

# Nonuniqueness of weak solutions to the Navier-Stokes equation

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## Abstract

For initial datum of finite kinetic energy, Leray has proven in 1934 that there exists at least one global in time finite energy weak solution of the 3D Navier-Stokes equations. In this paper we prove that weak solutions of the 3D Navier-Stokes equations are not unique in the class of weak solutions with finite kinetic energy. Moreover, we prove that Hölder continuous dissipative weak solutions of the 3D Euler equations may be obtained as a strong vanishing viscosity limit of a sequence of finite energy weak solutions of the 3D Navier-Stokes equations.

## 1. Introduction

In this paper we consider the 3D incompressible Navier-Stokes equation

$$(1.1a) \quad \partial_t v + \operatorname{div}(v \otimes v) + \nabla p - \nu \Delta v = 0,$$

$$(1.1b) \quad \operatorname{div} v = 0$$

posed on  $\mathbb{T}^3 \times \mathbb{R}$ , with periodic boundary conditions in  $x \in \mathbb{T}^3 = \mathbb{R}^3/2\pi\mathbb{Z}^3$ . We consider solutions normalized to have zero spatial mean, i.e.,  $\int_{\mathbb{T}^3} v(x, t) dx = 0$ . The constant  $\nu \in (0, 1]$  is the kinematic viscosity. We define weak solutions to the Navier-Stokes equations [49, Definition 1], [19, pp. 226]:

*Definition 1.1.* We say  $v \in C^0(\mathbb{R}; L^2(\mathbb{T}^3))$  is a weak solution of (1.1) if for any  $t \in \mathbb{R}$  the vector field  $v(\cdot, t)$  is weakly divergence free, has zero mean, and (1.1a) is satisfied in  $\mathcal{D}'(\mathbb{T}^3 \times \mathbb{R})$ , i.e.,

$$\int_{\mathbb{R}} \int_{\mathbb{T}^3} v \cdot (\partial_t \varphi + (v \cdot \nabla) \varphi + \nu \Delta \varphi) dx dt = 0$$

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holds for any test function  $\varphi \in C_0^\infty(\mathbb{T}^3 \times \mathbb{R})$  such that  $\varphi(\cdot, t)$  is divergence-free for all  $t$ .

As a direct result of the work of Fabes-Jones-Riviere [19], since the weak solutions defined above lie in  $C^0(\mathbb{R}; L^2(\mathbb{T}^3))$ , they are in fact solutions of the integral form of the Navier-Stokes equations

$$(1.2) \quad v(\cdot, t) = e^{\nu t \Delta} v(\cdot, 0) + \int_0^t e^{\nu(t-s)\Delta} \mathbb{P} \operatorname{div} (v(\cdot, s) \otimes v(\cdot, s)) ds,$$

and are sometimes called *mild* or *Oseen* solutions (cf. [19] and [39, Def. 6.5]). Here  $\mathbb{P}$  is the Leray projector and  $e^{t\Delta}$  denotes convolution with the heat kernel.

1.1. *Previous works.* In [40], Leray considered the Cauchy-problem for (1.1) for initial datum of finite kinetic energy,  $v_0 \in L^2$ . Leray proved that for any such datum, there exists a global in time weak solution  $v \in L_t^\infty L_x^2$ , which additionally has the regularity  $L_t^2 \dot{H}_x^1$ , and obeys the energy inequality  $\|v(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla u(s)\|_{L^2}^2 ds \leq \|v_0\|_{L^2}^2$ . Hopf [24] established a similar result for the equations posed in a smooth bounded domain, with Dirichlet boundary conditions. To date, the question of uniqueness of Leray-Hopf weak solutions for the 3D Navier-Stokes equations remains however open.

Based on the natural scaling of the equations

$$v(x, t) \mapsto v_\lambda(x, t) = \lambda v(\lambda x, \lambda^2 t),$$

a number of partial regularity results have been established [45], [7], [41], [37], [53], [35]; the local existence for the Cauchy problem has been proven in scaling-invariant spaces [30], [32], [28]; and conditional regularity has been established under geometric structure assumptions [11] or assuming a signed pressure [47]. The conditional regularity and weak-strong uniqueness results known under the umbrella of Ladyzhenskaya-Prodi-Serrin conditions [31], [43], [48], state that if a Leray-Hopf weak solution also lies in  $L_t^p L_x^q$ , with  $2/p + 3/q \leq 1$ , then the solution is unique and smooth in positive time. These conditions and their generalizations have culminated with the work of Escauriaza-Seregin-Šverák [27] who proved the  $L_t^\infty L_x^3$  endpoint. The uniqueness of mild/Oseen solutions is also known under the Ladyzhenskaya-Prodi-Serrin conditions, cf. [19] for  $p > 3$ , and [21], [42], [38], [33] for  $p = 3$ . Note that the regularity of Leray-Hopf weak solutions, or of bounded energy weak solutions, is consistent with the scaling  $2/p + 3/q = 3/2$ . In contrast, the additional regularity required to ensure that the energy equality holds in the Navier-Stokes equations is consistent with  $2/4 + 3/4 = 5/4$  for  $p = q = 4$  [50], [34]. See [12], [52], [38], [44], [39] for surveys of results on the Navier-Stokes equations.

The gap between the scaling of the kinetic energy and the natural scaling of the equations leaves open the possibility of nonuniqueness of weak solutions to (1.1). In [28], [29] Jia-Šverák proved that nonuniqueness of Leray-Hopf

weak solutions in the regularity class  $L_t^\infty L_x^{3,\infty}$  holds if a certain spectral assumption holds for a linearized Navier-Stokes operator. While a rigorous proof of this spectral condition remains open, very recently Guillod-Šverák [23] have provided compelling numerical evidence of it, using a scenario related to the example of Ladyzhenskaya [36]. Thus, the works [29], [23] strongly suggest that the Ladyzhenskaya-Prodi-Serrin regularity criteria are sharp.

**1.2. Main results.** In this paper we prove that weak solutions to (1.1) (in the sense of Definition 1.1) are not unique within the class of weak solutions with bounded kinetic energy. We establish the stronger result<sup>1</sup>:

**THEOREM 1.2** (Nonuniqueness of weak solutions). *There exists  $\beta > 0$ , such that for any nonnegative smooth function  $e(t): [0, T] \rightarrow \mathbb{R}_{\geq 0}$ , there exists  $v \in C_t^0([0, T]; H_x^\beta(\mathbb{T}^3))$  a weak solution of the Navier-Stokes equations, such that  $\int_{\mathbb{T}^3} |v(x, t)|^2 dx = e(t)$  for all  $t \in [0, T]$ . Moreover, the associated vorticity  $\nabla \times v$  lies in  $C_t^0([0, T]; L_x^1(\mathbb{T}^3))$ .*

In particular, the above theorem shows that  $v \equiv 0$  is not the only weak solution which vanishes at a time slice, thereby implying the nonuniqueness of weak solutions. Theorem 1.2 shows that weak solutions may come to rest in finite time, a question posed by Serrin [49, pp. 88]. Moreover, by considering  $e_1(t), e_2(t) > 0$  which are nonincreasing, such that  $e_1(t) = e_2(t)$  for  $t \in [0, T/2]$ , and  $e_1(T) < e_2(T)$ , the construction used to prove Theorem 1.2 also proves the nonuniqueness of dissipative weak solutions.

From the proof of Theorem 1.2 it is clear that the constructed weak solutions  $v$  also have regularity in time, i.e. there exists  $\gamma > 0$  such that  $v \in C_t^\gamma([0, T]; L_x^2(\mathbb{T}^3))$ . Thus,  $v \otimes v$  lies in  $C_t^\gamma L_x^1 \cap C_t^0 L_x^{1+\gamma}$ , and the fact that  $\nabla v \in C_t^0 L_x^1$  follows from (1.2) and the maximal regularity of the heat equation.

We note that while the weak solutions Theorem 1.2 may attain any smooth energy profile, at the moment we do not prove that they are Leray-Hopf weak solutions, i.e., they do not obey the energy inequality or have  $L_t^2 \dot{H}_x^1$  integrability. Moreover, the regularity parameter  $\beta > 0$  cannot be expected to be too large, since at  $\beta = 1/2$  one has weak-strong uniqueness [12]. We expect that the ideas used to prove Theorem 1.2 will in the future lead to a proof of nonuniqueness of weak solutions in  $C_t^0 L_x^p$ , for any  $2 \leq p < 3$ , and the nonuniqueness of Leray-Hopf weak solutions.

The proof of Theorem 1.2 builds on several of the fundamental ideas pioneered by De Lellis-Székelyhidi Jr. [15], [16]. These ideas were used to tackle the Onsager conjecture for the Euler equation [18], [10], [8] (set  $\nu = 0$  in (1.1))

<sup>1</sup>We denote by  $H^\beta$  the  $L^2$ -based Sobolev space with regularity index  $\beta$ . Clearly  $C_t^0 H_x^\beta \subset C_t^0 L_x^2$ .

via convex integration methods [46], [51], [2], [17], [1], [3], leading to the resolution of the conjecture by Isett [26], [25], using a key ingredient by Daneri and Székelyhidi Jr. [14]. The construction of dissipative Euler solutions below the Onsager regularity threshold was proven by authors of this paper jointly with De Lellis and Székelyhidi Jr. in [4], building on the ideas in [14], [26]. In order to treat the dissipative term  $-\nu\Delta$ , not present in the Euler system, we cannot proceed as in [6], [9], since in these works Hölder continuous weak solutions are constructed, which is possible only by using building blocks which are sparse in the frequency variable and for small fractional powers of the Laplacian. Instead, the main idea, which is also used in [5], is to use building blocks for the convex integration scheme which are “intermittent”. That is, the building blocks we use are spatially inhomogeneous, and have different scaling in different  $L^p$  norms. At high frequency, these building blocks attempt to saturate the Bernstein inequalities from Littlewood-Paley theory. Since they are built by adding eigenfunctions of curl in a certain geometric manner, we call these building blocks *intermittent Beltrami flows*. In particular, the proof of [Theorem 1.2](#) breaks down in  $2D$ , as is expected, since there are not enough spatial directions to oscillate in. The proof of [Theorem 1.2](#) is given in [Section 2](#) below.

The idea of using intermittent building blocks can be traced back to classical observations in hydrodynamic turbulence, see for instance [20]. Moreover, in view of the aforementioned works on the Onsager conjecture for the Euler equations, we are naturally led to consider the set of accumulation points in the vanishing viscosity limit  $\nu \rightarrow 0$  of the family of weak solutions to the Navier-Stokes equations which we constructed in [Theorem 1.2](#). We prove in this paper that this set of accumulation points, in the  $C_t^0 L_x^2$  topology, contains all the Hölder continuous weak solutions of the 3D Euler equations:

**THEOREM 1.3** (Dissipative Euler solutions arise in the vanishing viscosity limit). *For  $\bar{\beta} > 0$  let  $u \in C_{t,x}^{\bar{\beta}}(\mathbb{T}^3 \times [-2T, 2T])$  be a zero-mean weak solution of the Euler equations. Then there exists  $\beta > 0$ , a sequence  $\nu_n \rightarrow 0$ , and a uniformly bounded sequence  $v^{(\nu_n)} \in C_t^0([0, T]; H_x^\beta(\mathbb{T}^3))$  of weak solutions to the Navier-Stokes equations, with  $v^{(\nu_n)} \rightarrow u$  strongly in  $C_t^0([0, T]; L_x^2(\mathbb{T}^3))$ .*

In particular, [Theorem 1.3](#) shows that the nonconservative weak solutions to the Euler equations obtained in [26, 4] arise in the vanishing viscosity limit of weak solutions to the Navier-Stokes equations. Thus, being a strong limit of weak solutions to the Navier-Stokes equations, in the sense of [Definition 1.1](#), cannot serve as a selection criterion for weak solutions of the Euler equation. Whether similar vanishing viscosity results hold for sequences of Leray-Hopf weak solutions, or for suitable weak solutions of (1.1), remains a challenging open problem. The proof of [Theorem 1.3](#) is closely related to that of [Theorem 1.2](#), and is also given in [Section 2](#) below.

## 2. Outline of the convex integration scheme

In this section we sketch the proof of [Theorem 1.2](#). For every integer  $q \geq 0$  we will construct a solution  $(v_q, p_q, \mathring{R}_q)$  to the Navier-Stokes-Reynolds system

$$(2.1a) \quad \partial_t v_q + \operatorname{div}(v_q \otimes v_q) + \nabla p_q - \nu \Delta v_q = \operatorname{div} \mathring{R}_q,$$

$$(2.1b) \quad \operatorname{div} v_q = 0.$$

where the Reynolds stress  $\mathring{R}_q$  is assumed to be a trace-free symmetric matrix.

**2.1. Parameters.** Throughout the proof we fix a sufficiently large, universal constant  $b \in 16\mathbb{N}$ , and depending on  $b$  we fix a regularity parameter  $\beta > 0$  such that  $\beta b^2 \leq 4$  and  $\beta b \leq 1/40$ . We remark that it is sufficient to take  $b = 2^9$  and  $\beta = 2^{-16}$ .

The relative size of the approximate solution  $v_q$  and the Reynolds stress error  $\mathring{R}_q$  will be measured in terms of a frequency parameter  $\lambda_q$  and an amplitude parameter  $\delta_q$  defined as

$$\begin{aligned} \lambda_q &= a^{(b^q)}, \\ \delta_q &= \lambda_1^{3\beta} \lambda_q^{-2\beta} \end{aligned}$$

for some integer  $a \gg 1$  to be chosen suitably.

**2.2. Inductive estimates.** By induction, we will assume the following estimates<sup>2</sup> on the solution of (2.1) at level  $q$ :

$$(2.2) \quad \|v_q\|_{C_{x,t}^1} \leq \lambda_q^4,$$

$$(2.3) \quad \|\mathring{R}_q\|_{L^1} \leq \lambda_q^{-\varepsilon_R} \delta_{q+1},$$

$$(2.4) \quad \|\mathring{R}_q\|_{C_{x,t}^1} \leq \lambda_q^{10}.$$

We additionally assume

$$(2.5) \quad 0 \leq e(t) - \int_{\mathbb{T}^3} |v_q|^2 \, dx \leq \delta_{q+1}$$

and

$$(2.6) \quad e(t) - \int_{\mathbb{T}^3} |v_q(x, t)|^2 \, dx \leq \frac{\delta_{q+1}}{100} \Rightarrow v_q(\cdot, t) \equiv 0 \quad \text{and} \quad \mathring{R}_q(\cdot, t) \equiv 0.$$

for all  $t \in [0, T]$ .

<sup>2</sup>Here and throughout the paper we use the notation:  $\|f\|_{L^p} = \|f\|_{L_t^\infty L_x^p}$ , for  $1 \leq p \leq \infty$ ,  $\|f\|_{C^N} = \|f\|_{L_t^\infty C_x^N} = \sum_{0 \leq |\alpha| \leq N} \|D^\alpha f\|_{L^\infty}$ ,  $\|f\|_{C_{x,t}^N} = \sum_{0 \leq n+|\alpha| \leq N} \|\partial_t^n D^\alpha f\|_{L^\infty}$ , and  $\|f\|_{W^{s,p}} = \|f\|_{L_t^\infty W_x^{s,p}}$ , for  $s > 0$ , and  $1 \leq p \leq \infty$ .

**2.3. The main proposition and iterative procedure.** In addition to the sufficiently large universal constant  $b$ , and the sufficiently small regularity parameter  $\beta = \beta(b) > 0$  fixed earlier, we fix the constant  $M_e = \|e\|_{C_t^1}$ . The following iteration lemma states the existence of a solution of (2.1) at level  $q + 1$ , which obeys suitable bounds.

**PROPOSITION 2.1.** *There exists a universal constant  $M > 0$ , a sufficiently small parameter  $\varepsilon_R = \varepsilon_R(b, \beta) > 0$  and a sufficiently large parameter  $a_0 = a_0(b, \beta, \varepsilon_R, M, M_e) > 0$  such that for any integer  $a \geq a_0$ , which is a multiple of the  $N_\Lambda$  of Remark 3.3, the following holds: Let  $(v_q, p_q, \mathring{R}_q)$  be a triple solving the Navier-Stokes-Reynolds system (2.1) in  $\mathbb{T}^3 \times [0, T]$  satisfying the inductive estimates (2.2)–(2.6). Then there exists a second triple  $(v_{q+1}, p_{q+1}, \mathring{R}_{q+1})$  solving (2.1) and satisfying the (2.2)–(2.6) with  $q$  replaced by  $q + 1$ . In addition we have that*

$$(2.7) \quad \|v_{q+1} - v_q\|_{L^2} \leq M\delta_{q+1}^{1/2}.$$

The principal new idea in the proof of Proposition 2.1 is to construct the perturbation  $v_{q+1} - v_q$  as a sum of terms of the form

$$(2.8) \quad a_{(\xi)} \mathbb{W}_{(\xi)}$$

where  $\mathbb{W}_{(\xi)}$  is an intermittent Beltrami wave (cf. (3.12) below) with frequency support centered at frequency  $\xi\lambda_{q+1}$  for  $\xi \in \mathbb{S}^2$ . While these intermittent Beltrami waves have similar properties (cf. Proposition 3.4) to the usual Beltrami flows used in the previous convex integration constructions [16, 2, 17, 1, 3] for the Euler equations, they are fundamentally different since their  $L^1$  norm is much smaller than their  $L^2$  norm (cf. Proposition 3.5). The gain comes from the fact that the Reynolds stress has to be estimated in  $L^1$  rather than  $L^2$ , and that the term  $\nu\Delta v$  is linear in  $v$ . At the technical level, one difference with respect to [26], [4] is the usage of very large gaps between consecutive frequency parameters (i.e.,  $b \gg 1$ ), which is consistent with a small regularity parameter  $\beta$ . Next, we show that Proposition 2.1 implies the main theorems of the paper.

**2.4. Proof of Theorem 1.2.** Choose all the parameters from the statement of Proposition 2.1, except for  $a$ , which we may need to be larger (so that it is still larger than  $a_0$ ).

For  $q = 0$  we note that the identically zero solution trivially satisfies (2.1) with  $\mathring{R}_0 = 0$ , and the inductive assumptions (2.2), (2.3), and (2.4) hold. Moreover, by taking  $a$  sufficiently large such that it is in the range of Proposition 2.1 (i.e.  $a \geq a_0$ ) we may ensure that

$$|e(t)| \leq \|e\|_{C_t^1} = M_e \leq \frac{\lambda_1^\beta}{100} = \frac{\delta_1}{100}.$$

Then the zero solution also satisfies (2.5) and (2.6).

For  $q \geq 1$ , we inductively apply Proposition 2.1. The bound (2.7) and interpolation implies<sup>3</sup>

$$\begin{aligned}
 \sum_{q=0}^{\infty} \|v_{q+1} - v_q\|_{H^{\beta'}} &\lesssim \sum_{q=0}^{\infty} \|v_{q+1} - v_q\|_{L^2}^{1-\beta'} (\|v_{q+1}\|_{C^1} + \|v_q\|_{C^1})^{\beta'} \\
 &\lesssim \sum_{q=0}^{\infty} M^{1-\beta'} \lambda_1^{3\beta \frac{1-\beta'}{2}} \lambda_{q+1}^{-\beta \frac{1-\beta'}{2}} \lambda_{q+1}^{4\beta'} \\
 (2.9) \qquad &\lesssim M^{1-\beta'} \lambda_1^{3\beta \frac{1-\beta'}{2}},
 \end{aligned}$$

for  $\beta' < \beta/(8+\beta)$ , and hence the sequence  $\{v_q\}_{q \geq 0}$  is uniformly bounded  $C_t^0 H_x^{\beta'}$ , for such  $\beta'$ . Furthermore, by taking  $a$  sufficiently large (depending on  $b$ ,  $\beta$  and  $\beta'$ ) the implicit constant in (2.9) can be made to be universal. From (2.1), (2.3), the previously established uniform boundedness in  $C_t^0 L_x^2$ , and the embedding  $W_x^{2,1} \subset L_x^2$  we obtain that

$$\begin{aligned}
 \|\partial_t v_q\|_{H^{-3}} &\lesssim \|\mathbb{P} \operatorname{div} (v_q \otimes v_q) - \nu \Delta v_q - \mathbb{P} \operatorname{div} \dot{R}_q\|_{H^{-3}} \\
 &\lesssim \|v_q \otimes v_q\|_{L^1} + \|v_q\|_{L^2} + \|\dot{R}_q\|_{L^1} \\
 &\lesssim M^2 \lambda_1^{3\beta},
 \end{aligned}$$

where  $\mathbb{P}$  is the Leray projector. Thus, the sequence  $\{v_q\}_{q \geq 0}$  is uniformly bounded in  $C_t^1 H_x^{-3}$ . It follows that for any  $0 < \beta'' < \beta'$  the sum

$$\sum_{q \geq 0} (v_{q+1} - v_q) =: v$$

converges in  $C_t^0 H_x^{\beta''}$ , and since  $\|\dot{R}_q\|_{L^1} \rightarrow 0$  as  $q \rightarrow \infty$ ,  $v$  is a  $C_t^0 H_x^{\beta''}$  weak solution of the Navier-Stokes equation. Lastly, in view of (2.5) we have that the kinetic energy of  $v(\cdot, t)$  is given by  $e(t)$  for all  $t \in [0, T]$ , concluding the proof of the theorem.

**2.5. Proof of Theorem 1.3.** Fix  $\bar{\beta} > 0$  and a weak solution  $u \in C_{t,x}^{\bar{\beta}}$  to the Euler equation on  $[-2T, 2T]$ . The existence of such solutions is guaranteed in view of the results of [26], [4] for  $\bar{\beta} < 1/3$ , and for  $\bar{\beta} > 1$  from the classical local existence results. Let  $M_u = \|u\|_{C^{\bar{\beta}}}$ . Pick an integer  $n \geq 1$ .

Choose all the parameters as in Proposition 2.1, except for  $a \geq a_0$ , which we may take even larger, depending also on  $M_u$  and  $\beta'$  which obeys  $0 < \beta' < \min(\bar{\beta}/2, \beta/(8+\beta))$ . We make  $a$  even larger, depending also on  $\beta'$ , so that in

<sup>3</sup>Throughout this paper, we will write  $A \lesssim B$  to denote that there exists a sufficiently large constant  $C$ , which is independent of  $q$ , such that  $A \leq CB$ .

view of (2.9) we may ensure that

$$(2.10) \quad \sum_{q=n}^{\infty} M^{1-\beta'} \lambda_1^{3\beta \frac{1-\beta'}{2}} \lambda_{q+1}^{-\beta \frac{1-\beta'}{2}} \lambda_{q+1}^{4\beta'} \leq \frac{1}{2Cn}$$

where  $C$  is the implicit constant in (2.9).

Let  $\{\phi_\varepsilon\}_{\varepsilon>0}$  be a family of standard compact support (of width 2) Friedrichs mollifiers on  $\mathbb{R}^3$  (space), and  $\{\varphi_\varepsilon\}_{\varepsilon>0}$  be a family of standard compact support (of width 2) Friedrichs mollifiers on  $\mathbb{R}$  (time). We define

$$v_n = (u *_x \phi_{\lambda_n^{-1}}) *_t \varphi_{\lambda_n^{-1}}$$

to be a mollification of  $u$  in space and time, at length scale and time scale  $\lambda_n^{-1}$ , restricted to the temporal range  $[0, T]$ . Also, on  $[0, T]$  define the energy function

$$e(t) = \int_{\mathbb{T}^3} |v_n(x, t)|^2 dx + \frac{\delta_n}{2}$$

that ensures (2.5) and (2.6) hold for  $q = n$ .

Since  $u$  is a solution of the Euler equations, there exists a mean-free  $p_n$  such that

$$\partial_t v_n + \operatorname{div}(v_n \otimes v_n) + \nabla p_n - \lambda_n^{-2} \Delta v_n = \operatorname{div}(\mathring{R}_n),$$

where  $\mathring{R}_n$  is the traceless symmetric part of the tensor

$$(v_n \otimes v_n) - ((u \otimes u) *_x \phi_{\lambda_n^{-1}}) *_t \varphi_{\lambda_n^{-1}} - \lambda_n^{-2} \nabla v_n.$$

Using a version of the commutator estimate introduced in [10], which may for instance be found in [13, Lemma 1], we obtain that

$$(2.11) \quad \|\mathring{R}_n\|_{L^1} \lesssim \|\mathring{R}_n\|_{C^0} \lesssim \lambda_n^{-1} M_u + \lambda_n^{-2\bar{\beta}} M_u^2.$$

In addition, from a similar argument it follows that

$$(2.12) \quad \|\mathring{R}_n\|_{C_{t,x}^1} \lesssim M_u + \lambda_n^{1-2\bar{\beta}} M_u^2,$$

$$(2.13) \quad \|v_n\|_{C_{t,x}^1} \lesssim \lambda_n^{1-\bar{\beta}} M_u.$$

Setting

$$\nu := \nu_n := \lambda_n^{-1},$$

then with  $a$  sufficiently large, depending on  $M_u$  and  $\bar{\beta}$ , we may ensure the pair  $(v_n, \mathring{R}_n)$  obey the inductive assumptions (2.2)–(2.4) for  $q = n$ . Additionally, we may also choose  $a$  sufficiently large, depending on  $M_u$  and  $\bar{\beta}$ , so that

$$(2.14) \quad \lambda_n^{\bar{\beta}-\beta'} M_u \leq \frac{1}{2n|\mathbb{T}^3|^{1/2}}.$$



At this stage we may start the inductive [Proposition 2.1](#), and as in the proof of [Theorem 1.2](#), we obtain a weak solution  $u^{(\nu_n)}$  of the Navier-Stokes equations, with the desired regularity, such that

$$\left\| v^{(\nu_n)} - u \right\|_{H^{\beta'}} \leq \left\| v^{(\nu_n)} - v_n \right\|_{H^{\beta'}} + |\mathbb{T}^3|^{1/2} \|u - v_n\|_{C^{\beta'}} \leq \frac{1}{n}$$

in view of (2.10) and (2.14). Since  $n$  was arbitrary, this concludes the proof of the theorem.

### 3. Intermittent Beltrami waves

In this section we will describe in detail the construction of the *intermittent Beltrami waves* which will form the building blocks of our convex integration scheme. Very roughly, intermittent Beltrami waves are approximate Beltrami waves (approximate eigenfunctions to the curl operator) whose  $L^1$  norm is significantly smaller than their  $L^2$  norm.

**3.1. Beltrami waves.** We first recall from Proposition 3.1 and Lemma 3.2 in [15] the construction of Beltrami waves (see also the summary given in [2]). In order to better suit our later goal of defining intermittent Beltrami waves, the statements of these propositions are slightly modified from the form they appear in [2], by making the substitution  $\frac{k}{|k|} \mapsto \xi$ .

**PROPOSITION 3.1.** *Given  $\xi \in \mathbb{S}^2 \cap \mathbb{Q}^3$ , let  $A_\xi \in \mathbb{S}^2 \cap \mathbb{Q}^3$  be such that*

$$A_\xi \cdot \xi = 0, \quad |A_\xi| = 1, \quad A_{-\xi} = A_\xi.$$

*Furthermore, let*

$$B_\xi = \frac{1}{\sqrt{2}} (A_\xi + i\xi \times A_\xi).$$

*Let  $\Lambda$  be a given finite subset of  $\mathbb{S}^2 \cap \mathbb{Q}^3$  such that  $-\Lambda = \Lambda$ , and let  $\lambda \in \mathbb{Z}$  be such that  $\lambda\Lambda \subset \mathbb{Z}^3$ . Then for any choice of coefficients  $a_\xi \in \mathbb{C}$  with  $\bar{a}_\xi = a_{-\xi}$  the vector field*

$$(3.1) \quad W(x) = \sum_{\xi \in \Lambda} a_\xi B_\xi e^{i\lambda\xi \cdot x}$$

*is real-valued, divergence-free and satisfies*

$$(3.2) \quad \operatorname{div} (W \otimes W) = \nabla \frac{|W|^2}{2}.$$

*Furthermore, since  $B_\xi \otimes B_{-\xi} + B_{-\xi} \otimes B_\xi = \operatorname{Id} - \xi \otimes \xi$ , we have*

$$(3.3) \quad \int_{\mathbb{T}^3} W \otimes W \, dx = \frac{1}{2} \sum_{\xi \in \Lambda} |a_\xi|^2 (\operatorname{Id} - \xi \otimes \xi).$$

PROPOSITION 3.2. *For every  $N \in \mathbb{N}$ , we can choose  $\varepsilon_\gamma > 0$  and  $\lambda > 1$  with the following property. Let  $B_{\varepsilon_\gamma}(\text{Id})$  denote the ball of symmetric  $3 \times 3$  matrices, centered at  $\text{Id}$ , of radius  $\varepsilon_\gamma$ . Then, there exist pairwise disjoint subsets*

$$\Lambda_\alpha \subset \mathbb{S}^2 \cap \mathbb{Q}^3 \quad \alpha \in \{1, \dots, N\},$$

*with  $\lambda \Lambda_\alpha \in \mathbb{Z}^3$ , and smooth positive functions*

$$\gamma_\xi^{(\alpha)} \in C^\infty(B_\varepsilon(\text{Id})) \quad \alpha \in \{1, \dots, N\}, \xi \in \Lambda_\alpha,$$

*with derivatives that are bounded independently of  $\lambda$ , such that*

- (a)  $\xi \in \Lambda_\alpha$  implies  $-\xi \in \Lambda_\alpha$  and  $\gamma_\xi^{(\alpha)} = \gamma_{-\xi}^{(\alpha)}$ ;
- (b) for each  $R \in B_{\varepsilon_\gamma}(\text{Id})$ , we have the identity

$$(3.4) \quad R = \frac{1}{2} \sum_{\xi \in \Lambda_\alpha} \left( \gamma_\xi^{(\alpha)}(R) \right)^2 (\text{Id} - \xi \otimes \xi).$$

*Remark 3.3.* Throughout the construction, the parameter  $N$  is bounded by a universal constant; for instance one can take  $N = 2$ . Moreover, for each  $\alpha$  the cardinality of the set  $\Lambda_\alpha$  is also bounded by a universal constant; for instance one may take  $|\Lambda_\alpha| = 12$ . Consequently, the set of direction vectors  $\cup_{\alpha=1}^N \cup_{\xi \in \Lambda_\alpha} \{\xi, A_\xi, \xi \times A_\xi\} \subset \mathbb{S}^2 \cap \mathbb{Q}^3$  also has a universally bounded cardinality. Therefore, there exists a universal sufficiently large natural number  $N_\Lambda \geq 1$  such that we have

$$\{N_\Lambda \xi, N_\Lambda A_\xi, N_\Lambda \xi \times A_\xi\} \subset N_\Lambda \mathbb{S}^2 \cap \mathbb{Z}^3$$

for all vectors  $\xi$  in the construction.

It is also convenient to introduce a sufficiently small geometric constant  $c_\Lambda \in (0, 1)$  such that

$$\xi + \xi' \neq 0 \quad \Rightarrow \quad |\xi + \xi'| \geq 2c_\Lambda$$

for all  $\xi, \xi' \in \Lambda_\alpha$  and all  $\alpha \in \{1, \dots, N\}$ . In view of the aforementioned cardinality considerations, the geometric constant  $c_\Lambda$  is universal to the construction.

The implicit constants in the  $\lesssim$  of the below estimates are allowed to depend on  $N_\Lambda$  and  $c_\Lambda$ , but we will not emphasize this dependence, since these are universal constants.

3.2. *Intermittent Beltrami waves.* Recall cf. [22, §3] that the Dirichlet kernel  $D_n$  is defined as

$$(3.5) \quad D_n(x) = \sum_{\xi=-n}^n e^{ix\xi} = \frac{\sin((n+1/2)x)}{\sin(x/2)}$$

and has the property that for any  $p > 1$  it obeys the estimate

$$\|D_n\|_{L^p} \sim n^{1-1/p},$$

where the implicit constant only depends only on  $p$ . Replacing the sum in (3.5) by a sum of frequencies in a 3D integer cube

$$\Omega_r := \left\{ \xi = (j, k, \ell) : j, k, \ell \in \{-r, \dots, r\} \right\}$$

and normalizing to unit size in  $L^2$ , we obtain a kernel

$$D_r(x) := \frac{1}{(2r+1)^{3/2}} \sum_{\xi \in \Omega_r} e^{i\xi \cdot x} = \frac{1}{(2r+1)^{3/2}} \sum_{j,k,\ell \in \{-r, \dots, r\}} e^{i(jx_1 + kx_2 + \ell x_3)}$$

such that for  $1 < p \leq \infty$ , we have

$$(3.6) \quad \|D_r\|_{L^2}^2 = (2\pi)^3, \quad \text{and} \quad \|D_r\|_{L^p} \lesssim r^{3/2-3/p},$$

where the implicit constant depends only on  $p$ . Note that  $-\Omega_r = \Omega_r$ .

The principal idea in the construction of intermittent Beltrami waves is to modify the Beltrami waves of the previous section by adding oscillations that mimic the structure of the kernels  $D_r$  in order to construct approximate Beltrami waves with small  $L^p$  norm for  $p$  close to 1. The large parameter  $r$  will parametrize the number of frequencies along edges of the cube  $\Omega_r$ . We introduce a small parameter  $\sigma$ , such that  $\lambda\sigma \in \mathbb{N}$  parametrizes the spacing between frequencies, or equivalently such that the resulting rescaled kernel is  $(\mathbb{T}/\lambda\sigma)^3$ -periodic. We assume throughout the paper that

$$(3.7) \quad \sigma r \leq c_\Lambda / (10N_\Lambda),$$

where  $c_\Lambda \in (0, 1)$  and  $N_\Lambda \geq 1$  are the parameters from Remark 3.3. Lastly, we introduce a large parameter  $\mu \in (\lambda, \lambda^2)$ , which measures the amount of temporal oscillation in our building blocks. The parameters  $\lambda, r, \sigma$  and  $\mu$  are chosen in Section 4 below.

We recall from Propositions 3.1 and 3.2 that for  $\xi \in \Lambda_\alpha$ , the vectors  $\{\xi, A_\xi, \xi \times A_\xi\}$  form an orthonormal basis of  $\mathbb{R}^3$ , and by Remark 3.3 we have

$$N_\Lambda \xi, N_\Lambda A_\xi, N_\Lambda \xi \times A_\xi \in \mathbb{Z}^3 \quad \text{for all} \quad \xi \in \Lambda_\alpha, \alpha \in \{1, \dots, N\}.$$

Therefore, for  $\xi \in \Lambda_\alpha^+$ , we may define a directed and rescaled  $(\mathbb{T}/\lambda\sigma)^3 = (\mathbb{R}/2\pi\lambda\sigma\mathbb{Z})^3$ -periodic Dirichlet kernel by

$$(3.8) \quad \begin{aligned} \eta_{(\xi)}(x, t) &= \eta_{\xi, \lambda, \sigma, r, \mu}(x, t) \\ &= D_r(\lambda\sigma N_\Lambda(\xi \cdot x + \mu t), \lambda\sigma N_\Lambda A_\xi \cdot x, \lambda\sigma N_\Lambda(\xi \times A_\xi) \cdot x). \end{aligned}$$

For  $\xi \in \Lambda_\alpha^-$ , we define  $\eta_{(\xi)}(x, t) := \eta_{(-\xi)}(x, t)$ . The periodicity of  $\eta_{(\xi)}$  follows from the fact that  $D_r$  is  $\mathbb{T}^3$ -periodic, and the definition of  $N_\Lambda$ . We emphasize that we have the important identity

$$(3.9) \quad \frac{1}{\mu} \partial_t \eta_{(\xi)}(x, t) = \pm(\xi \cdot \nabla) \eta_{(\xi)}(x, t), \quad \text{for all} \quad \xi \in \Lambda_\alpha^\pm$$

as a consequence of the fact that the vectors  $A_\xi$  and  $\xi \times A_\xi$  are orthogonal to  $\xi$ .

Note that the map

$$(x_1, x_2, x_3) \mapsto (\lambda \sigma N_\Lambda (\xi \cdot x + \mu t), \lambda \sigma N_\Lambda A_\xi \cdot x, \lambda \sigma N_\Lambda (\xi \times A_\xi) \cdot x)$$

is the composition of a rotation by a rational orthogonal matrix which maps  $(e_1, e_2, e_3)$  to  $(\xi, A_\xi, \xi \times A_\xi)$ , a rescaling by  $\lambda \sigma N_\Lambda$ , and a translation by  $\lambda \sigma N_\Lambda \mu t e_1$ . These are all volume preserving transformations on  $\mathbb{T}^3$ , and thus by our choice of normalization for (3.6) we have that

$$(3.10) \quad \oint_{\mathbb{T}^3} \eta_{(\xi)}^2(x, t) dx = 1, \quad \text{and} \quad \|\eta_{(\xi)}\|_{L^p(\mathbb{T}^3)} \lesssim r^{3/2-3/p}$$

for all  $1 < p \leq \infty$ , pointwise in time.

Letting  $W_{(\xi)}$  be the Beltrami plane wave at frequency  $\lambda$ , namely

$$(3.11) \quad W_{(\xi)}(x) = W_{\xi, \lambda}(x) = B_\xi e^{i\lambda \xi \cdot x},$$

we have

$$\operatorname{curl} W_{(\xi)} = \lambda W_{(\xi)} \quad \text{and} \quad \operatorname{div} W_{(\xi)} = 0.$$

We take  $\lambda$  to be a multiple of  $N_\Lambda$ , so that  $W_{(\xi)}$  is  $\mathbb{T}^3$ -periodic. Finally, we define the *intermittent Beltrami wave*  $\mathbb{W}_{(\xi)}$  as

$$(3.12) \quad \mathbb{W}_{(\xi)}(x, t) = \mathbb{W}_{\xi, \lambda, \sigma, r, \mu}(x, t) = \eta_{\xi, \lambda, \sigma, r, \mu}(x, t) W_{\xi, \lambda}(x) = \eta_{(\xi)}(x, t) W_{(\xi)}(x).$$

We first make a few comments concerning the frequency support of  $\mathbb{W}_{(\xi)}$ . In view of (3.7) and the definition of  $\eta_{(\xi)}$ , which yields  $\mathbb{P}_{\leq 2\lambda \sigma r N_\Lambda} \eta_{(\xi)} = \eta_{(\xi)}$ , we have that

$$(3.13) \quad \mathbb{P}_{\leq 2\lambda} \mathbb{P}_{\geq \lambda/2} \mathbb{W}_{(\xi)} = \mathbb{W}_{(\xi)},$$

while for  $\xi' \neq -\xi$ , by the definition of  $c_\Lambda$  in Remark 3.3 we have

$$(3.14) \quad \mathbb{P}_{\leq 4\lambda} \mathbb{P}_{\geq c_\Lambda \lambda} (\mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(\xi')}) = \mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(\xi')}.$$

Note that the vector  $\mathbb{W}_{(\xi)}$  is not anymore divergence free, nor is it an eigenfunction of curl. These properties only hold to leading order:

$$\begin{aligned} \left\| \frac{1}{\lambda} \operatorname{div} \mathbb{W}_{(\xi)} \right\|_{L^2} &= \frac{1}{\lambda} \|B_\xi \cdot \nabla \eta_{(\xi)}\|_{L^2} \lesssim \frac{\lambda \sigma r}{\lambda} = \sigma r, \\ \left\| \frac{1}{\lambda} \operatorname{curl} \mathbb{W}_{(\xi)} - \mathbb{W}_{(\xi)} \right\|_{L^2} &= \frac{1}{\lambda} \|\nabla \eta_{(\xi)} \times B_\xi\|_{L^2} \lesssim \frac{\lambda \sigma r}{\lambda} = \sigma r \end{aligned}$$

and the parameter  $\sigma r$  will be chosen to be small. Moreover, from Propositions 3.1 and 3.2 we have:

**PROPOSITION 3.4.** *Let  $\mathbb{W}_{(\xi)}$  be as defined above, and let  $\Lambda_\alpha, \varepsilon_\gamma, \gamma_{(\xi)} = \gamma_\xi^{(\alpha)}$  be as in Proposition 3.2. If  $a_\xi \in \mathbb{C}$  are constants chosen such that  $\bar{a}_\xi = a_{-\xi}$ , the vector field*

$$\sum_{\alpha} \sum_{\xi \in \Lambda_\alpha} a_\xi \mathbb{W}_{(\xi)}(x)$$

is real valued. Moreover, for each  $R \in B_{\varepsilon_\gamma}(\text{Id})$  we have the identity

$$(3.15) \quad \left( \sum_{\xi \in \Lambda_\alpha} \gamma_{(\xi)}(R) \right)^2 \int_{\mathbb{T}^3} \mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(-\xi)} dx = \left( \sum_{\xi \in \Lambda_\alpha} \gamma_{(\xi)}(R) \right)^2 B_\xi \otimes B_{-\xi} = R.$$

*Proof of Proposition 3.4.* The first statement follows from the fact that  $\eta_{(-\xi)}(x, t) = \eta_{(\xi)}(x, t)$ . Identity (3.15) follows from (3.4) upon noting that  $2\text{Re}(B_\xi \otimes B_{-\xi}) = \text{Id} - \xi \otimes \xi$ , and the normalization (3.10).  $\square$

For the purpose of estimating the oscillation error in Section 5, it is useful to derive a replacement of identity (3.2), in the case of intermittent Beltrami waves. For this purpose, we first recall the vector identity

$$(A \cdot \nabla)B + (B \cdot \nabla)A = \nabla(A \cdot B) - A \times \text{curl} B - B \times \text{curl} A.$$

Hence, for  $\xi, \xi' \in \Lambda_\alpha$  we may rewrite

$$(3.16) \quad \begin{aligned} & (\text{div } \mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(\xi')} + \mathbb{W}_{(\xi')} \otimes \mathbb{W}_{(\xi)}) \\ &= (W_{(\xi)} \otimes W_{(\xi')} + W_{(\xi')} \otimes W_{(\xi)}) \nabla \cdot (\eta_{(\xi)} \eta_{(\xi')}) \\ &\quad + \eta_{(\xi)} \eta_{(\xi')} (W_{(\xi)} \cdot \nabla) W_{(\xi')} + (W_{(\xi')} \cdot \nabla) W_{(\xi)} \\ &= (W_{(\xi')} \cdot \nabla) (\eta_{(\xi)} \eta_{(\xi')}) W_{(\xi)} + (W_{(\xi)} \cdot \nabla) (\eta_{(\xi)} \eta_{(\xi')}) W_{(\xi')} \\ &\quad + \eta_{(\xi)} \eta_{(\xi')} \nabla \cdot (W_{(\xi)} \cdot W_{(\xi')}) - \lambda \eta_{(\xi)} \eta_{(\xi')} (W_{(\xi)} \times W_{(\xi')} + W_{(\xi')} \times W_{(\xi)}) \\ &= (W_{(\xi')} \cdot \nabla) (\eta_{(\xi)} \eta_{(\xi')}) W_{(\xi)} + (W_{(\xi)} \cdot \nabla) (\eta_{(\xi)} \eta_{(\xi')}) W_{(\xi')} \\ &\quad + \eta_{(\xi)} \eta_{(\xi')} \nabla \cdot (W_{(\xi)} \cdot W_{(\xi')}). \end{aligned}$$

In the last equality we have used that the cross-product is antisymmetric.

Let us now restrict to the case  $\xi + \xi' = 0$ . Recall that  $W_{(\xi)} = \frac{1}{\sqrt{2}}(A_\xi + i\xi \times A_\xi)e^{i\lambda\xi \cdot x}$ ,  $\xi \cdot A_\xi = 0$ , and  $|A_\xi| = 1$ . Therefore, when  $\xi' = -\xi$  the last term on the right side of (3.16) is zero, as  $W_{(\xi)} \cdot W_{(-\xi)} = 1$ . Thus we obtain

$$\begin{aligned} & (\text{div } \mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(-\xi)} + \mathbb{W}_{(-\xi)} \otimes \mathbb{W}_{(\xi)}) \\ &= (W_{(-\xi)} \cdot \nabla) \eta_{(\xi)}^2 W_{(\xi)} + (W_{(\xi)} \cdot \nabla) \eta_{(\xi)}^2 W_{(-\xi)} \\ &= (A_\xi \cdot \nabla) \eta_{(\xi)}^2 A_\xi + ((\xi \times A_\xi) \cdot \nabla) \eta_{(\xi)}^2 (\xi \times A_\xi) \\ &= \nabla \eta_{(\xi)}^2 - (\xi \cdot \nabla) \eta_{(\xi)}^2 \xi. \end{aligned}$$

In the last equality above we have used that  $\{\xi, A_\xi, \xi \times A_\xi\}$  is an orthonormal basis of  $\mathbb{R}^3$ . The above identity and property (3.9) of  $\eta_{(\xi)}$  shows that

$$(3.17) \quad (\text{div } \mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(-\xi)} + \mathbb{W}_{(-\xi)} \otimes \mathbb{W}_{(\xi)}) = \nabla \eta_{(\xi)}^2 - \frac{\xi}{\mu} \partial_t \eta_{(\xi)}^2.$$

which is the key identity that motivates the introduction of temporal oscillations in the problem.

Recall, the intermittent Beltrami waves were designed to include additional oscillations that cancel in order to minimize their  $L^1$  norm, in a way that is analogous to the cancellations of the Dirichlet kernel. In this direction, an immediate consequence of [property \(3.10\)](#) of  $\eta_{(\xi)}$ , of the frequency localization in the spatial variable [\(3.13\)](#), and of the frequency of the temporal oscillations, are the following bounds for  $\eta_{(\xi)}$  and the intermittent Beltrami waves  $\mathbb{W}_{(\xi)}$ :

PROPOSITION 3.5. *Let  $\mathbb{W}_{(\xi)}$  be defined as above. The bound*

$$(3.18) \quad \left\| \nabla^N \partial_t^K \mathbb{W}_{(\xi)} \right\|_{L^p} \lesssim \lambda^N (\lambda \sigma r \mu)^K r^{3/2-3/p},$$

$$(3.19) \quad \left\| \nabla^N \partial_t^K \eta_{(\xi)} \right\|_{L^p} \lesssim (\lambda \sigma r)^N (\lambda \sigma r \mu)^K r^{3/2-3/p}$$

for any  $1 < p \leq \infty$ ,  $N \geq 0$  and  $K \geq 0$ . The implicit constant depends only on  $N, K$  and  $p$ .

*Remark 3.6.* We note that while in the above proposition we state estimates for all orders of derivatives ( $N$  and  $K$ ), only derivatives up to a fixed order, which is independent of  $q$ , appear in the entire proof of [Proposition 2.1](#). Hence the implicit constants that depend on the number of derivatives taken are independent of  $q$ . This remark also applies to estimates in later parts of the paper (e.g. mollification estimates).

3.3.  *$L^p$  decorrelation.* We now introduce a crucial lemma from [\[5\]](#) that will be used throughout the paper. Suppose we wish to estimate

$$\left\| f \mathbb{W}_{(\xi)} \right\|_{L^1}$$

for some arbitrary function  $f : \mathbb{T}^3 \rightarrow \mathbb{R}$ . The trivial estimate is

$$\left\| f \mathbb{W}_{(\xi)} \right\|_{L^1} \lesssim \|f\|_{L^2} \left\| \mathbb{W}_{(\xi)} \right\|_{L^2}.$$

Such an estimate does not however take advantage of the special structure of the  $(2\pi\lambda\sigma)^{-1}$  periodic function  $\mathbb{W}_{(\xi)} e^{-i\lambda\xi \cdot x}$ . It turns out that if say  $f$  has frequency contained in a ball of radius  $\mu$  and  $\lambda\sigma \gg \mu$  then one obtains the improved estimate

$$\left\| f \mathbb{W}_{(\xi)} \right\|_{L^1} \lesssim \|f\|_{L^1} \left\| \mathbb{W}_{(\xi)} \right\|_{L^1},$$

which gives us the needed gain because  $\left\| \mathbb{W}_{(\xi)} \right\|_{L^1} \ll \left\| \mathbb{W}_{(\xi)} \right\|_{L^2}$ . This idea is one of the key insights of [\[5\]](#) and is summarized in [Lemma 3.7](#) below. For convenience we include the proof in [Appendix A](#).

LEMMA 3.7. *Fix integers  $M, \kappa, \lambda \geq 1$  such that*

$$\frac{2\pi\sqrt{3}\lambda}{\kappa} \leq \frac{1}{3} \quad \text{and} \quad \lambda^4 \frac{(2\pi\sqrt{3}\lambda)^M}{\kappa^M} \leq 1.$$

Let  $p \in \{1, 2\}$ , and let  $f$  be a  $\mathbb{T}^3$ -periodic function such that there exists a constant  $C_f$  such that

$$\|D^j f\|_{L^p} \leq C_f \lambda^j$$

for all  $1 \leq j \leq M + 4$ . In addition, let  $g$  be a  $(\mathbb{T}/\kappa)^3$ -periodic function. Then we have that

$$\|fg\|_{L^p} \lesssim C_f \|g\|_{L^p}$$

holds, where the implicit constant is universal.

#### 4. The perturbation

In this section we will construct the perturbation  $w_{q+1}$ .

**4.1. Mollification of  $v_q$ .** In order to avoid a loss of derivative, we replace  $v_q$  by a mollified velocity field  $v_\ell$ . Let  $\{\phi_\varepsilon\}_{\varepsilon>0}$  be a family of standard Friedrichs mollifiers (of compact support of radius 2) on  $\mathbb{R}^3$  (space), and  $\{\varphi_\varepsilon\}_{\varepsilon>0}$  be a family of standard Friedrichs mollifiers (of compact support of width 2) on  $\mathbb{R}$  (time). We define a mollification of  $v_q$  and  $\dot{R}_q$  in space and time, at length scale and time scale  $\ell$  (which is defined in (4.16) below) by

$$(4.1) \quad \begin{aligned} v_\ell &= (v_q *_x \phi_\ell) *_t \varphi_\ell, \\ \dot{R}_\ell &= (\dot{R}_q *_x \phi_\ell) *_t \varphi_\ell. \end{aligned}$$

Then using (2.1) we obtain that  $(v_\ell, \dot{R}_\ell)$  obey

$$(4.2a) \quad \partial_t v_\ell + \operatorname{div}(v_\ell \otimes v_\ell) + \nabla p_\ell - \Delta v_\ell = \operatorname{div}(\dot{R}_\ell + \widetilde{R}_{\text{commutator}}),$$

$$(4.2b) \quad \operatorname{div} v_\ell = 0,$$

where the new pressure  $p_\ell$  and the traceless symmetric commutator stress  $\widetilde{R}_{\text{commutator}}$  are given by

$$(4.3) \quad \begin{aligned} \widetilde{p}_\ell &= (p_q *_x \phi_\ell) *_t \varphi_\ell - |v_\ell|^2 - (|v_q|^2 *_x \phi_\ell) *_t \varphi_\ell, \\ \widetilde{R}_{\text{commutator}} &= (v_\ell \otimes v_\ell) - ((v_q \otimes v_q) *_x \phi_\ell) *_t \varphi_\ell. \end{aligned}$$

Here we have used  $a \otimes b$  to denote the traceless part of the tensor  $a \otimes b$ .

Note that in view of (2.2) the commutator stress  $\widetilde{R}_{\text{commutator}}$  obeys the lossy estimate

$$(4.4) \quad \|\widetilde{R}_{\text{commutator}}\|_{L^\infty} \lesssim \ell \|v_q \otimes v_q\|_{C^1} \lesssim \ell \|v_q\|_{C^1} \|v_q\|_{L^\infty} \lesssim \ell \lambda_q^8.$$

The parameter  $\ell$  will be chosen (cf. (4.16) below) to satisfy

$$(4.5) \quad (\sigma \lambda_{q+1})^{-1/2} \ll \ell \ll \lambda_q^{-19} \delta_{q+1}.$$

In particular,  $\mathring{R}_\ell$  inherits the  $L^1$  bound of  $\mathring{R}_q$  from (2.3), and in view (2.4) and the upper bound on  $\ell$  in (4.5), we have that

$$(4.6) \quad \left\| \mathring{R}_\ell \right\|_{C_{t,x}^N} \lesssim \lambda_q^{10} \ell^{-N+1} \lesssim \ell^{-N}.$$

Moreover, from (2.2) and the upper bound on  $\ell$  from (4.5) we obtain the bounds

$$(4.7) \quad \|v_q - v_\ell\|_{L^\infty} \lesssim \ell \|v_q\|_{C^1} \lesssim \ell \lambda_q^4,$$

$$(4.8) \quad \|v_\ell\|_{C_{x,t}^N} \lesssim \ell^{1-N} \|v_q\|_{C^1} \lesssim \ell^{1-N} \lambda_q^4 \lesssim \ell^{-N}.$$

4.2. *Stress cutoffs.* Because the Reynolds stress  $\mathring{R}_\ell$  is not spatially homogeneous, we introduce stress cutoff functions. We let  $0 \leq \tilde{\chi}_0, \tilde{\chi} \leq 1$  be bump functions adapted to the intervals  $[0, 4]$  and  $[1/4, 4]$  respectively, such that together they form a partition of unity:

$$(4.9) \quad \tilde{\chi}_0^2(y) + \sum_{i \geq 1} \tilde{\chi}_i^2(y) \equiv 1, \quad \text{where} \quad \tilde{\chi}_i(y) = \tilde{\chi}(4^{-i}y),$$

for any  $y > 0$ . We then define

$$(4.10) \quad \chi_{(i)}(x, t) = \chi_{i,q+1}(x, t) = \tilde{\chi}_i \left( \left\langle \frac{\mathring{R}_\ell(x, t)}{100 \lambda_q^{-\varepsilon_R} \delta_{q+1}} \right\rangle \right)$$

for all  $i \geq 0$ . Here and throughout the paper we use the notation  $\langle A \rangle = (1 + |A|^2)^{1/2}$  where  $|A|$  denotes the Euclidean norm of the matrix  $A$ . By definition the cutoffs  $\chi_{(i)}$  form a partition of unity

$$(4.11) \quad \sum_{i \geq 0} \chi_{(i)}^2 \equiv 1$$

and we will show in Lemma 4.1 below that there exists an index  $i_{\max} = i_{\max}(q)$ , such that  $\chi_{(i)} \equiv 0$  for all  $i > i_{\max}$ , and moreover that  $4^{i_{\max}} \lesssim \ell^{-1}$ .

4.3. *The definition of the velocity increment.* Define the coefficient function  $a_{\xi,i,q+1}$  by

$$(4.12) \quad a_{(\xi)} := a_{\xi,i,q+1} := \rho_i^{1/2} \chi_{i,q+1} \gamma_{(\xi)} \left( \text{Id} - \frac{\mathring{R}_\ell}{\rho_i(t)} \right).$$

where for  $i \geq 1$ , the parameters  $\rho_i$  are defined by

$$(4.13) \quad \rho_i := \lambda_q^{-\varepsilon_R} \delta_{q+1} 4^{i+c_0}$$

where  $c_0 \in \mathbb{N}$  is a sufficiently large constant, which depends on the  $\varepsilon_\gamma$  in Proposition 3.4. The addition of the factor  $4^{c_0}$  ensures that the argument of  $\gamma_{(\xi)}$  is in the range of definition. The definition  $\rho_0$  is slightly more complicated and as such its definition will be delayed to Section 4.4 below; see (4.25) and



(4.26). Modulo the definition of  $\rho_0$ , we note that as a consequence of (3.14), (3.15), (4.11), and (4.12) we have

$$(4.14) \quad \sum_{i \geq 0} \sum_{\xi, \xi' \in \Lambda_{(i)}} a_{(\xi)}^2 \left( \oint_{\mathbb{T}^3} \mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(\xi')} \right) dx = \sum_{i \geq 0} \rho_i \chi_{(i)}^2 \text{Id} - \mathring{R}_\ell,$$

which justifies the definition of the amplitude functions  $a_{(\xi)}$ .

By a slight abuse of notation, let us now fix  $\lambda, \sigma, r$ , and  $\mu$  for the shorthand notation  $\mathbb{W}_{(\xi)}$ ,  $W_{(\xi)}$  and  $\eta_{(\xi)}$  introduced in Section 3.2 (cf. (3.8), (3.11), (3.12)):

$$\mathbb{W}_{(\xi)} := \mathbb{W}_{\xi, \lambda_{q+1}, \sigma, r, \mu}, \quad W_{(\xi)} := W_{\xi, \lambda_{q+1}} \quad \text{and} \quad \eta_{(\xi)} := \eta_{\xi, \lambda_{q+1}, \sigma, r, \mu},$$

where the integer  $r$ , the parameter  $\sigma$ , and the parameter  $\mu$  are defined by

$$(4.15) \quad r = \lambda_{q+1}^{3/4}, \quad \sigma = \lambda_{q+1}^{-15/16} \quad \text{and} \quad \mu = \lambda_{q+1}^{5/4}.$$

The fact that  $\lambda_{q+1}\sigma \in \mathbb{N}$  is ensured by our choices  $a \in \mathbb{N}$  and  $b \in 16\mathbb{N}$ . In order to ensure  $\lambda_{q+1}$  is a multiple of  $N_\Lambda$ , we need to choose  $a$  which is a multiple of  $N_\Lambda$ . Moreover, at this stage we fix

$$(4.16) \quad \ell = \lambda_q^{-20},$$

which in view of the choice of  $\sigma$  in (4.15), ensures that (4.5) holds, upon taking  $\lambda_0$  sufficiently large. In view of (4.16), throughout the rest of the paper we may use either  $\ell^\varepsilon \leq \lambda_0^{-20\varepsilon}$  or  $\lambda_q^{-\varepsilon} \leq \lambda_0^{-\varepsilon}$ , with  $\varepsilon > 0$  arbitrarily small, to absorb any of the constants (which are  $q$ -independent) appearing due to  $\lesssim$  signs in the below inequalities. This is possible by choosing  $\lambda_0 = a$ , sufficiently large.

The *principal part* of  $w_{q+1}$  is defined as

$$(4.17) \quad w_{q+1}^{(p)} := \sum_i \sum_{\xi \in \Lambda_{(i)}} a_{(\xi)} \mathbb{W}_{(\xi)},$$

where the sum is over  $0 \leq i \leq i_{\max}(q)$ . The sets  $\Lambda_{(i)}$  are defined as follows. In Lemma 3.2 it suffices to take  $N = 2$ , so that  $\alpha \in \{\alpha_0, \alpha_1\}$ , and we define  $\Lambda_{(i)} = \Lambda_{\alpha_{i \bmod 2}}$ . This choice is allowable since  $\chi_i \chi_j \equiv 0$  for  $|i - j| \geq 2$ . In order to fix the fact that  $w_{q+1}^{(p)}$  is not divergence free, we define an *incompressibility corrector* by

$$(4.18) \quad w_{q+1}^{(c)} := \frac{1}{\lambda_{q+1}} \sum_i \sum_{\xi \in \Lambda_{(i)}} \left( \nabla \cdot a_{(\xi)} \eta_{(\xi)} \right) \times W_{(\xi)}.$$

Using that  $\text{div } W_{(\xi)} = 0$ , we then have

$$(4.19) \quad w_{q+1}^{(p)} + w_{q+1}^{(c)} = \frac{1}{\lambda_{q+1}} \sum_i \sum_{\xi \in \Lambda_{(i)}} \left( \text{curl} \cdot a_{(\xi)} \eta_{(\xi)} W_{(\xi)} \right) = \frac{1}{\lambda_{q+1}} \text{curl} (w_{q+1}^{(p)}),$$

and thus

$$\text{div} \left( w_{q+1}^{(p)} + w_{q+1}^{(c)} \right) = 0.$$

In addition to the incompressibility corrector  $w_{q+1}^{(c)}$ , we introduce a *temporal corrector*  $w_{q+1}^{(t)}$ , which is defined by

$$(4.20) \quad w_{q+1}^{(t)} := \frac{1}{\mu} \sum_i \sum_{\xi \in \Lambda_{(i)}^+} \mathbb{P}_H \mathbb{P}_{\neq 0} a_{(\xi)}^2 \eta_{(\xi)}^2 \xi.$$

Here we have denoted by  $\mathbb{P}_{\neq 0}$  the operator which projects a function onto its nonzero frequencies  $\mathbb{P}_{\neq 0} f = f - \mathcal{F}_{\mathbb{T}^3}^{-1} \mathcal{F}_{\mathbb{T}^3} f$ , and have used  $\mathbb{P}_H$  for the usual Helmholtz (or Leray) projector onto divergence-free vector fields,  $\mathbb{P}_H f = f - \nabla(\Delta^{-1} \operatorname{div} f)$ . The purpose of the corrector  $w_{q+1}^{(t)}$  becomes apparent upon recalling (3.17). Indeed, if we multiply identity (3.17) by  $a_{(\xi)}^2$ , remove the mean and a suitable pressure gradient, the leading order term left is

$$-(1/\mu) \mathbb{P}_H \mathbb{P}_{\neq 0} (\xi a_{(\xi)}^2 \partial_t \eta_{(\xi)}^2);$$

see (5.13) below. This term is not of high frequency (proportional to  $\lambda_{q+1}$ ). Moreover, upon writing this term as the divergence of a symmetric stress, the size of this stress term in  $L^1$  is  $\delta_{q+1}$ , instead of  $\delta_{q+2}$ ; thus this term does not obey a favorable estimate and has to be cancelled altogether. The corrector  $w_{q+1}^{(t)}$  is designed such that its *time derivative* achieves precisely this goal, of cancelling the  $-(1/\mu) \mathbb{P}_H \mathbb{P}_{\neq 0} (\xi a_{(\xi)}^2 \partial_t \eta_{(\xi)}^2)$  term.

Finally, we define the velocity increment  $w_{q+1}$  by

$$(4.21) \quad w_{q+1} := w_{q+1}^{(p)} + w_{q+1}^{(c)} + w_{q+1}^{(t)},$$

which is by construction mean zero and divergence-free. The new velocity field  $v_{q+1}$  is then defined as

$$(4.22) \quad v_{q+1} = v_\ell + w_{q+1}.$$

4.4. *The definition of  $\rho_0$ .* It follows from (4.14) that with the  $\rho_i$  defined above we have

$$(4.23) \quad \begin{aligned} \sum_{i \geq 1} \int_{\mathbb{T}^3} \left| \sum_{\xi \in \Lambda_{(i)}} a_{(\xi)} \mathbb{W}_{(\xi)} \right|^2 dx &= \sum_{i \geq 1} \sum_{\xi, \xi' \in \Lambda_{(i)}} \int_{\mathbb{T}^3} a_{(\xi)} a_{(\xi')} \operatorname{tr} (\mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(-\xi')}) dx \\ &= \sum_{i \geq 1} \sum_{\xi \in \Lambda_{(i)}} \int_{\mathbb{T}^3} a_{(\xi)}^2 \operatorname{tr} \mathcal{F}_{\mathbb{T}^3} (\mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(-\xi)}) dx + \text{error} \\ &= 3 \sum_{i \geq 1} \rho_i \int_{\mathbb{T}^3} \chi_{(i)}^2 dx + \text{error}, \end{aligned}$$

where the error term can be made arbitrarily small since the spatial frequency of the  $a_{(\xi)}$ 's is  $\ell^{-1}$ , while the minimal separation of frequencies of  $\mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(-\xi')}$  is  $\lambda_{q+1} \sigma \gg \ell^{-1}$ . The term labeled as *error* on the right side of (4.23) above will

be estimated precisely in [Section 6](#) below. We will show in the next section, [Lemma 4.3](#) that

$$(4.24) \quad \sum_{i \geq 1} \rho_i \int_{\mathbb{T}^3} \chi_{(i)}^2 dx \lesssim \delta_{q+1} \lambda_q^{-\varepsilon_R}.$$

In order to ensure (2.5) is satisfied for  $q+1$ , we design  $\rho_0$  such that

$$\int_{\mathbb{T}^3} \left| \sum_{\xi \in \Lambda(0)} a_{(\xi)} \mathbb{W}_{(\xi)} \right|^2 dx \approx \tilde{e}(t) := e(t) - \int_{\mathbb{T}^3} |v_q|^2 dx - 3 \sum_{i \geq 1} \rho_i \int_{\mathbb{T}^3} \chi_i^2 dx.$$

We thus define the auxiliary function

$$(4.25) \quad \rho(t) := \frac{1}{3|\mathbb{T}^3|} \left( \int_{\mathbb{T}^3} \chi_0^2 dx \right)^{-1} \left( \max \left( \tilde{e}(t) - \frac{\delta_{q+2}}{2}, 0 \right) \right).$$

The term  $-\delta_{q+2}/2$  is added to ensure that we leave room for future corrections and the max is in place to ensure that we do not correct the energy when the energy of  $v_q$  is already sufficiently close to the prescribed energy profile. This later property will allow us to take energy profiles with compact support. Finally, in order to ensure  $\rho_0^{1/2}$  is sufficiently smooth, we define  $\rho_0$  as the square of the mollification of  $\rho^{1/2}$  at time scale  $\ell$

$$(4.26) \quad \rho_0 = (\rho^{1/2}) *_t \varphi_\ell^2.$$

We note that (2.5) and (4.34) below imply that

$$(4.27) \quad \|\rho_0\|_{C_t^0} \leq 2\delta_{q+1} \quad \text{and} \quad \|\rho_0^{1/2}\|_{C_t^N} \lesssim \delta_{q+1}^{1/2} \ell^{-N}$$

for  $N \geq 1$ . By a slight abuse of notation we will denote

$$\frac{\dot{R}_\ell}{\rho_0(t)} = \begin{cases} \frac{\dot{R}_\ell}{\rho_0(t)} & \text{if } \chi_0 \neq 0 \text{ and } \dot{R}_\ell \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that if  $\chi_0 \neq 0$  and  $\dot{R}_\ell \neq 0$ , then (2.6) and (4.24) ensure that  $\rho_0 > 0$ . In order to ensure that  $\text{Id} - \frac{\dot{R}_\ell}{\rho_0(t)}$  is in the domain of the functions  $\gamma_{(\xi)}$  from [Proposition 3.4](#), we will need to ensure that

$$(4.28) \quad \left\| \frac{\dot{R}_\ell}{\rho_0(t)} \right\|_{L^\infty(\text{supp } \chi_{(0)})} \leq \varepsilon_\gamma.$$

We give the proof of (4.28) next. Owing to the estimate

$$\left| e(t) - \int_{\mathbb{T}^3} |v_q(x, t)|^2 dx - e(t') - \int_{\mathbb{T}^3} |v_q(x, t')|^2 dx \right| \lesssim \ell^{1/2}$$

for  $t' \in (t - \ell, t + \ell)$  which follows from Lemma 6.1 in Section 6, and the inequality  $\ell^{1/2} \ll \delta_{q+1}$ , we may apply (2.6) to conclude that it is sufficient to check the above condition when

$$e(t) - \int_{\mathbb{T}^3} |v_q(x, t)|^2 dx \geq \frac{\delta_{q+1}}{200}.$$

Then by (4.24), the above lower bound implies

$$\tilde{e}(t) \geq \frac{\delta_{q+1}}{400},$$

and thus

$$\rho(t) \geq \left( \frac{1}{|\mathbb{T}^3|} \frac{\delta_{q+1}}{400} - \frac{\delta_{q+2}}{2} \right) \geq \frac{\delta_{q+1}}{500},$$

where we used (4.34) from Lemma 4.3 below to bound the integral. Finally, using the estimate (6.4) from Section 6 we obtain

$$\rho_0(t) \geq \frac{\delta_{q+1}}{600}.$$

Since on the support of  $\chi_0$  we have  $|\mathring{R}_\ell| \leq 1000\lambda_q^{-\varepsilon_R}\delta_{q+1}$ , we obtain (4.28).

4.5. *Estimates of the perturbation.* We first collect a number of estimates concerning the cutoffs  $\chi_{(i)}$  defined in (4.10).

LEMMA 4.1. *For  $q \geq 0$ , there exists  $i_{\max}(q) \geq 0$ , determined by (4.31) below, such that*

$$\chi_{(i)} \equiv 0 \quad \text{for all } i > i_{\max}.$$

Moreover, we have that for all  $0 \leq i \leq i_{\max}$

$$(4.29) \quad \rho_i \lesssim 4^{i_{\max}} \lesssim \ell^{-1}$$

where the implicit constants can be made independent of other parameters. Moreover, we have

$$(4.30) \quad \sum_{i=0}^{i_{\max}} \rho_i^{1/2} 2^{-i} \leq 3\delta_{q+1}^{1/2}.$$

*Proof of Lemma 4.1.* Let  $i \geq 1$ . By the definition of  $\tilde{\chi}_i$  we have that  $\chi_{(i)} = 0$  for all  $(x, t)$  such that

$$\langle 100^{-1}\lambda_q^{\varepsilon_R}\delta_{q+1}^{-1}\mathring{R}_\ell(x, t) \rangle < 4^{i-1}.$$

Using the inductive assumption (2.4), we have that

$$\|\mathring{R}_\ell\|_{L^\infty} \lesssim \|\mathring{R}_\ell\|_{C^1} \lesssim \|\mathring{R}_q\|_{C^1} \leq C_{\max}\lambda_q^{10} \leq \lambda_q^{10+\varepsilon_R}$$

since the implicit constant  $C_{\max}$  is independent of  $q$  and of  $\varepsilon_R$  (it only depends on norms of the mollifier  $\phi$  used to define  $\check{R}_\ell$ ), and thus we have  $C_{\max} \leq \lambda_q^{\varepsilon_R}$ . Therefore, if  $i \geq 1$  is large enough such that

$$\langle 100^{-1} \delta_{q+1}^{-1} \lambda_q^{10+2\varepsilon_R} \rangle \leq 4^{i-2},$$

then  $\chi_{(i)} \equiv 0$ . Therefore, as  $\varepsilon_R \leq 1/4$  and since we have

$$\langle 100^{-1} \delta_{q+1}^{-1} \lambda_q^{10+2\varepsilon_R} \rangle \leq \langle \delta_{q+1}^{-1} \lambda_q^{10+1/2} \rangle \leq \delta_{q+1}^{-1} \lambda_q^{11}$$

for all  $q \geq 0$  (since  $\beta b$  is small), we may define  $i_{\max}$  by

$$(4.31) \quad i_{\max}(q) = \min \{i \geq 0: 4^{i-2} \geq \lambda_q^{11} \delta_{q+1}^{-1}\}.$$

Observe that, the first inequality of (4.29) follows trivially from the definition of  $\rho_i$  for  $i \geq 1$  and (4.27) for  $i = 0$ . The second inequality follows from the fact that  $\lambda_q^{11} \delta_{q+1}^{-1} \leq \ell^{-1}$ , which is a consequence of  $b\beta$  being small. Finally, from the definition (4.13) and the bound (4.27) we have

$$\begin{aligned} \sum_{i=0}^{i_{\max}} \rho_i^{1/2} 2^{-i} &\leq 2\delta_{q+1}^{1/2} + 2^{c_0} \sum_{i=1}^{i_{\max}} \lambda_q^{-\varepsilon_R/2} \delta_{q+1}^{1/2} \\ &\leq \delta_{q+1}^{1/2} (2 + 2^{c_0} \lambda_q^{-\varepsilon_R/2} (3 + \log_4(\lambda_q^{11} \delta_{q+1}^{-1}))). \end{aligned}$$

Since  $\lambda_q^{11} \delta_{q+1}^{-1} \leq \lambda_q^{20}$ , which is a consequence of  $\beta b$  being small, we can bound the second term above as

$$2^{c_0} \lambda_q^{-\varepsilon_R/2} (3 + \log_4(\lambda_q^{11} \delta_{q+1}^{-1})) \leq 2^{c_0} \lambda_q^{-\varepsilon_R/2} (3 + 20 \log_4(\lambda_q)) \leq 1$$

by taking  $a$  (and hence  $\lambda_q$ ) to be sufficiently large, depending on  $\varepsilon_R$  and  $c_0$ . This finishes the proof of (4.30).  $\square$

The size and derivative estimate for the  $\chi_{(i)}$  are summarized in the following lemma

LEMMA 4.2. *Let  $0 \leq i \leq i_{\max}$ . Then we have*

$$(4.32) \quad \|\chi_{(i)}\|_{L^2} \lesssim 2^{-i},$$

$$(4.33) \quad \|\chi_{(i)}\|_{C_{x,t}^N} \lesssim \lambda_q^{10} \ell^{1-N} \lesssim \ell^{-N}$$

for all  $N \geq 1$ .

*Proof of Lemma 4.2.* We prove that

$$\|\chi_{(i)}\|_{L^1} \lesssim 4^{-i},$$

so that the bound (4.32) follows since  $\chi_{(i)} \leq 1$ , upon interpolating the  $L^2$  norm between the  $L^1$  and the  $L^\infty$  norms.

When  $i = 0, 1$ , we have that  $\|\chi_{(i)}\|_{L^1} \leq |\mathbb{T}^3| \|\chi_{(0)}\|_{L^\infty} \lesssim 1 \lesssim 4^{-i}$ . For  $i \geq 2$ , we use the Chebyshev's inequality and the inductive assumption (2.3) to conclude

$$\begin{aligned} \|\chi_{(i)}\|_{L^1} &\leq \sup_t \left| \left\{ x: 4^{i-1} \leq \langle \lambda_q^{\varepsilon_R} \delta_{q+1}^{-1} \dot{R}_\ell(x, t) / 100 \rangle \leq 4^{i+1} \right\} \right| \\ &\leq \sup_t \left| \left\{ x: 4^{i-1} \leq \lambda_q^{\varepsilon_R} \delta_{q+1}^{-1} |\dot{R}_\ell(x, t)| / 100 + 1 \right\} \right| \\ &\leq \sup_t \left| \left\{ x: 100 \lambda_q^{-\varepsilon_R} \delta_{q+1} 4^{i-2} \leq |\dot{R}_\ell(x, t)| \right\} \right| \\ &\lesssim \lambda_q^{\varepsilon_R} \delta_{q+1}^{-1} 4^{-i} \|\dot{R}_\ell\|_{L_t^\infty L_x^1} \\ &\lesssim \lambda_q^{\varepsilon_R} \delta_{q+1}^{-1} 4^{-i} \|\dot{R}_q\|_{L_t^\infty L_x^1} \lesssim 4^{-i}, \end{aligned}$$

proving the desired  $L^1$  bound. In order to prove the estimate (4.33) we appeal to [2, Prop. C.1], which yields

$$\begin{aligned} \|\chi_{(i)}\|_{C_{t,x}^N} &\lesssim \left\| \langle \lambda_q^{\varepsilon_R} \delta_{q+1}^{-1} \dot{R}_\ell / 100 \rangle \right\|_{C_{t,x}^N} + \left\| \langle \lambda_q^{\varepsilon_R} \delta_{q+1}^{-1} \dot{R}_\ell / 100 \rangle \right\|_{C_{t,x}^1}^N \\ &\lesssim \ell^{-N+1} \|\dot{R}_\ell\|_{C_{t,x}^1} + \|\dot{R}_\ell\|_{C_{t,x}^1}^N \\ &\lesssim \ell^{-N+1} \|\dot{R}_q\|_{C_{t,x}^1} + \|\dot{R}_q\|_{C_{t,x}^1}^N \\ &\lesssim \lambda_q^{10} \ell^{-N+1} + \lambda_q^{10N} \lesssim \lambda_q^{10} \ell^{1-N}, \end{aligned}$$

where we have used that  $\delta_{q+1} \lesssim 1$  and (2.4).  $\square$

LEMMA 4.3. *We have that the following lower and upper bounds hold:*

$$(4.34) \quad \int_{\mathbb{T}^3} \chi_{(0)}^2 dx \geq \frac{|\mathbb{T}^3|}{2},$$

$$(4.35) \quad \sum_{i \geq 1} \rho_i \int_{\mathbb{T}^3} \chi_{(i)}^2(x, t) dx \lesssim \lambda_q^{-\varepsilon_R} \delta_{q+1}.$$

*Proof of Lemma 4.3.* By Chebyshev's inequality, we have

$$\left| \left\{ x: |\dot{R}_\ell| \geq 2 \lambda_q^{-\varepsilon_R} \delta_{q+1} |\mathbb{T}^3|^{-1} \right\} \right| \leq \frac{|\mathbb{T}^3| \|\dot{R}_\ell\|_{L^1}}{2 \lambda_q^{-\varepsilon_R} \delta_{q+1}} \leq \frac{|\mathbb{T}^3| \|\dot{R}_q\|_{L^1}}{2 \lambda_q^{-\varepsilon_R} \delta_{q+1}} \leq \frac{|\mathbb{T}^3|}{2},$$

where we have used (2.3). Then from the definition of  $\chi_{(0)}$  we obtain (4.34).

Observe that by definition,

$$\begin{aligned} \sum_{i \geq 1} \rho_i \int_{\mathbb{T}^3} \chi_{(i)}^2 dx &\lesssim \sum_{i \geq 1} (4^i \lambda_q^{-\varepsilon_R} \delta_{q+1}) \tilde{\chi}^2 \left( \frac{1}{4^i} \left\langle \frac{\dot{R}_\ell}{100 \lambda_q^{-\varepsilon_R} \delta_{q+1}} \right\rangle \right) dx \\ &\lesssim \|\dot{R}_\ell\|_{L^1} \lesssim \|\dot{R}_q\|_{L^1} \lesssim \lambda_q^{-\varepsilon_R} \delta_{q+1}, \end{aligned}$$

from which we conclude (4.35).  $\square$

LEMMA 4.4. *The bounds*

$$(4.36) \quad \|a_{(\xi)}\|_{L^2} \lesssim \rho_i^{1/2} 2^{-i} \lesssim \delta_{q+1}^{1/2},$$

$$(4.37) \quad \|a_{(\xi)}\|_{L^\infty} \lesssim \rho_i^{1/2} \lesssim \delta_{q+1}^{1/2} 2^i,$$

$$(4.38) \quad \|a_{(\xi)}\|_{C_{x,t}^N} \lesssim \ell^{-N}$$

hold for all  $0 \leq i \leq i_{\max}$  and  $N \geq 1$ .

*Proof of Lemma 4.4.* The bound (4.37) follows directly from the definitions (4.12), (4.13) together with the boundedness of the functions  $\gamma_{(\xi)}$  and  $\rho_0$  given in (4.27). Using (4.32) additionally, the estimate (4.36) follows similarly. For (4.38), we apply derivatives to (4.12), use [2, Prop. C.1], estimate (4.33), Lemma 4.1, the bound (4.6) for  $\dot{R}_\ell$ , the definition (4.16) of  $\ell$ , and the bound (4.29) to obtain the following estimate for the case  $i \geq 1$ :

$$(4.39) \quad \begin{aligned} \|a_{(\xi)}\|_{C_{x,t}^N} &\lesssim \left( \rho_i^{1/2} \|\chi_{(i)}\|_{L^\infty} \left\| \left( \gamma_{(\xi)} \text{Id} - \rho_i^{-1} \dot{R}_\ell \right) \right\|_{C_{x,t}^N} \right. \\ &\quad \left. + \|\chi_{(i)}\|_{C_{x,t}^N} \left\| \left( \gamma_{(\xi)} \text{Id} - \rho_i^{-1} \dot{R}_\ell \right) \right\|_{L^\infty} \right) \\ &\lesssim \left( \rho_i^{1/2} \rho_i^{-1} \|\dot{R}_\ell\|_{C_{x,t}^N} + \rho_i^{-1} \|\dot{R}_\ell\|_{C_{t,x}^1}^N + \lambda_q^{10} \ell^{1-N} \right) \\ &\lesssim \left( \rho_i^{1/2} \rho_i^{-1} \ell^{-N+1} \|\dot{R}_\ell\|_{C_{x,t}^1} + \rho_i^{-1} \|\dot{R}_\ell\|_{C_{t,x}^1}^N + \rho_i^{-1/2} \ell^{-N} \right) \\ &\lesssim \rho_i^{1/2} \left( \rho_i^{-1} \ell^{-N+1} \lambda_q^{10} + \rho_i^{-1} \lambda_q^{10N} + \rho_i^{-1/2} \ell^{-N} \right) \\ &\lesssim \ell^{-N}. \end{aligned}$$

For  $i = 0$ , the time derivative may land on  $\rho_0^{1/2}$ . We use (4.27) to estimate this contribution similarly, by loosing an  $\ell^{-1}$  for each time derivative.  $\square$

PROPOSITION 4.5. *The principal part of the velocity perturbation, the incompressibility, and the temporal correctors obey the bounds*

$$(4.40) \quad \|w_{q+1}^{(p)}\|_{L^2} \leq \frac{M}{2} \delta_{q+1}^{1/2},$$

$$(4.41) \quad \|w_{q+1}^{(c)}\|_{L^2} + \|w_{q+1}^{(t)}\|_{L^2} \lesssim r^{3/2} \ell^{-1} \mu^{-1} \delta_{q+1}^{1/2},$$

$$(4.42) \quad \|w_{q+1}^{(p)}\|_{W^{1,p}} + \|w_{q+1}^{(c)}\|_{W^{1,p}} + \|w_{q+1}^{(t)}\|_{W^{1,p}} \lesssim \ell^{-2} \lambda_{q+1} r^{3/2-3/p},$$

$$(4.43) \quad \|\partial_t w_{q+1}^{(p)}\|_{L^p} + \|\partial_t w_{q+1}^{(c)}\|_{L^p} \lesssim \ell^{-2} \lambda_{q+1} \sigma \mu r^{5/2-3/p},$$

$$(4.44) \quad \|w_{q+1}^{(p)}\|_{C_{x,t}^N} + \|w_{q+1}^{(c)}\|_{C_{x,t}^N} + \|w_{q+1}^{(t)}\|_{C_{x,t}^N} \leq \frac{1}{2} \lambda_{q+1}^{(3+5N)/2}$$

for  $N \in \{0, 1, 2, 3\}$  and  $p > 1$ , where  $M$  is a universal constant.

*Proof of Proposition 4.5.* For  $i \geq 0$ , from (4.36) and (4.38) we may estimate

$$\|D^N a_{(\xi)}\|_{L^2} \lesssim \delta_{q+1}^{1/2} \ell^{-2N},$$

where we have used that  $\ell \delta_{q+1}^{-1/2} = \lambda_q^{-20+\beta b} \lesssim 1$ , which follows from the restriction imposed on the smallness of  $\beta b$ . Since  $\mathbb{W}_{(\xi)}$  is  $(\mathbb{T}/\lambda_{q+1}\sigma)^3$  periodic, and the condition (4.5) gives that  $\ell^{-2} \ll \lambda_{q+1}\sigma$ , we may apply (3.18) with  $N = K = 0$  and Lemma 3.7 to conclude

$$\|a_{(\xi)} \mathbb{W}_{(\xi)}\|_{L^2} \lesssim \rho_i^{1/2} 2^{-i} \|\mathbb{W}_{(\xi)}\|_{L^2} \lesssim \rho_i^{1/2} 2^{-i}.$$

Upon summing over  $i \in \{0, \dots, i_{\max}\}$ , and appealing to (4.30), we obtain (4.40) for some fixed constant  $M$  independent of any parameter.

In order to bound the  $L^2$  norm of  $w_{q+1}^{(c)}$ , we use (3.19) and Lemma 4.4 to estimate

$$\begin{aligned} & \left\| \frac{1}{\lambda_{q+1}} \nabla (a_{(\xi)} \eta_{(\xi)}) \times W_{(\xi)} \right\|_{L^2} \\ & \lesssim \frac{1}{\lambda_{q+1}} \left( \|\nabla a_{(\xi)}\|_{L^\infty} \|\eta_{(\xi)}\|_{L^2} + \|a_{(\xi)}\|_{L^\infty} \|\nabla \eta_{(\xi)}\|_{L^2} \right) \\ & \lesssim \frac{1}{\lambda_{q+1}} \left( \ell^{-1} + \delta_{q+1}^{1/2} 2^i \lambda_{q+1} \sigma r \right) \\ & \lesssim \delta_{q+1}^{1/2} 2^i \sigma r, \end{aligned}$$

where we have used that  $\ell^{-1} \leq \lambda_{q+1} \delta_{q+1}^{1/2} \sigma r$ , which follows from (4.15)–(4.16) since  $b$  is sufficiently large. Analogously, bounding the summands in the definition of  $w_{q+1}^{(t)}$ , we have

$$\left\| \frac{1}{2\mu} \mathbb{P}_{H \setminus \mathbb{P} \neq 0} (a_{(\xi)}^2 \eta_{(\xi)}^2 \xi) \right\|_{L^2} \lesssim \frac{1}{\mu} \|a\|_{L^\infty}^2 \|\eta_{(\xi)}\|_{L^4}^2 \lesssim \frac{\delta_{q+1} 4^i r^{3/2}}{\mu}.$$

Summing in  $i \in \{0, \dots, i_{\max}\}$  and  $\xi$ , and employing (4.29), we obtain

$$\|w_{q+1}^{(c)}\|_{L^2} + \|w_{q+1}^{(t)}\|_{L^2} \lesssim \frac{\delta_{q+1}^{1/2} \sigma r}{\ell^{1/2}} + \frac{\delta_{q+1} r^{3/2}}{\ell \mu} \lesssim \frac{r^{3/2}}{\ell \mu} \delta_{q+1}^{1/2}.$$

In the above bound we have used the inequality  $\ell^{1/2} \mu \leq \sigma^{-1} r^{1/2}$ , which follows from (4.15)–(4.16).

Now consider (4.42). Observe that by definition (4.17), estimate (3.18), and Lemma 4.4, we have

$$\begin{aligned} (4.45) \quad & \|w_{q+1}^{(p)}\|_{W^{1,p}} \lesssim \sum_i \sum_{\xi \in \Lambda(i)} \|a_\xi\|_{C_{x,t}^1} \|\mathbb{W}_{(\xi)}\|_{W^{1,p}} \\ & \lesssim \sum_i \sum_{\xi \in \Lambda(i)} \ell^{-1} \lambda_{q+1} r^{3/2-3/p} \\ & \lesssim \ell^{-2} \lambda_{q+1} r^{3/2-3/p}. \end{aligned}$$



Here we have also used (4.29) in order to sum over  $i$ . For the analogous bound on  $w_{q+1}^{(c)}$ , using (3.19) and Lemma 4.4 we arrive at

$$\begin{aligned}
& \left\| \frac{1}{\lambda_{q+1}} (\nabla \cdot a_{(\xi)} \eta_{(\xi)}) \times W_{(\xi)} \right\|_{W^{1,p}} \\
& \lesssim \frac{1}{\lambda_{q+1}} \left( \left\| \nabla^2 a_{(\xi)} \eta_{(\xi)} \right\|_{L^p} + \lambda_{q+1} \left\| \nabla a_{(\xi)} \eta_{(\xi)} \right\|_{L^p} \right) \\
& \lesssim \frac{\|a_{(\xi)}\|_{C^2}}{\lambda_{q+1}} \left( \left\| \eta_{(\xi)} \right\|_{W^{2,p}} + \lambda_{q+1} \left\| \eta_{(\xi)} \right\|_{W^{1,p}} \right) \\
& \lesssim \frac{\ell^{-2}}{\lambda_{q+1}} (\lambda_{q+1} \sigma r)^2 r^{3/2-3/p} + \frac{\ell^{-2}}{\lambda_{q+1}} \lambda_{q+1}^2 \sigma r^{5/2-3/p} \\
& \lesssim \ell^{-2} \lambda_{q+1} r^{3/2-3/p} (\sigma r),
\end{aligned}$$

where we used  $\lambda_{q+1} \sigma r \leq \lambda_{q+1}$ . The above bound is consistent with (4.42) for  $w_{q+1}^{(c)}$  since summing over  $i$  and  $\xi$  loses an extra factor of  $\ell^{-1}$  that may be absorbed since  $\ell^{-1} \sigma r < 1$ . Similarly, in order to estimate  $w_{q+1}^{(t)}$  we use the bound (3.19) and Lemma 4.4 to obtain

$$\begin{aligned}
& \left\| \frac{1}{\mu} \mathbb{P}_{H \setminus \{0\}} (a_{(\xi)}^2 \eta_{(\xi)}^2 \xi) \right\|_{W^{1,p}} \\
& \lesssim \frac{1}{\mu} \|a_{(\xi)}\|_{C^1} \|a_{(\xi)}\|_{L^\infty} \left( \left\| \nabla \eta_{(\xi)} \right\|_{L^{2p}} \left\| \eta_{(\xi)} \right\|_{L^{2p}} + \left\| \eta_{(\xi)} \right\|_{L^{2p}}^2 \right) \\
& \lesssim \frac{1}{\mu} \ell^{-1} \delta_{q+1}^{1/2} 2^i (\lambda_{q+1} \sigma r) r^{3-3/p}.
\end{aligned}$$

Summing in  $i$  and  $\xi$  and using (4.29) we obtain

$$\left\| w_{q+1}^{(t)} \right\|_{W^{1,p}} \lesssim \frac{1}{\mu} \ell^{-3/2} \delta_{q+1}^{1/2} (\lambda_{q+1} \sigma r) r^{3-3/p} \lesssim \ell^{-2} \lambda_{q+1} r^{3/2-3/p} \frac{\sigma r^{5/2}}{\mu}.$$

Thus (4.42) also holds for  $w_{q+1}^{(t)}$ , as a consequence of the inequality  $\sigma r^{5/2} \leq \mu$ , which holds by (4.15).

Now consider the  $L^p$  estimates of the time derivatives of  $w_{q+1}^{(p)}$  and  $w_{q+1}^{(c)}$ . Estimates (3.18) and (4.38) yield

$$\begin{aligned}
\left\| \partial_t w_{q+1}^{(p)} \right\|_{L^p} & \lesssim \sum_i \sum_{\xi \in \Lambda(i)} \|a_{(\xi)}\|_{C_{x,t}^1} \left\| \partial_t \mathbb{W}_{(\xi)} \right\|_{L^p} \\
& \lesssim \sum_i \sum_{\xi \in \Lambda(i)} \ell^{-1} (\lambda_{q+1} \sigma r \mu) r^{3/2-3/p} \\
& \lesssim \ell^{-2} \lambda_{q+1} \sigma \mu r^{5/2-3/p}.
\end{aligned}$$

Similarly, using (3.19) and (4.38), we obtain

$$\begin{aligned}
\|\partial_t w_{q+1}^{(c)}\|_{L^p} &\lesssim \sum_i \sum_{\xi \in \Lambda(i)} \left\| \frac{1}{\lambda_{q+1}} \left( \partial_t \nabla a_{(\xi)} \eta_{(\xi)} \right) \times W_{(\xi)} \right\|_{L^p} \\
&\lesssim \frac{1}{\lambda_{q+1}} \sum_i \sum_{\xi \in \Lambda(i)} \left\| \partial_t \nabla a_{(\xi)} \eta_{(\xi)} \right\|_{L^p} \\
&\lesssim \frac{1}{\lambda_{q+1}} \sum_i \sum_{\xi \in \Lambda(i)} \|a_{(\xi)}\|_{C_{x,t}^2} \left( \|\partial_t \eta_{(\xi)}\|_{W^{1,p}} + \|\eta_{(\xi)}\|_{W^{1,p}} \right) \\
&\lesssim \ell^{-3} \lambda_{q+1} \sigma^2 \mu r^{7/2-3/p},
\end{aligned}$$

which is a bound that is consistent with (4.43), upon noting that  $\ell^{-1} \sigma r \leq 1$  holds.

For  $N = 0$ , the bound (4.44) holds for  $w_{q+1}^{(p)}$  in view of (3.18), (4.38), (4.29), and the fact that  $\ell^{-1} r^{3/2} \ll \lambda_{q+1}^{3/2}$ . For the derivative bounds of  $w_{q+1}^{(p)}$ , we use (3.18) and (4.38) to conclude

$$\|a_{(\xi)} \mathbb{W}_{(\xi)}\|_{C_{x,t}^N} \lesssim \|a_{(\xi)}\|_{C_{x,t}^N} \|\mathbb{W}_{(\xi)}\|_{C_{x,t}^N} \lesssim \ell^{-N} (\lambda_{q+1} \sigma r \mu)^N r^{3/2},$$

from which the first part of (4.44) immediately follows in view of our parameter choices (4.15)–(4.16). Indeed, (4.15) gives  $\lambda_{q+1} \sigma r \mu = \lambda_{q+1}^{33/16} = \lambda_{q+1}^2 \lambda_{q+1}^{1/16}$  and  $r^{3/2} = \lambda_{q+1}^{9/8}$ . The bound for the  $C_{x,t}^N$  norm of  $w_{q+1}^{(c)}$  and  $w_{q+1}^{(t)}$  follows *mutatis mutandis*.  $\square$

In view of

- the definitions of  $w_{q+1}$  and  $v_{q+1}$  in (4.21) and (4.22);
- the estimates (4.7) and (4.8);
- the identity  $v_{q+1} - v_q = w_{q+1} + (v_\ell - v_q)$ ;
- the bound  $\ell \lambda_q^4 \delta_{q+1}^{-1/2} + \ell^{-1} r^{3/2} \mu^{-1} \ll 1$ , which holds since  $b$  was taken to be sufficiently large,

the estimates in Proposition 4.5 directly imply

COROLLARY 4.6. *For  $N \in \{0, 1, 2, 3\}$  and  $p > 1$ , we have*

$$(4.46) \quad \|w_{q+1}\|_{L^2} \leq \frac{3M}{4} \delta_{q+1}^{1/2},$$

$$(4.47) \quad \|v_{q+1} - v_q\|_{L^2} \leq M \delta_{q+1}^{1/2},$$

$$(4.48) \quad \|w_{q+1}\|_{W^{1,p}} \lesssim \ell^{-2} \lambda_{q+1} r^{3/2-3/p},$$

$$(4.49) \quad \|w_{q+1}\|_{C_{x,t}^N} \leq \frac{1}{2} \lambda_{q+1}^{(3+5N)/2},$$

$$(4.50) \quad \|v_{q+1}\|_{C_{x,t}^N} \leq \lambda_{q+1}^{(3+5N)/2}.$$

Therefore, setting  $N = 1$  in the above estimate for  $v_{q+1}$  we have proven that (2.2) holds with  $q$  replaced by  $q + 1$ . Also, (4.47) proves the velocity increment bound we have claimed in (2.7).

## 5. Reynolds stress

The main result of this section may be summarized as follows:

**PROPOSITION 5.1.** *There exist a  $p > 1$  sufficiently close to 1 and an  $\varepsilon_R > 0$  sufficiently small, depending only on  $b$  and  $\beta$  (in particular, independent of  $q$ ), such that there exist a traceless symmetric 2 tensor  $\tilde{R}$  and a scalar pressure field  $\tilde{p}$ , defined implicitly in (5.7) below, satisfying*

$$(5.1) \quad \partial_t v_{q+1} + \operatorname{div} (v_{q+1} \otimes v_{q+1}) + \nabla \tilde{p} - \nu \Delta v_{q+1} = \operatorname{div} \tilde{R}$$

and the bound

$$(5.2) \quad \|\tilde{R}\|_{L^p} \lesssim \lambda_{q+1}^{-2\varepsilon_R} \delta_{q+2},$$

where the constant depends on the choice of  $p$  and  $\varepsilon_R$ .

An immediate consequence of Proposition 5.1 is that the desired inductive estimates (2.3)–(2.4) hold for a suitably defined Reynolds stress  $\mathring{R}_{q+1}$ ; see (5.5) below. We emphasize that compared to  $\tilde{R}$ , the stress  $\mathring{R}_{q+1}$  constructed below also obeys a satisfactory  $C^1$  estimate.

**COROLLARY 5.2.** *There exists a traceless symmetric 2 tensor  $\mathring{R}_{q+1}$  and a scalar pressure field  $p_{q+1}$  such that*

$$\partial_t v_{q+1} + \operatorname{div} (v_{q+1} \otimes v_{q+1}) + \nabla p_{q+1} - \nu \Delta v_{q+1} = \operatorname{div} \mathring{R}_{q+1}.$$

Moreover, the following bounds hold:

$$(5.3) \quad \|\mathring{R}_{q+1}\|_{L^1} \leq \lambda_{q+1}^{-\varepsilon_R} \delta_{q+2},$$

$$(5.4) \quad \|\mathring{R}_{q+1}\|_{C^1_{x,t}} \leq \lambda_{q+1}^{10}.$$

Before giving the proof of the corollary, we recall from [2, Def. 1.4] the 2-tensor valued elliptic operator  $\mathcal{R}$  that has the property that  $\mathcal{R}v(x)$  is a symmetric trace-free matrix for each  $x \in \mathbb{T}^3$ , and  $\mathcal{R}$  is a right inverse of the div operator, i.e.,

$$\operatorname{div} \mathcal{R}v = v - \int_{\mathbb{T}^3} v(x) dx$$

for any smooth  $v$ . Moreover, we have the classical Calderón-Zygmund bound  $\|\nabla|\mathcal{R}\|\|_{L^p \rightarrow L^p} \lesssim 1$ , and the Schauder estimates  $\|\mathcal{R}\|_{L^p \rightarrow L^p} + \|\mathcal{R}\|_{C^0 \rightarrow C^0} \lesssim 1$ , for  $p \in (1, \infty)$ . Since throughout the proof the value of  $p > 1$  is independent of  $q$ , the implicit constants in these inequalities are uniformly bounded.

*Proof of Corollary 5.2.* With  $\tilde{R}$  and  $\tilde{p}$  defined in Proposition 5.1, we let

$$(5.5) \quad \mathring{R}_{q+1} = \mathcal{R}(\mathbb{P}_H \operatorname{div} \tilde{R}) \quad \text{and} \quad p_{q+1} = \tilde{p} - \Delta^{-1} \operatorname{div} \operatorname{div} \tilde{R}.$$

With the parameter  $p > 1$  from [Proposition 5.1](#), using that  $\|\mathcal{R}\operatorname{div}\|_{L^p \rightarrow L^p} \lesssim 1$  we directly bound

$$\|\mathring{R}_{q+1}\|_{L^1} \lesssim \|\mathring{R}_{q+1}\|_{L^p} \lesssim \|\widetilde{R}\|_{L^p} \lesssim \lambda_{q+1}^{-2\varepsilon_R} \delta_{q+2}.$$

The estimate (5.3) then follows since the factor  $\lambda_{q+1}^{-\varepsilon_R}$  can absorb any constant if we assume  $a$  is sufficiently large.

Now consider (5.4). Using [equation \(5.1\)](#) and the bounds of [Corollary 4.6](#) we obtain

$$\begin{aligned} \|\mathring{R}_{q+1}\|_{C^1} &= \|\mathcal{R}\mathbb{P}_H(\operatorname{div} \widetilde{R})\|_{C^1} \\ &\lesssim \|\partial_t v_{q+1} + \operatorname{div}(v_{q+1} \otimes v_{q+1}) - \nu \Delta v_{q+1}\|_{C^1} \\ &\lesssim \|\partial_t v_{q+1}\|_{C^1} + \|v_{q+1} \otimes v_{q+1}\|_{C^2} + \|v_{q+1}\|_{C^3} \\ &\lesssim \lambda_{q+1}^9 \end{aligned}$$

by using the Schauder estimates  $\|\mathcal{R}\mathbb{P}_H\|_{C^0 \rightarrow C^0} \lesssim 1$ . Similarly, we have that

$$\begin{aligned} \|\partial_t \mathring{R}_{q+1}\|_{L^\infty} &\lesssim \|\partial_t(\partial_t v_{q+1} + \operatorname{div}(v_{q+1} \otimes v_{q+1}) - \nu \Delta v_{q+1})\|_{C^0} \\ &\lesssim \|\partial_t^2 v_{q+1}\|_{C^0} + \|\partial_t v_{q+1} \otimes v_{q+1}\|_{C^1} + \|\partial_t v_{q+1}\|_{C^2} \\ &\lesssim \lambda_{q+1}^9, \end{aligned}$$

which concludes the proof of (5.4) upon using the leftover power of  $\lambda_{q+1}$  to absorb all  $q$  independent constants.  $\square$

**5.1. Proof of [Proposition 5.1](#).** Recall that  $v_{q+1} = w_{q+1} + v_\ell$ , where  $v_\ell$  is defined in [Section 4.1](#). Using (4.2) and (4.21), we obtain

$$\begin{aligned} (5.6) \quad \operatorname{div} \widetilde{R} - \nabla \widetilde{p} &= -\nu \Delta w_{q+1} + \partial_t(w_{q+1}^{(p)} + w_{q+1}^{(c)}) + \operatorname{div}(v_\ell \otimes w_{q+1} + w_{q+1} \otimes v_\ell) \\ &\quad + \operatorname{div}\left((w_{q+1}^{(c)} + w_{q+1}^{(t)}) \otimes w_{q+1} + w_{q+1}^{(p)} \otimes (w_{q+1}^{(c)} + w_{q+1}^{(t)})\right) \\ &\quad + \operatorname{div}(w_{q+1}^{(p)} \otimes w_{q+1}^{(p)} + \mathring{R}_\ell) + \partial_t w_{q+1}^{(t)} \\ &\quad + \operatorname{div}(\widetilde{R}_{\text{commutator}}) - \nabla p_\ell \\ (5.7) \quad &= (\operatorname{div} \widetilde{R}_{\text{linear}} + \widetilde{R}_{\text{corrector}} + \widetilde{R}_{\text{oscillation}} + \widetilde{R}_{\text{commutator}}) \\ &\quad + \nabla(P - p_\ell). \end{aligned}$$

Here, the symmetric trace-free stresses  $\widetilde{R}_{\text{linear}}$  and  $\widetilde{R}_{\text{corrector}}$  are defined by applying the inverse divergence operator  $\mathcal{R}$  to the first and respectively second lines of (5.6). The stress  $\widetilde{R}_{\text{commutator}}$  was defined previously in (4.3), while the stress  $\widetilde{R}_{\text{oscillation}}$  is defined in [Section 5.3](#) below. The pressure  $P$  is given by (5.11) below.

Besides the already used inequalities between the parameters,  $\ell$ ,  $r$ ,  $\sigma$ , and  $\lambda_{q+1}$ , we shall use the following bound in order to achieve (5.2):

$$(5.8) \quad \begin{aligned} & \ell^{-2} \sigma \mu r^{5/2-3/p} + (r^{3/2} \ell^{-1} \mu^{-1})^{1/p} \lambda_{q+1}^{3(1-1/p)} \\ & + \frac{r^{3-3/p}}{\ell^3 \lambda_{q+1} \sigma} + \frac{\sigma r^{4-3/p}}{\ell^3} + \lambda_q^{-10} \lesssim \lambda_{q+1}^{-2\varepsilon_R} \delta_{q+2}. \end{aligned}$$

In view of (4.15)–(4.16), the above inequality holds for  $b$  sufficiently large,  $\beta$  sufficiently small depending on  $b$ , parameters  $\varepsilon_R, p-1 > 0$  sufficiently small depending on  $b$  and  $\beta$ , and for  $\lambda_0 = a$  sufficiently large depending on all these parameters and on  $M$ .

In view of the bound  $\ell \lambda_q^6 \lesssim \lambda_q^{-10}$ , the estimate (4.4) for  $\widetilde{R}_{\text{commutator}}$  is consistent with (5.2). Hence it remains to consider the linear, corrector and oscillation errors in (5.7).

5.2. *The linear and corrector errors.* In view of (5.2), we estimate contributions to the  $\widetilde{R}$  coming from the first line in (5.6) as

$$(5.9) \quad \begin{aligned} \|\widetilde{R}_{\text{linear}}\|_{L^p} & \lesssim \|\mathcal{R}(\nu \Delta w_{q+1})\|_{L^p} + \|\mathcal{R}(\partial_t(w_{q+1}^{(p)} + w_{q+1}^{(c)}))\|_{L^p} \\ & \quad + \|\mathcal{R} \operatorname{div}(v_\ell \otimes w_{q+1} + w_{q+1} \otimes v_\ell)\|_{L^p} \\ & \lesssim \|w_{q+1}\|_{W^{1,p}} + \frac{1}{\lambda_{q+1}} \|\partial_t \mathcal{R} \operatorname{curl}(w_{q+1}^{(p)})\|_{L^p} + \|v_\ell\|_{L^\infty} \|w_{q+1}\|_{L^p} \\ & \lesssim \lambda_q^4 \|w_{q+1}\|_{W^{1,p}} + \frac{1}{\lambda_{q+1}} \|\partial_t w_{q+1}^{(p)}\|_{L^p} \\ & \lesssim \lambda_q^4 \ell^{-2} \lambda_{q+1} r^{3/2-3/p} + \frac{1}{\lambda_{q+1}} \ell^{-2} \lambda_{q+1} \sigma \mu r^{5/2-3/p} \\ & \lesssim \ell^{-2} \sigma \mu r^{5/2-3/p}, \end{aligned}$$

where we have used  $\nu \leq 1$ ,  $\lambda_q^4 \lambda_{q+1} \leq \sigma \mu r$ , the identity (4.19), the inductive estimate (2.2) to bound  $\|v_\ell\|_{L^\infty} \lesssim \|v_\ell\|_{C^1} \lesssim \|v_q\|_{C^1}$ , estimates (4.43) and (4.48). Next we turn to the errors involving correctors, for which we appeal to  $L^p$  interpolation, the Poincaré inequality, and Proposition 4.5:

$$\begin{aligned} \|\widetilde{R}_{\text{corrector}}\|_{L^p} & \leq \|\mathcal{R} \operatorname{div}((w_{q+1}^{(c)} + w_{q+1}^{(t)}) \otimes w_{q+1} + w_{q+1}^{(p)} \otimes (w_{q+1}^{(c)} + w_{q+1}^{(t)}))\|_{L^p} \\ & \lesssim \|(w_{q+1}^{(c)} + w_{q+1}^{(t)}) \otimes w_{q+1}\|_{L^1}^{1/p} \|(w_{q+1}^{(c)} + w_{q+1}^{(t)}) \otimes w_{q+1}\|_{L^\infty}^{1-1/p} \\ & \quad + \|w_{q+1}^{(p)} \otimes (w_{q+1}^{(c)} + w_{q+1}^{(t)})\|_{L^1}^{1/p} \|w_{q+1}^{(p)} \otimes (w_{q+1}^{(c)} + w_{q+1}^{(t)})\|_{L^\infty}^{1-1/p} \\ & \lesssim (r^{3/2} \ell^{-1} \mu^{-1})^{1/p} \delta_{q+1}^{1/p} \lambda_{q+1}^{3(1-1/p)} \\ & \lesssim (r^{3/2} \ell^{-1} \mu^{-1})^{1/p} \lambda_{q+1}^{3(1-1/p)}. \end{aligned}$$

Due to (5.8) this estimate is sufficient.

5.3. *Oscillation error.* In this section we estimate the remaining error,  $\widetilde{R}_{\text{oscillation}}$ , which obeys

$$(5.10) \quad (\operatorname{div} \widetilde{R}_{\text{oscillation}}) + \nabla P = \operatorname{div} \left( w_{q+1}^{(p)} \otimes w_{q+1}^{(p)} + \mathring{R}_\ell \right) + \partial_t w_{q+1}^{(t)},$$

where the pressure term  $P$  is given by

$$(5.11) \quad P = \sum_{i \geq 0} \rho_i \chi_i^2 + \frac{1}{2} \sum_{i,j} \sum_{\xi \in \Lambda_{(i)}, \xi' \in \Lambda_{(j)}} a_{(\xi)} a_{(\xi')} \mathbb{P}_{\neq 0} \mathbb{W}_{(\xi)} \cdot \mathbb{W}_{(\xi')} \\ - \sum_i \sum_{\xi \in \Lambda_{(i)}^+} \frac{1}{\mu} \Delta^{-1} \operatorname{div} \partial_t \left( a_{(\xi)}^2 \eta_{(\xi)}^2 \xi \right).$$

Recall that from the definition of  $w_{q+1}^{(p)}$  and of the coefficients  $a_{(\xi)}$ , via (4.14) we have

$$(5.12) \quad \operatorname{div} \left( w_{q+1}^{(p)} \otimes w_{q+1}^{(p)} \right) + \operatorname{div} \mathring{R}_\ell \\ = \sum_{i,j} \sum_{\xi \in \Lambda_{(i)}, \xi' \in \Lambda_{(j)}} \left( \operatorname{div} a_{(\xi)} a_{(\xi')} \mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(\xi')} \right) + \operatorname{div} \mathring{R}_\ell \\ = \sum_{i,j} \sum_{\xi \in \Lambda_{(i)}, \xi' \in \Lambda_{(j)}} \left( \operatorname{div} a_{(\xi)} a_{(\xi')} \left( \mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(\xi')} - \int_{\mathbb{T}^3} \mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(\xi')} dx \right) \right) \\ + \left( \nabla \sum_{i \geq 0} \rho_i \chi_i^2 \right) \\ = \sum_{i,j} \sum_{\xi \in \Lambda_{(i)}, \xi' \in \Lambda_{(j)}} \underbrace{\left( \operatorname{div} a_{(\xi)} a_{(\xi')} \mathbb{P}_{\geq \lambda_{q+1}\sigma/2} \mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(\xi')} \right)}_{E_{(\xi, \xi')}} \\ + \left( \nabla \sum_{i \geq 0} \rho_i \chi_i^2 \right).$$

Here we use that the minimal separation between active frequencies of  $\mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(\xi')}$  and the 0 frequency is given by  $\lambda_{q+1}\sigma$  for  $\xi' = -\xi$ , and by  $c_\Lambda \lambda_{q+1} \geq \lambda_{q+1}\sigma$  for  $\xi' \neq -\xi$ . We proceed to estimate each symmetrized summand  $E_{(\xi, \xi')} + E_{(\xi', \xi)}$  individually. We split

$$E_{(\xi, \xi')} + E_{(\xi', \xi)} = \left( \mathbb{P}_{\neq 0} \mathbb{P}_{\geq \lambda_{q+1}\sigma/2} \left( \mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(\xi')} + \mathbb{W}_{(\xi')} \otimes \mathbb{W}_{(\xi)} \right) \nabla a_{(\xi)} a_{(\xi')} \right) \\ + \left( \mathbb{P}_{\neq 0} a_{(\xi)} a_{(\xi')} \left( \operatorname{div} \mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(\xi')} + \mathbb{W}_{(\xi')} \otimes \mathbb{W}_{(\xi)} \right) \right) \\ =: E_{(\xi, \xi', 1)} + E_{(\xi, \xi', 2)}.$$

Here we used the fact that  $E_{(\xi, \xi')} + E_{(\xi', \xi)}$  has zero mean to subtract the mean from each of the two terms on the right-hand side of the above. We have also removed the unnecessary frequency projection  $P_{\geq \lambda_{q+1}\sigma/2}$  from the second term.

The term  $E_{(\xi, \xi', 1)}$  can easily be estimated using [Lemmas 4.4](#) and [B.1](#), estimate [\(B.2\)](#), with  $\lambda = \ell^{-1}$ ,  $C_a = \ell^{-2}$ ,  $\kappa = \lambda_{q+1}\sigma/2$ , and  $L$  sufficiently large, as

$$\begin{aligned} \|\mathcal{R}E_{(\xi, \xi', 1)}\|_{L^p} &\lesssim \| |\nabla|^{-1} E_{(\xi, \xi', 1)} \|_{L^p} \\ &\lesssim \| |\nabla|^{-1} (\mathbb{P}_{\neq 0} \mathbb{P}_{\geq \lambda_{q+1}\sigma/2} \mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(\xi')}) \nabla a_{(\xi)} a_{(\xi')} ) \|_{L^p} \\ &\lesssim \frac{1}{\ell^2 \lambda_{q+1} \sigma} \left( 1 + \frac{1}{\ell^L (\lambda_{q+1} \sigma)^{L-2}} \right) \| \mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(\xi')} \|_{L^p} \\ &\lesssim \frac{1}{\ell^2 \lambda_{q+1} \sigma} \| \mathbb{W}_{(\xi)} \|_{L^{2p}} \| \mathbb{W}_{(\xi')} \|_{L^{2p}} \\ &\lesssim \frac{r^{3-3/p}}{\ell^2 \lambda_{q+1} \sigma}. \end{aligned}$$

In the last inequality above we have used the estimate [\(3.18\)](#), and in the second to last inequality we have used that for  $b$  sufficiently large and  $L$  sufficiently large, we have  $\ell^{-L} (\lambda_{q+1} \sigma)^{2-L} \lesssim 1$ . Indeed, this inequality holds under the conditions  $L \geq 3$  and  $b(L-2)/16 \geq 20L$ . After summing in  $i$  and  $\xi$  we incur an additional loss of  $\ell^{-1}$ . By [\(5.8\)](#) this bound is consistent with [\(5.2\)](#).

For the term  $E_{(\xi, \xi', 2)}$ , we split into two cases:  $\xi + \xi' \neq 0$  and  $\xi + \xi' = 0$ . Let us first consider the case  $\xi + \xi' \neq 0$ . Applying the identity [\(3.16\)](#) and using [\(3.14\)](#), we have

$$\begin{aligned} &a_{(\xi)} a_{(\xi')} (\operatorname{div} \mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(\xi')} + \mathbb{W}_{(\xi')} \otimes \mathbb{W}_{(\xi)}) \\ &= a_{(\xi)} a_{(\xi')} (\mathbb{W}_{(\xi')} \cdot \nabla \eta_{(\xi)} \eta_{(\xi')}) (\mathbb{W}_{(\xi)} + \mathbb{W}_{(\xi)} \cdot \nabla \eta_{(\xi)} \eta_{(\xi')}) \mathbb{W}_{(\xi')} \\ &\quad + a_{(\xi)} a_{(\xi')} \eta_{(\xi)} \eta_{(\xi')} \nabla \mathbb{W}_{(\xi)} \cdot \mathbb{W}_{(\xi')} \\ &= a_{(\xi)} a_{(\xi')} \mathbb{P}_{\geq c_\Lambda \lambda_{q+1}} (\nabla \eta_{(\xi)} \eta_{(\xi')}) (\mathbb{W}_{(\xi')} \otimes \mathbb{W}_{(\xi)} + \mathbb{W}_{(\xi)} \otimes \mathbb{W}_{(\xi')}) \\ &\quad + (\nabla a_{(\xi)} a_{(\xi')} \mathbb{W}_{(\xi)} \cdot \mathbb{W}_{(\xi')}) (\nabla a_{(\xi)} a_{(\xi')}) \mathbb{P}_{\geq c_\Lambda \lambda_{q+1}} \mathbb{W}_{(\xi)} \cdot \mathbb{W}_{(\xi')} \\ &\quad - a_{(\xi)} a_{(\xi')} \mathbb{P}_{\geq c_\Lambda \lambda_{q+1}} (\mathbb{W}_{(\xi)} \cdot \mathbb{W}_{(\xi')}) \nabla (\eta_{(\xi)} \eta_{(\xi')}). \end{aligned}$$

The second term is a pressure, and to the remaining terms we apply the inverse divergence operator  $\mathcal{R}$ . We estimate  $\mathcal{R}$  applied to the third term analogously to  $E_{(\xi, \xi', 1)}$ , and  $\mathcal{R}$  applied to the fourth term can be estimated similarly to the first term. Thus it suffices to estimate  $\mathcal{R}$  applied to the first term. Applying [\(3.19\)](#), [Lemma 4.4](#), estimate [\(B.2\)](#) of [Lemma B.1](#), with  $\lambda = \ell^{-1}$ ,  $C_a = \ell^{-2}$ ,  $\kappa = c_\Lambda \lambda_{q+1}$ , and for  $b$  and  $L$  sufficiently large ( $L \geq 3$  and  $b(L-2) \geq 20L$ )

suffices), we obtain

$$\begin{aligned} & \left( \left\| \mathcal{R} \left( a_{(\xi)} a_{(\xi')} \mathbb{P}_{\geq c_\Lambda \lambda_{q+1}} \left( \nabla \eta_{(\xi)} \eta_{(\xi')} \right) (W_{(\xi')} \otimes W_{(\xi)} + W_{(\xi)} \otimes W_{(\xi')}) \right) \right\|_{L^p} \right) \\ & \lesssim \ell^{-2} \left( 1 + \frac{1}{\ell^L \lambda_{q+1}^{L-2}} \right) \frac{\left\| \nabla \eta_{(\xi)} \eta_{(\xi')} \right\|_{L^p}}{\lambda_{q+1}} \\ & \lesssim \frac{\sigma r^{4-3/p}}{\ell^2}. \end{aligned}$$

Summing in  $\xi$  and  $i$  we lose an additional  $\ell^{-1}$  factor. By (5.8) this bound is consistent with (5.2).

Now let us consider  $E_{(\xi, \xi', 2)}$  for the case  $\xi + \xi' = 0$ . Applying the identity (3.17), we have

$$\begin{aligned} (5.13) \quad E_{(\xi, -\xi, 2)} &= \left( \mathbb{P}_{\neq 0} \left( a_{(\xi)}^2 \nabla \eta_{(\xi)}^2 - a_{(\xi)}^2 \frac{(\xi)}{\mu} \partial_t \eta_{(\xi)}^2 \right) \right) \\ &= \left( \nabla \left( a_{(\xi)}^2 \mathbb{P}_{\geq \lambda_{q+1} \sigma/2} \eta_{(\xi)}^2 \right) \right) - \left( \mathbb{P}_{\neq 0} \left( \mathbb{P}_{\geq \lambda_{q+1} \sigma/2} \eta_{(\xi)}^2 \right) \nabla a_{(\xi)}^2 \right) \\ &\quad - \frac{1}{\mu} \partial_t \left( \mathbb{P}_{\neq 0} \left( a_{(\xi)}^2 \eta_{(\xi)}^2 \xi \right) \right) + \frac{1}{\mu} \left( \mathbb{P}_{\neq 0} \left( \eta_{(\xi)}^2 \partial_t a_{(\xi)}^2 \right) \xi \right). \end{aligned}$$

Here we have used that  $\mathbb{P}_{\neq 0} \eta_{(\xi)}^2 = \mathbb{P}_{\geq \lambda_{q+1} \sigma/2} \eta_{(\xi)}^2$ , which holds since  $\eta_{(\xi)}$  is  $(\mathbb{T}/\lambda_{q+1} \sigma)^3$ -periodic. Hence, summing in  $\xi$  and  $i$ , using that  $\eta_{(\xi)} = \eta_{(-\xi)}$ , pairing with the  $\partial_t w_{q+1}^{(t)}$  present in (5.10), recalling the definition of  $w_{q+1}^{(t)}$  in (4.20), and noting that  $\text{Id} - \mathbb{P}_H = \nabla \Delta^{-1} \text{div}$ , we obtain

$$\begin{aligned} (5.14) \quad & \sum_i \sum_{\xi \in \Lambda_{(i)}^+} E_{(\xi, -\xi, 2)} + \partial_t w_{q+1}^{(t)} \\ &= \left( \nabla \sum_i \sum_{\xi \in \Lambda_{(i)}^+} a_{(\xi)}^2 \mathbb{P}_{\geq \lambda_{q+1} \sigma/2} \eta_{(\xi)}^2 \right) \\ &\quad - \sum_i \sum_{\xi \in \Lambda_{(i)}^+} \left( \mathbb{P}_{\neq 0} \left( \mathbb{P}_{\geq \lambda_{q+1} \sigma/2} \eta_{(\xi)}^2 \right) \nabla a_{(\xi)}^2 \right) \\ &\quad - \left( \nabla \sum_i \sum_{\xi \in \Lambda_{(i)}^+} \frac{1}{\mu} \Delta^{-1} \text{div} \left( \partial_t a_{(\xi)}^2 \eta_{(\xi)}^2 \xi \right) \right) \\ &\quad + \frac{1}{\mu} \sum_i \sum_{\xi \in \Lambda_{(i)}^+} \left( \mathbb{P}_{\neq 0} \left( \eta_{(\xi)}^2 \partial_t a_{(\xi)}^2 \right) \xi \right). \end{aligned}$$

The first and third terms are pressure terms, and to the remaining terms we apply the inverse divergence operator  $\mathcal{R}$ . Thus it suffices to estimate  $\mathcal{R}$  applied to the second and the last term above. We split the second term of (5.14) into its summands, apply  $\mathcal{R}$ , and estimate each term individually, similarly to the



estimate of  $\mathcal{R}E_{(\xi, \xi', 1)}$ . Using (3.19), Lemma 4.4 and Lemma B.1, estimate (B.2), with  $\lambda = \ell^{-1}$ ,  $C_a = \ell^{-2}$ ,  $\kappa = \lambda_{q+1}\sigma/2$ , and for  $b$  and  $L$  sufficiently large ( $L \geq 3$  and  $b(L-2)/16 \geq 20L$ ), we obtain

$$(5.15) \quad \begin{aligned} \left\| \mathcal{R} \mathbb{P}_{\geq \lambda_{q+1}\sigma/2}(\eta_{(\xi)}^2) \nabla a_{(\xi)}^2 \right\|_{L^p} &\lesssim \frac{1}{\ell^2 \lambda_{q+1} \sigma} \left( 1 + \frac{1}{\ell^L (\lambda_{q+1} \sigma)^{L-2}} \right) \|\eta_{(\xi)}^2\|_{L^p} \\ &\lesssim \frac{r^{3-3/p}}{\ell^2 \lambda_{q+1} \sigma}. \end{aligned}$$

Summing over  $\xi$  and  $i$  we lose a factor of  $\ell^{-1}$ . Lastly, applying  $\mathcal{R}$  to the last term on the right side of (5.14), the bound on each summand is a simple consequence of (3.19) and Lemma 4.4:

$$(5.16) \quad \begin{aligned} &\left\| \mathcal{R} \frac{1}{\mu} \sum_i \sum_{\xi \in \Lambda_{(i)}^+} \left( \mathbb{P}_{\neq 0} \partial_t(a_{(\xi)}^2) \eta_{(\xi)}^2 \xi \right) \right\|_{L^p} \\ &\lesssim \frac{1}{\mu} \sum_i \sum_{\xi \in \Lambda_{(i)}^+} \left\| \partial_t(a_{(\xi)}^2) \eta_{(\xi)}^2 \xi \right\|_{L^p} \\ &\lesssim \frac{1}{\mu} \sum_i \sum_{\xi \in \Lambda_{(i)}^+} \|a_{(\xi)}\|_{C_{t,x}^1} \|a_{(\xi)}\|_{L^\infty} \|\eta_{(\xi)}\|_{L^{2p}}^2 \\ &\lesssim \frac{1}{\mu} \sum_i \ell^{-1} \delta_{q+1}^{1/2} 2^i r^{3-3/p} \\ &\lesssim \frac{\ell^{-2} r^{3-3/p}}{\mu} \lesssim \frac{r^{3-3/p}}{\ell^2 \lambda_{q+1} \sigma}, \end{aligned}$$

where we have used that  $\lambda_{q+1}\sigma \leq \mu$ . Using (5.8), the bounds (5.15) and (5.16) are consistent with (5.2), which concludes the proof of Proposition 5.1.

## 6. The energy iterate

LEMMA 6.1. *For all  $t$  and  $t'$  satisfying  $|t - t'| \leq 2\ell$ , and all  $i \geq 0$ , we have*

$$(6.1) \quad |e(t') - e(t'')| \lesssim \ell^{1/2},$$

$$(6.2) \quad \left| \int_{\mathbb{T}^3} |v_q(x, t)|^2 dx - \int_{\mathbb{T}^3} |v_q(x, t')|^2 dx \right| \lesssim \ell^{1/2},$$

$$(6.3) \quad \left| \int_{\mathbb{T}^3} \chi_i^2(x, t) - \chi_i^2(x, t') dx \right| \lesssim \ell^{1/2},$$

$$(6.4) \quad |\rho(t) - \rho(t')| \lesssim \ell^{1/2}.$$

*Proof of Lemma 6.1.* In the proof of the lemma, we crudely use a factor of  $\lambda_q$  to absorb constants. First note that (6.1) follows immediately from the assumed estimate  $\|e\|_{C_t^1} \leq M_e$ . Using (2.2) we have

$$\left| \int_{\mathbb{T}^3} |v_q(x, t)|^2 dx - \int_{\mathbb{T}^3} |v_q(x, t')|^2 dx \right| \lesssim \ell \|v_q\|_{C_{t,x}^1}^2 \lesssim \lambda_q^8 \ell,$$

which implies (6.2). The estimate (6.3) follows in a similar fashion, from Lemma 4.2. Finally, (6.4) follows directly from the definition of  $\rho(t)$ , (4.34) and the bounds (6.1)–(6.3) above.  $\square$

LEMMA 6.2. *If  $\rho_0(t) \neq 0$ , then the energy of  $v_{q+1}$  satisfies the following estimate:*

$$(6.5) \quad \left| e(t) - \int_{\mathbb{T}^3} |v_{q+1}(x, t)|^2 dx - \frac{\delta_{q+2}}{2} \right| \leq \frac{\delta_{q+2}}{4}.$$

Note that the above lemma implies that if  $\rho_0(t) \neq 0$ , then

$$e(t) - \int_{\mathbb{T}^3} |v_q(x, t)|^2 dx > \frac{\delta_{q+1}}{100},$$

and thus (2.6) is an empty statement for such times.

*Proof of Lemma 6.2.* By definition we have

$$(6.6) \quad \begin{aligned} \int_{\mathbb{T}^3} |v_{q+1}(x, t)|^2 dx &= \int_{\mathbb{T}^3} |v_\ell(x, t)|^2 dx \\ &\quad + 2 \int_{\mathbb{T}^3} w_{q+1}(x, t) \cdot v_\ell(x, t) dx + \int_{\mathbb{T}^3} |w_{q+1}(x, t)|^2 dx. \end{aligned}$$

Using (4.14), similarly to (4.23), we have that

$$\begin{aligned} \int_{\mathbb{T}^3} |w_{q+1}^{(p)}(x, t)|^2 dx - 3 \sum_{i \geq 0} \rho_i \int_{\mathbb{T}^3} \chi_{(i)}^2(x, t) dx \\ = \sum_{i, j \geq 0} \sum_{\xi \in \Lambda_{(i)}, \xi' \in \Lambda_{(j)}} \int_{\mathbb{T}^3} a_{(\xi)} a_{(\xi')} \mathbb{P}_{\geq \lambda_{q+1}\sigma/2}(\mathbb{W}_{(\xi)} \cdot \mathbb{W}_{(-\xi')}) dx =: E_\rho(t). \end{aligned}$$

Using the standard integration by parts argument

$$\left| \int_{\mathbb{T}^3} f \mathbb{P}_{\geq \mu} g dx \right| = \left| \int_{\mathbb{T}^3} |\nabla|^L f |\nabla|^{-L} \mathbb{P}_{\geq \mu} g dx \right| \lesssim \|g\|_{L^2} \mu^{-L} \|f\|_{C^L},$$

with  $L$  sufficiently large, we obtain from (4.38), since  $\ell^{-1} \ll \lambda_{q+1}\sigma$ , that

$$(6.7) \quad \left| \int_{\mathbb{T}^3} |w_{q+1}^{(p)}(x, t)|^2 dx - 3 \sum_{i \geq 0} \rho_i \int_{\mathbb{T}^3} \chi_{(i)}^2(x, t) dx \right| = |E_\rho(t)| \leq \ell^{1/2}.$$

We consider two sub-cases:  $\rho(t) \neq 0$  and  $\rho(t) = 0$ . First consider the case  $\rho(t) \neq 0$ ; then using the definition of  $\rho$ , we obtain

$$\begin{aligned} & 3 \sum_{i \geq 0} \rho_i \int_{\mathbb{T}^3} \chi_{(i)}^2(x, t) dx \\ &= 3\rho(t) \int_{\mathbb{T}^3} \chi_{(0)}^2(x, t) dx \\ &+ 3(\rho_0(t) - \rho(t)) \int_{\mathbb{T}^3} \chi_{(0)}^2(x, t) dx + 3 \sum_{i \geq 1} \rho_i \int_{\mathbb{T}^3} \chi_{(i)}^2(x, t) dx \\ &= e(t) - \int_{\mathbb{T}^3} |v_q(x, t)|^2 dx + 3(\rho_0(t) - \rho(t)) \int_{\mathbb{T}^3} \chi_0^2(x, t) dx - \frac{\delta_{q+2}}{2}. \end{aligned}$$

For the case that  $\rho(t) = 0$ , we have that by continuity for some  $t' \in (t - \ell, t + \ell)$ ,

$$e(t') - \int_{\mathbb{T}^3} |v_q(x, t')|^2 dx - 3 \sum_{i \geq 1} \rho_i \int_{\mathbb{T}^3} \chi_i^2(x, t') dx - \frac{\delta_{q+2}}{2} = 0.$$

Thus applying [Lemma 6.1](#) we conclude that for either case  $\rho(t) \neq 0$  or  $\rho(t) = 0$ ,

$$(6.8) \quad \left| 3 \sum_{i \geq 0} \rho_i \int_{\mathbb{T}^3} \chi_{(i)}^2(x, t) dx - e(t) + \int_{\mathbb{T}^3} |v_q(x, t)|^2 dx + \frac{\delta_{q+2}}{2} \right| \lesssim \ell^{1/2}.$$

When  $\rho(t) \neq 0$ , in the above estimate we have used the bound  $|\rho_0(t) - \rho(t)| \lesssim \ell^{1/2}$ , which follows from the definition of  $\rho_0$  in [\(4.26\)](#) and the estimate [\(6.4\)](#) established earlier.

Observe that using [\(2.2\)](#), the definition of  $v_\ell$ , and [\(4.7\)](#) we have

$$(6.9) \quad \left| \int_{\mathbb{T}^3} |v_q(x, t)|^2 dx - \int_{\mathbb{T}^3} |v_\ell(x, t)|^2 dx \right| \lesssim \|v_q\|_{L^\infty} \|v_\ell - v_q\|_{L^\infty} \lesssim \lambda_q^8 \ell \lesssim \ell^{1/2}.$$

Further, using also [\(3.13\)](#), [\(4.8\)](#), [\(4.38\)](#), and integration by parts, we obtain

$$\begin{aligned} \left| \int_{\mathbb{T}^3} w_{q+1}(x, t) \cdot v_\ell(x, t) dx \right| &\leq \|w_{q+1}^{(c)} + w_{q+1}^{(t)}\|_{L^2} \|v_\ell\|_{C^1} \\ &+ \sum_{i \geq 0} \sum_{\xi \in \Lambda_{(i)}} \left| \int_{\mathbb{T}^3} a_{(\xi)} \mathbb{W}_{(\xi)}(x, t) \cdot v_\ell(x, t) dx \right| \\ &\lesssim \ell^{-2} r^{3/2} \mu^{-1} + \sum_{i \geq 0} \sum_{\xi \in \Lambda_{(i)}} \lambda_{q+1}^{-N} \|a_{(\xi)} v_\ell\|_{C^N} \\ &\lesssim \ell^{-2} \lambda_{q+1}^{-1/8} + \lambda_{q+1}^{-N} \ell^{-N-2}. \end{aligned}$$

In the last line we have used [\(4.15\)](#) and the fact that summing over  $\xi$  and  $i$  costs at most an extra  $\ell^{-1}$ . Taking  $N$  sufficiently large, we obtain

$$(6.10) \quad \left| \int_{\mathbb{T}^3} w_{q+1}(x, t) \cdot v_\ell(x, t) dx \right| \lesssim \ell^{-2} \lambda_{q+1}^{-1/8} \lesssim \ell^{1/2},$$

upon taking  $b \geq 400$ . Using (4.40) and (4.41) yields

$$\begin{aligned}
 (6.11) \quad & \left| \int_{\mathbb{T}^3} |w_{q+1}(x, t)|^2 dx - \int_{\mathbb{T}^3} |w_{q+1}^{(p)}(x, t)|^2 dx \right| \\
 & \lesssim \left( \|w_{q+1}\|_{L^2} + \|w_{q+1}^{(p)}\|_{L^2} \right) \left( \|w_{q+1}^{(c)}\|_{L^2} + \|w_{q+1}^{(t)}\|_{L^2} \right) \\
 & \lesssim \delta_{q+1} \ell^{-1} \mu^{-1} \\
 & \lesssim \ell^{1/2}.
 \end{aligned}$$

Thus we conclude from (6.6), (6.7), (6.8), (6.10) and (6.11) that

$$\left| e(t) - \int_{\mathbb{T}^3} |v_{q+1}(x, t)|^2 dx - \frac{\delta_{q+2}}{2} \right| \lesssim \ell^{1/2},$$

from which (6.5) immediately follows.  $\square$

LEMMA 6.3. *If  $\rho_0(t) = 0$ , then  $v_{q+1}(\cdot, t) \equiv 0$ ,  $\dot{R}_{q+1}(\cdot, t) \equiv 0$  and*

$$(6.12) \quad e(t) - \int_{\mathbb{T}^3} |v_{q+1}(x, t)|^2 dx \leq \frac{3\delta_{q+2}}{4}.$$

*Proof of Lemma 6.3.* Since  $\rho_0(t) = 0$ , it follows from the definition of  $\rho_0$  and  $\rho$  that for all  $t' \in (t - \ell, t + \ell)$ , we have

$$e(t') - \int_{\mathbb{T}^3} |v_q(x, t')|^2 dx - 3 \sum_{i \geq 1} \rho_i \int_{\mathbb{T}^3} \chi_i^2(x, t') dx \leq \frac{\delta_{q+2}}{2}.$$

Using (4.35), this implies that

$$(6.13) \quad e(t') - \int_{\mathbb{T}^3} |v_q(x, t')|^2 dx - \frac{\delta_{q+2}}{2} \lesssim \lambda_q^{-\varepsilon_R} \delta_{q+1}.$$

Using that  $\lambda_q^{-\varepsilon_R}$  and the ratio  $\delta_{q+2}\delta_{q+1}^{-1}$  can absorb any constant, from (2.6) we conclude that  $v_q(\cdot, t') \equiv 0$  and  $\dot{R}_q(\cdot, t') \equiv 0$  for all  $t' \in (t - \ell, t + \ell)$ . Hence  $v_\ell(\cdot, t) \equiv 0$  and  $\dot{R}_\ell(\cdot, t) \equiv 0$ . This in turn implies that  $\chi_i(\cdot, t) \equiv 0$  for all  $i \geq 1$ . Since in addition  $\rho_0(t) = 0$ , it follows by (4.12) that  $a_{(\xi)}(\cdot, t) \equiv 0$  for all  $i \geq 0$ , and thus that  $w_{q+1}(\cdot, t) \equiv 0$ . Hence we have that  $v_{q+1}(\cdot, t) \equiv 0$ . Moreover, since the  $\{\chi_i(x, t)\}_{i \geq 1}$  and  $\rho_0^{1/2}(t)$  are nonnegative smooth functions, it follows that  $\partial_t \chi_i(\cdot, t) \equiv 0$  for all  $i \geq 1$  and that  $\partial_t \rho_0^{1/2}(t) = 0$ . Hence we also obtain  $\partial_t a_{(\xi)}(\cdot, t) \equiv 0$ , and from the definition of  $w_{q+1}$  we have  $\partial_t w_{q+1}(\cdot, t) \equiv 0$ . Since  $v_q$  vanishes on  $(t - \ell, t + \ell)$  and  $v_\ell, w_{q+1}, \partial_t w_{q+1}, \dot{R}_\ell$ , all vanish at time  $t$ , it follows from (4.3) and (5.7) that  $\widetilde{R}(\cdot, t) \equiv 0$ , and therefore  $\dot{R}_{q+1}(\cdot, t) \equiv 0$ .

Using (6.13) and (6.9) (note that  $\ell^{-1/4}$  may be used to absorb constants), we obtain

$$\begin{aligned}
 e(t) - \int_{\mathbb{T}^3} |v_{q+1}(x, t)|^2 dx &= e(t) - \int_{\mathbb{T}^3} |v_q(x, t)|^2 dx + \int_{\mathbb{T}^3} |v_q(x, t)|^2 dx - \int_{\mathbb{T}^3} |v_\ell(x, t)|^2 dx \\
 &\leq \frac{5\delta_{q+2}}{8} + \ell^{1/4} \leq \frac{3\delta_{q+2}}{4}.
 \end{aligned}$$

In the last inequality we used that  $\beta b^2$  is sufficiently small. Hence we obtain (6.12).  $\square$

We conclude this section by using Lemmas 6.2 and 6.3 to conclude (2.5) and (2.6) for  $q+1$ . Observe that the estimates (6.5) and (6.12), together imply (2.5) for  $q+1$ . From (6.5), if

$$e(t) - \int_{\mathbb{T}^3} |v_{q+1}(x, t)|^2 dx \leq \frac{\delta_{q+2}}{100},$$

then  $\rho_0(t) = 0$ . Hence from Lemma 6.3 we obtain that  $v_{q+1} \equiv 0$  and  $\dot{R}_{q+1} \equiv 0$ , from which we conclude (2.6).

### Appendix A. $L^p$ product estimate

*Proof of Lemma 3.7.* For convenience we give here the proof from [5]. We first consider the case  $p = 1$ . With these assumptions we have

$$\|fg\|_{L^1} \leq \sum_j \int_{T_j} |fg|,$$

where  $T_j$  are cubes of side-length  $\frac{2\pi}{\kappa}$ . For any function  $h$ , let  $\bar{h}_j$  denote its mean on the cube  $T_j$ . Observe that for  $x \in T_j$ , we have

$$\begin{aligned} |f(x)| &= |\bar{f}_j + f(x) - \bar{f}_j| \\ &\leq |\bar{f}_j| + \sup_{T_j} |f(x) - \bar{f}_j| \\ &\leq |\bar{f}_j| + \frac{2\pi\sqrt{3}}{\kappa} \sup_{T_j} |Df| \\ &\leq |\bar{f}_j| + \frac{2\pi\sqrt{3}}{\kappa} |\overline{Df_j}| + \frac{2\pi\sqrt{3}}{\kappa} \sup_{T_j} |Df - \overline{Df_j}| \\ &\leq |\bar{f}_j| + \frac{2\pi\sqrt{3}}{\kappa} |\overline{Df_j}| + \frac{6\pi}{\kappa^2} \sup_{T_j} |D^2 f| \\ &\leq |\bar{f}_j| + \frac{2\pi\sqrt{3}}{\kappa} |\overline{Df_j}| + \frac{6\pi}{\kappa^2} \sup_{T_j} |\overline{D^2 f_j}| + \frac{6\pi}{\kappa^2} \sup_{T_j} |D^2 f - \overline{D^2 f_j}|. \end{aligned}$$

Iterating this procedure  $M$  times we see that on  $T_j$  we have the pointwise estimate

$$|f| \leq \sum_{m=0}^M (2\pi\sqrt{3}\kappa^{-1})^m |\overline{D^m f_j}| + (2\pi\sqrt{3}\kappa^{-1})^M \|D^M f\|_{L^\infty}.$$

Upon multiplying the above by  $|g|$  and integrating over  $T_j$ , and then summing over  $j$ , we obtain

$$\begin{aligned} \|fg\|_{L^1(\mathbb{T}^3)} &\leq \sum_j \int_{T_j} \left( |g| \sum_{m=0}^M (2\pi\sqrt{3}\kappa^{-1})^m |\overline{D^m f_j}| \right) dx \\ &\quad + (2\pi\sqrt{3}\kappa^{-1})^M \|D^M f\|_{L^\infty} \|g\|_{L^1} \\ &\leq \sum_{m=0}^M (2\pi\sqrt{3}\kappa^{-1})^m \sum_j \frac{1}{|T_j|} \|D^m f\|_{L^1(T_j)} \|g\|_{L^1(T_j)} \\ &\quad + (2\pi\sqrt{3}\kappa^{-1})^M \|D^M f\|_{L^\infty} \|g\|_{L^1}. \end{aligned}$$

Since  $g$  is a  $T_j$ -periodic function, we have

$$\|g\|_{L^1(\mathbb{T}^3)} = \frac{|\mathbb{T}^3|}{|T_j|} \|g\|_{L^1(T_j)}$$

for any value of  $j$ , and since the interiors of the  $\{T_j\}$  are mutually disjoint, based on the assumption on the  $L^1$  cost of a derivative acting on  $f$  and the Sobolev embedding, we conclude from the above that (here we used the Sobolev embedding of  $W^{d+1,1} \subset L^\infty$ )

$$\begin{aligned} \|fg\|_{L^1(\mathbb{T}^3)} &\leq \frac{1}{|\mathbb{T}^3|} \|g\|_{L^1(\mathbb{T}^3)} \sum_{m=0}^M (2\pi\sqrt{3}\kappa^{-1})^m \|D^m f\|_{L^1(\mathbb{T}^3)} \\ &\quad + (2\pi\sqrt{3}\kappa^{-1})^M \|D^{M+4} f\|_{L^1} \|g\|_{L^1} \\ &\leq \frac{1}{|\mathbb{T}^3|} \|g\|_{L^1(\mathbb{T}^3)} \sum_{m=0}^M (2\pi\sqrt{3}\kappa^{-1})^m \lambda^m C_f \\ &\quad + (2\pi\sqrt{3}\kappa^{-1})^M \lambda^{M+4} C_f \|g\|_{L^1} \\ &\leq (1 + 2|\mathbb{T}^3|) C_f \|g\|_{L^1(\mathbb{T}^3)}. \end{aligned}$$

The case  $p = 2$  follows from the case  $p = 1$  applied to the functions  $f^2$  and  $g^2$ , and from the bound

$$\|D^m(f^2)\|_{L^1} \leq \left( \sum_{k=0}^m \binom{m}{k} \right) \|D^k f\|_{L^2} \|D^{m-k} f\|_{L^2} \leq \left( \sum_{k=0}^m \binom{m}{k} \right) \lambda^m C_f^2 = (2\lambda)^m C_f^2.$$

Here we are thus using that  $4\pi\sqrt{3}\lambda\kappa^{-1} \leq 2/3 < 1$  so that we have a geometric sum.  $\square$

## Appendix B. Commutator estimate

LEMMA B.1. *Fix  $\kappa \geq 1$ ,  $p \in (1, 2]$ , and a sufficiently large  $L \in \mathbb{N}$ . Let  $a \in C^L(\mathbb{T}^3)$  be such that there exists  $1 \leq \lambda \leq \kappa$ , and  $C_a > 0$  with*

$$(B.1) \quad \|D^j a\|_{L^\infty} \leq C_a \lambda^j$$

for all  $0 \leq j \leq L$ . Assume furthermore that  $\int_{\mathbb{T}^3} a(x) \mathbb{P}_{\geq \kappa} f(x) dx = 0$ . Then we have

$$(B.2) \quad \left\| |\nabla|^{-1} (a \mathbb{P}_{\geq \kappa} f) \right\|_{L^p} \lesssim \left( C_a \left( 1 + \frac{\lambda^L}{\kappa^{L-2}} \right) \right) \frac{\|f\|_{L^p}}{\kappa}$$

for any  $f \in L^p(\mathbb{T}^3)$ , where the implicit constant depend on  $p$  and  $L$ .

*Proof of Lemma B.1.* We have that

$$(B.3) \quad \begin{aligned} |\nabla|^{-1} (a \mathbb{P}_{\geq \kappa} f) &= |\nabla|^{-1} (\mathbb{P}_{\leq \kappa/2} a \mathbb{P}_{\geq \kappa} f) + |\nabla|^{-1} (\mathbb{P}_{\geq \kappa/2} a \mathbb{P}_{\geq \kappa} f) \\ &= (\mathbb{P}_{\geq \kappa/2} |\nabla|^{-1}) (\mathbb{P}_{\leq \kappa/2} a \mathbb{P}_{\geq \kappa} f) + |\nabla|^{-1} (\mathbb{P}_{\geq \kappa/2} a \mathbb{P}_{\geq \kappa} f). \end{aligned}$$

Note that  $\int_{\mathbb{T}^3} \mathbb{P}_{\geq \kappa/2} g(x) dx = 0$  for any function  $g$ , and thus the assumption that  $a \mathbb{P}_{\geq \kappa} f$  has zero mean on  $\mathbb{T}^3$  implies that  $\mathbb{P}_{\geq \kappa/2} a \mathbb{P}_{\geq \kappa} f$  also has zero mean on  $\mathbb{T}^3$ . We then use

$$\left\| |\nabla|^{-1} \mathbb{P}_{\geq \kappa/2} \right\|_{L^p \rightarrow L^p} \lesssim \frac{1}{\kappa},$$

which is a direct consequence of the Littlewood-Paley decomposition, and the bound

$$\left\| |\nabla|^{-1} \mathbb{P}_{\neq 0} \right\|_{L^p \rightarrow L^p} \lesssim 1,$$

which is a direct consequence of Schauder estimates (see [22]). Combining these facts and appealing to the embedding  $W^{1,4}(\mathbb{T}^3) \subset L^\infty(\mathbb{T}^3)$ , we obtain

$$\begin{aligned} \left\| |\nabla|^{-1} (a \mathbb{P}_{\geq \kappa} f) \right\|_{L^p} &\lesssim \frac{1}{\kappa} \left\| \mathbb{P}_{\leq \kappa/2} a \mathbb{P}_{\geq \kappa} f \right\|_{L^p} + \left\| \mathbb{P}_{\geq \kappa/2} a \mathbb{P}_{\geq \kappa} f \right\|_{L^p} \\ &\lesssim \left( \|a\|_{L^\infty} + \kappa \left\| D \mathbb{P}_{\geq \kappa/2} a \right\|_{L^4} \right) \frac{\|f\|_{L^p}}{\kappa} \\ &\lesssim \left( \|a\|_{L^\infty} + \kappa^{2-L} \left\| D^L \mathbb{P}_{\geq \kappa/2} a \right\|_{L^4} \right) \frac{\|f\|_{L^p}}{\kappa} \\ &\lesssim \left( \|a\|_{L^\infty} + \kappa^2 \frac{\left\| D^L a \right\|_{L^\infty}}{\kappa^L} \right) \frac{\|f\|_{L^p}}{\kappa}. \end{aligned}$$

The proof of (B.2) is concluded in view of assumption (B.1).  $\square$

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