1610.00676, to appear in *Communications on Pure and Applied Mathematics*.

[3] D. Chae, P. Constantin, D. Córdoba, F. Gancedo, J. Wu. Generalized surface quasi-geostrophic equations with singular velocities. *Comm. Pure Appl. Math.*, **65** (2012) No. 8, 1037-1066.

[4] A. Cheskidov, M. C. Lopes Filho; H. J. Nussenzveig Lopes; R. Shvydkoy. Energy conservation in two-dimensional incompressible ideal fluids. *Comm. Math. Phys.* 348 (2016), no. 1, 129–143.

[5] P. Constantin, D. Cordoba, J. Wu. On the critical dissipative quasi-geostrophic equation. *Indiana Univ. Math. J.*, 50 (Special Issue): 97–107, 2001. Dedicated to Professors Ciprian Foias and Roger Temam (Bloomington, IN, 2000).

[6] P. Constantin, M. Ignatova. Remarks on the fractional Laplacian with Dirichlet boundary conditions and applications. *Internat. Math. Res. Notices*, (2016), 1-21.

[7] P. Constantin, M. Ignatova. Critical SQG in bounded domains. *Ann. PDE* (2016) 2:8.

[8] P. Constantin, H.Q. Nguyen. Global weak solutions for SQG in bounded domains. *Comm. Pure Appl. Math*, 71 (2018), no. 11, 2323-2333.

[9] P. Constantin, H. Q. Nguyen. Local and global strong solutions for SQG in bounded domains. *Phys. D* Vol. 376-378 (2018), Special Issue in Honor of Edriss Titi, 195-203.

[10] P. Constantin, A.J. Majda, and E. Tabak. Formation of strong fronts in the 2-D quasigeostrophic thermal active scalar. *Nonlinearity*, 7(6) (1994), 1495–1533.

[11] P. Constantin, A. Tarfulea, V. Vicol. Absence of anomalous dissipation of energy in forced two dimensional fluid equations. *Arch. Ration. Mech. Anal.* **212** (2014), 875-903.

[12] F. Marchand. Existence and Regularity of Weak Solutions to the Quasi-Geostrophic Equations in the Spaces L^p or $\dot{H}^{-1/2}$. *Comm. Math. Phys.* (2008) 277(1): 45–67.

[13] I.M. Held, R.T. Pierrehumbert, S.T. Garner, and K.L. Swanson. Surface quasi-geostrophic dynamics. *J. Fluid Mech.*, 282 (1995),1–20.

[14] P. Isset and V. Vicol. Hölder continuous solutions of active scalar equations. *Ann. PDE* 1 (2015), no. 1, 1–77.

[15] H. Q. Nguyen. Global weak solutions for generalized SQG in bounded domains. *Anal. PDE*, Vol. 11 (2018), No. 4, 1029–1047.

[16] J.L. Lions, *Quelque methodes de résolution des problemes aux limites non linéaires*. Paris: Dunod-Gauth, 1969.

[17] J. L. Lions, E. Magenes, *Non-homogeneous boundary value problems and applications. Vol. I.* Translated from the French by P. Kenneth. Die Grundlehren der mathematischen Wissenschaften, Band 181. Springer-Verlag, New York-Heidelberg, 1972.

[18] S. Resnick, *Dynamical problems in nonlinear advective partial differential equations*. ProQuest LLC, Ann Arbor, MI, 1995, Thesis (Ph.D.)–The University of Chicago.

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