

EXPLORING THE EFFECTS OF UNCERTAINTY IN PARAMETER TRACKING ESTIMATES FOR THE TIME-VARYING EXTERNAL VOLTAGE PARAMETER IN THE FITZHUGH-NAGUMO MODEL

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SUMMARY

This study explores how uncertainty in time-varying parameter estimates obtained using nonlinear filtering algorithms with parameter tracking affects corresponding model output predictions. Results are demonstrated on a numerical example estimating the time-varying external voltage parameter in the FitzHugh-Nagumo system for modeling the spiking dynamics of neurons.

Key words: *time-varying parameter estimation, parameter uncertainty, nonlinear filtering*

1 INTRODUCTION

Many applications in modern-day science involve system parameters that are unknown and must be estimated using little to no prior information. The problem of estimating and quantifying uncertainty in such parameters remains a big challenge in computational and applied mathematics. A subset of these parameter estimation problems involve system parameters that change with time but do not have known evolution models. One example includes the external voltage parameter in modeling the spiking dynamics of neurons [1], which may exhibit periodic behavior.

This study explores the effects of uncertainty in time-varying parameter estimates obtained using nonlinear filtering algorithms with parameter tracking, with particular focus on how uncertainty in the parameter estimates affect corresponding model output predictions. As will be demonstrated on a numerical example estimating the time-varying external voltage parameter in the FitzHugh-Nagumo model, uncertainty in this setting results from the choice of the drift variance in the parameter tracking algorithm, along with the inability of parameter tracking to maintain the structural characteristics (such as periodicity) of the parameter. Results show how parameter uncertainties have a noticeable effect on the corresponding FitzHugh-Nagumo model predictions.

2 METHODOLOGY

2.1 The Parameter Estimation Inverse Problem

The parameter estimation inverse problem is to estimate the unknown (or poorly known) parameters of a system given noisy, sometimes partial observations of the system states (or some function of the system states). More specifically, assume the system dynamics are described by an ordinary differential equation model

$$\frac{dx}{dt} = f(t, x, \theta), \quad x(0) = x_0, \quad (1)$$

where $x = x(t) \in \mathbb{R}^d$ is the vector of model states, $\theta \in \mathbb{R}^p$ is the vector of unknown parameters, $f : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^p \rightarrow \mathbb{R}^d$ is the known model function, and x_0 is the vector of initial values (which may also be unknown or poorly known). Further assume discrete, noisy observations

$$y_k = g(x(t_k), \theta) + w_k, \quad 0 < t_1 < t_2 < \dots < t_T \quad (2)$$

where $g : \mathbb{R}^d \times \mathbb{R}^p \rightarrow \mathbb{R}^m$, $m \leq d$, is the known observation function and w_k represents the observation error. The inverse problem is therefore to estimate the parameters θ and model states $x(t)$ at given times from the observations y_k .

2.2 Parameter Tracking via Nonlinear Filtering

While there are many approaches to solving the classic parameter estimation inverse problem as posed, nonlinear filtering algorithms provide a natural framework for estimating time-varying parameters given time series data, as considered in this study. Nonlinear filtering methods fall under the category of Bayesian approaches to solving the inverse problem, where the model states x and parameters θ are treated as random variables with probability distributions, and their joint posterior density is assembled using Bayes' theorem

$$\pi(x, \theta | y) \propto \pi(y | x, \theta) \pi(x, \theta) \quad (3)$$

with likelihood function $\pi(y | x, \theta)$ and prior distribution $\pi(x, \theta)$. Nonlinear filters (also known as sequential Monte Carlo methods) rely on the use of discrete-time stochastic equations describing the model states and observations to sequentially update the joint posterior density $\pi(x, \theta | y)$. Denoting by D_j the accumulated observations up to time t_j , the posterior distribution $\pi(x_j, \theta | D_j)$ is updated sequentially using a two-step, predictor-corrector-type scheme:

$$\pi(x_j, \theta | D_j) \longrightarrow \pi(x_{j+1}, \theta | D_j) \longrightarrow \pi(x_{j+1}, \theta | D_{j+1}). \quad (4)$$

The first step (the prediction step) uses the state evolution model to predict the values of the states at time t_{j+1} without knowledge of the data, while the second step (the analysis step or observation update) uses the observation equation to correct that prediction by taking into account the data at t_{j+1} . Starting with a prior density $\pi(x_0, \theta | D_0)$, $D_0 = \emptyset$, this updating scheme is repeated until the final joint posterior density is obtained when $j = T$.

Various nonlinear filtering algorithms exist in the literature, including particle filters [4, 5, 6] and ensemble Kalman filters (EnKFs) [7, 8, 9]. In standard formulations of these algorithms, the parameters θ are assumed to be static, i.e., $d\theta/dt = 0$, and are artificially evolved over time. However, in cases where the parameter values change slowly over time, a drift can be added to the parameters in the prediction step in order to track the change in the parameters over time, modeling the change as a random walk

$$\theta_{j+1} = \theta_j + \xi_{j+1}, \quad \xi_{j+1} \sim \mathcal{N}(0, E_{j+1}). \quad (5)$$

The covariance E_{j+1} of the drift term, typically modeled as $E_{j+1} = \sigma_\xi^2 I$ with drift variance σ_ξ^2 , must be carefully chosen a priori for each application considered in order to avoid filter divergence and obtain a useful parameter estimate. The choice of σ_ξ^2 has a direct effect on the amount of uncertainty in the resulting parameter estimate. Parameter tracking also does not guarantee that structural characteristics (such as periodicity) are maintained, introducing additional uncertainty into the parameter estimate.

3 NUMERICAL RESULTS

To explore the effects of uncertainty in the parameter tracking estimates of time-varying parameters, consider the following numerical example of estimating the external voltage parameter in the FitzHugh-Nagumo model [2], which is commonly used as a simplified version of the Hodgkin-Huxley model [3] for studying the spiking dynamics of neurons. The FitzHugh-Nagumo system is given by

$$\frac{dx_1}{dt} = c \left(x_2 + x_1 - \frac{x_1^3}{3} + v(t) \right) \quad (6)$$

$$\frac{dx_2}{dt} = -\frac{1}{c} \left(x_1 - a + bx_2 \right) \quad (7)$$

where x_1 describes the measurable membrane potential of the neuron and x_2 is an unobservable combined effect of different ionic currents. While the parameters a , b , and c are typically assigned as some known constants, the time-varying external voltage parameter $v(t)$ is unknown.

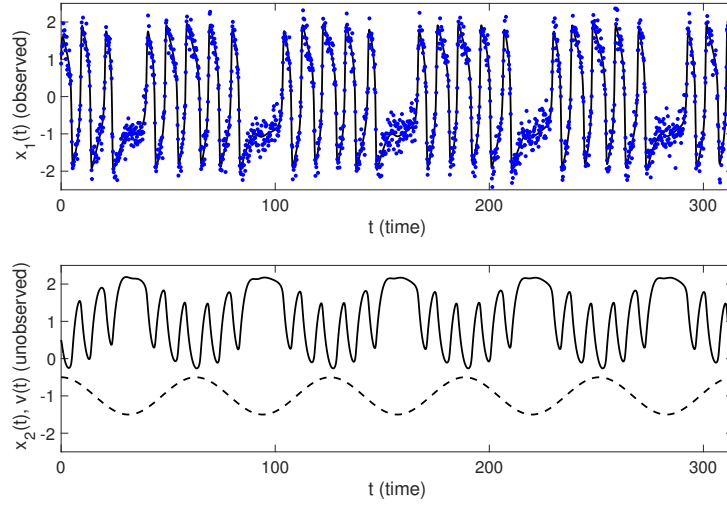


Figure 1: **States and external voltage parameter of the FitzHugh-Nagumo system.** Noisy observations of x_1 (top, blue markers) of the FitzHugh-Nagumo system (6)–(7), along with the unobserved state x_2 (bottom, solid black) and external voltage parameter $v(t)$ (bottom, dashed black).

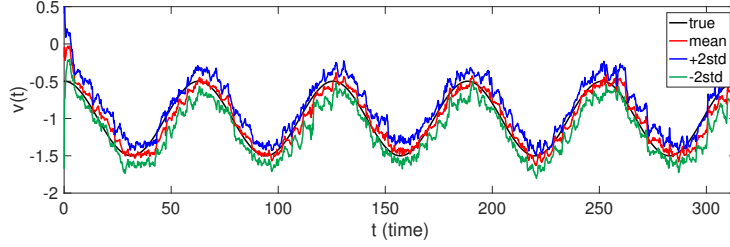


Figure 2: **Parameter tracking estimates of the external voltage parameter.** EnKF with parameter tracking estimates of the voltage parameter $v(t)$ in the FitzHugh-Nagumo system (6)–(7). The estimated EnKF mean is plotted in solid red, and the ± 2 standard deviation curves are plotted in blue and green, respectively. The true sinusoidal voltage function used to generate the synthetic data is plotted in black.

Figure 1 shows synthetic data generated from the system (6)–(7) with initial conditions $x_1(0) = 1$ and $x_2(0) = 0.5$, fixed parameters $a = 0.7$, $b = 0.8$, and $c = 3$, and modeling the time-varying external voltage parameter as a periodic, sinusoidal function $v(t) = 0.5 \sin(\omega t + \pi/2) - 1$ with frequency $\omega = 0.1$, varying more slowly than the system dynamics; a similar example was considered in [1, 10]. Data was generated by taking measurements of x_1 at 1571 equidistant time instances over the interval from $t = 0$ to $t = 314$, which covers five periods of $v(t)$. Observations were corrupted with zero-mean Gaussian noise with standard deviation assigned to be 20% of the standard deviation of x_1 .

Figure 2 shows the resulting mean and ± 2 standard deviation curves estimating $v(t)$ via parameter tracking with an augmented EnKF in the style of [9, 10] using $n = 15$ ensemble members and drift standard deviation $\sigma_\xi = 0.01$. While the resulting curves are able to track the general shape of the underlying sinusoidal function, periodicity is not maintained. Further, model predictions using these curves result in widely different time series estimates of x_1 and x_2 , shown in Figure 3.

4 CONCLUSIONS

The results shown in Figure 3 demonstrate the sensitivity of the FitzHugh-Nagumo system (6)–(7) to uncertainty in the estimate of the external voltage parameter $v(t)$. While the parameter tracking approach is able to capture the general behavior of the parameter over time, it is not able to maintain the periodicity intrinsic to the parameter. Uncertainty in the external voltage parameter estimates result in model predictions that are unable to capture fully the true underlying model states and differ significantly between the mean estimate and ± 2 standard deviation curves. This motivates future work

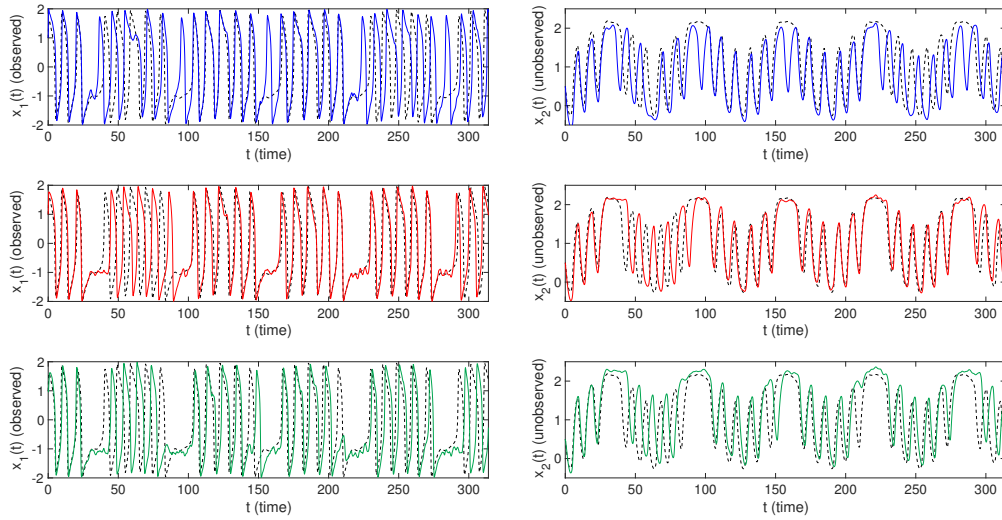


Figure 3: **Model predictions using parameter tracking estimates of the external voltage parameter.** Model predictions obtained using the parameter tracking estimates of the external voltage parameter $v(t)$ in the FitzHugh-Nagumo system (6)–(7). Predictions of the model states x_1 and x_2 using the EnKF mean estimate are plotted in solid red (middle), while the ± 2 standard deviation curves are plotted in blue (top) and green (bottom), respectively. The true model states used to generate the synthetic data are plotted in dashed black.

to develop parameter tracking algorithms that are able to better maintain structural characteristics while estimating time-varying parameters.

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