# Exploring Finite Phononic Materials using Linear Systems Theory and Pole-zero Distributions

**Mostafa Nouh**[1]

[1] *Department of Mechanical and Aerospace Engineering, University at Buffalo (SUNY), Buffalo, NY 14260, USA*
*mnouh@buffalo.edu*

**Abstract:** This talk describes the system dynamics of finite realizations of phononic materials in an attempt to understand phenomenon such as Bragg band gaps, truncation resonances, as well as local resonance effects in finite phononic crystals and acoustic metamaterials. Through a rigorous mathematical scheme, we show the morphing of the pole-zero characteristics of finite lattices in the frequency domain as such gaps evolve, materialize and become noticeable.

Phononic Materials (PMs) are synthetically formed by spatially arranging materials of different acoustic/elastic properties in a periodic pattern[1]. The interest in such PMs stems from their ability to manipulate incident waves across different length and time scales culminating in band gaps, or frequency ranges within which wave propagations are not permitted. Band gaps in PMs take place by virtue of different mechanisms, including Bragg scattering and local resonances in acoustic metamaterials. Owing to such unique dispersion attributes, the use of PMs opens up new avenues in vibration mitigation, wave cloaking, acoustic guidance, and refraction, to name a few. This talk focuses on the system dynamics of finite realizations of such periodic structures in an attempt to understand, and further exploit, band gap related phenomena in PMs. Despite their prediction in theoretically infinite systems, band gaps materialize in finite realizations of the phononic unit cells as demonstrated numerically and backed up by experimental evidence. This implies the presence of a set of adaptations in the dynamics of such systems, as the number of cells grow, which explain the band gap formation and evolution at the finite level. Through a rigorous mathematical scheme, we show the morphing of the PM's pole-zero characteristics in the frequency domain as Bragg and resonance band gaps evolve and become noticeable. The analysis provides a unique interpretation of band gaps at the nexus of wave physics and systems theory and, as will be shown, bridges the gap between both in the limiting case.

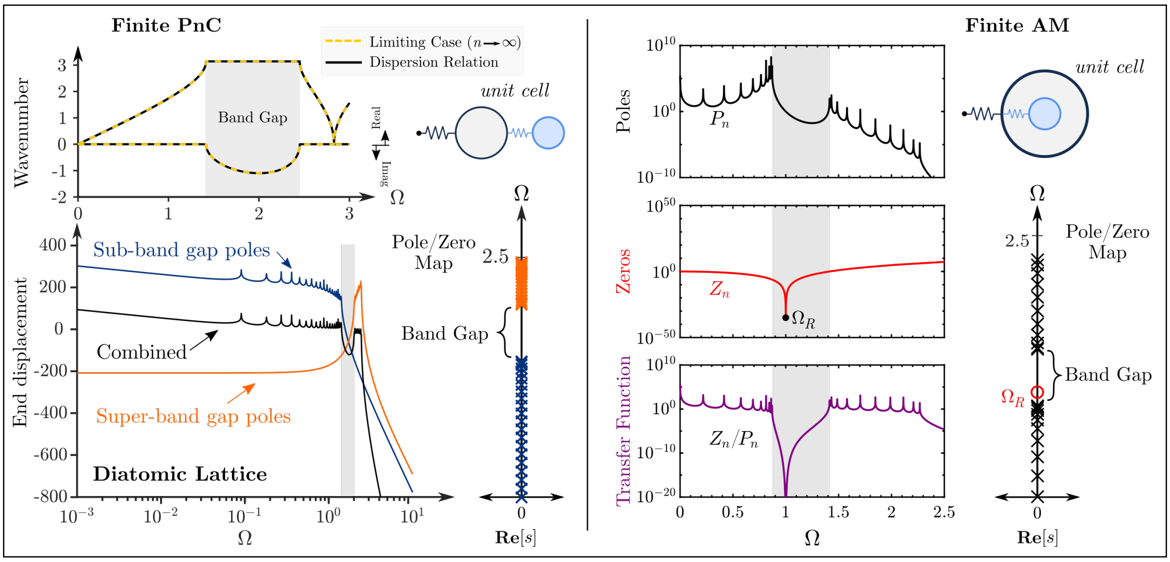


**Figure 1** Band gap formation in a finite phononic crystal (left) and a locally resonant acoustic metamaterial (right). Top left corner shows the convergence of the finite analysis to the dispersion curves at the limiting case corresponding to $n \rightarrow \infty$. Bode plots in the bottom left corner reveal the separate contributions of the sub- and super-band gap poles on the response of a finite phononic lattice. Frequency response functions in the bottom right corner show characteristic polynomials of the poles, zeros as well as the overall transfer function; highlighting the mechanism of a local resonance band gap creation.

We start by considering lumped mass realizations of PMs with two types of periodicities: (1) Alternating materials and (2) Periodic elastic supports. Following which, a set of analytical formulae depicting the eigenvalues of both systems, in their unconstrained form, as functions of any prescribed parameters are derived. A block diagram reduction approach is then used to extract closed-form expressions for the end-to-end transfer function of a finite PM with any arbitrary inertial or stiffness values. The latter reveals a consistent pattern which resembles the *mathematical continuous fraction*[2]. Using the derived transfer functions, the limiting case of the continuous fraction model as the number of cells $n$ approaches infinity is presented which is shown to perfectly match the band structure of an infinite PM obtained via a conventional Bloch-wave solution.

In the second thrust of this study, we utilize Bode plots to interpret the mechanism of Bragg band gap creation in finite PMs in the frequency domain. We show that such gaps are formed as a result of a two-way split of the system's natural frequencies (poles) on both sides of the gap's frequency range. The two groups are denoted the sub- and super-band gap poles, respectively. Sub-band gap poles contribute to the band gap attenuation which hits a maximum value at a central point $\Omega_{max}$, an expression for which is analytically derived. At higher frequencies on the opposite side of the band gap, a set of super-band gap poles lift up the Bode magnitude and eventually close the band gap at its upper frequency bound. Numerical examples are used to illustrate these features and highlight the minor differences between the system dynamics of both PM types considered. It is shown that stronger attenuation within the band gap can be realized when the sub- and super-band gap poles are placed further apart, provided that the lower band gap bound and the number of cells are kept constant. We draw the contrast between the set of conditions leading to the formation of a Bragg band gap in a Phononic Crystal (PnC) and a local resonance band gap in an Acoustic Metamaterial (AM)[3]. The latter being the culmination of: (1) A multiplicity of identical zeros (anti-resonances) at a single frequency combined with (2) The existence of permanent poles at two distinct frequencies which annihilate the multiplicity effect and from the upper and lower band gap bounds.

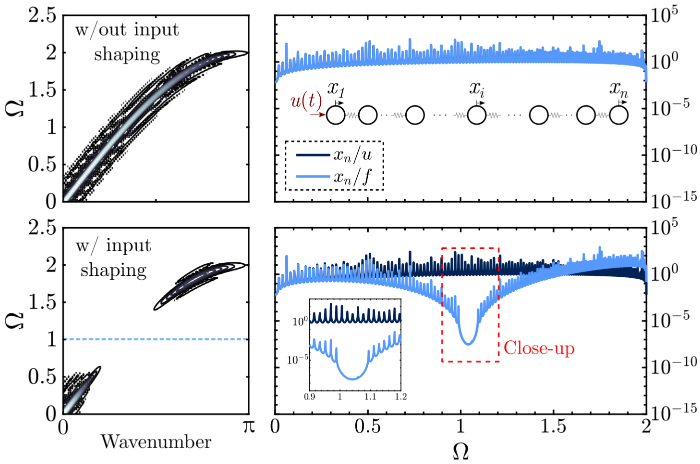


**Figure 2** Artificial band gap synthesis and tuning using input shaping in finite monatomic spring-mass lattices

In the third and final thrust of the study, conditions leading to the appearance of a truncation pole which emerges *inside* a band gap (depending on the chosen parameters), as well as its notable effect on magnitude attenuation, are thoroughly detailed. Finally, we briefly show a couple of examples on how such understanding informs the process of phononic material design and explore the idea of band gap synthesis using input shaping in simple monatomic lattices[4].

## References

[1] M. I. Hussein, M. J. Leamy, and M. Ruzzene, *Appl. Mech. Rev.* **66**, 040802 (2014).
[1] H. Al Ba'ba'a, M. Nouh, and T. Singh, *J. Acoust. Soc. Am.* **142**, 1399-1412 (2017).
[3] H. Al Ba'ba'a, M. Nouh, and T. Singh, *J. Sound Vib.* **410**, 429-446 (2017).
[4] H. Al Ba'ba'a, J. Callanan, M. Nouh and T. Singh, Meccanica 53, 3105 - 3122 (2018)