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A note on a Newtonian approximation in a Schwarzschild background

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We consider a (very) simple version of the restricted three body problem in general relativity. The background geometry is given by a Schwarzschild solution governing the motion of two bodies of masses m_1 and m_2. We assume that corrections to the trajectory of the body of mass m_1 due to the presence of the mass m_2 are given by a Newtonian approximation where Poisson's equation is solved with respect to the Schwarzschild background geometry. Under these assumptions, we derive approximate equations of motion for the corrections to the trajectory of the body of mass m_1.

1. Introduction

In this note, we consider a simplified version of the restricted three-body problem in general relativity with orbital corrections to the farthermost body given in a Newtonian type framework, and with all interactions other than gravity neglected. More precisely, consider three point bodies of masses M, m_1, and m_2. Assume that $M \gg m_2 \gg m_1$, so that the dynamics of the body of mass M is unaffected by the presence of the other two masses, and the effect of the body of mass m_1 on that of mass m_2 is neglected as well. Let the bodies of masses m_1 and m_2 orbit the body of mass M along trajectories $y_1 = y_1(\tau)$ and $x_2 = x_2(\tau)$, respectively, where τ is an affine parameter. Assume further that the body of mass M is described by the Schwarzschild solution to Einstein's equations:

$$g_{\text{Sch}} = -\left(1-\frac{r_s}{r}\right)dt^2 + \left(1-\frac{r_s}{r}\right)^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2(\theta)d\phi^2 \qquad (1)$$

where t denotes a time parameter, (r,θ,ϕ) are spherical coordinates, and $r_s = 2GM$ (the Schwarzschild radius), with G being Newton's constant. Because of the ensuing assumptions, only the exterior region of the Schwarzschild solution is relevant here, so we can think of the body of mass M as a static black hole as well as a massive star where any interior dynamics is neglected. Finally, we assume that $y_1(\tau)$ and $x_2(\tau)$ form closed orbits that do not intersect, and that on each time slice $t = constant$ we have $\text{dist}_g(x_2(\tau), r_s) > 0$ and $\text{dist}_g\big(y_1(\tau), x_2(\tau)\big) \gg 1$, where g is the spatial metric induced on $t = constant$ time-slices, given by

$$g = \left(1-\frac{r_s}{r}\right)^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2(\theta)d\phi^2 \qquad (2)$$

and dist_g is the distance in the metric g. This situation is illustrated in Figure 1 below.

Under the above assumptions, the motion of the body of mass m_2 is given by a closed geodesic in the metric (1). The motion of the body of mass m_1 is given by a

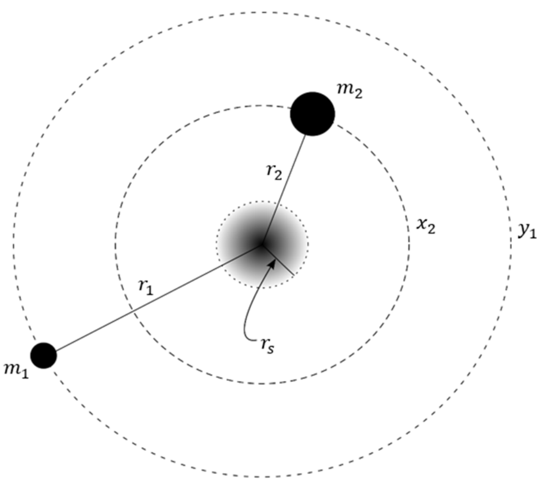

Figure 1. A schematic representation of the system.

closed geodesic in the metric (1) plus corrections due to the to the presence of the mass m_2. A simple derivation of the equations of motion governing these corrections is the goal of this note. We suppose that these corrections are given by nearly Newtonian dynamics of a test particle of mass m_1 subject to the gravitational attraction of the body of mass m_2, in the following sense. Write $y_1(\tau) = w_1(\tau) + x_1(\tau)$, where $w_1 = w_1(\tau)$ corresponds to the trajectory due to the presence of the mass M and $x_1 = x_1(\tau)$ due to the presence of the mass m_2. The form of w_1 is known (see below) and we are interested in determining a simple form for the equations of motion satisfied by x_1, at least approximately. By saying that x_1 is given by nearly Newtonian dynamics we mean that the acceleration of the body of mass m_1 due to the mass m_2 is obtained upon solving Poisson's equation for the Newtonian potential Ψ:

$$\Delta_g \Psi(x_1, x_2) = 4\pi q \delta(x_1 - x_2) \qquad (3)$$

but with the Laplacian in (3) given by the Laplacian in the metric (2) rather than the Euclidean Laplacian. Explicitly:

$$\Delta_g = \frac{1}{\sqrt{|g|}}\partial_\alpha\left(\sqrt{|g|}g^{\alpha\beta}\partial_\beta\right)$$

with $g_{\alpha\beta}$ the components of the metric (2), $g^{\alpha\beta}$ the components of the inverse of g, and $|g| = \det(g)$. Above, δ is the Dirac-delta function, and q is a coupling constant given by $q = m_2 G$. The equation of motion of the trajectory $x_1 = x_1(\tau)$ is then given by

$$\nabla_{\dot{x}_1}\dot{x}_1 = -\nabla\Psi(x_1, x_2) \tag{4}$$

where $\dot{} = \frac{d}{d\tau}$, and ∇ represents the covariant derivatives associated with (2). This situation obviously reduces to the example of a test body of mass m_1 attracted by m_2 when $M = 0$. The resulting (approximate) equations of motion are given in (12), (13), and *(14)*, for one particular approximation involving logarithmic terms, and (16), (17), and *(18)*, for another type of approximation involving powers of $\frac{1}{r}$.

We believe that considering an approximation of this type is very natural. To the best of our knowledge, however, it has not appeared in the literature, despite similar ideas in different settings [1, 2], and the extended literature on post-Newtonian approximations. (The literature on post-Newtonian approximation is quite large, thus a complete list cannot be given here. See, e.g., [3] and references therein for standard results, or [4, 5] for some more recent developments).

Hence, the main purpose of this note is to document that the approach here considered is a possible avenue to study general relativistic corrections in a simplified version of the three-body problem. Therefore, we have not striven to generality or applications, rather focusing on the equations of motions and some calculations that illustrate how corrections can be computed.

2. Geodesic and Newtonian dynamics

For a particle with position x with spherical coordinates (r, θ, ϕ) in the slice metric (2). For $\alpha, \beta, \gamma \in \{r, \theta, \phi\}$, we have the components of the acceleration $\nabla_{\dot{x}}\dot{x}$:

$$\nabla_{\dot{x}}\dot{x}^\alpha = \ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma$$

where $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols of the second kind associated with the metric (2), given by:

$$\Gamma^r = \begin{pmatrix} -\frac{r_s}{2r^2}\left(1-\frac{r_s}{r}\right)^{-1} & 0 & 0 \\ 0 & -r\left(1-\frac{r_s}{r}\right) & 0 \\ 0 & 0 & -r\sin^2(\theta)\left(1-\frac{r_s}{r}\right) \end{pmatrix} \quad \Gamma^\theta = \begin{pmatrix} 0 & \frac{1}{r} & 0 \\ \frac{1}{r} & 0 & 0 \\ 0 & 0 & -\sin(\theta)\cos(\theta) \end{pmatrix}$$

$$\Gamma^\phi = \begin{pmatrix} 0 & 0 & \frac{1}{r} \\ 0 & 0 & \cot(\theta) \\ \frac{1}{r} & \cot(\theta) & 0 \end{pmatrix}$$

The entries in the above matrices follow the order of the coordinates (r, θ, ϕ). By denoting the position x_1 in coordinates (r_1, θ_1, ϕ_1) and with the same convention for x_2, we can now write explicitly (4) as

$$\ddot{r}_1 - \frac{r_s}{2r_1^2}\left(1-\frac{r_s}{r_1}\right)^{-1}\dot{r}_1^2 - r_1\left(1-\frac{r_s}{r_1}\right)\dot{\theta}_1^2 - r_1\sin^2(\theta_1)\left(1-\frac{r_s}{r_1}\right)\dot{\phi}_1^2 = -\nabla^r\Psi(x_1, x_2) \tag{5}$$

$$\ddot{\theta}_1 + \frac{2}{r_1}\dot{r}_1\dot{\theta}_1 - \sin(\theta_1)\cos(\theta_1)\,\dot{\phi}_1^2 = -\nabla^\theta\Psi(x_1, x_2) \tag{6}$$

$$\ddot{\phi}_1 + \frac{2}{r_1}\dot{r}_1\dot{\phi}_1 + 2\cot(\theta_1)\,\dot{\theta}_1\dot{\phi}_1 = -\nabla^\phi\Psi(x_1, x_2) \tag{7}$$

As mentioned, the motion of the body of mass m_2 is given by a closed geodesic in the Schwarzschild background, which corresponds to solving equations (5), (6), and (7) with the indices 1 and 2 reversed and $\Psi \equiv 0$. Although the form of closed geodesics in a Schwarzschild background is well-known (see, e.g., [6]), we derive it here for the reader's convenience.

We are interested in closed orbits, and in this case, we can assume without loss of generality that $\theta_2 = \frac{\pi}{2}$.

The radial and angular part of the geodesic equation are given, respectively, by

$$\dot{r}_2^2 + \left(1-\frac{r_s}{r_2}\right)\left(1+\frac{L^2}{r_2^2}\right) = E^2 \tag{8}$$

$$\dot{\phi}_2 = \frac{L}{r_2^2} \tag{9}$$

where E and L are constants of motion (L is the orbital angular momentum).

Equation (8) can be viewed as the one-dimensional motion of a particle of unit mass subject to the potential

$$V(r) = \frac{1}{2}\left(1 - \frac{r_s}{r}\right)\left(1 + \frac{L^2}{r^2}\right)$$

evaluated at $r = r_2$. We see that V has a local minimum at

$$r_+ = \frac{L^2}{r_s}\left(1 + \frac{\sqrt{L^2 - 3r_s^2}}{L}\right)$$

provided that $L^2 - 3r_s^2 > 0$. Combining with (9) and our assumptions, we see that the sought closed orbit $x_2(\tau) = \left(r_2(\tau), \theta_2(\tau), \phi_2(\tau)\right)$ is given by

$$\begin{cases} r_2(\tau) = \dfrac{L^2}{r_s}\left(1 + \dfrac{\sqrt{L^2 - 3r_s^2}}{L}\right) \\ \theta_2(\tau) = \dfrac{\pi}{2}, \\ \phi_2(\tau) = \dfrac{r_s^2 \tau}{L^3\left(1 + \dfrac{\sqrt{L^2 - 3r_s^2}}{L}\right)^2} + \phi_2(0) \end{cases} \quad (10)$$

Write

$$A = \frac{r_s^2 \tau}{L^3\left(1 + \dfrac{\sqrt{L^2 - 3r_s^2}}{L}\right)^2}$$

If we take the initial position $\phi_2(0) = 0$, we can describe the ϕ_2 position in the orbit more succinctly as $\phi_2(\tau) = A\tau$. Notice that the trajectory $w_2(\tau)$ has the same form as (10), only with different constants of motion.

3. $\frac{1}{r}$ and logarithmic approximations for the Newtonian potential

We will consider two approximations for the equations of motion governing $x_1(\tau)$. One consists in an expansion of the first few terms of Ψ in powers of inverse of r_1. This seems to be the more natural expansion under our assumptions. We also consider, however, an expansion in logarithmic terms. This approximation is algebraically more complicated than the one involving powers of $\frac{1}{r_1}$, but it is in a sense more robust in that the series expansion of $\log\left(1 - \frac{r_s}{r_1}\right)$ contains terms in $\frac{1}{r_1}$ to all orders.

A general solution to (3) can be found via separation of variables and reads (see, e.g., [7])

$$\Psi(x_1, x_2) = \frac{8\pi q}{r_s} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{l,m}^*(\theta_2, \phi_2) \cdot Y_{l,m}(\theta_1, \phi_1) P_l\left(1 - \frac{2r_<}{r_s}\right) Q_l\left(1 - \frac{2r_>}{r_s}\right) \quad (11)$$

Above, P_l and Q_l are, respectively, Legendre functions of the first and second kind; $Y_{l,m}$ are spherical harmonics, * is complex conjugation, $r_< = \min\{r_1, r_2\}$, and $r_> = \max\{r_1, r_2\}$. Recall that we have the ordering in the r coordinate of the metric given by $r_s < r_2 \equiv r_+ = r_< < r_> = r_1$.

Using standard properties of Legendre functions (see, e.g., [8]), the Newtonian potential $\Psi(x_1, x_2)$ can also be written as

$$\Psi(x_1, x_2) = -q \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} Y_{l,m}^*(\theta_2, \phi_2) \cdot Y_{l,m}(\theta_1, \phi_1) \left(\sum_{j=0}^{\infty} \sum_{k=0}^{l} C_{k,j}^l r_s^{l+j-k} \frac{r_<^k}{r_>^{l+j+1}} \right)$$

where

$$C_{k,j}^l = \frac{(-1)^{l+k}(2l+1)(l+k)!\,[(l+j)!]^2}{(l-k)!\,(k!)^2 j!\,(2l+j+1)!}$$

4. Logarithmic approximation

Using the expression for Ψ given in (11), we will keep the only the terms $l = 0$ and $l = 1$ in (11)

The specific Legendre functions P_0, P_1, Q_0, Q_1 are given, for $z \in \mathbb{C}$, by

$$P_0(z) = 1, \quad Q_0(z) = \frac{1}{2}\mathrm{Log}\left(\frac{1+z}{1-z}\right)$$

$$P_1(z) = z, \quad Q_1(z) = \frac{z}{2}\mathrm{Log}\left(\frac{1+z}{1-z}\right)$$

Since $r_> = r_1$, the argument of Q_l is given by

$$\frac{1 + \left(1 - \frac{2r_1}{r_s}\right)}{1 - \left(1 - \frac{2r_1}{r_s}\right)} = \frac{2 - \frac{2r_1}{r_s}}{\frac{2r_1}{r_s}} = \frac{r_s - r_1}{r_1} = \frac{r_s}{r_1} - 1$$

and similarly, for the argument of P_l since $r_< = r_+$. Because $r_s < r_1$, we know that $\frac{r_s - r_1}{r_1} < 0$, so

$$\left|\frac{r_s}{r_1} - 1\right| = 1 - \frac{r_s}{r_1}$$

It remains to choose the correct branch of the Log function corresponding to the argument $\frac{r_s}{r_1} - 1$. Recall that for $z = re^{i\theta} \in \mathbb{C}$, we have that

$$\mathrm{Log}(z) = \ln|z| + i\mathrm{Arg}(z) = \ln(r) + i\theta.$$

We know that $\frac{r_s}{r_1} - 1$ is a purely real negative number, so then

$$\mathrm{Log}\left(\frac{r_s}{r_1}-1\right)=\ln\left|\frac{r_s}{r_1}-1\right|=\ln\left(1-\frac{r_s}{r_1}\right).$$

Therefore, we conclude that

$$Q_0\left(1-\frac{2r_1}{r_s}\right)=\frac{1}{2}\mathrm{Log}\left(\frac{r_s}{r_1}-1\right)=\frac{1}{2}\ln\left(1-\frac{r_s}{r_1}\right)$$

$$Q_1\left(1-\frac{2r_1}{r_s}\right)=\frac{1}{2}\left(1-\frac{2r_1}{r_s}\right)\mathrm{Log}\left(\frac{r_s}{r_1}-1\right)-1$$
$$=\frac{1}{2}\left(1-\frac{2r_1}{r_s}\right)\ln\left(1-\frac{r_s}{r_1}\right)-1$$

Now we state the well-known spherical harmonics $Y_{0,0}(\theta,\phi), Y_{1,-1}(\theta,\phi), Y_{1,0}(\theta,\phi), Y_{1,1}(\theta,\phi)$ and their conjugates:

$$Y_{0,0}(\theta,\phi)=Y_{0,0}^*(\theta,\phi)=\frac{1}{2}\sqrt{\frac{1}{\pi}}$$

$$Y_{1,-1}(\theta,\phi)=Y_{1,-1}^*(\theta,\phi)=\frac{1}{2}\sqrt{\frac{3}{\pi}}\sin(\theta)\sin(\phi)$$

$$Y_{1,0}(\theta,\phi)=Y_{1,0}^*(\theta,\phi)=\frac{1}{2}\sqrt{\frac{3}{\pi}}\cos(\theta)$$

$$Y_{1,1}(\theta,\phi)=Y_{1,1}^*(\theta,\phi)=\frac{1}{2}\sqrt{\frac{3}{\pi}}\sin(\theta)\cos(\phi)$$

Since we are taking $\theta_2=\frac{\pi}{2}$ and $\phi_2=A\tau$, we have the following:

$$Y_{0,0}^*(\theta_2,\phi_2)=\frac{1}{2}\sqrt{\frac{1}{\pi}}$$

$$Y_{1,-1}^*(\theta_2,\phi_2)=\frac{1}{2}\sqrt{\frac{3}{\pi}}\sin(A\tau)$$

$$Y_{1,0}^*(\theta_2,\phi_2)=0,$$

$$Y_{1,1}^*(\theta_2,\phi_2)=\frac{1}{2}\sqrt{\frac{3}{\pi}}\cos(A\tau)$$

Now we will compute the individual terms

$$S(l,m)\coloneqq Y_{l,m}^*(\theta_2,\phi_2)Y_{l,m}(\theta_1,\phi_1)$$
$$\cdot P_l\left(1-\frac{2r_<}{r_s}\right)Q_l\left(1-\frac{2r_>}{r_s}\right)$$

for $l=0,1$. Since $r_<=r_2$ and $r_>=r_1$ we substitute these values into the aforementioned terms and get the following:

$$S(0,0)=\frac{1}{8\pi}\log\left(1-\frac{r_s}{r_1}\right)$$

$$S(1,-1)=\frac{3}{4\pi}\sin(\theta_1)\sin(\phi_1)\sin(A\tau)$$
$$\cdot\left(1-\frac{2r_2}{r_s}\right)\left[\frac{1}{2}\left(1-\frac{2r_1}{r_s}\right)\log\left(1-\frac{r_s}{r_1}\right)-1\right]$$

$$S(1,0)=0,$$

$$S(1,1)=\frac{3}{4\pi}\sin(\theta_1)\cos(\phi_1)\cos(A\tau)$$
$$\left(1-\frac{2r_2}{r_s}\right)\left[\frac{1}{2}\left(1-\frac{2r_1}{r_s}\right)\log\left(1-\frac{r_s}{r_1}\right)-1\right]$$

Then

$$\sum_{l=0}^{1}\sum_{m=-l}^{l}S(l,m)=\frac{1}{8\pi}\log\left(1-\frac{r_s}{r_1}\right)+$$
$$\frac{3}{8\pi}\sin(\theta_1)\left(\sin(\phi_1)\sin(A\tau)+\cos(\phi_1)\cos(A\tau)\right)$$
$$\cdot\left[\left(1-\frac{2r_1}{r_s}\right)\log\left(1-\frac{r_s}{r_1}\right)-2\right]$$
$$=\frac{1}{8\pi}\log\left(1-\frac{r_s}{r_1}\right)+\frac{3}{8\pi}\sin(\theta_1)\cos(\phi_1-A\tau)$$
$$\cdot\left[\left(1-\frac{2r_2}{r_s}\right)\left(1-\frac{2r_1}{r_s}\right)\log\left(1-\frac{r_s}{r_1}\right)-2\right]$$

So our approximation for Ψ becomes

$$\Psi(x_1,x_2)\approx\frac{8\pi q}{r_s}\left(\frac{1}{8\pi}\log\left(1-\frac{r_s}{r_1}\right)\right.$$
$$+\frac{3}{8\pi}\sin(\theta_1)\cos(\phi_1-A\tau)\left(1-\frac{2r_2}{r_s}\right)$$
$$\left.\cdot\left[\left(1-\frac{2r_1}{r_s}\right)\log\left(1-\frac{r_s}{r_1}\right)-2\right]\right)$$
$$=-\frac{q}{r_s}\log\left(1-\frac{r_s}{r_1}\right)-\frac{3q}{r_s}\sin(\theta_1)\cos(\phi_1-A\tau)\left(1-\frac{2r_2}{r_s}\right)$$
$$\cdot\left[\left(1-\frac{2r_1}{r_s}\right)\log\left(1-\frac{r_s}{r_1}\right)-2\right]$$

We now wish to compute the components of the Schwarzschild gradient $\nabla\Psi$. In order to do this, we first compute

$$\frac{\partial\Psi}{\partial r_1}=\frac{q}{1-\frac{r_s}{r_1}}\frac{1}{r_1^2}-\frac{3q}{r_s}\sin(\theta_1)\cos(\phi_1-A\tau)\left(1-\frac{2r_2}{r_s}\right)$$
$$\cdot\left[-\frac{2}{r_s}\log\left(1-\frac{r_s}{r_1}\right)+\left(1-\frac{2r_1}{r_s}\right)\frac{1}{1-\frac{r_s}{r_1}}\frac{r_s}{r_1^2}\right]$$

$$\frac{\partial\Psi}{\partial\theta_1}=\frac{3q}{r_s}\cos(\theta_1)\cos(A\tau-\phi_1)\left(1-\frac{2r_2}{r_s}\right)$$
$$\cdot\left[\left(1-\frac{2r_1}{r_s}\right)\log\left(1-\frac{r_s}{r_1}\right)-2\right]$$

$$\frac{\partial\Psi}{\partial\phi_1}=\frac{3q}{r_s}\sin(\theta_1)\sin(A\tau-\phi_1)\left(1-\frac{2r_2}{r_s}\right)$$
$$\cdot\left[\left(1-\frac{2r_1}{r_s}\right)\log\left(1-\frac{r_s}{r_1}\right)-2\right]$$

Recall that the components of the Schwarzschild gradient are given by the following formulas:

$$\nabla^r\Psi=g^{rr}\frac{\partial\Psi}{\partial r}+g^{r\theta}\frac{\partial\Psi}{\partial\theta}+g^{r\phi}\frac{\partial\Psi}{\partial\phi}$$
$$\nabla^\theta\Psi=g^{\theta r}\frac{\partial\Psi}{\partial r}+g^{\theta\theta}\frac{\partial\Psi}{\partial\theta}+g^{\theta\phi}\frac{\partial\Psi}{\partial\phi}$$
$$\nabla^\phi\Psi=g^{\phi r}\frac{\partial\Psi}{\partial r}+g^{\phi\theta}\frac{\partial\Psi}{\partial\theta}+g^{\phi\phi}\frac{\partial\Psi}{\partial\phi}$$

Since the slice metric is given by

$$g=\begin{pmatrix}\left(1-\frac{r_s}{r}\right)^{-1} & 0 & 0\\ 0 & r^2 & 0\\ 0 & 0 & r^2\sin^2(\theta)\end{pmatrix}$$

we have that its inverse is

$$g^{-1}=\begin{pmatrix}1-\frac{r_s}{r} & 0 & 0\\ 0 & \frac{1}{r^2} & 0\\ 0 & 0 & \frac{1}{r^2\sin(\theta)}\end{pmatrix}$$

It follows that

$$\nabla^r\Psi(x_1,x_2)=\left(1-\frac{r_s}{r_1}\right)\frac{\partial\Psi}{\partial r_1}(x_1,x_2)$$

$$\nabla^\theta\Psi(x_1,x_2)=\frac{1}{r_1^2}\frac{\partial\Psi}{\partial\theta_1}(x_1,x_2)$$

$$\nabla^\phi\Psi(x_1,x_2)=\frac{1}{r_1^2\sin^2(\theta_1)}\frac{\partial\Psi}{\partial\phi_1}(x_1,x_2)$$

So combining these with our expressions for $\frac{\partial\Psi}{\partial r_1},\frac{\partial\Psi}{\partial\theta_1},\frac{\partial\Psi}{\partial\phi_1}$, we have that the components of the Schwarzschild gradient of $\Psi(x_1,x_2)$ are as follows:

$$\nabla^r\Psi(x_1,x_2)=\frac{q}{r_1^2}+\frac{3q}{r_s}\sin(\theta_1)\cos(\phi_1-A\tau)\left(1-\frac{2r_2}{r_s}\right)\cdot\left[-\frac{2}{r_s}\left(1-\frac{r_s}{r_1}\right)\log\left(1-\frac{r_s}{r_1}\right)+\left(1-\frac{2r_1}{r_s}\right)\frac{r_s}{r_1^2}\right]$$

$$\nabla^\theta\Psi(x_1,x_2)=\frac{3q}{r_s}\frac{(\cos(\theta_1)\cos(A\tau-\phi_1))}{r_1^2}\left(1-\frac{2r_2}{r_s}\right)\cdot\left[\left(1-\frac{2r_1}{r_s}\right)\log\left(1-\frac{r_s}{r_1}\right)-2\right]$$

$$\nabla^\phi(x_1,x_2)=\frac{3q}{r_s}\frac{\sin(A\tau-\phi_1)}{r_1^2\sin(\theta_1)}\left(1-\frac{2r_2}{r_s}\right)\cdot\left[\left(1-\frac{2r_1}{r_s}\right)\log\left(1-\frac{r_s}{r_1}\right)-2\right]$$

Combining the above with equations (5), (6), and (7), we finally obtain that $x_2(\tau)=(r_2(\tau),\theta_2(\tau),\phi_2(\tau))$ satisfies, in this approximation,

$$\begin{aligned}&\ddot r_1-\frac{r_s}{2r_1^2}\left(1-\frac{r_s}{r_1}\right)^{-1}\dot r_1^2-r_1\left(1-\frac{r_s}{r_1}\right)\dot\theta_1^2\\&-r_1\sin^2(\theta_1)\left(1-\frac{r_s}{r_1}\right)\dot\phi_1^2=-\frac{q}{r_1^2}\\&-\frac{3q}{r_s}\sin(\theta_1)\cos(\phi_1-A\tau)\left(1-\frac{2r_2}{r_s}\right)\\&\cdot\left[\left(1-\frac{2r_1}{r_s}\right)\log\left(1-\frac{r_s}{r_1}\right)-2\right]\end{aligned}\tag{12}$$

$$\begin{aligned}&\ddot\theta_1+\frac{2}{r_1}\dot r_1\dot\theta_1-\sin(\theta_1)\cos(\theta_1)\dot\phi_1^2\\&=-\frac{3q}{r_s}\frac{\sin(A\tau-\phi_1)}{r_1^2\sin(\theta_1)}\left(1-\frac{2r_2}{r_s}\right)\\&\cdot\left[\left(1-\frac{2r_1}{r_s}\right)\log\left(1-\frac{r_s}{r_1}\right)-2\right]\end{aligned}\tag{13}$$

$$\begin{aligned}&\ddot\phi_1+\frac{2}{r_1}\dot r_1\dot\phi_1+2\cot(\theta)\dot\theta_1\dot\phi_1\\&=-\frac{3q}{r_s}\frac{\sin(A\tau-\phi_1)}{r_1^2\sin(\theta_1)}\left(1-\frac{2r_2}{r_s}\right)\\&\left[\left(1-\frac{2r_1}{r_s}\right)\log\left(1-\frac{r_s}{r_1}\right)-2\right]\end{aligned}\tag{14}$$

5. $\frac{1}{r}$ approximation

Here we are concerned with the terms $\frac{1}{r^n}$ with least exponents $n\in\mathbb{N}$, we will approximate Ψ in (11) by summing $l,j=0,1$ only. In other words, this approximation is given by

$$\Psi(x_1,x_2)\approx-q\sum_{l=0}^{1}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}Y^*_{l,m}(\theta_2,\phi_2)\cdot Y_{l,m}(\theta_1,\phi_1)\left(\sum_{j=0}^{1}\sum_{k=0}^{l}C^l_{k,j}r_s^{l+j-k}\frac{r_<^k}{r_>^{l+j+1}}\right)$$

First, we focus on the term

$$\sum_{j=0}^{1}\sum_{k=0}^{l}C^l_{k,j}r_s^{l+j-k}\frac{r_<^k}{r_>^{l+j+1}}\tag{15}$$

Recall that $r_<=r_2$ and $r_>=r_1$, so this term becomes $\sum_{j=0}^1\sum_{k=0}^1C^1_{k,j}r_s^{1+j-k}\frac{r_2^k}{r_1^{1+j+1}}$. When $l=0$, (15) becomes

$$\sum_{j=0}^{1}\sum_{k=0}^{0}C^0_{k,j}r_s^{0+j-k}\frac{r_2^k}{r_1^{0+j+1}}=\sum_{j=0}^{1}C^0_{0,j}r_s^{j-0}\frac{r_2^0}{r_1^{j+1}}=C^0_{0,0}\frac{1}{r_1}+C^0_{0,1}r_s\frac{1}{r_1^2}$$

When $l=1$, (15) becomes

$$\begin{aligned}&\sum_{j=0}^{1}\sum_{k=0}^{1}C^1_{k,j}r_s^{1+j-k}\frac{r_2^k}{r_1^{1+j+1}}\\&=\sum_{j=0}^{1}\left(C^1_{0,j}r_s^{1+j-0}\frac{r_2^0}{r_1^{j+2}}+C^1_{1,j}r_s^{1+j-1}\frac{r_2^1}{r_1^{j+2}}\right)\\&=C^1_{0,0}r_s\frac{1}{r_1^2}+C^1_{1,0}\frac{r_2}{r_1^2}+C^1_{0,1}r_s^2\frac{1}{r_1^3}+C^1_{1,1}r_s\frac{r_2}{r_1^3}\end{aligned}$$

For the coefficients $C^1_{k,j}$, we have the following:

$$C_{0,0}^{0} = 1 \quad C_{0,1}^{0} = \frac{1}{2}$$
$$C_{0,0}^{1} = -\frac{1}{2} \quad C_{1,0}^{1} = 1$$
$$C_{0,1}^{1} = -\frac{1}{2} \quad C_{1,1}^{1} = 1$$

Further, we have the following products of spherical harmonics:

$$Y_{0,0}^{*}(\theta_2, \phi_2) Y_{0,0}(\theta_1, \phi_1) = \frac{1}{4\pi}$$
$$Y_{1,-1}^{*}(\theta_2, \phi_2) Y_{(1,-1)}(\phi_1, \phi_1) = \frac{3}{4\pi} \sin(A\tau) \sin(\theta_1) \sin(\phi_1)$$
$$Y_{1,0}^{*}(\theta_2, \phi_2) Y_{1,0}(\theta_1, \phi_1) = 0$$
$$Y_{1,1}^{*}(\theta_2, \phi_2) Y_{1,1}(\theta_1, \phi_1) = \frac{3}{4\pi} \cos(A\tau) \sin(\theta_1) \cos(\phi_1)$$

Now consider the terms in the sum of $\Psi(x_1, x_2)$. When $l = 0$:

$$4\pi Y_{0,0}^{*}(\theta_2, \phi_2) Y_{0,0}(\theta_1, \phi_1) \left(C_{0,0}^{0} \frac{1}{r_1} + C_{0,1}^{0} \frac{r_s}{r_1^2} \right)$$
$$= \frac{1}{r_1} + \frac{r_s}{2} \frac{1}{r_1^2}$$

When $l = 1$:

$$\sum_{m=-1}^{1} \frac{4\pi}{3} Y_{1,m}^{*}(\theta_2, \phi_2) Y_{(1,m)(\theta_1,\phi_1)}$$
$$\left(C_{0,0}^{1} r_s \frac{1}{r_1^2} + C_{1,0}^{1} \frac{r_2}{r_1^2} + C_{0,1}^{1} r_s^2 \frac{1}{r_1^3} + C_{1,1}^{1} r_s \frac{r_2}{r_1^3} \right)$$
$$= \frac{4\pi}{3} \big(Y_{(1,-1)}^{*}(\theta_2, \phi_2) Y_{1,-1}(\theta_1, \phi_1)$$
$$+ Y_{1,0}^{*}(\theta_2, \phi_2) Y_{1,0}(\theta_1, \phi_1) + Y_{1,1}^{*}(\theta_2, \phi_2) Y_{1,1}(\theta_1, \phi_1) \big)$$
$$\cdot \left(C_{0,0}^{1} \, r_s \frac{1}{r_1^2} + C_{1,0}^{1} \frac{r_2}{r_1^2} + C_{0,1}^{1} r_s^2 \frac{1}{r_1^3} + C_{1,1}^{1} r_s \frac{r_2}{r_1^3} \right)$$
$$= \sin(\theta_1) \cos(A\tau - \phi_1) \left(\frac{2r_2 - r_s}{2} \frac{1}{r_1^2} + \frac{2r_s r_2 - r_s^2}{2} \frac{1}{r_1^3} \right)$$

Our resulting approximation of $\Psi(x_1, x_2)$ becomes

$$\Psi(x_1, x_2) \approx -q \frac{1}{r_1} - q \frac{r_s}{2} \frac{1}{r_1^2}$$
$$-q \sin(\theta_1) \cos(A\tau - \phi_1)$$
$$\cdot \left(\frac{2r_2 - r_s}{2} \frac{1}{r_1^2} + \frac{2r_s r_2 - r_s^2}{2} \frac{1}{r_1^3} \right)$$

Now we have

$$\frac{\partial \Psi}{\partial r_1}(x_1, x_2) = \frac{q}{r_1^2} + q r_s \frac{1}{r_1^3}$$
$$+ q \sin(\theta_1) \cos(A\tau - \phi_1)$$
$$\cdot \left((2r_2 - r_s) \frac{1}{r_1^3} + \frac{3}{2} (2r_s r_2 - r_s^2) \frac{1}{r_1^4} \right)$$
$$\frac{\partial \Psi}{\partial \theta_1}(x_1, x_2) = -q \cos(\theta_1) \cos(A\tau - \phi_1)$$
$$\cdot \left(\frac{2r_2 - r_s}{2} \frac{1}{r_1^2} + \frac{2r_{sr_2} - r_s^2}{2} \frac{1}{r_1^3} \right)$$
$$\frac{\partial \Psi}{\partial \phi_1}(x_1, x_2) = -q \sin(\theta_1) \sin(A\tau - \phi_1)$$
$$\cdot \left(\frac{2r_2 - r_s}{2} \frac{1}{r_1^2} + \frac{2r_s r_2 - r_s^2}{2} \frac{1}{r_1^3} \right)$$

The components of the gradient are

$$\nabla^{r}(x_1, x_2) = \frac{q}{r_1^2} + q r_s \frac{1}{r_1^3} + q \sin(\theta_1) \cos(A\tau - \phi_1)$$
$$\cdot \left((2r_2 - r_s) \frac{1}{r_1^3} + \frac{3}{2} (2r_s r_2 - r_s^2) \frac{1}{r_1^4} \right)$$
$$-r_s q \frac{1}{r_1^3} - q r_s \frac{1}{r_1^4} - q r_s \sin(\theta_1) \cos(A\tau - \phi_1)$$
$$\cdot \left((2r_2 - r_s) \frac{1}{r_1^4} + \frac{3}{2} (2r_s r_2 - r_s^2) \frac{1}{r_1^5} \right)$$
$$= q \frac{1}{r_1^2} + q \sin(\theta_1) \cos(A\tau - \phi_1) (2r_2 - r_s) \frac{1}{r_1^3}$$
$$+ q \left(\frac{1}{2} r_s \sin(\theta_1) \cos(A\tau - \phi_1) (2r_2 - r_s) - r_s^2 \right) \frac{1}{r_1^4}$$
$$-q \left(\frac{3}{2} r_s^2 (2r_2 - r_s) \sin(\theta_1) \cos(A\tau - \phi_1) \right) \frac{1}{r_1^5}$$
$$\nabla^{\theta} \Psi(x_1, x_2) = -q \cos(\theta_1) \cos(A\tau - \phi_1)$$
$$\cdot \left(\frac{2r_2 - r_s}{2} \frac{1}{r_1^4} + \frac{2r_s r_2 - r_s^2}{2} \frac{1}{r_1^5} \right)$$
$$\nabla^{\phi} \Psi(x_1, x_2) = -q \frac{\sin(A\tau - \phi_1)}{\sin(\theta_1)} \left(\frac{2r_s - r_s}{2} \frac{1}{r_1^4} \right.$$
$$\left. + \frac{2r_s r_2 - r_s^2}{2} \frac{1}{r_1^5} \right)$$

Combining the above with equations (5), (6), and (7), we finally obtain that $x_2(\tau) = \big(r_2(\tau), \theta_2(\tau), \phi_2(\tau)\big)$ satisfies, in this approximation,

$$\ddot{r}_1 - \frac{r_s}{2r_1^2}\left(1-\frac{r_s}{r_1}\right)^{-1}\dot{r}_1^2 - r_1\left(1-\frac{r_s}{r_1}\right)\dot{\theta}_1^2 - r_1\sin^2(\theta_1)\left(1-\frac{r_s}{r_1}\right)\dot{\theta}_1^2 = -q\left(\frac{1}{2}r_s\sin(\theta_1)\cos(A\tau-\phi_1)(2r_2-r_s)-r_s^2\right)\frac{1}{r_1^4} + q\left(\frac{3}{2}r_s^2(2r_2-r_s)\sin(\theta_1)\cos(A\tau-\phi_1)\right)\frac{1}{r_1^5} \tag{16}$$

$$\ddot{\theta}_1 + \frac{2}{r_1}\dot{r}_1\dot{\theta}_1 - \sin(\theta_1)\cos(\theta_1)\dot{\phi}_1^2 = q\cos(\theta_1)\cos(A\tau-\phi_1)\cdot\left(\frac{2r_2-r_s}{2}\frac{1}{r_1^4}+\frac{2r_sr_2-r_s^2}{2}\frac{1}{r_1^5}\right) \tag{17}$$

$$\ddot{\phi}_1 + \frac{2}{r_1}\dot{r}_1\dot{\phi}_1 + 2\cot(\theta_1)\dot{\theta}_1\dot{\phi}_1 = q\frac{\sin(A\tau-\phi_1)}{\sin(\theta_1)}\cdot\left(\frac{2r_2-r_s}{2}\frac{1}{r_1^4}+\frac{2r_sr_2-r_s^2}{2}\frac{1}{r_1^5}\right) \tag{18}$$

Since $r_1 \gg r_2 > r_s$, we have that the distance between the bodies of masses m_1 and m_2 is of the order r_1. The highest contribution to (16), $\frac{q}{r_1^2}$, is therefore consistent with the Newtonian gravitational interactions between these bodies, as it should in light of our assumptions.

6. Discussion

In this work, we have derived an approximation for the orbital corrections to the equations of motion of the outermost object in a restricted three body-type of problem on a Schwarzschild background. The approximation is sufficiently simple to be treated by elementary methods, yet it provides insight on the behavior of such corrections without appealing to the heavy machinery of post-Newtonian approximations used to describe more realistic scenarios. Therefore, equations (12), (13), and (14) and (16), (17), and (18) provide a quick assessment of the qualitative behavior of the system that can be used as a starting point to more thorough and quantitative studies.

The method relies essentially on an exact solution to Poisson's equation for a Schwarzschild metric and can thus be adapted to other backgrounds where the Poisson equation can be solved exactly or approximately. In particular, as long as an approximate solution to Poisson's equation can be written in terms of a series expansion that asymptotes to an exact solution, a procedure similar to the one here presented can be used.

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