NON-UNIQUENESS OF LERAY-HOPF WEAK SOLUTIONS OF THE 3D HALL-MHD SYSTEM

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Abstract. Non-uniqueness of Leray-Hopf type of solutions is obtained for the three dimensional magneto-hydrodynamics with Hall effect. It seems to be the first result in the literature on non-uniqueness of Leray-Hopf weak solutions for dissipative equations. As for the proof, we adapted the widely appreciated convex integration framework developed in a recent work of Buckmaster and Vicol [5] for the Navier-Stokes equation, and with deep roots in a sequence of breakthrough papers for the Euler equation.

KEY WORDS: Hall-magneto-hydrodynamics; Leray-Hopf weak solutions; non-uniqueness; convex integration.
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1. Introduction

To capture the fast process of the magnetic reconnection phenomena in the nature, the following two-fluid model of the incompressible magneto-hydrodynamics (MHD) with Hall effect

$$\begin{aligned} u_t + u\cdot\nabla u - B\cdot\nabla B + \nabla p &= \Delta u,\\ B_t + u\cdot\nabla B - B\cdot\nabla u + \zeta\nabla\times((\nabla\times B)\times B) &= \Delta B,\\ \nabla\cdot u &= 0, \end{aligned} \tag{1.1}$$

was proposed by astrophysicists. In (1.1), u, p and B represent the fluid velocity field, the scalar pressure, and the magnetic field, respectively; they are the unknown functions on the spacial-time domain $\Omega\times[0,\infty)$. In the present paper, we take $\Omega=\mathbb{T}^3$. The parameter ζ in front of the Hall term indicates the strength of the Hall effect. For mathematical study on this model which was started not a long time ago, we refer to [1, 8, 9, 10, 12, 14, 17, 21] and references therein.

We notice that system (1.1) with $\zeta = 0$ is the usual MHD model. In this case, one also observes that the magnetic field equation is essentially linear in B, while the velocity equation is obviously the Navier-Stokes equation (NSE) with a Lorentz force term. Due to the linear character of the magnetic field equation, it is expected that the properties of solutions to the MHD system do not seriously deviate from those of the solutions to the NSE. In fact, a vast amount of work for the MHD and the NSE have shown this consistence.

However, for the Hall MHD system (1.1) with $\zeta > 0$, the situation is drastically different, comparing to the usual MHD system. On one hand, the equation of B is nonlinear with a strong nonlinear Hall term which is actually more singular than $u\cdot\nabla u$ in the NSE; on the other hand, a natural scaling does not exist for the Hall MHD system, while the MHD system shares the same natural scaling as for the

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NSE. More discussion on the scaling analysis will be provided at a later point. Due to the obvious difference of the two systems, a natural question is that: how does the presence of the Hall term change the behavior of solutions? Since the Hall term is more singular than other nonlinear terms in the system, one expectation is that it is probably more approachable to construct wild solutions and to discover severe ill-posedness for the Hall-MHD system. Searching wild solutions and justifying ill-posedness for fluid equations remains mathematically interesting and physically important before one can give a positive answer to the global regularity problem of these equations.

As for the 3D NSE, Leray's conjecture regarding the appearance of singularity at finite time has been a long-standing open problem; the uniqueness of Leray-Hopf weak solutions is not known either. Since the time of these problems raised in 1930s, much effort has been taken to tackle them from the negative side in the means of constructing blow-up solutions, wild solutions, or wild data-to-solution maps. In [25], Jia and Šverák showed non-uniqueness of Leray-Hopf weak solutions in $L^\infty(L^{3,\infty})$ with the assumption that certain spectral condition holds for a linearized Navier-Stokes operator. For an averaged NSE with modified nonlinear term, Tao constructed a smooth solution which blows up in finite time in [29]; moreover, the author proposed a program for adapting the blowup construction to the true NSE. Very recently, in the groundbreaking paper [5], Buckmaster and Vicol constructed nontrivial weak solutions for the 3D NSE by developing the convex integration scheme using intermittent Beltrami flows, which leads to non-uniqueness of weak solutions. It is a significant progress towards settling the problem of non-uniqueness of Leray-Hopf weak solutions, although the weak solutions constructed there belong to the space $C^0(0,T;H^\beta(\mathbb{T}^3))$ for a very small number $\beta>0$.

The convex integration method developed in [5] dates back to a sequence of breakthrough work for the Euler equation in the last decade. It was first introduced by De Lellis and Székelyhidi in [19, 20] to study the non-uniqueness of weak solutions and the existence of dissipative continuous solutions for the Euler equation. The framework of convex integration was further developed in [2, 3, 18, 23] and eventually leads to a complete resolution of the second half of Onsager's conjecture by Isett [24], and Buckmaster, De Lellis, Székelyhidi and V. Vicol [4].

Back to the dissipative equations, as mentioned above, the non-uniqueness of Leray-Hopf weak solutions to the 3D NSE is still open. Following the convex integration method in [5], one may expect to construct non-trivial solutions in $C^0(H^\beta)$ for $\beta<1/2$ and close enough to $1/2$; while crossing $1/2$ spacial regularity would be a major barrier. The reason is that $\dot H^{1/2}$ is critical for the 3D NSE, in which the regularity implies uniqueness. When the dissipation is weak, as for the hyperviscous Navier-Stokes equation with fractional Laplacian $(-\Delta)^\theta$ with $\theta\in(0,1/5)$ in [13], Colombo, De Lellis and De Rosa showed the non-uniqueness of Leray weak solutions, that is, solutions with finite energy and in the space $C^0(H^\theta)$.

Regarding the hyperviscous NSE with $\theta<5/4$, adapting the convex integration techniques of [5], Luo and Titi in [28] established the non-uniqueness of weak solutions by slightly simplifying the original construction of Buckmaster and Vicol. In another work [6], Buckmaster, Colombo, and Vicol constructed wild solutions for the 3D NSE, whose singular set in time has Hausdorff dimension strictly less than 1. Moreover, the result holds for the hyperviscous NSE with $\theta<5/4$ as well. Thus, along with the uniqueness result for $\theta \geq 5/4$ by Lions [27], the work of [28] and [6] indicates the well-posedness criticality of the exponent $\theta = 5/4$.

The main purpose of this paper is to address the problem of non-uniqueness of Leray-Hopf weak solutions for the Hall MHD system (1.1) with $\zeta > 0$. A scaling analysis will be helpful to demonstrate why it is approachable to study this problem by adapting the convex integration techniques. We first look at the MHD system, that is (1.1) with $\zeta = 0$. If the triplet $(u(x,t), B(x,t), p(x,t))$ solves the MHD system with data $(u_0(x), B_0(x))$, the following scaled functions

$$u_\lambda = \lambda u(\lambda x, \lambda^2 t), \quad B_\lambda = \lambda B(\lambda x, \lambda^2 t), \quad p_\lambda = \lambda^2 p(\lambda x, \lambda^2 t), \tag{1.2}$$

solve the MHD system as well with scaled data $(\lambda u_0(\lambda x), \lambda B_0(\lambda x))$. In the case of vanishing magnetic field B, such scaling holds for the NSE. Under scaling (1.2), the space $H^{1/2}(\mathbb{T}^3) \times H^{1/2}(\mathbb{T}^3)$ is critical for the 3D MHD system. It is known that regularity and hence uniqueness holds in subcritical spaces H^s with $s > 1/2$. Since the MHD system with zero magnetic field reduces to the NSE, the non-uniqueness result of the 3D NSE in [5] immediately provides a proof of non-uniqueness of weak solutions for the 3D MHD system. Similarly as for the 3D NSE, the uniqueness of Leray-Hopf solutions to the 3D MHD remains an open problem. The attempt to construct non-unique Leray-Hopf solutions via the convex integration method might not succeed since the criticality of 1/2 spacial regularity would be a crucial obstacle to overcome.

Now we turn to the Hall MHD system (1.1) with $\zeta > 0$, a natural scaling no longer holds due to the presence of the Hall term $\nabla \times ((\nabla \times B) \times B)$. One can see that the Hall term is more singular than other nonlinear terms in the system and the most singular one in the magnetic equation. This motivates us to consider the magnetic equation with vanishing velocity field as the first step. Thus we analyze the so-called Hall equation,

$$B_t + \nabla \times ((\nabla \times B) \times B) = \Delta B \tag{1.3}$$

which has the natural scaling

$$B_\lambda = B(\lambda x, \lambda^2 t). \tag{1.4}$$

We observe that if $\nabla \cdot B(x,0) = 0$, $\nabla \cdot B(x,t) = 0$ holds for all time $t > 0$. The basic energy law for the Hall equation (1.3) is

$$\frac{1}{2}\frac{d}{dt}\|B(t)\|_{L^2}^2 + \|\nabla B\|_{L^2}^2 = 0.$$

A Leray-Hopf type of weak solution to (1.3) is a function $B \in L^\infty(L^2) \cap L^2(H^1)$ which satisfies (1.3) in the distributional sense. On the other hand, under scaling (1.4), the Sobolev space $\dot{H}^{3/2}$ (the same as $H^{3/2}$ on periodic domains) is critical for (1.3) in three dimensions. One can expect global regularity of solution in $\dot{H}^{3/2}$ and uniqueness in the space as a consequence. Since the critical index 3/2 of regularity is larger than the Leray-Hopf weak solution regularity index 1, non-uniqueness of Leray-Hopf weak solutions in $C^0(H^1)$ constructed by the convex integration method would not contradict with anything according to the scaling properties.

Inspired by the aforementioned analysis, we adapt the convex integration scheme to the Hall equation (1.3) and establish the first main result as follows.

Theorem 1.1. *For any nonnegative smooth function $E(t) : [0,T] \to \mathbb{R}_{\geq 0}$, there exists a weak solution $B \in L^\infty([0,T]; L^2(\mathbb{T}^3)) \cap C^0([0,T]; H^1(\mathbb{T}^3))$ to the Hall equation*

(1.3), such that

$$\int_{\mathbb{T}^3} |\nabla \times B|^2 \mathrm{d}x = E(t), \quad t \in [0, T].$$

The statement implies non-uniqueness of Leray-Hopf weak solutions of the Hall equation (1.3). Indeed, the vorticity of the weak solutions can have any nonnegative energy profiles and thus a constant (in particular, zero) is not the only weak solution.

Concerning the strategy to prove Theorem 1.1, we take the curl of the Hall equation and apply the convex integration method to the resulted equation of the current density $J = \nabla \times B$. Section 4 will be devoted to this purpose.

Once we have the convex integration scheme for the Hall equation, we turn to the coupled Hall MHD system. At each level of the convex integration which produces B_q, we solve the velocity field equation – the NSE with the Lorentz force $(B_q \cdot \nabla) B_q$. We show that there exists a Leray-Hopf weak solution u_q to the NSE based on the estimates on B_q. In the end, we illustrate that the sequence $\{u_q, B_q\}$ converges to a pair of functions $\{u, B\}$ which is a Leray-Hopf weak solution of the Hall MHD system (1.1). Therefore, we are able to prove the second main result stated below.

Theorem 1.2. *For any nonnegative smooth function $E(t) : [0, T] \to \mathbb{R}_{\geq 0}$, there exists a Leray-Hopf weak solution (u, B, p) to the Hall MHD system (1.1) with $\zeta > 0$ on $[0, T]$, such that*

$$\int_{\mathbb{T}^3} |\nabla \times B|^2 \mathrm{d}x = E(t), \quad t \in [0, T].$$

Analogously, Theorem 1.2 suggests $(0, 0, p)$ is not the only Leray-Hopf weak solution of (1.1). Thus we provide a construction of non-unique Leray-Hopf weak solutions for the 3D Hall MHD system. The proof of Theorem 1.2 will be laid out in Section 5.

We conclude this section by a few well-posedness results for the Hall MHD system. In a previous paper [14], the author showed that system (1.1) with $\zeta > 0$ is locally well-posed in the Sobolev space $H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$ with $s > n/2$. Eventually in [15], the author established the local well-posedness of the system in $H^s(\mathbb{R}^n) \times H^{s+1}(\mathbb{R}^n)$ with $s > n/2 - 1$, which appears to be optimal in regards to the scaling of the NSE and scaling (1.4) for the Hall equation.

2. Preliminaries

2.1. **Notation.** For the sake of brevity, we first fix some notations. We denote by: $A \lesssim B$ an estimate of the form $A \leq CB$ with an absolute constant C; $A \sim B$ an estimate of the form $C_1 B \leq A \leq C_2 B$ with absolute constants C_1, C_2.

2.2. **The Hall equation.** To analyze the effect of the Hall term, we first consider the Hall equation, which is recalled here

$$B_t + \nabla \times ((\nabla \times B) \times B) = \Delta B. \tag{2.5}$$

Note that $\nabla \cdot B(t) = 0$ for all $t \geq 0$ if $\nabla \cdot B(0) = 0$. It is easy to verify that a smooth solution of the Hall equation satisfies the energy identity,

$$\frac{1}{2} \frac{d}{dt} \|B(t)\|_{L^2}^2 + \|\nabla B(t)\|_{L^2}^2 = 0.$$

Definition 2.1. We say that B is a Leray-Hopf weak solution of (2.5), if for any $\varphi \in C_c^\infty([0,T]\times\mathbb{T}^3)$, the following integral equation

$$\int_0^T\int_{\mathbb{T}^3} B\cdot\varphi_t + (B\otimes B):\nabla\nabla\times\varphi \,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_{\mathbb{T}^3}\nabla B:\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t$$

is satisfied, and B belongs to $L^\infty(0,T;L^2(\mathbb{T}^3))\cap L^2(0,T;H^1(\mathbb{T}^3))$.

Note that the definition is valid, since the vector identity

$$(\nabla\times B)\times B = \nabla\cdot(B\otimes B)$$

is valid for divergence free vector field B. The existence of Leray-Hopf weak solutions to (2.5) is trivial; for instance, it can be established by the standard Galerkin's approximating method.

Taking curl on the Hall equation leads to

$$(\nabla\times B)_t + \nabla\times\nabla\times((\nabla\times B)\times B) = \Delta\nabla\times B.$$

By introducing the vorticity of the magnetic field, current density, $J=\nabla\times B$, we give two formulations of the equation. The first one reads as

$$J_t + \nabla\times\nabla\times(J\times B) = \Delta J. \tag{2.6}$$

By applying a few vector calculus identities, see Section 6, the current density equation can be formulated in a more symmetric way, namely

$$J_t + \nabla\cdot((\nabla\times J)\otimes B + B\otimes(\nabla\times J) - \nabla(J\times B)) - \nabla\frac{|J|^2}{2} = \Delta J. \tag{2.7}$$

In the rest of the paper, we will need to refer to both formulations to have a more complete vision of the structure of the Hall equation.

2.3. **Leray-Hopf weak solution of the Hall-MHD.**

Definition 2.2. We say that (u,p,B) is a Leray-Hopf weak solution of (1.1), if for any $\psi,\varphi\in C_c^\infty([0,T]\times\mathbb{T}^3)$, the following integral equations

$$\int_0^T\int_{\mathbb{T}^3} u\cdot\psi_t + (u\otimes u):\nabla\psi - (B\otimes B):\nabla\psi\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_{\mathbb{T}^3}\nabla u:\nabla\psi\,\mathrm{d}x\,\mathrm{d}t,$$

$$\int_0^T\int_{\mathbb{T}^3} B\cdot\varphi_t + (u\otimes B):\nabla\varphi - (B\otimes u):\nabla\varphi + \zeta(B\otimes B):\nabla\nabla\times\varphi\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^T\int_{\mathbb{T}^3}\nabla B:\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t$$

are satisfied; and

$$(u,B)\in\left(L^\infty(0,T;L^2(\mathbb{T}^3))\cap L^2(0,T;H^1(\mathbb{T}^3))\right)^2.$$

The existence of Leray-Hopf weak solutions of (1.1) can be found in [7].

2.4. **Estimates for periodic functions and anti-derivative operator.** The following lemma regards Hölder's inequality for two periodic functions with frequencies far apart.

Lemma 2.3. [5] *Assume integers $M, \kappa, \lambda \geq 1$ satisfy*

$$\frac{2\sqrt{3}\pi\lambda}{\kappa} \leq \frac{1}{3} \quad and \quad \lambda^4 \frac{(2\sqrt{3}\pi\lambda)^M}{\kappa^M} \leq 1.$$

Let $p \in \{1, 2\}$ and f be a $\mathbb{T}^3$-periodic function with the property: there exists a constant C_f such that

$$\|D^j f\|_{L^p} \leq C_f \lambda^j$$

holds for all $1 \leq j \leq M+4$. In addition, let g be a $(\mathbb{T}/\kappa)^3$-periodic function. Then the following inequality

$$\|fg\|_{L^p} \lesssim C_f \|g\|_{L^p}$$

holds, where the implicit constant is universal.

A type of commutator estimate for periodic functions is introduced below.

Lemma 2.4. [5] *Assume $\kappa \geq 1$, $p \in (1, 2]$ and $L \in \mathbb{N}$ is sufficiently large. Let function $a \in C^L(\mathbb{T}^3)$ be such that there exists $1 \leq \lambda \leq \kappa$ and $C_a > 0$ with*

$$\|D^j a\|_{L^\infty} \leq C_a \lambda^j$$

for all $0 \leq j \leq L$. Assume in addition that $\int_{\mathbb{T}^3} a(x)\mathbb{P}_{\geq\kappa} f(x)\, \mathrm{d}x = 0$. Then the estimate

$$\||\nabla|^{-1}(a\mathbb{P}_{\geq\kappa} f)\|_{L^p} \lesssim C_a \left(1 + \frac{\lambda^L}{\kappa^{L-2}}\right) \frac{\|f\|_{L^p}}{\kappa}$$

holds for any $f \in L^p(\mathbb{T}^3)$, where the implicit constant depends on p and L.

We also introduce an estimate for the symmetric anti-divergence operator.

Lemma 2.5. [19] *There exists a linear operator $\mathcal{R}$ of order -1, such that*

$$\nabla \cdot \mathcal{R}(u) = u - \fint_{\mathbb{T}^3} u\, \mathrm{d}x.$$

It satisfies the Calderon-Zygmund and Schauder estimates, for $1 < p < \infty$,

$$\|\mathcal{R}\|_{L^p \to W^{1,p}} \lesssim 1, \quad \|\mathcal{R}\|_{C^0 \to C^0} \lesssim 1, \quad \|\mathcal{R}\mathbb{P}_{\neq 0} u\|_{L^p} \lesssim \||\nabla|^{-1}\mathbb{P}_{\neq 0} u\|_{L^p}.$$

3. The Hall equation and intermittent Beltrami flows

In this part, we analyze the structure of the equation of the current density $J = \nabla \times B$ and lay out the intermittent Beltrami flows introduced in [5]. The analysis will reveal the fact that the equation of the current density is analogous to the NSE near the intermittent Beltrami flows.

3.1. **Analyzing the equation.** If we apply the convex integration scheme directly to equation (2.6), we would consider the approximating equation

$$\partial_t J_q + \nabla\times\nabla\times(J_q\times B_q) = \Delta J_q + \nabla\times\nabla\times M_q, \tag{3.8}$$

with $J_q = \nabla\times B_q$, and M_q being certain vector with the property that $M_q\to 0$ in an appropriate sense as $q\to\infty$. The main idea would be to construct building blocks for the increments $v_{q+1} = B_{q+1}-B_q$ and $w_{q+1} = J_{q+1}-J_q$, which give rise to a new pair (B_{q+1}, J_{q+1}) and consequently a new vector M_{q+1} according to equation (3.8) at the level of $q+1$. Most importantly, the construction should be designed in such a way that: at level $q+1$, the major contribution of nonlinear interaction to the new vector M_{q+1} cancels M_q; and hence the sequence $\{M_q\}$ converges to zero eventually.

However, we realize that it has certain advantages to apply the convex integration scheme to the slightly more symmetric equation (2.7). In fact, we will work with the approximating form of (2.7)

$$\partial_t J_q+\nabla\cdot((\nabla\times J_q)\otimes B_q+B_q\otimes(\nabla\times J_q)-\nabla(J_q\times B_q))-\nabla\frac{|J_q|^2}{2} = \Delta J_q+\nabla\cdot R_q, \tag{3.9}$$

where R_q is recognized as an error stress tensor. The main element is that we need to design building blocks for the increments v_{q+1} and w_{q+1}, which in turn yield the triplet $(B_{q+1}, J_{q+1}, R_{q+1})$ with the property: the significantly large part of R_{q+1} from the nonlinear interaction represented by $(\nabla\times J)\otimes B+B\otimes(\nabla\times J)-\nabla(J\times B)$ cancels the previous level of stress tensor R_q. A crucial observation is that:

- if we take $B = W(x)$ as the Beltrami wave defined in Section 3.2 and $J = \nabla\times W(x) = \lambda W(x)$, then we can verify
$$\nabla\cdot((\nabla\times J)\otimes B + B\otimes(\nabla\times J)-\nabla(J\times B)) - \nabla\frac{|J|^2}{2} = \nabla\cdot(J\otimes J);$$
- if we take $B = \eta(x,t)W(x)$ as the intermittent Beltrami wave and $J = \nabla\times(\eta(x,t)W(x))$, with an approriate choice of $\eta(t,x)$ defined in Section 3.2, one can make sure the difference
$$\left[\nabla\cdot((\nabla\times J)\otimes B + B\otimes(\nabla\times J)-\nabla(J\times B)) - \nabla\frac{|J|^2}{2}\right] - \nabla\cdot(J\otimes J) \tag{3.10}$$
is small.

This indicates that, the stationary Beltrami wave is a solution of the Hall equation; while near certain intermittent Beltrami waves, equation (2.7) is “close” to

$$J_t + \nabla\cdot(J\otimes J) = \Delta J$$

which is the NSE without a pressure term (or constant pressure). An important motivation we obain is that an analogous construction scheme by using the convex integration method as for the NSE in [5] would possibly lead to the non-uniqueness of weak solutions of equation (2.7) with $J\in C^0(0,T;H^\beta)$ for a small $\beta>0$; hence it implies $B\in C^0(0,T;\dot H^1)$ since B is divergence free. Of course, in our case, two functions J and B are simultaneously involved in the construction; and the relation $J = \nabla\times B = \nabla\times(\eta W)$ will generate many error terms. On the other hand, it is also crucial to determine how to apply the important geometric lemma in the current context.

We will describe the convex integration scheme in detail for equation (2.7) by considering its approximation sequence (3.9) in the following section.

3.2. **Building blocks.** We adapt the construction idea of [5] using intermittent Beltrami flows. While we have to keep in mind that, rather than dealing with one function satisfying the NSE, we deal with the pair (B, J) with $J = \nabla \times B$ satisfying (2.7) in our context.

We first fix ξ, A_ξ, B_ξ, and a_ξ as defined in [5]:

$$\xi \in \mathbb{S}^2 \cap \mathbb{Q}^3, \quad A_\xi \in \mathbb{S}^2 \cap \mathbb{Q}^3, \quad A_\xi \cdot \xi = 0, \quad A_\xi = A_{-\xi},$$
$$a_\xi \in \mathbb{C}, \quad \bar{a}_\xi = a_{-\xi},$$
$$B_\xi = \frac{1}{\sqrt{2}} \left(A_\xi + i\xi \times A_\xi \right).$$

The stationary Beltrami wave is taken as

$$W(x) = \sum_{\xi \in \Lambda} a_\xi W_\xi := \sum_{\xi \in \Lambda} a_\xi B_\xi e^{i\lambda \xi \cdot x},$$

where Λ is a given finite subset of $\mathbb{S}^2$ such that $\Lambda = -\Lambda$, and λ is an integer such that $\lambda\Lambda \subset \mathbb{Z}^3$. One can verify that $W(x)$ is real-valued and satifies

$$\nabla \cdot W = 0, \quad \nabla \times W = \lambda W, \quad \nabla \cdot (W \otimes W) = \nabla \frac{|W|^2}{2},$$
$$\fint_{\mathbb{T}^3} W \otimes W \, \mathrm{d}x = \frac{1}{2} \sum_{\xi \in \Lambda} |a_\xi|^2 (\mathrm{Id} - \xi \otimes \xi).$$

Lemma 3.1. [5] *For any $N \in \mathbb{N}$, we can find $\varepsilon_\gamma > 0$ and $\lambda > 1$ with the following property. Let $B_{\varepsilon_\gamma}(\mathrm{Id})$ be the ball of symmetric 3×3 matrices, centered at* Id *of radius ε_γ. There exists pairwise disjoint subsets*

$$\Lambda_\alpha \subset \mathbb{S}^2 \cap \mathbb{Q}^3, \quad \alpha \in \{1, ..., N\},$$

with $\lambda\Lambda_\alpha \in \mathbb{Z}^3$, and smooth positive functions

$$\gamma_\xi^\alpha \in C^\infty(B_{\varepsilon_\gamma}(\mathrm{Id})), \quad \alpha \in \{1, ..., N\}, \quad \xi \in \Lambda_\alpha,$$

with derivatives that are bounded independently of λ, such that:

1. *$\xi \in \Lambda_\alpha$ implies $-\xi \in \Lambda_\alpha$ and $\gamma_\xi^\alpha = \gamma_{-\xi}^\alpha$;*
2. *For each $R \in B_{\varepsilon_\gamma}(\mathrm{Id})$ we have the identity*

$$R = \frac{1}{2} \sum_{\xi \in \Lambda_\alpha} \left(\gamma_\xi^\alpha(R) \right)^2 (\mathrm{Id} - \xi \otimes \xi).$$

Next we describe the intermittent Beltrami flows by adding oscillations to the Beltrami waves. We start with the Dirichlet kernel D_n

$$D_n(x) = \sum_{\xi=-n}^{n} e^{ix\xi} = \frac{\sin((n + \frac{1}{2})x)}{\sin(\frac{x}{2})}$$

which satisfies for $p > 1$

$$\|D_n\|_{L^p} \sim n^{1-\frac{1}{p}}.$$

We define the lattice cube

$$\Omega_r := \{\xi = (j, k, l) : j, k, l \in \{-r, ..., r\}\}$$

and the 3D normalized Dirichlet kernel

$$D_r(x) := \frac{1}{(2r+1)^{\frac{3}{2}}} \sum_{\xi \in \Omega_r} e^{i\xi \cdot x}$$

satisfying

$$\|D_r\|_{L^2}^2 = (2\pi)^3, \quad \|D_r\|_{L^p} \lesssim r^{\frac{3}{2}-\frac{3}{p}}, \quad p > 1,$$

where the implicit constant depends only on p. The parameter r refers to the number of frequencies along edges of Ω_r.

We shall define a directed and rescaled periodic Dirichlet kernel with period $(\mathbb{T}/(\lambda\sigma))^3$. The small constant σ is chosen such that $\lambda\sigma \in \mathbb{N}$ which parameterizes the spacing between frequencies; and $\sigma r \ll 1$. We fix an integer $N_0 \geq 1$ such that

$$\{N_0\xi, N_0 A_\xi, N_0\xi \times A_\xi\} \subset N_0\mathbb{S}^2 \cap \mathbb{Z}^3$$

for all $\xi \in \Lambda_\alpha$ and $\alpha \in \{1, ..., N\}$. We also introduce a parameter $\mu \in (\lambda, \lambda^2)$, which adjusts the temporal oscillation. It is then ready to define the modified Dirichlet kernel

$$\eta_{\xi,\lambda,\sigma,r,\mu}(x,t) = D_r\left(\lambda\sigma N_0(\xi \cdot x + \mu t), \lambda\sigma N_0 A_\xi \cdot x, \lambda\sigma N_0(\xi \times A_\xi) \cdot x\right) \tag{3.11}$$

for $\xi \in \Lambda_\alpha^+$; while $\eta_{\xi,\lambda,\sigma,r,\mu}(x,t) = \eta_{-\xi,\lambda,\sigma,r,\mu}(x,t)$ for $\xi \in \Lambda_\alpha^-$. We take the short notation $\eta_\xi(x,t) = \eta_{\xi,\lambda,\sigma,r,\mu}(x,t)$. It is important to notice that

$$\mu^{-1}\partial_t \eta_\xi(x,t) = \pm(\xi \cdot \nabla)\eta_\xi(x,t), \ \forall \xi \in \Lambda_\alpha^\pm, \tag{3.12}$$

which is the crucial identity used to design temporal oscillation in the increments later.

One also observe that

$$\fint_{\mathbb{T}^3} \eta_\xi^2(x,t)\,\mathrm{d}x = \fint_{\mathbb{T}^3} D_r^2(x)\,\mathrm{d}x = 1, \quad \|\eta_\xi(\cdot,t)\|_{L^p} = \|D_r\|_{L^p} \lesssim r^{\frac{3}{2}-\frac{3}{p}}, \tag{3.13}$$

for all $1 < p \leq \infty$.

Now we are ready to introduce the intermittent wave $\mathbb{W}_\xi$:

$$\mathbb{W}_\xi(x,t) = \eta_\xi(x,t) B_\xi e^{i\lambda\xi\cdot x}. \tag{3.14}$$

It is worth to point out that $\mathbb{W}_\xi$ is supported on certain frequencies. Indeed, we have

$$\begin{aligned} \mathbb{P}_{\leq 2\lambda\sigma r N_0}\eta_\xi &= \eta_\xi, \\ \mathbb{P}_{\leq 2\lambda}\mathbb{P}_{\geq \lambda/2}\mathbb{W}_\xi &= \mathbb{W}_\xi, \\ \mathbb{P}_{\leq 4\lambda}\mathbb{P}_{\geq c_0\lambda}(\mathbb{W}_\xi \otimes \mathbb{W}_{\xi'}) &= \mathbb{W}_\xi \otimes \mathbb{W}_{\xi'} \end{aligned}$$

where c_0 is a small constant and $\xi' \neq -\xi$.

Another important fact regarding $\mathbb{W}_\xi$ is given by

$$\nabla \cdot (\mathbb{W}_\xi \otimes \mathbb{W}_{-\xi} + \mathbb{W}_{-\xi} \otimes \mathbb{W}_\xi) = \nabla\eta_\xi^2 - (\xi \cdot \nabla)\eta_\xi^2\xi = \nabla\eta_\xi^2 - \frac{\xi}{\mu}\partial_t\eta_\xi^2.$$

It is the main motivation that we need to include the temporal oscillation w_{q+1}^t into the construction later.

Different from the Beltrami wave $W_\xi(x) = B_\xi e^{i\lambda\xi\cdot x}$, the intermittent Beltrami wave $\mathbb{W}_\xi$ is not divergence free or an eigenfunction of curl, i.e.

$$\nabla \cdot \mathbb{W}_\xi \neq 0, \quad \nabla \times \mathbb{W}_\xi \neq \lambda\mathbb{W}_\xi.$$

Instead, we have

$$\begin{aligned} \nabla \cdot \mathbb{W}_\xi &= \nabla\eta_\xi \cdot W_\xi, \\ \nabla \times \mathbb{W}_\xi &= \lambda\mathbb{W}_\xi + \nabla\eta_\xi \times W_\xi. \end{aligned}$$

Parameters λ, σ, r, and μ will be chosen in an appropriate way such that $\nabla\eta_\xi \cdot W_\xi$ and $\nabla\eta_\xi \times W_\xi$ are sufficiently small.

For such intermittent Beltrami waves $\mathbb{W}_\xi$ and $\Lambda_\alpha, \varepsilon_\gamma, \gamma_\xi$ as in Lemma 3.1, we have the following geometric lemma, which is a key ingredient in the construction.

Lemma 3.2. [5] *Assume $a_\xi \in \mathbb{C}$ are constants satisfying $\bar{a}_\xi = a_{-\xi}$. The vector field*

$$\sum_\alpha \sum_{\xi\in\Lambda_\alpha} a_\xi \mathbb{W}_\xi(x)$$

is real valued. Moreover, for each matrix $R \in B_{\varepsilon_\gamma}(\mathrm{Id})$ we have

$$\sum_{\xi\in\Lambda_\alpha} (\gamma_\xi(R))^2 \fint_{\mathbb{T}^3} \mathbb{W}_\xi \otimes \mathbb{W}_{-\xi}\,\mathrm{d}x = \sum_{\xi\in\Lambda_\alpha} (\gamma_\xi(R))^2 B_\xi \otimes B_{-\xi} = R. \tag{3.15}$$

3.3. **Analogy of equation (2.7) with the NSE near intermittent Beltrami flows.** In this part, we further analyze the structure of the nonlinearity of equation (2.7) by comparing it with the NSE near the intermittent Beltrami flows introduced above. We can take the magnetic field B as

$$\mathbb{W}^B_\xi = \frac{1}{\lambda}\mathbb{W}_\xi = \frac{1}{\lambda}\eta_\xi W_\xi.$$

An important observation is that

$$\nabla \times \mathbb{W}^B_\xi = \eta_\xi W_\xi + \frac{1}{\lambda}\nabla\eta_\xi \times W_\xi = \mathbb{W}_\xi + \frac{1}{\lambda}\nabla\eta_\xi \times W_\xi$$

and

$$\|\frac{1}{\lambda}\nabla\eta_\xi \times W_\xi\|_{L^2} \lesssim \sigma r,$$

where the upper bound σr can be sufficiently small by choosing the parameters appropriately. We denote $\mathbb{W}_\varepsilon =: \frac{1}{\lambda}\nabla\eta_\xi \times W_\xi$ to be the small error term between $\nabla \times \mathbb{W}^B_\xi$ and $\mathbb{W}_\xi$. Thus we can naturally adapt $J = \nabla \times \mathbb{W}^B_\xi$.

Now we show that the difference (3.10) is actually small near the intermittent Beltrami flows. Namely, by taking $B = \mathbb{W}^B_\xi = \frac{1}{\lambda}\mathbb{W}_\xi$ and $J = \nabla \times \mathbb{W}^B_\xi = \mathbb{W}_\xi + \mathbb{W}_\varepsilon$, a straight forward computation shows the difference

$$\begin{aligned}
&\left[\nabla\cdot((\nabla\times J)\otimes \mathbb{W}^B_\xi + \mathbb{W}^B_\xi \otimes (\nabla\times J) - \nabla(J\times \mathbb{W}^B_\xi)) - \nabla\frac{|J|^2}{2}\right] - \nabla\cdot(J\otimes J)\\
=&\nabla\cdot((\nabla\times(\mathbb{W}_\xi+\mathbb{W}_\varepsilon))\otimes\frac{1}{\lambda}\mathbb{W}_\xi + \frac{1}{\lambda}\mathbb{W}_\xi\otimes(\nabla\times(\mathbb{W}_\xi+\mathbb{W}_\varepsilon)) - \nabla((\mathbb{W}_\xi+\mathbb{W}_\varepsilon)\times\frac{1}{\lambda}\mathbb{W}_\xi))\\
&-\nabla\frac{|\mathbb{W}_\xi+\mathbb{W}_\varepsilon|^2}{2} - \nabla\cdot((\mathbb{W}_\xi+\mathbb{W}_\varepsilon)\otimes(\mathbb{W}_\xi+\mathbb{W}_\varepsilon))\\
\sim&\nabla\cdot((\nabla\times\mathbb{W}_\xi)\otimes\frac{1}{\lambda}\mathbb{W}_\xi + \frac{1}{\lambda}\mathbb{W}_\xi\otimes(\nabla\times\mathbb{W}_\xi) - \nabla(\mathbb{W}_\xi\times\frac{1}{\lambda}\mathbb{W}_\xi)) - \nabla\frac{|\mathbb{W}_\xi|^2}{2} - \nabla\cdot(\mathbb{W}_\xi\otimes\mathbb{W}_\xi)\\
\sim&\nabla\cdot(\lambda\mathbb{W}_\xi\otimes\frac{1}{\lambda}\mathbb{W}_\xi + \frac{1}{\lambda}\mathbb{W}_\xi\otimes\lambda\mathbb{W}_\xi) - 2\nabla\cdot(\mathbb{W}_\xi\otimes\mathbb{W}_\xi)\\
=&0.
\end{aligned}$$

Thus, near the intermittent Beltrami flows $(B, J) = (\mathbb{W}^B_\xi, \nabla\times\mathbb{W}^B_\xi)$, equation (2.7) (the curl of the Hall equation) is indeed “close” to the NSE. Also, an obvious fact is that $J = \nabla\times B$ scales as the velocity field in the NSE. This is the main motivation to investigate the problem of non-uniqueness of Leray-Hopf weak solutions for the Hall-MHD system by adhering to what has been done for the NSE in [5]. Of course, new difficulties arise in the construction. In particular, rather than one function, involved here are a pair of functions B and J, which are related through $J = \nabla\times B$.

On the other hand, to apply the rigid geometric lemma, one has to regroup the nonlinear interactions in a suitable way such that error terms can be controlled. It is also non-trivial to determine how to introduce the temporal oscillation. In the end, to show non-uniqueness of Leray-Hopf weak solutions for the Hall-MHD system, we need to design a scheme of combining the convex integration method for the magnetic field equation and the classical regularity theory for the NSE. We will address all of these problems in the rest of the article.

4. Convex integration for the Hall equation

In this part, we adapt the convex integration method to construct Leray-Hopf weak solutions of the Hall equation with nonnegative energy profiles for the current density field. The main strategy is to design an iteration scheme for the approximating equation (3.9) illustrated in Proposition 4.1.

We start with fixing several parameters: for large enough constants $a \gg 1$ and $b \gg 1$, and small enough positive constant $\beta \ll 1$, we define:

$$\lambda_q = a^{b^q}, \quad \delta_q = \lambda_1^{3\beta}\lambda_q^{-2\beta}, \tag{4.16}$$

$$r = \lambda_{q+1}^{\frac{3}{4}}, \ \sigma = \lambda_q^{-\frac{15}{16}}, \ \mu = \lambda_{q+1}^{\frac{5}{4}}, \ \ell = \lambda_q^{-20}. \tag{4.17}$$

It is easy to see that $\lambda_{q+1} = \lambda_q^b$.

Proposition 4.1. *There exists an absolute constant $C > 0$ and a sufficiently small parameter ε_R depending on b and β such that the following inductive statement holds. Let (B_q, J_q, R_q) be a solution of the approximating equation (3.9) on $\mathbb{T}^3 \times [0,T]$ satisfying:*

$$\|B_q\|_{C^1_{x,t}} \le \lambda_q^3, \tag{4.18}$$

$$\|J_q\|_{C^1_{x,t}} \le \lambda_q^4, \tag{4.19}$$

$$0 \le E(t) - \int_{\mathbb{T}^3} |J_q|^2 \,\mathrm{d}x \le \delta_{q+1}, \tag{4.20}$$

and

$$E(t) - \int_{\mathbb{T}^3} |J_q|^2 \,\mathrm{d}x \le \frac{\delta_{q+1}}{100} \quad \textit{implies} \quad J_q(\cdot,t) \equiv 0 \textit{ and } R_q(\cdot,t) \equiv 0. \tag{4.21}$$

In addition, we assume

$$\nabla \cdot R_q = \nabla \cdot \tilde{R}_q + \nabla \times \nabla \times \tilde{M}_q + \nabla \cdot \nabla \tilde{Q}_q + \nabla \tilde{p}_{q+1} \tag{4.22}$$

for a traceless symmetric tensor $\tilde{R}_q$, vector field $\tilde{M}_q$ and $\tilde{Q}_q$, and a scalar pressure function $\tilde{p}_{q+1}$, which satisfy

$$\|\tilde{R}_q\|_{L^\infty(L^1)} + \|\tilde{M}_q\|_{L^\infty(L^1)} + \|\tilde{Q}_q\|_{L^\infty(L^1)} \le \lambda_q^{-\varepsilon_R}\delta_{q+1}, \tag{4.23}$$

$$\|R_q\|_{C^1_{x,t}} \le \lambda_q^{12}. \tag{4.24}$$

Then we can find another solution $(B_{q+1}, J_{q+1}, R_{q+1})$ of (3.9) satisfying (4.18)-(4.24) with q replaced by $q+1$. Moreover, the increments $v_{q+1} = B_{q+1} - B_q$ and $w_{q+1} = J_{q+1} - J_q$ satisfy

$$\|v_{q+1}\|_{L^2} \le C\lambda_{q+1}^{-1}\delta_{q+1}^{1/2}, \quad \|w_{q+1}\|_{L^2} \le C\delta_{q+1}^{1/2}. \tag{4.25}$$

This proposition leads to a proof of Theorem 1.1 immediately.

Proof of Theorem 1.1: At the first step, we take $(B_0, J_0, R_0) = (0, 0, 0)$ which satisfies (4.18)-(4.23), and (4.20)-(4.21) for large enough $a > 0$. For $q \geq 1$, we apply Proposition 4.1 to obtain a sequence of approximating solutions $\{(B_q, J_q, R_q)\}$ satisfying (4.18)-(4.21). It follows from (4.25) that

$$\sum_{q\geq 0} \|J_{q+1} - J_q\|_{L^2} = \sum_{q\geq 0} \|w_{q+1}\|_{L^2} \lesssim \sum_{q\geq 0} \delta_{q+1}^{1/2} < \infty.$$

which implies the strong convergence of $J_q = \nabla \times B_q$ to a function J in $C^0(0, T; L^2)$, and the strong convergence of B_q to a function B in $C^0(0, T; H^1)$ with $J = \nabla \times B$ and $\nabla \cdot B = 0$.

While $\|\tilde{R}_q\|_{L^\infty(0,T;L^1)} \to 0$ and $\|\tilde{M}_q\|_{L^\infty(0,T;L^1)} \to 0$ as $q \to \infty$, we conclude J is a weak solution of (2.7), and B is a weak solution of (2.5); moreover, it is obvious that $B \in L^\infty(0, T; L^2(\mathbb{T}^3)) \cap L^2(0, T; H^1(\mathbb{T}^3))$, since B is divergence free.

□

The proof of Proposition 4.1 will be carried out in Sections 4.1 – 4.5 below.

4.1. **Construction of the perturbation** (v_{q+1}, w_{q+1})**.** Based on the building blocks introduced in Section 3.2, we proceed to construct the perturbation $v_{q+1} = B_{q+1} - B_q$,

$$v_{q+1} := v_{q+1}^p + v_{q+1}^c + v_{q+1}^t$$

where v_{q+1}^p and v_{q+1}^c are defined as

$$\begin{aligned} v_{q+1}^p &= \sum_{\xi\in\Lambda_\alpha} a_\xi \mathbb{W}_\xi^B = \lambda_{q+1}^{-1} \sum_{\xi\in\Lambda_\alpha} a_\xi \eta_\xi W_\xi, \\ v_{q+1}^c &= \lambda_{q+1}^{-2} \sum_{\xi\in\Lambda_\alpha} \nabla(a_\xi \eta_\xi) \times W_\xi, \end{aligned}$$

while v_{q+1}^t will be defined through w_{q+1}^t later. One can verify that

$$\nabla \cdot (v_{q+1}^p + v_{q+1}^c) = \lambda_{q+1}^{-2} \sum_{\xi\in\Lambda_\alpha} \nabla \cdot (\nabla \times (a_\xi \eta_\xi W_\xi)) = 0.$$

We now define the perturbation $w_{q+1} = J_{q+1} - J_q$ as

$$w_{q+1} = w_{q+1}^p + w_{q+1}^c + w_{q+1}^t$$

with

$$\begin{aligned} w_{q+1}^p &= \nabla \times v_{q+1}^p, \ \ w_{q+1}^c = \nabla \times v_{q+1}^c, \\ w_{q+1}^t &= \mu^{-1} \sum_{\xi} \mathbb{P}_H \mathbb{P}_{\neq 0}(a_\xi^2 \eta^2 \xi). \end{aligned}$$

In the end, we define v_{q+1}^t through $w_{q+1}^t = \nabla \times v_{q+1}^t$ up to a gradient which we can take as zero. Indeed, for $v_{q+1}^t \in L^2$, we can decompose v_{q+1}^t as

$$v_{q+1}^t = v_{q+1,0}^t + \nabla\phi, \quad \text{with} \quad \nabla \cdot v_{q+1,0}^t = 0.$$

In our case, we simply take $v_{q+1,0}^t$ to be v_{q+1}^t, since $\nabla \times \nabla\phi = 0$. Thus, $\nabla \cdot v_{q+1}^t = 0$ holds. Along with the fact $\nabla \cdot (v_{q+1}^p + v_{q+1}^c) = 0$, we have

$$\nabla \cdot v_{q+1} = 0.$$

On the other hand, it is obvious that

$$\nabla \cdot w_{q+1}^p = \nabla \cdot w_{q+1}^c = \nabla \cdot w_{q+1}^t = \nabla \cdot w_{q+1} = 0$$

and

$$w_{q+1} = \nabla \times v_{q+1}.$$

4.2. **Estimates of building blocks.** The main purpose of adding the oscillation η_ξ to the Beltrami waves is to make sure the L^1 norm of the waves is significantly smaller than the L^2 norm. This can be seen in the following lemma.

Lemma 4.2. [5] *The bounds*

$$\|\nabla^N \partial_t^K \mathbb{W}_\xi\|_{L^p} \lesssim \lambda^N (\lambda\sigma r\mu)^K r^{\frac{3}{2}-\frac{3}{p}}, \tag{4.26}$$

$$\|\nabla^N \partial_t^K \eta_\xi\|_{L^p} \lesssim (\lambda\sigma r)^N (\lambda\sigma r\mu)^K r^{\frac{3}{2}-\frac{3}{p}} \tag{4.27}$$

hold for all $1 < p \le \infty$.

We point out that, following [5], in order to avoid a loss of derivative, the pair (v_q, w_q) at each level needs to be regularized by using standard FriedDrichs mollifiers. Moreover, the corresponding stress tensor R_q is not spatially homogenous. To fix it, cutoff functions that form a partition of unity can be introduced to decompose R_q into slices. The two steps involve delicate computations, which will be omitted in our presentation. Rather, we do adapt the regularization parameter ℓ from the first step. We also adapt the partition of unity: let $0 \le \tilde{\chi}_0, \tilde{\chi} \le 1$ be smooth functions supported on $[0,4]$ and $[\frac{1}{4}, 4]$ respectively; and $\tilde{\chi}_i(z) = \tilde{\chi}(4^{-i}z)$ satisfying

$$\tilde{\chi}_0^2(z) + \sum_{i\ge 1} \tilde{\chi}_i(z) \equiv 1, \quad \forall z > 0.$$

Then we define the amplitude function a_ξ for the intermittent Beltrami flows as,

$$a_{\xi,i,q+1} = \rho_i^{\frac{1}{2}} \chi_{i,q+1} \gamma(\xi) \left(\mathrm{Id} - \frac{R_q}{\rho_i} \right) \tag{4.28}$$

where ρ_i and $\chi_{i,q+1}$ are defined as

$$\rho_i = \lambda_q^{-\varepsilon_R} \delta_{q+1} 4^{i+c_0}, \quad i \ge 1,$$
$$\chi_{i,q+1}(x,t) = \tilde{\chi}_i \left(\left\langle \frac{R_q(x,t)}{100\lambda_q^{-\varepsilon_R}\delta_{q+1}} \right\rangle \right).$$

Here we use the notation $\langle A \rangle = (1 + |A|^2)^{\frac{1}{2}}$ with $|\cdot|$ being the Euclidean norm of a matrix. Referring to [5], we have

$$4^{\max\{i\}} \lesssim \ell^{-1}. \tag{4.29}$$

To make sure the inequality (4.20) holds, we need to choose ρ_0 as follows,

$$\rho(t) = \frac{1}{3|\mathbb{T}^3|} \left(\int_{\mathbb{T}^3} \chi_0^2 \,\mathrm{d}x \right)^{-1} \max \left(E(t) - \int_{\mathbb{T}^3} |J_q|^2 \,\mathrm{d}x - 3 \sum_{i\ge 1} \rho_i \int_{\mathbb{T}^3} \chi_i^2 \,\mathrm{d}x - \frac{\delta_{q+1}}{2}, 0 \right)$$

$$\rho_0 = \left((\rho^{1/2}) * \varphi_\ell \right)^2,$$

where φ_ℓ is the standard Friedrichs mollifer at time scale ℓ. It was shown in [5], such defined ρ_0 satisfies

$$\|\rho_0\|_{C_t^0} \le 2\delta_{q+1}, \quad \|\rho_0^{1/2}\|_{C_t^N} \lesssim \delta_{q+1}^{1/2} \ell^{-N}, \text{ for } N \ge 1.$$

Below is a collection of estimates satisfied by the amplitude function a_ξ.

Lemma 4.3. *The following bounds hold*

$$\|\chi_i\|_{L^1} \lesssim 4^{-i} \tag{4.30}$$

$$\|a_\xi\|_{L^2} \lesssim \delta_{q+1}^{\frac{1}{2}}, \tag{4.31}$$

$$\|a_\xi\|_{L^\infty} \lesssim \delta_{q+1}^{\frac{1}{2}}\ell^{-\frac{1}{2}}, \tag{4.32}$$

$$\|a_\xi\|_{L^p} \lesssim \delta_{q+1}^{\frac{1}{2}}\ell^{-\frac{1}{2}(1-\frac{1}{p})}, \quad for \quad p\geq 1, \tag{4.33}$$

$$\|a_\xi\|_{C^N_{x,t}} \lesssim \ell^{-N}, \quad for \quad N\geq 1. \tag{4.34}$$

Proof: We only need to show (4.33), since other ones were shown in [5]. In view of (4.30), we deduce

$$\|a_\xi\|_{L^1} \lesssim \rho_i^{\frac{1}{2}}\|\chi_i\|_{L^1} \lesssim \lambda_q^{-\varepsilon_R/2}\delta_{q+1}^{\frac{1}{2}}.$$

Thus, by interpolation we obtain

$$\|a_\xi\|_{L^p} \lesssim \|a_\xi\|_{L^\infty}^{\frac{p-1}{p}}\|a_\xi\|_{L^1}^{\frac{1}{p}} \lesssim \delta_{q+1}^{\frac{1}{2}(1-\frac{1}{p})}\ell^{-\frac{1}{2}(1-\frac{1}{p})}\lambda_q^{-\varepsilon_R\frac{1}{2p}}\delta_{q+1}^{\frac{1}{2p}} \lesssim \delta_{q+1}^{\frac{1}{2}}\ell^{-\frac{1}{2}(1-\frac{1}{p})}.$$

□

4.3. **Estimates of the perturbation.**

Lemma 4.4. *The increment $v_{q+1} = B_{q+1} - B_q$ satisfies the following estimates*

$$\|v^p_{q+1}\|_{L^2} \lesssim \lambda_{q+1}^{-1}\delta_{q+1}^{\frac{1}{2}}, \tag{4.35}$$

$$\|v^c_{q+1}\|_{L^2} \lesssim \ell^{-1}\mu^{-1}\lambda_{q+1}^{-1}\delta_{q+1}^{\frac{1}{2}}r^{\frac{3}{2}}, \tag{4.36}$$

$$\|v^t_{q+1}\|_{L^2} \lesssim \ell^{-1}\mu^{-1}(\lambda_{q+1}\sigma)^{-1}\delta_{q+1}r^{\frac{3}{2}}, \tag{4.37}$$

$$\|v^p_{q+1}\|_{L^p} \lesssim \lambda_{q+1}^{-1}\delta_{q+1}^{\frac{1}{2}}\ell^{-\frac{1}{2}(1-\frac{1}{p})}r^{\frac{3}{2}-\frac{3}{p}}, \quad p\geq 1, \tag{4.38}$$

$$\|v^c_{q+1}\|_{L^p} \lesssim \lambda_{q+1}^{-1}\delta_{q+1}^{\frac{1}{2}}\ell^{-\frac{1}{2}(1-\frac{1}{p})}\sigma r^{\frac{5}{2}-\frac{3}{p}}, \quad p\geq 1, \tag{4.39}$$

$$\|v^t_{q+1}\|_{L^p} \lesssim \ell^{-1}\mu^{-1}(\lambda_{q+1}\sigma)^{-1}\delta_{q+1}r^{3-\frac{3}{p}}, \quad p\geq 1, \tag{4.40}$$

$$\|v^p_{q+1}\|_{W^{1,p}} + \|v^c_{q+1}\|_{W^{1,p}} \lesssim \ell^{-2}r^{\frac{3}{2}-\frac{3}{p}}, \quad p\geq 1, \tag{4.41}$$

$$\|v^t_{q+1}\|_{W^{1,p}} \lesssim \mu^{-1}\delta_{q+1}\ell^{-1}r^{4-\frac{3}{p}}, \quad p\geq 1, \tag{4.42}$$

$$\|v^p_{q+1}\|_{C^N_{x,t}} + \|v^c_{q+1}\|_{C^N_{x,t}} \lesssim \lambda_{q+1}^{\frac{1+5N}{2}}, \tag{4.43}$$

$$\|v^t_{q+1}\|_{C^N_{x,t}} \lesssim \lambda_{q+1}^{\frac{3+5N}{2}}, \tag{4.44}$$

$$\|B_{q+1}\|_{C^N_{x,t}} \lesssim \lambda_{q+1}^{\frac{3+5N}{2}}. \tag{4.45}$$

Proof: Adhering to Lemma 2.3, (4.33) and (4.26), we obtain for $1\leq p\leq\infty$

$$\begin{aligned}\|v^p_{q+1}\|_{L^p} &\leq \lambda_{q+1}^{-1}\sum_{\xi\in\Lambda_\alpha}\|a_\xi\mathbb{W}_\xi\|_{L^p}\\ &\lesssim \lambda_{q+1}^{-1}\sum_{\xi\in\Lambda_\alpha}\|a_\xi\|_{L^p}\|\mathbb{W}_\xi\|_{L^p}\\ &\lesssim \lambda_{q+1}^{-1}\delta_{q+1}^{\frac{1}{2}}\ell^{-\frac{1}{2}(1-\frac{1}{p})}r^{\frac{3}{2}-\frac{3}{p}}.\end{aligned}$$

In view of Lemma 2.3, (4.33), (4.34), and (4.27), and the choice of parameters (4.16)-(4.17), we obtain

$$\begin{aligned}\|v^c_{q+1}\|_{L^p} &\le \lambda_{q+1}^{-2}\sum_{\xi\in\Lambda_\alpha}\|\nabla(a_\xi\eta_\xi)\|_{L^p}\\ &\lesssim \lambda_{q+1}^{-2}\sum_{\xi\in\Lambda_\alpha}\|a_\xi\nabla\eta_\xi\|_{L^p}+\lambda_{q+1}^{-2}\sum_{\xi\in\Lambda_\alpha}\|\nabla a_\xi\eta_\xi\|_{L^p}\\ &\lesssim \lambda_{q+1}^{-2}\sum_{\xi\in\Lambda_\alpha}\|a_\xi\|_{L^p}\|\nabla\eta_\xi\|_{L^p}+\lambda_{q+1}^{-2}\sum_{\xi\in\Lambda_\alpha}\|\nabla a_\xi\|_{L^\infty}\|\eta_\xi\|_{L^p}\\ &\lesssim \lambda_{q+1}^{-2}\delta_{q+1}^{\frac12}\ell^{-\frac12(1-\frac1p)}\lambda_{q+1}\sigma r^{\frac52-\frac3p}+\lambda_{q+1}^{-2}\ell^{-1}r^{\frac32-\frac3p}\\ &\lesssim \lambda_{q+1}^{-1}\delta_{q+1}^{\frac12}\ell^{-\frac12(1-\frac1p)}\sigma r^{\frac52-\frac3p}.\end{aligned}$$

The proof of estimates on other norms of v^p_{q+1} and v^c_{q+1} can be found in [5] (by multiplying each estimate the factor λ_{q+1}^{-1}). We only need to show the estimates for v^t_{q+1}. Recall that v^t_{q+1} satisfies

$$\nabla\times v^t_{q+1}=w^t_{q+1}=\mu^{-1}\sum_{\xi\in\Lambda}\mathbb{P}_H\mathbb{P}_{\neq0}(a_\xi^2\eta_\xi^2\xi)$$

and v^t_{q+1} is divergence free. Thus by Lemma 2.3 we deduce

$$\begin{aligned}\|v^t_{q+1}\|_{L^2} \le&\mu^{-1}\left\|\sum_{\xi\in\Lambda}\mathrm{curl}^{-1}\mathbb{P}_H\mathbb{P}_{\neq0}(a_\xi^2\eta_\xi^2\xi)\right\|_{L^2}\\ \lesssim&\mu^{-1}\left\|\sum_{\xi\in\Lambda}\mathrm{curl}^{-1}\left(a_\xi^2\mathbb{P}_{\ge\lambda_{q+1}\sigma/2}(\eta_\xi^2\xi)\right)\right\|_{L^2}\\ \lesssim&\mu^{-1}\sum_{\xi\in\Lambda}\|a_\xi^2\|_{L^\infty}(\lambda_{q+1}\sigma)^{-1}\left(1+\frac{1}{\ell^L(\lambda_{q+1}\sigma)^{L-2}}\right)\|\eta_\xi^2\|_{L^2}\\ \lesssim&\mu^{-1}\delta_{q+1}\ell^{-1}(\lambda_{q+1}\sigma)^{-1}r^{\frac32}.\end{aligned}$$

In an analogous way, we can obtain

$$\begin{aligned}\|v^t_{q+1}\|_{L^p}\lesssim&\mu^{-1}\delta_{q+1}\ell^{-1}(\lambda_{q+1}\sigma)^{-1}r^{3-\frac3p},\\ \|v^t_{q+1}\|_{W^{1,p}}\lesssim&\mu^{-1}\delta_{q+1}\ell^{-1}r^{4-\frac3p}.\end{aligned}$$

Proof of inequality (4.44) can be referred to [5]; inequality (4.45) follows from (4.43) and (4.44).

□

Lemma 4.5. *The increment $w_{q+1}=J_{q+1}-J_q$ satisfies the following estimates,*

$$\|w^p_{q+1}\|_{L^2}\lesssim\delta_{q+1}^{\frac12},\tag{4.46}$$

$$\|w^c_{q+1}\|_{L^2}+\|w^t_{q+1}\|_{L^2}\lesssim\ell^{-1}\mu^{-1}\delta_{q+1}^{\frac12}r^{\frac32},\tag{4.47}$$

$$\|w_{q+1}\|_{L^p}\lesssim\delta_{q+1}^{\frac12}\ell^{-\frac12(1-\frac1p)}r^{\frac32-\frac3p},\quad p\ge1,\tag{4.48}$$

$$\|w^p_{q+1}\|_{W^{1,p}}+\|w^c_{q+1}\|_{W^{1,p}}+\|w^t_{q+1}\|_{W^{1,p}}\lesssim\ell^{-2}\lambda_{q+1}r^{\frac32-\frac3p},\quad p\ge1,\tag{4.49}$$

$$\|\partial_t w^p_{q+1}\|_{L^p}+\|\partial_t w^c_{q+1}\|_{L^p}\lesssim\ell^{-2}\lambda_{q+1}\sigma\mu r^{\frac52-\frac3p},\quad p\ge1,\tag{4.50}$$

$$\|w^p_{q+1}\|_{C^N_{x,t}} + \|w^c_{q+1}\|_{C^N_{x,t}} + \|w^t_{q+1}\|_{C^N_{x,t}} \lesssim \lambda_{q+1}^{\frac{3+5N}{2}}, \tag{4.51}$$

$$\||\nabla|^N w^p_{q+1}\|_{L^p} + \||\nabla|^N w^c_{q+1}\|_{L^p} + \||\nabla|^N w^t_{q+1}\|_{L^p} \lesssim \lambda^N_{q+1} r^{\frac{3}{2}-\frac{3}{p}}, \quad p \geq 1, \tag{4.52}$$

$$\||\nabla|^N J_{q+1}\|_{L^p} \lesssim \lambda_{q+1}^{N+\frac{3}{2}}, \quad p \geq 1. \tag{4.53}$$

Proof: Recall that

$$\begin{aligned} w^p_{q+1} = \nabla \times v^p_{q+1} &= \sum_{\xi\in\Lambda} a_\xi \mathbb{W}_\xi + \lambda_{q+1}^{-1} \sum_{\xi\in\Lambda} \nabla(a_\xi\eta_\xi) \times W_\xi \\ &=: \ \mathbb{W}^J + \mathbb{W}_{\epsilon,1} = \lambda_{q+1} v^p_{q+1} + \mathbb{W}_{\epsilon,1}; \end{aligned}$$

$$w^c_{q+1} = \nabla \times v^c_{q+1} = \lambda_{q+1}^{-2} \nabla \times \left(\sum_{\xi\in\Lambda} \nabla(a_\xi\eta_\xi) \times W_\xi \right) = \lambda_{q+1}^{-1} \nabla \times \mathbb{W}_{\epsilon,1}.$$

Note that $\mathbb{W}^J$ is the intermittent wave defined for the principle part of the velocity increment $u^p_{q+1} - u^p_q$ in [5]; while the temporal oscillation part w^t_{q+1} is defined the same way as in [5]. Thus the estimates on $\mathbb{W}^J$ and w^t_{q+1} can be adapted from [5]. Therefore, it is sufficient to estimate $\mathbb{W}_{\epsilon,1}$ and $\nabla \times \mathbb{W}_{\epsilon,1}$.

In addition, we notice that $\mathbb{W}_{\epsilon,1} = \lambda_{q+1} v^c_{q+1}$. It then follows from Lemma 4.4 that

$$\begin{aligned} \|\mathbb{W}_{\epsilon,1}\|_{L^2} &\leq \lambda_{q+1} \|v^c_{q+1}\|_{L^2} \lesssim \delta_{q+1}^{\frac{1}{2}}, \\ \|\mathbb{W}_{\epsilon,1}\|_{W^{1,p}} &\leq \lambda_{q+1} \|v^c_{q+1}\|_{W^{1,p}} \lesssim \ell^{-2} \lambda_{q+1} r^{\frac{3}{2}-\frac{3}{p}}, \\ \|\partial_t \mathbb{W}_{\epsilon,1}\|_{L^p} &\leq \lambda_{q+1} \|\partial_t v^c_{q+1}\|_{L^p} \lesssim \ell^{-2} \lambda_{q+1} \sigma\mu r^{\frac{5}{2}-\frac{3}{p}}, \\ \|\mathbb{W}_{\epsilon,1}\|_{C^N_{x,t}} &\leq \lambda_{q+1} \|v^c_{q+1}\|_{C^N_{x,t}} \lesssim \lambda_{q+1}^{\frac{3+5N}{2}}. \end{aligned}$$

The estimates of $\nabla \times \mathbb{W}_{\epsilon,1}$ and hence w^c_{q+1} are carried out as follows. First, a direct computation leads to

$$w^c_{q+1} = \lambda_{q+1}^{-2} \sum_{\xi\in\Lambda} \left(W_\xi \cdot \nabla\nabla(a_\xi\eta_\xi) - W_\xi \cdot \Delta(a_\xi\eta_\xi) - \nabla(a_\xi\eta_\xi) \cdot \nabla W_\xi \right) \tag{4.54}$$

where we used the fact that $\nabla \cdot W_\xi = 0$. Thus, we have

$$\begin{aligned} \|w^c_{q+1}\|_{L^2} &\lesssim \lambda_{q+1}^{-2} \left(\|W_\xi \nabla\nabla(a_\xi\eta_\xi)\|_{L^2} + \|\nabla(a_\xi\eta_\xi)\nabla W_\xi\|_{L^2} \right) \\ &\lesssim \lambda_{q+1}^{-2} \left(\|\nabla\nabla(a_\xi\eta_\xi)\|_{L^2} + \lambda_{q+1} \|\nabla(a_\xi\eta_\xi)\|_{L^2} \right) \\ &\lesssim \lambda_{q+1}^{-2} \left(\|a_\xi \nabla^2\eta_\xi\|_{L^2} + \|\nabla a_\xi \nabla\eta_\xi\|_{L^2} + \|\nabla^2 a_\xi \eta_\xi\|_{L^2} \right) \\ &\quad + \lambda_{q+1}^{-1} \left(\|a_\xi \nabla\eta_\xi\|_{L^2} + \|\nabla a_\xi \eta_\xi\|_{L^2} \right). \end{aligned}$$

Following Lemma 2.3 for L^2 norm, we obtain

$$\|a_\xi \nabla^2\eta_\xi\|_{L^2} \lesssim \|a_\xi\|_{L^2} \|\nabla^2\eta_\xi\|_{L^2} \lesssim \delta_{q+1}^{\frac{1}{2}} (\lambda_{q+1}\sigma r)^2 \lesssim \lambda_{q+1}^2 \ell^{-1} \mu^{-1} \delta_{q+1}^{\frac{1}{2}} r^{\frac{3}{2}}$$

due to (4.27) and (4.31), and the choice of parameters (4.16) and (4.17). The other terms are treated in an analogous way,

$$
\begin{aligned}
\|\nabla a_\xi \nabla \eta_\xi\|_{L^2} &\lesssim \|\nabla a_\xi\|_{L^\infty}\|\nabla \eta_\xi\|_{L^2} \lesssim \ell^{-1}\lambda_{q+1}\sigma r \lesssim \lambda_{q+1}^2 \ell^{-1}\mu^{-1}\delta_{q+1}^{\frac12} r^{\frac32};\\
\|\nabla^2 a_\xi \eta_\xi\|_{L^2} &\lesssim \|\nabla^2 a_\xi\|_{L^\infty}\|\eta_\xi\|_{L^2} \lesssim \ell^{-2} \lesssim \lambda_{q+1}^2 \ell^{-1}\mu^{-1}\delta_{q+1}^{\frac12} r^{\frac32};\\
\|a_\xi \nabla \eta_\xi\|_{L^2} &\lesssim \|a_\xi\|_{L^2}\|\nabla \eta_\xi\|_{L^2} \lesssim \delta_{q+1}^{\frac12}\lambda_{q+1}\sigma r \lesssim \lambda_{q+1} \ell^{-1}\mu^{-1}\delta_{q+1}^{\frac12} r^{\frac32};\\
\|\nabla a_\xi \eta_\xi\|_{L^2} &\lesssim \|\nabla a_\xi\|_{L^\infty}\|\eta_\xi\|_{L^2} \lesssim \ell^{-1} \lesssim \lambda_{q+1} \ell^{-1}\mu^{-1}\delta_{q+1}^{\frac12} r^{\frac32}.
\end{aligned}
$$

Combining the estimates above yields

$$
\|w_{q+1}^c\|_{L^2} \lesssim \ell^{-1}\mu^{-1}\delta_{q+1}^{\frac12} r^{\frac32}
$$

which concludes the proof of (4.47).

Now we estimate the L^p norm of w_{q+1}^p, w_{q+1}^c, and w_{q+1}^t. Again we recall that $w_{q+1}^p = \lambda_{q+1} v_{q+1}^p + \lambda_{q+1} v_{q+1}^c$. The estimates (4.38) and (4.39) give immediately

$$
\begin{aligned}
\|w_{q+1}^p\|_{L^p} &\lesssim \delta_{q+1}^{\frac12}\ell^{-\frac12(1-\frac1p)} r^{\frac32-\frac3p} + \delta_{q+1}^{\frac12}\ell^{-\frac12(1-\frac1p)}\sigma r^{\frac52-\frac3p}\\
&\lesssim \delta_{q+1}^{\frac12}\ell^{-\frac12(1-\frac1p)} r^{\frac32-\frac3p}.
\end{aligned}
$$

In an analogous way of estimating $\|w_{q+1}^c\|_{L^2}$, we can obtain

$$
\|w_{q+1}^c\|_{L^p} \lesssim \ell^{-1}\mu^{-1}\delta_{q+1}^{\frac12} r^{3-\frac3p}.
$$

While we deal with w_{q+1}^t as follows, by using (4.32) and (4.27)

$$
\begin{aligned}
\|w_{q+1}^t\|_{L^p} &\lesssim \mu^{-1}\sum_{\xi\in\Lambda_\alpha}\|a_\xi^2\eta_\xi^2\|_{L^p}\\
&\lesssim \mu^{-1}\sum_{\xi\in\Lambda_\alpha}\|a_\xi\|_{L^\infty}^2\|\eta_\xi\|_{L^{2p}}^2\\
&\lesssim \mu^{-1}\delta_{q+1}\ell^{-1} r^{3-\frac3p}.
\end{aligned}
$$

Combining the last three estimates yields

$$
\|w_{q+1}\|_{L^p} \lesssim \delta_{q+1}^{\frac12}\ell^{-\frac12(1-\frac1p)} r^{\frac32-\frac3p}
$$

which proves (4.48).

Next we estimate $\|w_{q+1}^c\|_{W^{1,p}}$. It follows from (4.54), Lemma 4.3, Lemma 4.5 and (4.16)-(4.17) that

$$
\begin{aligned}
\|w_{q+1}^c\|_{W^{1,p}} &\lesssim \lambda_{q+1}^{-2}\left(\|W_\xi\nabla^2(a_\xi\eta_\xi)\|_{W^{1,p}} + \|\nabla W_\xi\nabla(a_\xi\eta_\xi)\|_{W^{1,p}}\right)\\
&\lesssim \lambda_{q+1}^{-2}\left(\|\nabla^3(a_\xi\eta_\xi)\|_{L^p} + \lambda_{q+1}\|\nabla^2(a_\xi\eta_\xi)\|_{L^p}\right)\\
&\lesssim \lambda_{q+1}^{-2}\|a_\xi\|_{C^3}\left(\|\nabla^3\eta_\xi\|_{L^p} + \lambda_{q+1}\|\nabla^2\eta_\xi\|_{L^p}\right)\\
&\lesssim \lambda_{q+1}^{-2}\ell^{-3}\left((\lambda_{q+1}\sigma r)^3 + \lambda_{q+1}(\lambda_{q+1}\sigma r)^2\right) r^{\frac32-\frac3p}\\
&\lesssim \ell^{-2}\lambda_{q+1} r^{\frac32-\frac3p}.
\end{aligned}
$$

Thus, the proof of (4.49) is also complete. To prove (4.50), we proceed to estimate $\|\partial_t w^c_{q+1}\|_{L^p}$,

$$\begin{aligned}\|\partial_t w^c_{q+1}\|_{L^p} &\lesssim \lambda_{q+1}^{-2}\sum_{\xi\in\Lambda}\left(\|\partial_t\nabla^2(a_\xi\eta_\xi)\|_{L^p}+\lambda_{q+1}\|\partial_t\nabla(a_\xi\eta_\xi)\|_{L^p}\right)\\ &\lesssim \lambda_{q+1}^{-2}\sum_{\xi\in\Lambda}\|a_\xi\|_{C^3}\left(\|\partial_t\nabla^2\eta_\xi\|_{L^p}+\|\nabla^2\eta_\xi\|_{L^p}\right)\\ &\quad+\lambda_{q+1}^{-1}\sum_{\xi\in\Lambda}\|a_\xi\|_{C^2}\left(\|\partial_t\nabla\eta_\xi\|_{L^p}+\|\nabla\eta_\xi\|_{L^p}\right)\\ &\lesssim \lambda_{q+1}^{-2}\ell^{-3}\left(\lambda_{q+1}\sigma r\mu+1\right)(\lambda_{q+1}\sigma r)^2 r^{\frac32-\frac3p}\\ &\quad+\lambda_{q+1}^{-1}\ell^{-2}\left(\lambda_{q+1}\sigma r\mu+1\right)(\lambda_{q+1}\sigma r) r^{\frac32-\frac3p}\\ &\lesssim \ell^{-2}\lambda_{q+1}\sigma\mu r^{\frac52-\frac3p}.\end{aligned}$$

In the end, we estimate $\|w^c_{q+1}\|_{C^N_{x,t}}$,

$$\begin{aligned}\|w^c_{q+1}\|_{C^N_{x,t}} &\lesssim \lambda_{q+1}^{-2}\sum_{\xi\in\Lambda}\left(\|\nabla^2(a_\xi\eta_\xi)\|_{C^N_{x,t}}+\lambda_{q+1}\|\nabla(a_\xi\eta_\xi)\|_{C^N_{x,t}}\right)\\ &\lesssim \lambda_{q+1}^{-2}\sum_{\xi\in\Lambda}\left(\|a_\xi\|_{C^N_{x,t}}\|\nabla^2\eta_\xi\|_{C^N_{x,t}}+\|\nabla a_\xi\|_{C^N_{x,t}}\|\nabla\eta_\xi\|_{C^N_{x,t}}+\|\nabla^2 a_\xi\|_{C^N_{x,t}}\|\eta_\xi\|_{C^N_{x,t}}\right)\\ &\quad+\lambda_{q+1}^{-1}\sum_{\xi\in\Lambda}\left(\|a_\xi\|_{C^N_{x,t}}\|\nabla\eta_\xi\|_{C^N_{x,t}}+\|\nabla a_\xi\|_{C^N_{x,t}}\|\eta_\xi\|_{C^N_{x,t}}\right)\\ &\lesssim \lambda_{q+1}^{-2}\left(\ell^{-N}(\lambda_{q+1}\sigma r)^2+\ell^{-N-1}\lambda_{q+1}\sigma r+\ell^{-N-2}\right)(\lambda_{q+1}\sigma r\mu)^N r^{\frac32}\\ &\quad+\lambda_{q+1}^{-1}\left(\ell^{-N}\lambda_{q+1}\sigma r+\ell^{-N-1}\right)(\lambda_{q+1}\sigma r\mu)^N r^{\frac32}\\ &\lesssim \lambda_{q+1}^{\frac32+\frac{5N}{2}}.\end{aligned}$$

It completes the proof of the inequality (4.51).

Inequality (4.52) can be obtained analogously as (4.49); while (4.53) is implied by (4.52) and (4.17).

□

4.4. **Estimate of the stress tensor** R_{q+1}.

Lemma 4.6. *Consider the equation*

$$\partial_t J_{q+1}+\nabla\times\nabla\times(J_{q+1}\times B_{q+1})=\Delta J_{q+1}+\nabla\cdot R_{q+1}. \tag{4.55}$$

There exists a traceless symmetric tensor R_{q+1} *satisfying (4.55). Moreover, there exists another traceless symmetric tensor* $\tilde R_{q+1}$*, vector field* $\tilde M_{q+1}$ *and* $\tilde Q_{q+1}$*, and a scalar pressure function* $\tilde p_{q+1}$ *satisfying*

$$\nabla\cdot R_{q+1}=\nabla\cdot\tilde R_{q+1}+\nabla\times\nabla\times\tilde M_{q+1}+\nabla\cdot\nabla\tilde Q_{q+1}+\nabla\tilde p_{q+1}.$$

In addition, there exists $p>1$ *sufficiently close to 1, and a sufficiently small* $\varepsilon_R>0$ *independent of* q *such that*

$$\|\tilde R_{q+1}\|_{L^p}+\|\tilde M_{q+1}\|_{L^p}+\|\tilde Q_{q+1}\|_{L^p}\lesssim \lambda_{q+1}^{-2\varepsilon_R}\delta_{q+2} \tag{4.56}$$

holds for some implicit constant which depends on p *and* ε_R.

To estimate the stress tensor R_{q+1}, we first subtract the equation (3.9) at level of J_q from the equation at level of J_{q+1} to arrive

$$\begin{aligned}\partial_t w_{q+1} &+ \nabla\cdot((\nabla\times J_{q+1})\otimes B_{q+1} - (\nabla\times J_q)\otimes B_q)\\ &+ \nabla\cdot(B_{q+1}\otimes(\nabla\times J_{q+1}) - B_q\otimes(\nabla\times J_q))\\ &- \nabla\cdot(\nabla(J_{q+1}\times B_{q+1}) - \nabla(J_q\times B_q)) - \nabla p_{q+1}\\ &= \Delta w_{q+1} + \nabla\cdot R_{q+1} - \nabla\cdot R_q.\end{aligned}$$

Rearranging the terms we obtain

$$\begin{aligned}\nabla\cdot R_{q+1} = {}&\partial_t w_{q+1} - \Delta w_{q+1}\\ &+ \nabla\cdot(\nabla\times J_{q+1}\otimes v_{q+1} + \nabla\times w_{q+1}\otimes B_q)\\ &+ \nabla\cdot(B_{q+1}\otimes\nabla\times w_{q+1} + v_{q+1}\otimes\nabla\times J_q)\\ &+ \nabla\cdot(\nabla(J_{q+1}\times v_{q+1}) + \nabla(w_{q+1}\times B_q))\\ &+ \nabla\cdot R_q - \nabla p_{q+1}.\end{aligned}$$

We further classify the terms on the right hand side into linear, correction and oscillation terms:

$$\begin{aligned}\nabla\cdot R_{q+1} = {}&\{\nabla\cdot[\mathcal{R}(\partial_t w^p_{q+1} + \partial_t w^c_{q+1} - \Delta w_{q+1})]\\ &+ \nabla\cdot[\nabla\times J_q\otimes v_{q+1} + v_{q+1}\otimes\nabla\times J_q + \nabla\times w_{q+1}\otimes B_q + B_q\otimes\nabla\times w_{q+1}]\\ &+ \nabla\cdot[\nabla(J_q\times v_{q+1}) + \nabla(w_{q+1}\times B_q)]\}\\ &+ \{\nabla\cdot[\nabla\times w^p_{q+1}\otimes(v^c_{q+1}+v^t_{q+1}) + (v^c_{q+1}+v^t_{q+1})\otimes\nabla\times w_{q+1}]\\ &+ \nabla\cdot[v^p_{q+1}\otimes\nabla\times(w^c_{q+1}+w^t_{q+1}) + \nabla\times(w^c_{q+1}+w^t_{q+1})\otimes v_{q+1}]\\ &+ \nabla\cdot[\nabla(w^p_{q+1}\times(v^c_{q+1}+v^t_{q+1})) + \nabla((w^c_{q+1}+w^t_{q+1})\times v_{q+1})]\}\\ &+ \{\nabla\cdot[\nabla\times w^p_{q+1}\otimes v^p_{q+1} + v^p_{q+1}\otimes\nabla\times w^p_{q+1} - \nabla(w^p_{q+1}\times v^p_{q+1})]\\ &- \nabla\frac{|\nabla\times v^p_{q+1}|^2}{2} + (\nabla\cdot R_q + \partial_t w^t_{q+1})\}\\ =:{}& \nabla\cdot R_{\text{linear}} + \nabla\cdot R_{\text{corrector}} + \nabla\cdot R_{\text{oscillation}}.\end{aligned}$$

On the right hand side of the equation above, the first three lines correspond to linear terms, the middle three correspond to correction terms, and the last two lines correspond to oscillation terms. The estimates of them will be accomplished separately below.

4.4.1. *Linear terms.* The estimates of the linear terms are relatively easy.

Lemma 4.7. *For $p>1$ sufficiently close to 1, R_{linear} satisfies*

$$\|R_{\text{linear}}\|_{L^p} \lesssim \lambda_{q+1}^{-2\varepsilon_R}\delta_{q+2}.$$

Proof: It follows from Lemma 2.5 and (4.49) that,

$$\|\mathcal{R}\Delta w_{q+1}\|_{L^p} \lesssim \|w_{q+1}\|_{W^{1,p}} \lesssim \ell^{-2}\lambda_{q+1} r^{\frac{3}{2}-\frac{3}{p}};$$

while Lemma 2.5 and (4.50) together give

$$\begin{aligned}\|\mathcal{R}(\partial_t(w^p_{q+1}+w^c_{q+1}))\|_{L^p} &= \|\mathcal{R}(\partial_t\nabla\times(v^p_{q+1}+v^c_{q+1}))\|_{L^p}\\ &= \lambda_{q+1}^{-1}\|\mathcal{R}\partial_t\nabla\times\nabla\times v^p_{q+1}\|_{L^p}\\ &= \lambda_{q+1}^{-1}\|\mathcal{R}\partial_t\nabla\times w^p_{q+1}\|_{L^p}\\ &\lesssim \lambda_{q+1}^{-1}\|\partial_t w^p_{q+1}\|_{L^p}\\ &\lesssim \ell^{-2}\sigma\mu r^{\frac52-\frac3p}.\end{aligned}$$

We have, by (4.53) and (4.38)-(4.40),

$$\begin{aligned}&\|(\nabla\times J_q)\otimes v_{q+1}+v_{q+1}\otimes(\nabla\times J_q)\|_{L^p}\\ \lesssim&\|\nabla\times J_q\|_{L^\infty}\|v_{q+1}\|_{L^p}\\ \lesssim&\ \lambda_q^3\left(\lambda_{q+1}^{-1}\delta_{q+1}^{\frac12}\ell^{-\frac12(1-\frac1p)}r^{\frac32-\frac3p}+\ell^{-1}\mu^{-1}(\lambda_{q+1}\sigma)^{-1}\delta_{q+1}r^{3-\frac3p}\right)\\ \lesssim&\ \lambda_q^3\ell^{-1}\mu^{-1}(\lambda_{q+1}\sigma)^{-1}\delta_{q+1}r^{3-\frac3p};\end{aligned}$$

and similarly, by (4.45) and (4.49),

$$\begin{aligned}&\|(\nabla\times w_{q+1})\otimes B_q+B_q\otimes(\nabla\times w_{q+1})\|_{L^p}\\ \lesssim&\|B_q\|_{L^\infty}\|w_{q+1}\|_{W^{1,p}}\\ \lesssim&\ \lambda_q^3\ell^{-2}\lambda_{q+1}r^{\frac32-\frac3p}.\end{aligned}$$

Combining (4.53), (4.38)-(4.40), (4.41), (4.42), (4.45), (4.48), and (4.49) yields

$$\begin{aligned}&\|\nabla(J_q\times v_{q+1})+\nabla(w_{q+1}\times B_q)\|_{L^p}\\ \lesssim&\|\nabla J_q\|_{L^\infty}\|v_{q+1}\|_{L^p}+\|J_q\|_{L^\infty}\|v_{q+1}\|_{W^{1,p}}\\ &+\|\nabla B_q\|_{L^\infty}\|w_{q+1}\|_{L^p}+\|B_q\|_{L^\infty}\|w_{q+1}\|_{W^{1,p}}\\ \lesssim&\ \lambda_q^3\ell^{-1}\mu^{-1}\lambda_{q+1}^{-1}\sigma^{-1}r_{q+1}^{3-\frac3p}+\lambda_q^3\left(\ell^{-2}r^{\frac32-\frac3p}+\mu^{-1}\delta_{q+1}\ell^{-1}r^{4-\frac3p}\right)\\ &+\lambda_q^4\left(\delta_{q+1}^{\frac12}\ell^{-\frac12(1-\frac1p)}r^{\frac32-\frac3p}+\ell^{-2}\lambda_{q+1}r^{\frac32-\frac3p}\right).\end{aligned}$$

Summarizing the estimates above and taking into account the choice of parameters (4.16)-(4.17) concludes the proof.

□

4.4.2. *Correction terms.*

Lemma 4.8. *The corrector part $\nabla\cdot R_{\text{corrector}}$ of the stress tensor can be written as*

$$\nabla\cdot R_{\text{corrector}}=\nabla\cdot\tilde R_{\text{corrector}}+\nabla\times\nabla\times\tilde M_{q+1,1}+\nabla\cdot\nabla(2\tilde M_{q+1,1}+\tilde M_{q+1,2})+\nabla\tilde p_{q+1}$$

for certain tensor $\tilde R_{\text{corrector}}$, vector field $\tilde M_{q+1,1}$ and $\tilde M_{q+1,2}$, and a pressure term $\tilde p_{q+1}$. For $p>1$ close enough to 1, and a sufficiently small constant $\varepsilon_R>0$ depending on p, the following estimates hold:

$$\|\tilde R_{\text{corrector}}\|_{L^p}+\|\tilde M_{q+1,1}\|_{L^p}+\|\tilde M_{q+1,2}\|_{L^p}\lesssim\lambda_{q+1}^{-2\varepsilon_R}\delta_{q+1}.$$

Proof: Upon the choice of parameters (4.16)-(4.17), the upper bound of $\|v^t_{q+1}\|_{L^p}$ is larger than that of $\|v^c_{q+1}\|_{L^p}$, as in Lemma 4.4. Thus we have to handle the terms involving $\|v_{q+1}^t\|_{L^p}$ in a more delicate way. In order to do so, we rearrange $\nabla\cdot R_{\text{corrector}}$ as,

$$\begin{aligned}\nabla\cdot R_{\text{corrector}} =&\nabla\cdot[(\nabla\times w_{q+1}^p)\otimes(v_{q+1}^c+v_{q+1}^t)+(v_{q+1}^c+v_{q+1}^t)\otimes\nabla\times w_{q+1}]\\ &+\nabla\cdot[v_{q+1}^p\otimes\nabla\times(w_{q+1}^c+w_{q+1}^t)+\nabla\times(w_{q+1}^c+w_{q+1}^t)\otimes v_{q+1}]\\ &+\nabla\cdot[\nabla(w_{q+1}^p\times(v_{q+1}^c+v_{q+1}^t))+\nabla((w_{q+1}^c+w_{q+1}^t)\times v_{q+1})]\\ =&\nabla\cdot[(\nabla\times w_{q+1}^p)\otimes v_{q+1}^t+v_{q+1}^t\otimes\nabla\times w_{q+1}]\\ &+\nabla\cdot[\nabla\times(w_{q+1}^c+w_{q+1}^t)\otimes v_{q+1}^t]\\ &+\nabla\cdot[\nabla(w_{q+1}^p\times v_{q+1}^t)+\nabla((w_{q+1}^c+w_{q+1}^t)\times v_{q+1}^t)]\\ &+\nabla\cdot[(\nabla\times w_{q+1}^p)\otimes v_{q+1}^c+v_{q+1}^c\otimes\nabla\times w_{q+1}]\\ &+\nabla\cdot[v_{q+1}^p\otimes\nabla\times(w_{q+1}^c+w_{q+1}^t)+\nabla\times(w_{q+1}^c+w_{q+1}^t)\otimes(v_{q+1}^p+v_{q+1}^c)]\\ &+\nabla\cdot[\nabla(w_{q+1}^p\times v_{q+1}^c)+\nabla((w_{q+1}^c+w_{q+1}^t)\times(v_{q+1}^p+v_{q+1}^c))]\\ =&\nabla\cdot[(\nabla\times w_{q+1})\otimes v_{q+1}^t+v_{q+1}^t\otimes(\nabla\times w_{q+1})+\nabla(w_{q+1}\times v_{q+1}^t)]\\ &+\nabla\cdot[(\nabla\times w_{q+1})\otimes v_{q+1}^c+v_{q+1}^c\otimes(\nabla\times w_{q+1})]\\ &+\nabla\cdot[(\nabla\times(w_{q+1}^c+w_{q+1}^t))\otimes v_{q+1}^p+v_{q+1}^p\otimes(\nabla\times(w_{q+1}^c+w_{q+1}^t))]\\ &+\nabla\cdot\nabla(w_{q+1}\times v_{q+1}^c+(w_{q+1}^c+w_{q+1}^t)\times v_{q+1}^p)\\ =:&\nabla\cdot R_{\{\text{cor},1\}}+\nabla\cdot R_{\{\text{cor},2\}}+\nabla\cdot R_{\{\text{cor},3\}}+\nabla\cdot R_{\{\text{cor},4\}}.\end{aligned}$$

We notice that only $R_{\{\text{cor},1\}}$ involves with v_{q+1}^t. We can further rewrite $\nabla\cdot R_{\{\text{cor},1\}}$ into

$$\begin{aligned}\nabla\cdot R_{\{\text{cor},1\}} =&\nabla\cdot[(\nabla\times w_{q+1})\otimes v_{q+1}^t+v_{q+1}^t\otimes(\nabla\times w_{q+1})]+\nabla\cdot\nabla(w_{q+1}\times v_{q+1}^t)\\ =&\nabla\times\nabla\times(w_{q+1}\times v_{q+1}^t)+\nabla(w_{q+1}\cdot(\nabla\times v_{q+1}^t))+2\nabla\cdot\nabla(w_{q+1}\times v_{q+1}^t)\\ =&\nabla\times\nabla\times(w_{q+1}\times v_{q+1}^t)+\nabla(w_{q+1}\cdot w_{q+1}^t)+2\nabla\cdot\nabla(w_{q+1}\times v_{q+1}^t).\end{aligned}$$

Denote

$$\tilde M_{q+1,1}=w_{q+1}\times v_{q+1}^t,\quad \tilde p_{q+1}=w_{q+1}\cdot w_{q+1}^t$$

where $\tilde p_{q+1}$ can be seen as a dummy pressure term. It follows that

$$\nabla\cdot R_{\{\text{cor},1\}}=\nabla\times\nabla\times\tilde M_{q+1,1}+\nabla\cdot\nabla(2\tilde M_{q+1,1})+\nabla\tilde p_{q+1}. \tag{4.57}$$

While we can estimate $\tilde M_{q+1,1}$ as, in view of (4.48), (4.40) and (4.16)-(4.17)

$$\begin{aligned}\|\tilde M_{q+1,1}\|_{L^p}\le&\|w_{q+1}\|_{L^{2p}}\|v_{q+1}^t\|_{L^{2p}}\\ \lesssim&\,\delta_{q+1}^{\frac12}\ell^{-\frac12(1-\frac1p)}r^{\frac32-\frac{3}{2p}}\ell^{-1}\mu^{-1}\lambda_{q+1}^{-1}\sigma^{-1}\delta_{q+1}r^{3-\frac{3}{2p}}\\ \lesssim&\,\ell^{-3}\mu^{-1}\lambda_{q+1}^{-1}\sigma^{-1}\delta_{q+1}r^{\frac92-\frac3p}\\ \lesssim&\,\lambda_{q+1}^{-2\varepsilon_R}\delta_{q+1}.\end{aligned}$$

We turn to the estimates of $R_{\{\text{cor},2\}}$, $R_{\{\text{cor},3\}}$, and $R_{\{\text{cor},4\}}$ which are trivial. Following from (4.39) and (4.49), it has

$$\begin{aligned}\|R_{\{\text{cor},2\}}\|_{L^p}\le&\|w_{q+1}\|_{W^{1,2p}}\|v_{q+1}^c\|_{L^{2p}}\\ \lesssim&\,\lambda_{q+1}^{-1}\delta_{q+1}^{\frac12}\ell^{-\frac12(1-\frac1p)}\sigma r^{\frac52-\frac{3}{2p}}\ell^{-2}\lambda_{q+1}r^{\frac32-\frac{3}{2p}}\\ \lesssim&\,\ell^{-3}\delta_{q+1}^{\frac12}\sigma r^{4-\frac3p}\\ \lesssim&\,\lambda_{q+1}^{-2\varepsilon_R}\delta_{q+1}.\end{aligned}$$

By (4.49) and (4.38), we have, for $p>1$ sufficiently close to 1

$$\begin{aligned}\|R_{\{\mathrm{cor},3\}}\|_{L^p} &\le \|w^c_{q+1}+w^t_{q+1}\|_{W^{1,2p}}\|v^p_{q+1}\|_{L^{2p}}\\ &\lesssim \lambda_{q+1}^{-1}\delta_{q+1}^{\frac12}\ell^{-\frac12(1-\frac1p)}r^{\frac32-\frac{3}{2p}}\ell^{-2}\lambda_{q+1}r^{\frac32-\frac{3}{2p}}\\ &\lesssim \ell^{-3}\delta_{q+1}^{\frac12}r^{3-\frac3p}\\ &\lesssim \lambda_{q+1}^{-2\varepsilon_R}\delta_{q+1}.\end{aligned}$$

Now we estimate $R_{\{\mathrm{cor},4\}}$ by observing that

$$R_{\{\mathrm{cor},4\}}=\nabla\cdot\nabla\tilde M_{q+1,2},\quad \tilde M_{q+1,2}:=w_{q+1}\times v^c_{q+1}+(w^c_{q+1}+w^t_{q+1})\times v^p_{q+1};$$

and $\tilde M_{q+1,2}$ can be estimated as

$$\begin{aligned}\|\tilde M_{q+1,2}\|_{L^p} &\lesssim \|w_{q+1}\|_{L^{2p}}\|v^c_{q+1}\|_{L^{2p}}+\|w^c_{q+1}+w^t_{q+1}\|_{L^{2p}}\|v^p_{q+1}\|_{L^{2p}}\\ &\lesssim \delta_{q+1}^{\frac12}\ell^{-\frac12(1-\frac1p)}r^{\frac32-\frac{3}{2p}}\lambda_{q+1}^{-1}\delta_{q+1}^{\frac12}\ell^{-\frac12(1-\frac{1}{2p})}\sigma r^{\frac52-\frac{3}{2p}}\\ &\quad+\delta_{q+1}^{\frac12}\ell^{-\frac12(1-\frac1p)}r^{\frac32-\frac{3}{2p}}\lambda_{q+1}^{-1}\delta_{q+1}^{\frac12}\ell^{-\frac12(1-\frac{1}{2p})}r^{\frac32-\frac{3}{2p}}\\ &\lesssim \ell^{-3}\lambda_{q+1}^{-1}\delta_{q+1}^{\frac12}r^{3-\frac3p}.\end{aligned}$$

□

4.4.3. *Oscillation terms.*

Lemma 4.9. *The oscillation part $\nabla\cdot R_{\mathrm{oscillation}}$ of the stress tensor can be written as*

$$\nabla\cdot R_{\mathrm{oscillation}}=\nabla\cdot(\lambda_{q+1}v^p_{q+1}\otimes\lambda_{q+1}v^p_{q+1}+R_q+\mathcal{R}\partial_t w^t_{q+1})+\nabla\cdot\mathcal{D}_{q+1}$$

for a certain tensor $\mathcal{D}_{q+1}$. For a $p>1$ sufficiently close to 1 and an arbitrarily small constant $\varepsilon_R>0$, we have

$$\|\lambda_{q+1}v^p_{q+1}\otimes\lambda_{q+1}v^p_{q+1}+R_q+\mathcal{R}\partial_t w^t_{q+1}\|_{L^p}+\|\mathcal{D}_{q+1}\|_{L^p}\lesssim\lambda_{q+1}^{-2\varepsilon_R}\delta_{q+1}.$$

Proof: In fact the first four oscillation terms can be written as

$$\begin{aligned}&\nabla\cdot[\nabla\times w^p_{q+1}\otimes v^p_{q+1}+v^p_{q+1}\otimes\nabla\times w^p_{q+1}-\nabla(w_{q+1}\times v^p_{q+1})]-\nabla\frac{|\nabla\times v^p_{q+1}|^2}{2}\\ =&\nabla\times\nabla\times((\nabla\times v^p_{q+1})\times v^p_{q+1}).\end{aligned}$$

Thus we can write the oscillation part as

$$\begin{aligned}\nabla\cdot R_{\mathrm{oscillation}}=&\nabla\times\nabla\times((\nabla\times v^p_{q+1})\times v^p_{q+1})+\nabla\cdot R_q+\partial_t w^t_{q+1}\\ =&\nabla\times\nabla\times((\nabla\times v^p_{q+1})\times v^p_{q+1})-\nabla\cdot(\lambda_{q+1}v^p_{q+1}\otimes\lambda_{q+1}v^p_{q+1})\\ &+\nabla\cdot(\lambda_{q+1}v^p_{q+1}\otimes\lambda_{q+1}v^p_{q+1})+\nabla\cdot R_q+\partial_t w^t_{q+1}\end{aligned}\tag{4.58}$$

Notice that the last three terms together are exactly the oscillation part $\nabla\cdot\tilde R_{\mathrm{oscillation}}$ for the NSE in [5], and thus can be estimated the same way. Thus we are left to estimate the difference

$$\nabla\cdot\mathcal{D}_{q+1}=:\nabla\times\nabla\times((\nabla\times v^p_{q+1})\times v^p_{q+1})-\nabla\cdot(\lambda_{q+1}v^p_{q+1}\otimes\lambda_{q+1}v^p_{q+1})$$

which will be shown to be small enough. Indeed, we first recall that

$$w^p_{q+1}=\nabla\times v^p_{q+1}=\lambda_{q+1}v^p_{q+1}+\lambda_{q+1}^{-1}\sum_{\xi\in\Lambda}\nabla(a_\xi\eta_\xi)\times W_\xi:=\lambda_{q+1}v^p_{q+1}+\mathbb{W}_{\epsilon,1}.$$

Thus, we have

$$\lambda_{q+1} v_{q+1}^p = w_{q+1}^p - \mathbb{W}_{\epsilon,1}.$$

On the other hand, we notice that

$$\begin{aligned}\nabla \times w_{q+1}^p =&\lambda_{q+1} \sum_{\xi\in\Lambda} a_\xi \mathbb{W}_\xi + \sum_{\xi\in\Lambda} \nabla(a_\xi\eta_\xi)\times W_\xi + \nabla\times\mathbb{W}_{\epsilon,1}\\ =&\lambda_{q+1}\left(w_{q+1}^p - \mathbb{W}_{\epsilon,1}\right) + \lambda_{q+1}\mathbb{W}_{\epsilon,1} + \nabla\times\mathbb{W}_{\epsilon,1}\\ =&\lambda_{q+1} w_{q+1}^p + \nabla\times\mathbb{W}_{\epsilon,1}.\end{aligned}$$

Therefore, a straightforward computation leads to

$$\begin{aligned}\nabla\cdot\mathcal{D}_{q+1} =&\nabla\times\nabla\times(w_{q+1}^p\times v_{q+1}^p) - \nabla\cdot\left((w_{q+1}^p - \mathbb{W}_{\epsilon,1})\otimes(w_{q+1}^p - \mathbb{W}_{\epsilon,1})\right)\\ =&\nabla\cdot\left((\nabla\times w_{q+1}^p)\otimes v_{q+1}^p + v_{q+1}^p\otimes(\nabla\times w_{q+1}^p) - \nabla(w_{q+1}^p\times v_{q+1}^p)\right)\\ &- w_{q+1}^p\cdot\nabla w_{q+1}^p - \nabla\cdot\left(w_{q+1}^p\otimes w_{q+1}^p\right) - \nabla\cdot\left(\mathbb{W}_{\epsilon,1}\otimes w_{q+1}^p\right)\\ &- \nabla\cdot\left(w_{q+1}^p\otimes\mathbb{W}_{\epsilon,1}\right) + \nabla\cdot(\mathbb{W}_{\epsilon,1}\otimes\mathbb{W}_{\epsilon,1})\\ =&\nabla\cdot\left((\lambda_{q+1}w_{q+1}^p - \nabla\times\mathbb{W}_{\epsilon,1})\otimes v_{q+1}^p + v_{q+1}^p\otimes(\lambda_{q+1}w_{q+1}^p - \nabla\times\mathbb{W}_{\epsilon,1})\right)\\ &- \nabla\cdot\nabla\left((\lambda_{q+1}v_{q+1}^p + \mathbb{W}_{\epsilon,1})\times v_{q+1}^p\right) - w_{q+1}^p\cdot\nabla w_{q+1}^p - \nabla\cdot\left(w_{q+1}^p\otimes w_{q+1}^p\right)\\ &- \nabla\cdot\left(\mathbb{W}_{\epsilon,1}\otimes w_{q+1}^p\right) - \nabla\cdot\left(w_{q+1}^p\otimes\mathbb{W}_{\epsilon,1}\right) + \nabla\cdot(\mathbb{W}_{\epsilon,1}\otimes\mathbb{W}_{\epsilon,1})\\ =&\nabla\cdot\left(w_{q+1}^p\otimes\lambda_{q+1}v_{q+1}^p + \lambda_{q+1}v_{q+1}^p\otimes w_{q+1}^p\right)\\ &- \nabla\cdot\left(\nabla\times\mathbb{W}_{\epsilon,1}\otimes v_{q+1}^p + v_{q+1}^p\otimes\nabla\times\mathbb{W}_{\epsilon,1}\right) - \nabla\cdot\nabla\left(\mathbb{W}_{\epsilon,1}\times v_{q+1}^p\right)\\ &- w_{q+1}^p\cdot\nabla w_{q+1}^p - \nabla\cdot\left(w_{q+1}^p\otimes w_{q+1}^p\right)\\ &- \nabla\cdot\left(\mathbb{W}_{\epsilon,1}\otimes w_{q+1}^p\right) - \nabla\cdot\left(w_{q+1}^p\otimes\mathbb{W}_{\epsilon,1}\right) + \nabla\cdot(\mathbb{W}_{\epsilon,1}\otimes\mathbb{W}_{\epsilon,1})\\ =&\nabla\cdot\left(w_{q+1}^p\otimes(w_{q+1}^p - \mathbb{W}_{\epsilon,1}) + (w_{q+1}^p - \mathbb{W}_{\epsilon,1})\otimes w_{q+1}^p\right)\\ &- \nabla\cdot\left(\nabla\times\mathbb{W}_{\epsilon,1}\otimes v_{q+1}^p + v_{q+1}^p\otimes\nabla\times\mathbb{W}_{\epsilon,1}\right) - \nabla\cdot\nabla\left(\mathbb{W}_{\epsilon,1}\times v_{q+1}^p\right)\\ &- w_{q+1}^p\cdot\nabla w_{q+1}^p - \nabla\cdot\left(w_{q+1}^p\otimes w_{q+1}^p\right)\\ &- \nabla\cdot\left(\mathbb{W}_{\epsilon,1}\otimes w_{q+1}^p\right) - \nabla\cdot\left(w_{q+1}^p\otimes\mathbb{W}_{\epsilon,1}\right) + \nabla\cdot(\mathbb{W}_{\epsilon,1}\otimes\mathbb{W}_{\epsilon,1})\\ =&-\nabla\cdot\left(w_{q+1}^p\otimes\mathbb{W}_{\epsilon,1} + \mathbb{W}_{\epsilon,1}\otimes w_{q+1}^p\right)\\ &- \nabla\cdot\left(\nabla\times\mathbb{W}_{\epsilon,1}\otimes v_{q+1}^p + v_{q+1}^p\otimes\nabla\times\mathbb{W}_{\epsilon,1}\right) - \nabla\cdot\nabla\left(\mathbb{W}_{\epsilon,1}\times v_{q+1}^p\right)\\ &- \nabla\cdot\left(\mathbb{W}_{\epsilon,1}\otimes w_{q+1}^p\right) - \nabla\cdot\left(w_{q+1}^p\otimes\mathbb{W}_{\epsilon,1}\right) + \nabla\cdot(\mathbb{W}_{\epsilon,1}\otimes\mathbb{W}_{\epsilon,1})\end{aligned}$$

where we used the fact that $\nabla\cdot w_{q+1}^p = 0$. The next step is to estimate the terms of $\mathcal{D}_{q+1}$. We also notice that $\nabla\times\mathbb{W}_{\epsilon,1} = w_{q+1}^c$.

$$\begin{aligned}&\|w_{q+1}^p\otimes\mathbb{W}_{\epsilon,1} + \mathbb{W}_{\epsilon,1}\otimes w_{q+1}^p\|_{L^p}\\ \lesssim&\lambda_{q+1}^{-1}\left\|w_{q+1}^p\sum_{\xi\in\Lambda}\nabla(a_\xi\eta_\xi)\right\|_{L^p}\\ \lesssim&\lambda_{q+1}^{-1}\|w_{q+1}^p\|_{L^p}\left\|\sum_{\xi\in\Lambda}\nabla(a_\xi\eta_\xi)\right\|_{L^\infty}\\ \lesssim&\lambda_{q+1}^{-1}\delta_{q+1}^{\frac12}\ell^{-\frac12(1-\frac1p)}r^{\frac32-\frac3p}\left(\ell^{-1}r^{\frac32} + \delta_{q+1}^{\frac12}\ell^{-\frac12}\lambda_{q+1}\sigma r^{\frac52}\right)\\ \lesssim&\lambda_{q+1}^{-2\varepsilon_R}\delta_{q+2}.\end{aligned}$$

□

4.5. **The energy iteration.**

Lemma 4.10. *If $\rho_0(t) \neq 0$, then the energy of the current density J_{q+1} satisfies*

$$\left| E(t) - \int_{\mathbb{T}^3} |J_{q+1}(x,t)|^2 \, dx - \frac{\delta_{q+2}}{2} \right| \leq \frac{\delta_{q+2}}{4}.$$

Lemma 4.11. *If $\rho_0(t) = 0$, then $J_{q+1}(\cdot, t) \equiv 0$, $R_{q+1}(\cdot, t) \equiv 0$ and*

$$E(t) - \int_{\mathbb{T}^3} |J_{q+1}(x,t)|^2 \, dx \leq \frac{3}{4}\delta_{q+2}.$$

The proof of Lemma 4.10 and Lemma 4.11 follows closely as the proof of Lemma 6.2 and Lemma 6.3 in [5]. The two estimates in Lemma 4.10 and Lemma 4.11 immediately implies (4.20) for $q+1$. On the other hand, if

$$E(t) - \int_{\mathbb{T}^3} |J_{q+1}(x,t)|^2 \, dx \leq \frac{\delta_{q+2}}{100},$$

it follows from Lemma 4.10 that $\rho_0(t) = 0$. Thus, Lemma 4.11 guarantees $J_{q+1}(t) \equiv 0$ and $R_{q+1}(t) \equiv 0$, which shows (4.21) for $q+1$.

Now we can conclude that the proof of Proposition 4.1 is complete.

5. Non-uniqueness of the Hall MHD system

In this section, we come back to the 3D Hall-MHD system (1.1) with $\zeta = 1$ and demonstrate that non-unique Leray-Hopf weak solutions can be actually constructed for this coupled system of the NSE and the Hall equation. That is, we prove Theorem 1.2.

We consider the approximating system

$$\begin{aligned} \partial_t u_q + (u_q \cdot \nabla) u_q + \nabla p_q &= \Delta u_q + (B_q \cdot \nabla) B_q, \\ \partial_t J_q + \nabla \times \nabla \times (B_q \times u_q) + \nabla \times \nabla \times (J_q \times B_q) &= \Delta J_q + \nabla \cdot R_q^s \\ \nabla \cdot u_q &= 0. \end{aligned} \tag{5.59}$$

The plan is to apply convex integration framework only to the equation of the current density and solve the NSE at every level of the convex integration. The detailed scheme is described below:

- Start with $(u_0, B_0, J_0, R_0^s) = (0,0,0,0)$ which satisfies (5.59) automatically;
- Construct appropriate perturbations $w_1 = J_1 - J_0$ and $v_1 = B_1 - B_0$ for the J_1 equation of (5.59) in the spirit of convex integration applied to the pure Hall equation in Section 4; for such $B_1 = v_1 + B_0$, we solve the NSE of u_1 in (5.59);
- Take the subtraction of equation J_q in (5.59) at levels $q = 1$ and $q = 0$ and obtain R_1^s; thus we obtain (u_1, B_1, J_1, R_1^s) satisfying the system (5.59) at level $q = 1$;
- Repeat the last two steps iteratively to generate a sequence $\{(u_q, B_q, J_q, R_q^s)\}$ for $q \geq 0$;
- Prove that the sequence $\{(u_q, B_q, J_q, R_q^s)\}$ converges to $(u, B, J, 0)$ with functions u, B, J satisfying

  $$J = \nabla \times B, \quad u \in L^\infty(L^2) \cap L^2(H^1), \quad B \in L^\infty(L^2) \cap L^2(H^1),$$

  and (u, B) is a Leray-Hopf weak solution of the Hall-MHD system (1.1).

With the construction of perturbations at hand (refer to Section 4), the rest of the scheme involves two major bulks: solving the NSE of u_q and applying convex integration on the J_q equation. Details are demonstrated by proving the following iterative argument.

Proposition 5.1. *There exists an absolute constant $C > 0$ and a sufficiently small parameter ε_R depending on b and β such that the following inductive statement holds. Let $(u_q, p_q, B_q, J_q, R^s_q)$ be a solution of the approximating equation (5.59) on $\mathbb{T}^3 \times [0, T]$ satisfying:*

$$\|B_q\|_{C^1_{x,t}} \le \lambda_q^3, \tag{5.60}$$

$$\|J_q\|_{C^1_{x,t}} \le \lambda_q^4, \tag{5.61}$$

$$0 \le E(t) - \int_{\mathbb{T}^3} |J_q|^2 \, \mathrm{d}x \le \delta_{q+1}, \tag{5.62}$$

and

$$E(t) - \int_{\mathbb{T}^3} |J_q|^2 \, \mathrm{d}x \le \frac{\delta_{q+1}}{100} \quad \textit{implies} \quad J_q(\cdot, t) \equiv 0 \textit{ and } R^s_q(\cdot, t) \equiv 0. \tag{5.63}$$

In addition, we assume

$$\nabla \cdot R^s_q = \nabla \cdot R_q + \nabla \times \nabla \times M^\epsilon_q \tag{5.64}$$

with R_q being the stress tensor in (3.9) and M^ϵ_q being a vector field. Then we can find another solution $(u_{q+1}, p_{q+1}, B_{q+1}, J_{q+1}, R^s_{q+1})$ of (5.59) satisfying (5.60)-(5.64) with q replaced by $q+1$. Moreover, R_q satisfies the properties in Proposition 4.1; the increments $v_{q+1} = B_{q+1} - B_q$, $w_{q+1} = J_{q+1} - J_q$, $z_{q+1} = u_{q+1} - u_q$, and M^ϵ_q satisfy

$$\|v_{q+1}\|_{L^2} \le C\lambda_{q+1}^{-1}\delta_{q+1}^{1/2}, \quad \|w_{q+1}\|_{L^2} \le C\delta_{q+1}^{1/2}, \tag{5.65}$$

$$\lim_{q\to\infty} \|z_{q+1}\|_{L^p} = 0, \quad 1 \le p \le 2, \tag{5.66}$$

$$\lim_{q\to\infty} \|M^\epsilon_q\|_{L^p} = 0, \quad \textit{for } p > 1 \textit{ close enough to } 1. \tag{5.67}$$

In analogy with Proposition 4.1 and Theorem 1.1, a proof of Theorem 1.2 follows immediately from Proposition 5.1; thus the details are omitted.

Regarding the proof of Proposition 5.1, we emphasize again that we adapt the same construction for perturbations of $v_{q+1} = B_{q+1} - B_q$ and $w_{q+1} = J_{q+1} - J_q$ as for the Hall equation in Section 4; the stress tensor R_q in (5.64) is the same stress tensor in the approximating equation (3.9); while the vector M^ϵ_q comes from the nonlinear interaction of $u_q \times B_q$ and will be shown to be small. Thus the estimates for v_{q+1} and w_{q+1} in Lemma 4.4 and Lemma 4.5, respectively, are valid; and the estimates for R_{q+1} in Lemma 4.6 also hold. In particular, estimates (5.60), (5.61), and (5.65) automatically hold.

We focus on completing the proof of Proposition 5.1 in the two subsections below.

5.1. **Weak solution u_{q+1} of the NSE in $L^\infty(L^2) \cap L^2(H^1)$.** We consider the forced NSE

$$\partial_t u_{q+1} + (u_{q+1} \cdot \nabla)u_{q+1} + \nabla p_{q+1} = \Delta u_{q+1} + \nabla \cdot (B_{q+1} \otimes B_{q+1}). \tag{5.68}$$

By construction, we have

$$B_{q+1} = B_0 + \sum_{j=0}^{j=q} v_{j+1}, \quad J_{q+1} = J_0 + \sum_{j=0}^{j=q} w_{j+1}$$

with $\|v_{q+1}\|_{L^2} \le C\lambda_{q+1}^{-1}\delta_{q+1}^{1/2}$ and $\|w_{q+1}\|_{L^2} \le C\delta_{q+1}^{1/2}$. It is then obvious that $\|B_{q+1}\|_{L^2} \le C$ and $\|J_{q+1}\|_{L^2} \le C$ which implies $B_{q+1} \in L^\infty(0,T;H^1(\mathbb{T}^3))$, since B_{q+1} is divergence free.

It follows from the Sobolev embedding theorem that $B_{q+1}\otimes B_{q+1}$ is in $L^2(0,T;L^3(\mathbb{T}^3))$, and hence in $L^2(0,T;L^2(\mathbb{T}^3))$ as well. Thus we have $\nabla\cdot(B_{q+1}\otimes B_{q+1}) \in L^2(0,T;W^{-1,2})$. Then there exists a weak solution u_{q+1} of (5.68) with $u_{q+1} \in L^\infty(0,T;L^2(\mathbb{T}^3)) \cap L^2(0,T;H^1(\mathbb{T}^3))$, see [30].

Upon writing u_{q+1} as the sum of increments,

$$u_{q+1} = u_0 + \sum_{j=0}^{j=q}(u_{j+1}-u_j) = u_0 + \sum_{j=0}^{j=q} z_{j+1},$$

the fact $u_{q+1} \in L^\infty(0,T;L^2(\mathbb{T}^3))$ implies

$$\lim_{q\to\infty}\|z_{q+1}(t)\|_{L^2(\mathbb{T}^3)} = 0, \ t\in[0,T].$$

Moreover, we have $\lim_{q\to\infty}\|z_{q+1}(t)\|_{L^p(\mathbb{T}^3)} = 0$, $0 \le t \le T$, for all $p \in [1,2]$. Therefore, (5.66) is justified.

5.2. **Convex integration for the Maxwell equation.** With $v_{q+1} = B_{q+1} - B_q$ and $w_{q+1} = J_{q+1} - J_q$ constructed as in Section 4 and u_{q+1} obtained in Section 5.1, we operate the convex integration method on the J_{q+1} equation in (5.59). Compared to the J_q equation in (3.9), there is one extra term $\nabla\times\nabla\times(B_q\times u_q)$ in the J_q equation of (5.59). Thus, R^s_{q+1} will be different from R_{q+1} due to the interaction of this extra nonlinear term. In fact, taking the subtraction of the J_{q+1} equation and J_q equation in (5.59), it is not hard to see

$$\nabla\cdot R^s_{q+1} = \nabla\cdot R_{q+1} + \nabla\times\nabla\times(B_{q+1}\times u_{q+1} - B_q\times u_q)$$

and we denote $M^\epsilon_{q+1} = B_{q+1}\times u_{q+1} - B_q\times u_q$. An obvious rearrangement yields

$$M^\epsilon_{q+1} = B_{q+1}\times u_{q+1} - B_q\times u_q = v_{q+1}\times u_{q+1} + B_q\times z_{q+1}.$$

Therefore, we deduce from (4.38), (4.39), and (4.40), for $p>1$ close to 1,

$$\begin{aligned}
\|M^\epsilon_{q+1}\|_{L^p} &\le \|v_{q+1}\times u_{q+1}\|_{L^p} + \|B_q\times z_{q+1}\|_{L^p}\\
&\lesssim \|v_{q+1}\|_{L^{\frac{2p}{2-p}}}\|u_{q+1}\|_{L^2} + \|B_q\|_{L^{\frac{2p}{2-p}}}\|z_{q+1}\|_{L^2}\\
&\lesssim \|v_{q+1}\|_{L^{\frac{2p}{2-p}}}\|u_{q+1}\|_{L^2} + \|z_{q+1}\|_{L^2}\sum_{j=0}^{j=q-1}\|v_{j+1}\|_{L^{\frac{2p}{2-p}}}\\
&\lesssim \lambda_{q+1}^{-1}\delta_{q+1}^{1/2}r^{\frac32-\frac{3(2-p)}{2p}} + \|z_{q+1}\|_{L^2}\sum_{j=0}^{j=q-1}\lambda_{j+1}^{-1}\delta_{j+1}^{1/2}\lambda_{j+1}^{\frac34\left[\frac32-\frac{3(2-p)}{2p}\right]}\\
&\lesssim \lambda_{q+1}^{-1}\delta_{q+1}^{1/2}r^{3-\frac3p} + \|z_{q+1}\|_{L^2}.
\end{aligned}$$

Therefore, along with (5.66), we can conclude (5.67).

Regarding the energy iteration properties (5.62) and (5.63), they can be obtained in a similar way as of (4.20) and (4.21). Indeed, we notice that the J_q equation in (5.59) differs from the J_q equation (3.9) by the nonlinear term $\nabla\times\nabla\times(u_q\times B_q)$, which is smaller than the nonlinear portion of the Hall term $\nabla\times\nabla\times(J_q\times B_q)$ (up to scale λ_q^{-1}). Therefore, when deriving (5.62) and (5.63) for $q+1$, the nonlinear term $\nabla\times\nabla\times(u_{q+1}\times B_{q+1})$ can be treated as a small error term and hence absorbed by other terms in the estimates. Thus, slight modification of the proof of energy iteration in [5] will yield (5.62) and (5.63).

We conclude the proof of Proposition 5.1.

Proof of Theorem 1.2: We are left to show that the sequence $\{(u_q, B_q)\}_{q=1}^\infty$ converges to a pair $(u, B) \in \left(L^\infty(0, T; L^2(\mathbb{T}^3)) \cap L^2(0, T; H^1(\mathbb{T}^3))\right)^2$ which solves the Hall MHD (1.1).

For given (u_0, B_0, J_0, R_0), we apply Proposition 5.1 iteratively to obtain a sequence of approximating solutions $\{(u_q, B_q, J_q, R_q)\}$ satisfying (5.60)-(5.67). It follows from (5.65) that

$$\sum_{q\geq 0} \|J_{q+1} - J_q\|_{L^2} = \sum_{q\geq 0} \|w_{q+1}\|_{L^2} \lesssim \sum_{q\geq 0} \delta_{q+1}^{1/2} < \infty.$$

which implies the strong convergence of $J_q = \nabla \times B_q$ to a function J in $C^0(0, T; L^2)$, and the strong convergence of B_q to a function B in $C^0(0, T; H^1)$ with $J = \nabla \times B$ and $\nabla \cdot B = 0$.

According to the analysis above, we have $u_q \in L^\infty(0, T; L^2(\mathbb{T}^3)) \cap L^2(0, T; W^1(\mathbb{T}^3))$ for all $q \geq 1$. It follows the weak convergence of u_q to a function u in $L^\infty(0, T; L^2)$. Combining the fact of B_q converging to B strongly in $C^0(0, T; H^1)$, we obtain that (u, B) solves the NSE part of (1.1) in the weak sense.

On the other hand, the facts $\|R_q\|_{L^\infty(0,T;L^1)} \to 0$ and $\|M_q^\epsilon\|_{L^\infty(0,T;L^1)} \to 0$ as $q \to \infty$ lead to $\|R_q^s\|_{L^\infty(0,T;L^1)} \to 0$ as $q \to \infty$. Thus, (u, B) also solves the second equation of (1.1) in the weak sense. It indicates that (u, B) is a weak solution of (1.1).

To show convergence of u_q to a function u in $C^0(0, T; L^2)$, do we need to estimate $\partial_t u_q$?

$$\|\partial_t u_q\|_{H^{-2}} \lesssim \|\Delta u_q - \nabla \cdot (u_q \times u_q) + \nabla \cdot (B_q \times B_q)\|_{H^{-2}}$$

6. Appendix: Vector calculus identities

Let A and B be vector valued functions, and φ be a scalar function. The following identities hold:

$$\begin{aligned}
\nabla \times (\varphi A) &= \varphi(\nabla \times A) + (\nabla\varphi) \times A; \\
\nabla(A \cdot B) &= B \cdot \nabla A + A \cdot \nabla B; \\
\nabla \cdot (A \times B) &= (\nabla \times A) \cdot B - A \cdot (\nabla \times B); \\
\nabla \times (A \times B) &= A(\nabla \cdot B) - B(\nabla \cdot A) + B \cdot \nabla A - A \cdot \nabla B \\
&= \nabla \cdot (BA^T - AB^T); \\
\nabla \times (\nabla \times A) &= \nabla(\nabla \cdot A) - \nabla^2 A = \nabla(\nabla \cdot A) - \Delta A.
\end{aligned}$$

Applying the identities above, one can rewrite

$$\begin{aligned}
&\nabla \times \nabla \times [(\nabla \times B) \times B] \\
&= \nabla(\nabla \cdot [(\nabla \times B) \times B]) - \Delta[(\nabla \times B) \times B] \\
&= \nabla(\nabla \times (\nabla \times B) \cdot B - \nabla \times B \cdot \nabla \times B) - \Delta[(\nabla \times B) \times B] \\
&= \nabla \times (\nabla \times B) \cdot \nabla B + B \cdot \nabla\nabla \times (\nabla \times B) - \nabla \frac{|\nabla \times B|^2}{2} - \nabla \cdot \nabla[(\nabla \times B) \times B].
\end{aligned}$$

Assume B is divergence free. Let $J = \nabla \times B$, then $\nabla \cdot J = 0$ and $\nabla \cdot \nabla \times J = 0$. Thus we can further rewrite

$$\begin{aligned}&\nabla \times \nabla \times ((\nabla \times B) \times B)\\ =&\nabla \cdot [(\nabla \times J) \otimes B + B \otimes (\nabla \times J) - \nabla(J \times B)] - \nabla \frac{|J|^2}{2}.\end{aligned}$$

One can also derive the identity,

$$\Delta(\nabla \times B) = \nabla \times (\Delta B).$$

Thus, taking curl of the Hall equation

$$B_t + \nabla \times ((\nabla \times B) \times B) = \Delta B,$$

we obtain the equation of the current density $J = \nabla \times B$,

$$J_t + \nabla \cdot [(\nabla \times J) \otimes B + B \otimes (\nabla \times J) - \nabla(J \times B)] - \nabla \frac{|J|^2}{2} = \Delta J.$$

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