

SPECTRAL DIMENSION FOR β -ALMOST PERIODIC SINGULAR JACOBI OPERATORS AND THE EXTENDED HARPER'S MODEL

RUI HAN, FAN YANG AND SHIWEN ZHANG

ABSTRACT. We study fractal dimension properties of singular Jacobi operators. We prove quantitative lower spectral/quantum dynamical bounds for general operators with strong repetition properties and controlled singularities. For analytic quasiperiodic Jacobi operators in the positive Lyapunov exponent regime, we obtain a sharp arithmetic criterion of full spectral dimensionality. The applications include the extended Harper's model where we obtain arithmetic results on spectral dimensions and quantum dynamical exponents.

1. INTRODUCTION

In this paper, we study self-adjoint Jacobi operators on $\ell^2(\mathbb{Z})$ given by:

$$(1.1) \quad (Hu)_n = w_n u_{n+1} + \overline{w}_{n-1} u_{n-1} + v_n u_n, \quad n \in \mathbb{Z}$$

where $w_n \in \mathbb{C} \setminus \{0\}$ and $v_n \in \mathbb{R}$ are bounded sequences in n . If $w_n \equiv 1$, H is a discrete Schrödinger operator. We will focus on singular Jacobi operators, where the off-diagonal sequence w_n has an accumulation point 0 at $\pm\infty$. A prime example of such operator, both in math and in physics literature, is the extended Harper's model (EHM), see (2.13).

We are interested in the fractal decomposition of the spectral measure and quantitative spectral/quantum dynamical bounds. In a recent work of Jitomirskaya and Zhang [32], many quantitative criteria of fractal dimensions of spectral measure for lattice Schrödinger operators were obtained. Their criteria have various applications to quasiperiodic Schrödinger operators, e.g., the almost Mathieu operator or the Sturmian Hamiltonians. However, whether their results could be applied to the Jacobi case, in particular the singular Jacobi case, has remained a question. Indeed, many generalizations of the spectral theory from the Schrödinger case to the singular Jacobi case have shown to be highly nontrivial, e.g. [23, 28, 37, 22, 21, 3, 41].

In this paper we give general sufficient conditions for spectral continuity in the singular Jacobi case, see Theorem 2.1. We show spectral continuity follows if the parameters w_n, v_n of H satisfy: (i) strong repetition properties; (ii) control of the averaged closeness between w_n and 0. In particular, condition (ii) is imposed to control the strength of singularity. We also show such operators exist widely in the general context of quasiperiodic setting. In the positive Lyapunov exponent regime of analytic quasiperiodic Jacobi operators, the general statement leads to the first arithmetic if-and-only-if criterion for full spectral dimensionality. Notably, our results have applications to the extended Harper's model for both spectral and quantum dynamical properties.

Our proof is based on a general dynamical system approach, which has recently shown to be extremely powerful in the study of spectral properties, e.g. [35, 5, 34, 17, 12, 24, 36]. The eigenvalue equation of (1.1), is associated to a linear cocycle system, see (3.3). In the Schrödinger case the cocycles are $\mathrm{SL}(2, \mathbb{R})$ -valued, whereas in the singular Jacobi case the cocycles are $\mathrm{GL}(2, \mathbb{C})$ -valued with determinants approaching zero along a sub-sequence. This presents the main obstruction in [23, 28, 37, 22, 21, 3, 41] and in our paper.

It was shown in [32] that the fractal dimensions of spectral measures depend on the competition between the quality of repetitions and the growth of the Schrödinger cocycles. Such competition was

resolved in the $SL(2, \mathbb{R})$ setting involving delicate algebraic arguments, which are difficult to carry out directly in the $GL(2, \mathbb{C})$ setting due to the presence of singularity. To reduce to $SL(2, \mathbb{R})$ case, we employ a family of conjugacies which were first introduced in a recent work of Avila-Jitomirskaya-Marx [3]. Such regularization moves the singularity into the conjugate matrices. The main technical accomplishment of our work is to develop general quantitative estimates (see Lemmas 5.3, 5.4 and 5.5) of the conjugacy under assumptions (i) and (ii). The successful combination of these estimates with the mechanism in [32] proves the quantitative spectral continuity results for the singular Jacobi case.

We show the assumptions (i) and (ii) hold for singular Jacobi operators over a quasiperiodic base. In particular, the proof of (ii) is close in spirit to the characterization of singularity in [22] (see also [27, 31]). Here we need to study the finer decomposition of the singular spectral measure, thus a strengthened characterization is developed. Moreover, our estimates hold for general C^k sampling functions with finitely many non-degenerate zeros, which reduces the analytic regularity requirements in [27, 31, 22]. This part is also of independent interest in the study of uniform upper-semi continuity of the Lyapunov growth.

The rest of this paper is organized in the following way. In section 2, we give all the definitions and state our main results. After giving the preliminaries in section 3, we proceed to discuss the (Λ, β) bound in the quasiperiodic case in section 4. In section 5, we prove the general spectral continuity results. In section 6, we focus on the analytic quasiperiodic Jacobi operator and prove arithmetic if-and-only-if criterion for full spectral dimensionality. In the last section, we discuss the explicit parameter partitions for the extended Harper's model.

2. MAIN RESULTS

To formulate the main results, we introduce the following definitions.

Definition 2.1. A sequence $\{a_n\}_{n \in \mathbb{Z}}$ is said to be β - q almost periodic if there exist $\delta > 0$, $\beta > 0$, $q \in \mathbb{N}$, such that the following holds:

$$(2.1) \quad \max_{|m| \leq e^{\delta\beta q}} |a_m - a_{m \pm q}| \leq e^{-\beta q}.$$

We say $\{a_n\}_{n \in \mathbb{Z}}$ is β -almost periodic (about q_n) if there exists a sequence of positive integers $q_n \rightarrow \infty$, such that $\{a_n\}$ is β - q_n almost periodic.

Remark 2.1. The β -almost periodicity was first introduced in [32] to study quantitative spectral bounds in the Schrödinger case. Note that the β -almost periodicity does not imply the almost periodicity in the usual sense. A typical example is the sequence generated by skew-shift map $(x, y) \mapsto (x + y, y + 2\alpha)$ with a smooth sampling function $f(x, y)$ on $\mathbb{T}^2$. The sequence $v_n = f(x + ny + n(n-1)\alpha, y + 2n\alpha)$ is β -almost periodic for typical α , but not almost periodic for any α .

Definition 2.2. We say w_n is (Λ, β) - q bounded if there exist $\Lambda > 0$, $\beta > 0$, $\delta > 0$, and $q \in \mathbb{N}$, such that

$$(2.2) \quad \min_{|m| \leq e^{\delta\beta q}} \prod_{j=m}^{m+q-1} |w_j| > e^{-\Lambda q}.$$

We say w_n is (Λ, β) bounded (about q_n) if there exists a sequence of positive integers $q_n \rightarrow \infty$, such that w_n is (Λ, β) - q_n bounded.

Remark 2.2. If we only consider the maximum of (2.1) and (2.2) over $|m| \leq 2q_n$, then the standard Gordon-type argument will be enough to show the absence of point spectrum for the associated Jacobi operator, provided $\beta \gtrsim \Lambda$. Assume further the Lyapunov exponent is positive, then the operator

has purely singular continuous spectrum by Kotani theory [33], see e.g. [8, 6]. See more discussion on the Gordon-type argument and purely singular continuous spectrum in [9] and references therein.

Let μ be the spectral measure of the Jacobi operator given as in (1.1). The fractal properties of μ are closely related to the boundary behavior of its Borel transforms, see e.g. [13]. Let

$$(2.3) \quad M(E + i\varepsilon) = \int \frac{d\mu(E')}{E' - (E + i\varepsilon)}$$

be the (whole line) Weyl-Titchmarsh m -function of H . We are interested in the following fractal dimension of μ :

Definition 2.3. We say μ is (upper) γ -spectral continuous if for some $\gamma \in (0, 1)$ and μ a.e. E , we have

$$(2.4) \quad \liminf_{\varepsilon \downarrow 0} \varepsilon^{1-\gamma} |M(E + i\varepsilon)| < \infty.$$

Define the (upper) spectral dimension of μ to be

$$(2.5) \quad \dim_{\text{spe}}(\mu) = \sup \{ \gamma \in (0, 1) : \mu \text{ is } \gamma\text{-spectral continuous} \}.$$

Our first result is about spectral continuity and the lower bound on the spectral dimension.

Theorem 2.1. *Let H be given as in (1.1) and let μ be the spectral measure of H . Assume that there are positive constants Λ, β, δ and a sequence of positive integers $q_n \rightarrow \infty$ such that w_n, v_n are β -almost periodic and w_n is (Λ, β) bounded about q_n . There exists an explicit constant $C = C(\delta, \Lambda, \|w\|_\infty, \|v\|_\infty) > 0$, such that if $\beta > C$ and $\gamma < 1 - \frac{C}{\beta}$, then μ is γ -spectral continuous. Consequently, we have the following lower bound on the spectral dimension of μ :*

$$(2.6) \quad \dim_{\text{spe}}(\mu) \geq 1 - \frac{C}{\beta}.$$

We will formulate a more precise lower bound (specifying the dependence of C on $\delta, \Lambda, \|w\|_\infty, \|v\|_\infty$) in Theorem 5.2.

It is well known that periodicity implies absolute continuity. We actually prove a quantitative weakening version of this result: β -almost periodicity implies γ -spectral continuity. On the other hand, it is well known that Gordon condition implies absence of point spectrum, which predicts purely singular continuous spectrum in many situations. Our result distinguishes the singular continuous spectrum further according to their spectral dimensions. This can be viewed as a quantitative strengthening of Gordon-type results. Quantitative results directly linking easily formulated properties of the potential to dimensional/quantum dynamical results were first proved in [32] for the Schrödinger case. Theorem 2.1 was a further generalization of this type of estimates to more general singular Jacobi operators.

An important context where we have generic β -almost periodicity and (Λ, β) bound is the quasiperiodic Jacobi operators with smooth sampling functions defined as follows. Consider real and complex valued sampling functions $v : \mathbb{T} \mapsto \mathbb{R}$ and $c : \mathbb{T} \mapsto \mathbb{C}$. We also assume $\ln |c| \in L^1(\mathbb{T})$, which is the minimum requirement for the Lyapunov exponent to exist. Let $H_{\alpha, \theta} = H_{\alpha, \theta, c, v}$ be the Jacobi operator on $\ell^2(\mathbb{Z})$ given by:

$$(2.7) \quad (H_{\alpha, \theta} u)_n = c(\theta + n\alpha)u_{n+1} + \bar{c}(\theta + (n-1)\alpha)u_{n-1} + v(\theta + n\alpha)u_n, \quad n \in \mathbb{Z},$$

where $\theta \in \mathbb{T} := [0, 1]$ is the phase, $\alpha \in [0, 1] \setminus \mathbb{Q}$ and $\bar{c}(\theta)$ is the complex conjugate of $c(\theta)$ in the usual sense.

Given α , let p_n/q_n be the continued fraction approximants to α . Define

$$(2.8) \quad \beta(\alpha) := \limsup_n \frac{\ln q_{n+1}}{q_n} \in [0, \infty].$$

It is easy to check that for any Lipschitz continuous sampling functions v and c , the sequences $v(\theta + n\alpha), c(\theta + n\alpha)$ are β -almost periodic as defined in (2.1) for any $\theta \in \mathbb{T}$ and any $\beta < \beta(\alpha)/2$. See a proof about this simple fact in section 4. Furthermore, if we require some non-degenerate regularity near the zeros of c , then $c(\theta + n\alpha)$ will be (Λ, β) bounded for a.e. θ . As a consequence of Theorem 2.1, we have spectral continuity for a.e. θ for (2.7). More precisely, let $\mu_{\alpha, \theta}$ be the spectral measure of $H_{\alpha, \theta}$ (2.7), we have:

Corollary 2.2. *Assume $v(\theta)$ is Lipschitz continuous on $\mathbb{T}$ and $c(\theta)$ is C^k continuous on $\mathbb{T}$ with finitely many non-degenerate zeros¹. For all $k = 1, 2, \dots$, there exists an explicit constant $C = C(c, v, k) > 0$ and a full measure set $\Theta = \Theta(\alpha, c) \subsetneq \mathbb{T}$, only depending on α and the zeros of $c(\theta)$ with the following properties: suppose $\beta(\alpha) > C$, then for any $\theta \in \Theta$,*

- (a): $H_{\alpha, \theta}$ has no eigenvalues in the spectrum;
- (b): the spectral dimension of $\mu_{\alpha, \theta}$ is bounded from below as:

$$(2.9) \quad \dim_{\text{spe}}(\mu_{\alpha, \theta}) \geq 1 - \frac{C}{\beta}.$$

In particular, if $\beta(\alpha) = \infty$, then for a.e. θ , $\dim_{\text{spe}}(\mu_{\alpha, \theta}) = 1$.

We will prove (Λ, β) bound of $c(\theta + n\alpha)$ for a.e. θ in section 4 and then part (b) follows directly from Theorem 2.1. The main ingredient is one fundamental estimate (see Lemma 4.1) about the trigonometric product over irrational rotation in [1]. Similar arguments have been used in [27, 22, 31] to study the arithmetic criterion of purely singular continuous spectrum. In those papers, the authors considered periodic approximation based on Gordon-type argument. The growth of the transfer matrix only need to be controlled within at most two periods. In our case, the quantitative spectral continuity relies on (Λ, β) bound over exponentially many periods. The use of Lemma 4.1 is more delicate and involved. See more details in Lemma 4.2.

As mentioned before, the absence of point spectrum in part (a) is a direct consequence of the (Λ, β) boundedness of $c(\theta + n\alpha)$ and the standard Gordon-type argument. In view of Definition 2.3, it is easy to check that point measure has zero (spectral/Hausdorff/packing) dimension. (2.9) implies that the spectral measure $\mu_{\alpha, \theta}$ has positive spectral dimension for $\beta > C$. Part (a) can also be derived as a corollary of (2.9). An interesting question that remained here is whether the assumption on $c(\theta)$ can be weakened: For example, could any Lipschitz continuous function with finitely many zeros generate a (Λ, β) bounded sequence? Will the associated Jacobi operator have absence of point spectrum and full spectral dimension? We will not go further in this direction in the current paper. We are planning to answer some of these questions in another paper (under preparation).

It is clear that our general results (2.6) and (2.9) only go in one direction, as even absolute continuity of the spectral measures does not imply β -almost periodicity for $\beta > 0$. However, in the important context of analytic quasiperiodic operators (e.g. EHM) this leads to a sharp if-and-only-if result in the positive Lyapunov exponent regime.

Let $H_{\alpha, \theta}$ be the Jacobi operator on $\ell^2(\mathbb{Z})$ defined as in (2.7). The Lyapunov exponent of $H_{\alpha, \theta}$ at energy E is defined through the associated skew-product over irrational rotations (quasiperiodic cocycles). For any irrational α , the Lyapunov exponent is only a function of E, α and is independent of θ , therefore, denoted as $L(E, \alpha)$. See more basic properties and discussions about Lyapunov exponent in section 3.

¹We say $\theta_0 \in \mathbb{T}$ is a non-degenerate zero of $f \in C^k(\mathbb{T}, \mathbb{C})$ if $f(\theta_0) = 0$ and $f^{(k)}(\theta_0) \neq 0$.

Assume further v, c of $H_{\alpha, \theta}$ are analytic on $\mathbb{T}$ with real and complex values, respectively. Let $\mu_{\alpha, \theta, \Sigma_+}$ be the restriction of the spectral measure $\mu_{\alpha, \theta}$ of $H_{\alpha, \theta}$ on $\Sigma_+ := \{E \in \sigma(H_{\alpha, \theta}) : L(E, \alpha) > 0\}$. We have the following sharp estimate on the spectral dimension of $\mu_{\theta, \alpha, \Sigma_+}$:

Theorem 2.3. *For any $\alpha \in [0, 1]$, let $\beta(\alpha)$ be defined as in (2.8). For any analytic sampling functions v and c , there is a full Lebesgue measure set $\Theta = \Theta(\alpha, c) \subset \mathbb{T}$ explicitly depends on α and c such that for any $\theta \in \Theta$, $\dim_{\text{spe}}(\mu_{\alpha, \theta, \Sigma_+}) = 1$ if and only if $\beta(\alpha) = \infty$.*

The proof of Theorem 2.3 contains two parts. Clearly, the ‘if’ part of Theorem 2.3 is a direct consequence of spectral continuity and follows from Corollary 2.2. The ‘only if’ part is usually referred to as the so-called spectral singularity, defined through the singular boundary behavior of the m function. More precisely, we say the spectral measure μ is (upper) γ -spectral singular if for some $\gamma \in (0, 1)$ and μ a.e. E ,

$$(2.10) \quad \liminf_{\varepsilon \downarrow 0} \varepsilon^{1-\gamma} |M(E + i\varepsilon)| = +\infty.$$

Define

$$(2.11) \quad \dim_{\widehat{\text{spe}}} = \inf \{ \gamma \in (0, 1) : \mu \text{ is } \gamma\text{-spectral singular} \}.$$

Obviously, $\dim_{\text{spe}} \leq \dim_{\widehat{\text{spe}}}$. Theorem 2.3 also holds for $\dim_{\widehat{\text{spe}}}$. We actually can prove the following local quantitative upper bound of the spectral dimension which completes the sufficient part of Theorem 2.3.

Theorem 2.4. *Consider the quasiperiodic Jacobi operator defined in (2.7) with analytic sampling functions v, c . Let $L(E)$ be the associated Lyapunov exponent defined in (3.12). Assume $L(E) \geq a > 0$ on a compact set S . Consider the spectral measure $\mu_{\alpha, \theta}$ restricted on S , denoted by $\mu_{\alpha, \theta, S}$. Suppose $\beta(\alpha) < \infty$, then there is a $C = C(a, v, c, S) > 0$ and a full Lebesgue measure set $\Theta = \Theta(\alpha, c)$, such that for any $\theta \in \Theta$, $\mu_{\alpha, \theta}$ is γ -spectral singular for any $\gamma \geq \frac{1}{1+\frac{C}{\beta}}$. Consequently,*

$$(2.12) \quad \dim_{\text{spe}}(\mu_{\alpha, \theta, S}) \leq \dim_{\widehat{\text{spe}}}(\mu_{\alpha, \theta, S}) \leq \frac{1}{1 + \frac{C}{\beta}} < 1.$$

The spectral singularity can be viewed as “weak-type of localization”. It involves the decay of the Green’s function in a finite box with a low density (see Lemma 6.4). Such decay/localization density was previously known either with a strong non-resonance condition on ω (e.g. $\beta(\omega) = 0$, see [12]), or for a concrete example with $\beta(\omega) \lesssim L$ (see [4, 26]). Such a phenomenon was first found in [32] for general analytic quasiperiodic Schrödinger operators with extremely large β . Two crucial ingredients for the quantitative spectral singularity are:

- (1) quantitative subordinate theory (Jitomirskaya-Last inequality, Lemma 3.2);
- (2) existence of generalized eigenfunctions with sub-linear growth by Last-Simon estimate (Lemma 3.3).

Theorem 2.4 generalizes the result for Schrödinger operators in [32] to singular Jacobi operators. The techniques to deal with the singular Jacobi case are more involved and very delicate in view of the quantitative estimates (2.12). In section 6, we reduce the proof of Theorem 2.4 to a quantitative result (see Lemma 6.1) obtained in [32]. One key observation in [32] is that the norm of the analytic transfer matrix can be approximated by trigonometric polynomials with uniform linear degree. The generalization of this result to the meromorphic transfer matrix in our case (see Lemma 6.2) becomes an important part of the proof of Theorem 2.4.

2.1. Applications to the extended Harper's model. Quasiperiodic Jacobi operators arise naturally from the study of tight-binding electrons on a two-dimensional lattice exposed to a perpendicular magnetic field. A more general model is the extended Harper's model (EHM), defined as follow:

$$(2.13) \quad (H_{\lambda, \alpha, \theta} u)_n = c_\lambda(\theta + n\alpha)u_{n+1} + \bar{c}_\lambda(\theta + (n-1)\alpha)u_{n-1} + 2\cos 2\pi(\theta + n\alpha)u_n, \quad n \in \mathbb{Z}.$$

Here,

$$(2.14) \quad c_\lambda(\theta) = \lambda_1 e^{-2\pi i(\theta + \frac{\alpha}{2})} + \lambda_2 + \lambda_3 e^{2\pi i(\theta + \frac{\alpha}{2})},$$

$\bar{c}_\lambda(\theta)$ is the complex conjugate of $c_\lambda(\theta)$ in the usual sense and $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$ are real coupling constants. EHM was introduced by D.J.Thouless in 1983 [40], which includes the AMO as a special case.

The extended Harper's model is a prime example of quasiperiodic Jacobi matrix. It has attracted great attention from both mathematics and physics (see e.g. [7, 10, 20]) literature in the past several decades. Recent developments on the spectral theory of the AMO and EHM include: pure point spectrum for Diophantine frequencies in the positive Lyapunov exponent regime I° [23]; explicit formula for the Lyapunov exponent $L(E, \lambda)$ (see (7.3)) on the spectrum throughout all the three regions [28]; dry ten Martini problem for Diophantine frequencies in the self-dual regions [21]; complete spectral decomposition for all α and a.e. θ in the zero Lyapunov exponent regions [3]; and arithmetic spectral transition in α in the positive Lyapunov exponent regime [22].

As a central example of the analytic quasiperiodic singular Jacobi operators, Theorem 2.3 can be applied to the extended Harper's model $H_{\lambda, \alpha, \theta}$ defined in (2.13). As a consequence of the Lyapunov exponent formula of EHM in terms of the coupling constants $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, we have more explicit conclusions on the full spectral dimensionality of EHM. Moreover, our lower bounds in Theorem 2.3 are effective for $\beta > \max\{C \sup_{E \in \sigma(H)} L(E), 0\}$ by some simple scaling argument (see Lemma 4.2 and section 7). Thus the range of β is increased for smaller Lyapunov exponents. In particular, we obtain full spectral dimensionality as long as $\beta(\alpha) > 0$, when Lyapunov exponents are zero on the spectrum. This applies, in particular, to the critical EHM.

Consider the following three parameter regions of $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$:

$$\mathcal{R}_1 = \{\lambda \in \mathbb{R}^3 : 0 < \lambda_1 + \lambda_3 < 1, 0 < \lambda_2 < 1\}.$$

$$\mathcal{R}_2 = \{\lambda \in \mathbb{R}^3 : \lambda_2 > \max\{\lambda_1 + \lambda_3, 1\}, \lambda_1 + \lambda_3 \geq 0 \text{ or } \lambda_1 + \lambda_3 > \max\{\lambda_2, 1\}, \lambda_1 \neq \lambda_3, \lambda_2 > 0\}.$$

$$\mathcal{R}_3 = \{\lambda \in \mathbb{R}^3 : 0 \leq \lambda_1 + \lambda_3 \leq 1, \lambda_2 = 1 \text{ or } \lambda_1 + \lambda_3 \geq \max\{\lambda_2, 1\}, \lambda_1 = \lambda_3, \lambda_2 > 0\}.$$

Corollary 2.5. *Let $\mu_{\lambda, \alpha, \theta}$ be the spectral measure of EHM: $H_{\lambda, \alpha, \theta}$. For any $\alpha \in [0, 1]$, there is a full measure set $\Theta = \Theta(\alpha) \subset \mathbb{T}$ such that for all $\theta \in \Theta$, the following hold:*

- (1) *For $\lambda \in \mathcal{R}_1$, $\dim_{\text{spe}}(\mu_{\lambda, \alpha, \theta}) = 1$ if and only if $\beta(\alpha) = \infty$.*
- (2) *For $\lambda \in \mathcal{R}_2$ and for all $\alpha \in [0, 1]$, $\dim_{\text{spe}}(\mu_{\lambda, \alpha, \theta}) = 1$.*
- (3) *For $\lambda \in \mathcal{R}_3$, $\dim_{\text{spe}}(\mu_{\lambda, \alpha, \theta}) = 1$ if $\beta(\alpha) > 0$.*

We will see the explicit formula of the Lyapunov exponent and the spectral decomposition of EHM, in section 7. In region $\mathcal{R}_1$, EHM has positive Lyapunov for all α . Part (1) then follows from Theorem 2.3 directly. Region $\mathcal{R}_2$ is actually where EHM has purely absolutely continuous measure for all α and a.e. θ , see [3] and Theorem 7.2 in section 7. In view of Definition (2.3), it is well known that if a measure is absolutely continuous w.r.t. Lebesgue measure, then it has full spectral dimension. Part (2) is then a direct consequence of a.c. spectrum and this fact. We list part (2) here for completeness only. $\mathcal{R}_3$ is the region where EHM has zero Lyapunov exponent and purely singular continuous spectrum for almost all (θ, α) , part (3) follows from Theorem 2.1 and some technical improvements of the (Λ, β) bound for analytic sampling functions. We will discuss

in details about these three parts in section 7.

Full spectral dimensionality is defined through the boundary behavior of the Borel transform of the spectral measure. It implies a range of properties, in particular, maximal packing dimension and quasiballistic quantum dynamics. Thus our criterion links way a purely analytic property of the spectral measure to arithmetic property of the frequency in a sharp. In particular, consider $H_{\lambda,\alpha,\theta}$, the extended Harper's model (EHM) given in (2.13). In this part, we will focus on EHM and discuss the consequences of the full spectral dimensionality in terms of these explicit parameters.

Recall that the Hausdorff/packing dimension of a (Borel) measure μ , namely, $\dim_H(\mu)/\dim_P(\mu)$ is defined through the $\limsup/\liminf$ (μ almost everywhere) of its γ -derivative

$$\lim_{\varepsilon \downarrow 0} \frac{\ln \mu(E - \varepsilon, E + \varepsilon)}{\ln \varepsilon}.$$

If the $\liminf$ is replaced by $\limsup$ in the definition (2.3), we can define correspondingly the lower spectral dimension $\underline{\dim}_{\text{spe}}(\mu)$. It is well known (see e.g. [10, 14, 32]) the relation between these fractal dimensions is $\dim_H(\mu) = \underline{\dim}_{\text{spe}}(\mu) \leq \dim_{\text{spe}}(\mu) \leq \dim_P(\mu)$.² Therefore, lower bounds on spectral dimension lead to lower bounds on packing dimension, thus also for the packing/upper box counting dimensions of the spectrum as a set. We obtain corresponding non-trivial results for all the above quantities. The lower bounds also provide explicit examples where the spectral measure has different Hausdorff and packing dimension.

Lower bounds on spectral dimension also have immediate applications to the lower bounds on quantum dynamics. Let $\delta_j \in \ell^2(\mathbb{Z})$ be the delta vector in the usual sense. For $p > 0$, define

$$(2.15) \quad \langle |X|_{\delta_0}^p \rangle(T) = \frac{2}{T} \int_0^\infty e^{-2t/T} \sum_n |n|^p |\langle e^{-itH} \delta_0, \delta_n \rangle|^2.$$

The power law of $\langle |X|_{\delta_0}^p \rangle(T)$ characterizes the propagation rate of $e^{-itH} \delta_0$. Define the upper/lower transport exponents to be

$$(2.16) \quad \beta_{\delta_0}^+(p) = \limsup_{T \rightarrow \infty} \frac{\ln \langle |X|_{\delta_0}^p \rangle(T)}{p \ln T}, \quad \beta_{\delta_0}^-(p) = \liminf_{T \rightarrow \infty} \frac{\ln \langle |X|_{\delta_0}^p \rangle(T)}{p \ln T}.$$

$\beta_{\delta_0}^-(p) = 1$ for all $p > 0$ corresponds to ballistic motion, $\beta_{\delta_0}^+(p) = 1$ for all $p > 0$ corresponds to quasiballistic motion. $\beta_{\delta_0}^-(p) = 0$ sometimes is called quasilocalized motion. It was proved in [19] that $\beta_{\delta_0}^+(p) \geq \dim_P(\mu)$, $\forall p > 0$. In view of Corollary 2.5, we have:

Corollary 2.6. *Let $\mu_{\lambda,\alpha,\theta}$ be the spectral measure of EHM: $H_{\lambda,\alpha,\theta}$ defined in (2.13). For any $\alpha \in [0, 1]$, there is a full measure set $\Theta = \Theta(\alpha) \subset \mathbb{T}$ such that for any $\theta \in \Theta$, $H_{\lambda,\alpha,\theta}$ has full packing dimension of $\mu_{\lambda,\alpha,\theta}$ and quasiballistic motion if*

- (1) $\lambda \in \mathcal{R}_1$ and $\beta(\alpha) = \infty$.
- (2) $\lambda \in \mathcal{R}_2$ and for all $\alpha \in [0, 1]$.
- (3) $\lambda \in \mathcal{R}_3$ and $\beta(\alpha) > 0$.

Hausdorff dimension of the spectral measure is always equal to zero for a.e. phase for any ergodic operator [39] in the regime of positive Lyapunov exponents. Combining the Lyapunov exponent formula of $H_{\lambda,\alpha,\theta}$ (see (7.3)) with the result of Simon in [39], we have $\dim_H(\mu_{\lambda,\alpha,\theta}) = 0$ for $\lambda > 1$, a.e. θ and any α . In view of part (2) of Corollary 2.6, for $\lambda > 1$, a.e. θ and $\beta(\alpha) = \infty$, we have $0 = \dim_H(\mu_{\lambda,\alpha,\theta}) < \dim_P(\mu_{\lambda,\alpha,\theta}) = 1$.

²In contrast to the Hausdorff dimension, the relation for the packing dimension only goes in one direction.

We are also interested in the fractal dimensional properties of the density states measure and the dimension of the spectrum as a set. Let $dN_{\lambda,\alpha}$ be the density states measure and $\Sigma_{\lambda,\alpha}$ be the spectrum of $H_{\lambda,\alpha,\theta}$. For irrational α , they are both θ independent. It is well known that

$$(2.17) \quad dN_{\lambda,\alpha} = \mathbb{E}_\theta(\mu_{\lambda,\alpha,\theta})$$

and $\Sigma_{\lambda,\alpha} = \text{supp}_{\text{top}}(dN_{\lambda,\alpha})$. By these relations and the general properties of the packing dimension of a measure and its topological support (see e.g. [14]), Corollary 2.6 implies that

$$(2.18) \quad \dim_P(dN_{\lambda,\alpha}) = \dim_P(\Sigma_{\lambda,\alpha}) = 1$$

in the corresponding parameter regions where EHM has full packing dimension.

For the dynamical transport part, Last in [35] proved that almost Mathieu operator with an appropriate Liouville frequency has quasiballistic motion for the first time. In general, quasiballistic property is a G_δ in any regular space, see e.g. [38, 17], thus this was known for (unspecified) topologically generic frequencies. In [32], the authors gave a precise arithmetic condition on α for the quasiballistic motion depending on whether or not Lyapunov exponent vanishes in the quasiperiodic Schrödinger setting. Here, we provide the parametric conditions for the EHM. The conclusions can also be extended directly to more general singular Jacobi operators with analytic quasiperiodic potentials.

3. PRELIMINARIES

We recall some commonly used notations for reader's convenience. We denote $L^\infty(\mathbb{T}, \mathbb{R})$ and $L^\infty(\mathbb{T}, \mathbb{C})$ to be the space of all 1-periodic bounded functions, taking values in $\mathbb{R}$ and $\mathbb{C}$ respectively. Denote the usual L^∞ norm in both spaces by $\|f\|_\infty := \sup_{x \in \mathbb{T}} |f(x)|$. Note we only require the diagonal potential function v to be real valued functions, all the other sampling function are allowed to take value in $\mathbb{C}$. We do not emphasize the real/complex value anymore unless necessary. Denote $L^1(\mathbb{T}, \mathbb{C})$ to be the usual Lebesgue space with the 1-norm $\|f\|_1 := \int_{\mathbb{T}} |f(\theta)| d\theta$. Denote $C^\omega(\mathbb{T}, \mathbb{C})$ to be the space of all 1-periodic analytic functions and denote $C^k(\mathbb{T}, \mathbb{C})$ to be the space of all functions with continuous k -th order derivatives for all $k = 0, 1, \dots, \infty$. We denote $\text{Lip}(\mathbb{T}, \mathbb{C})$ to be the space of all 1-periodic Lipschitz continuous functions, induced with the Lipschitz norm given by:

$$(3.1) \quad \|f\|_{\text{Lip}} := \|f\|_\infty + \sup_{x,y \in \mathbb{T}} \frac{|f(x) - f(y)|}{|x - y|}.$$

We identify the sequence $u = \{u_n\}_{n \in \mathbb{Z}}$ with u_n whenever it is clear that n is the index. Denote the ℓ^∞ norm of $u \in \ell^\infty(\mathbb{Z}, \mathbb{C})$ by $\|u\|_\infty := \sup_{n \in \mathbb{Z}} |u_n|$. We will denote the distance on $\mathbb{T}^1$ by $\|\theta\|_{\mathbb{T}} := \inf_{n \in \mathbb{Z}} |\theta - n|$ and may drop the subindex $\|\cdot\| = \|\cdot\|_{\mathbb{T}}$ whenever it is clear.

3.1. Transfer matrices and Lyapunov exponents. Let H be given as in (1.1):

$$(3.2) \quad (Hu)_n = w_n u_{n+1} + \bar{w}_{n-1} u_{n-1} + v_n u_n, \quad n \in \mathbb{Z}.$$

The eigenvalue equation $Hu = Eu$ can be rewritten via the following skew product:

$$(3.3) \quad \begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = A_n(E) \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix},$$

where

$$(3.4) \quad A_n(E) = \frac{1}{w_n} D_n(E), \quad D_n(E) = \begin{pmatrix} E - v_n & -\bar{w}_{n-1} \\ w_n & 0 \end{pmatrix}.$$

For $n \in \mathbb{N}^+$ and $m \in \mathbb{Z}$, define the n -step transfer matrix at position m to be

$$(3.5) \quad A(n, m; E) = \prod_{j=m}^{n+m-1} A_j(E),$$

$$(3.6) \quad D(n, m; E) = \prod_{j=m}^{n+m-1} D_j(E).$$

We denote the scalar product of w_n by the similar notation:

$$(3.7) \quad w(n, m) = \prod_{j=m}^{n+m-1} w_j, \quad m \in \mathbb{Z}, \quad n \in \mathbb{N}^+$$

and denote

$$(3.8) \quad A(n; E) = A(n, 1; E), n > 0; \quad A(0; E) = Id; \quad A(n; E) = A^{-1}(-n, n+1; E), n < 0,$$

$$(3.9) \quad D(n; E) = D(n, 1; E), n > 0; \quad D(0; E) = Id; \quad D(n; E) = D^{-1}(-n, n+1; E), n < 0,$$

$$(3.10) \quad c(n) = c(n, 1), n > 0,$$

for simplicity.

The (upper) Lyapunov exponent characterizes the grow(decay) rate of the norm of the transfer matrix $\|A(n, m)\|$, it will be convenient to introduce the Lyapunov exponent by using the dynamical notations. We refer readers to [32, 2] and references therein for the general definition of the Lyapunov exponent of linear skew product. In this part, we will restrict ourselves to the quasiperiodic cocycles. Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $A : \mathbb{T} \mapsto \text{GL}(2, \mathbb{C})$. We call (α, A) a (complex) cocycle. In view of (3.5) and (3.8), denote the transfer matrix in the quasiperiodic cocycle case by

$$(3.11) \quad A(n; \theta, \alpha) = \prod_{j=1}^n A(\theta + (j-1)\alpha), \quad \theta \in \mathbb{T}, n \in \mathbb{N}^+.$$

The Lyapunov exponent is given by the formula:

$$(3.12) \quad L(A, \alpha) = \lim_{n \rightarrow +\infty} \frac{1}{n} \int_{\mathbb{T}} \ln \|A(n; \theta, \alpha)\| d\theta = \inf_{n > 0} \frac{1}{n} \int_{\mathbb{T}} \ln \|A(n; \theta, \alpha)\| d\theta.$$

For irrational α , the point-wise limit $L(A, \alpha) = \lim_{n \rightarrow +\infty} \frac{1}{n} \ln \|A(n; \theta, \alpha)\|$ also hold true for a.e. $\theta \in \mathbb{T}$ by subadditive ergodic theory.

By uniquely ergodicity of the irrational rotations we have the following uniform upper bound (in θ) for both matrix and scalar cases:

Lemma 3.1 (e.g. [15, 30]). *If $A \in C^0(\mathbb{T}, \text{GL}(2, \mathbb{C}))$, then*

$$(3.13) \quad \limsup_{n \rightarrow +\infty} \frac{1}{n} \ln \|A(n; \theta, \alpha)\| \leq L(A, \alpha)$$

uniformly in $\theta \in \mathbb{T}$.

If $a \in C^0(\mathbb{T}, \mathbb{C})$ and $\ln |a(\theta)| \in L^1(\mathbb{T})$, then

$$(3.14) \quad \limsup_{n \rightarrow +\infty} \frac{1}{n} \ln \left| \prod_{j=1}^n a(\theta + (j-1)\alpha) \right| \leq \int_{\mathbb{T}} \ln |a(\theta)| d\theta$$

uniformly in $\theta \in \mathbb{T}$.

Remark 3.1. If $a \in C^0(\mathbb{T}, \mathbb{C})$ has no zeros, then $\frac{1}{a(\theta)}$ is also continuous. By (3.14), we have

$$(3.15) \quad \frac{1}{n} \ln \left| \prod_{j=1}^n \frac{1}{a(\theta + (j-1)\alpha)} \right| \leq \int_{\mathbb{T}} \ln \left| \frac{1}{a(\theta)} \right| d\theta + \epsilon$$

$$(3.16) \quad \iff \prod_{j=1}^n |a(\theta + (j-1)\alpha)| \geq e^{n(\int_{\mathbb{T}} \ln |a(\theta)| d\theta - \epsilon)}$$

for $n > n_0(\epsilon)$ (uniform in θ). This immediately gives the desired lower bound in (2.2) in a uniform way. If $a(\theta)$ has zeros, there is no such uniform lower bound for the scalar product anymore. One technical achievement in the paper is, with some mild assumptions on the non-degeneracy of the zeros, we are able to get a weakened version of (3.15) (see Lemma 4.2), which will be sufficient for the spectral continuity.

3.2. The Weyl-Titchmarsh m -function and subordinacy theory. The boundary behavior of the m function is linked to the power law of the half line solution and the growth of the transfer matrix norm $A(n, m; E)$ via the well known Gilbert-Pearson subordinacy theory [16, 18]. We give a brief review on m -function and the subordinacy theory. More details can be found, e.g., in [7].

Let H be as in (1.1) and $z = E + i\varepsilon \in \mathbb{C}$. Consider equation

$$(3.17) \quad Hu = zu.$$

with the family of normalized phase boundary conditions:

$$(3.18) \quad u_0^\varphi \cos \varphi + u_1^\varphi \sin \varphi = 0, \quad -\pi/2 < \varphi < \pi/2, \quad |u_0^\varphi|^2 + |u_1^\varphi|^2 = 1.$$

Let $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$ and $\mathbb{Z}^- = \{\dots, -2, -1, 0\}$. Denote by $u^\varphi = \{u_j^\varphi\}_{j \geq 0}$ the right half line solution on $\mathbb{Z}^+$ of (3.17) with boundary condition (3.18) and by $u^{\varphi, -} = \{u_j^{\varphi, -}\}_{j \leq 0}$ the left half line solution on $\mathbb{Z}^-$ of the same equation. Also denote by v^φ and $v^{\varphi, -}$ the right and left half line solutions of (3.17) with the orthogonal boundary conditions to u^φ and $u^{\varphi, -}$, i.e., $v^\varphi = u^{\varphi+\pi/2}$, $v^{\varphi, -} = u^{\varphi+\pi/2, -}$. For any function $u : \mathbb{Z}^+ \rightarrow \mathbb{C}$ we denote by $\|u\|_\ell$ the norm of u over a lattice interval of length ℓ ; that is

$$(3.19) \quad \|u\|_\ell = \left[\sum_{n=1}^{[\ell]} |u_n|^2 + (\ell - [\ell]) |u_{[\ell]+1}|^2 \right]^{1/2}.$$

Similarly, for $u : \mathbb{Z}^- \rightarrow \mathbb{C}$, we define

$$(3.20) \quad \|u\|_\ell = \left[\sum_{n=1}^{[\ell]-1} |u_{-n}|^2 + (\ell - [\ell]) |u_{-[\ell]}|^2 \right]^{1/2}.$$

For any $\varepsilon > 0$, let $\ell = \ell(\varphi, \varepsilon, E)$ be

$$(3.21) \quad \|u^\varphi\|_{\ell(\varphi, \varepsilon)} \|v^\varphi\|_{\ell(\varphi, \varepsilon)} = \frac{1}{2\varepsilon}.$$

$\ell^-(\varphi)$ is defined through the same equation by $u^{\varphi, -}, v^{\varphi, -}$. It is easy to check

$$(3.22) \quad \|u^\varphi\|_\ell \cdot \|v^\varphi\|_\ell \geq \frac{1}{2}([\ell] - 1).$$

Let $m_\varphi(z) : \mathbb{C}^+ \mapsto \mathbb{C}^+$ and $m_\varphi^-(z) : \mathbb{C}^+ \mapsto \mathbb{C}^+$ the right and left Weyl-Titchmarsh m -functions (half line) associated with the boundary condition (3.18). Let $m = m_0$ and $m^- = m_0^-$ be the half line m -functions corresponding to the Dirichlet boundary conditions. The following quantitative subordinate theory was proved in [24], well known as Jitomirskaya-Last inequality.

Lemma 3.2 (Jitomirskaya-Last inequality, Theorem 1.1 in [24]). *For $E \in \mathbb{R}$ and $\varepsilon > 0$, the following inequality holds for any $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2}]$:*

$$(3.23) \quad \frac{5 - \sqrt{24}}{|m_\varphi(E + i\varepsilon)|} < \frac{\|u^\varphi\|_{\ell(\varphi, \varepsilon)}}{\|v^\varphi\|_{\ell(\varphi, \varepsilon)}} < \frac{5 + \sqrt{24}}{|m_\varphi(E + i\varepsilon)|}.$$

There is also one general statement about the existence of generalized eigenfunctions with sub-linear growth in its ℓ -norm:

Lemma 3.3 ([36]). *For μ_θ -a.e. E , there exists $\varphi \in (-\pi/2, \pi/2]$ such that u^φ and $u^{\varphi,-}$ both obey*

$$(3.24) \quad \limsup_{\ell \rightarrow \infty} \frac{\|u\|_\ell}{\ell^{1/2} \ln \ell} < \infty.$$

This inequality provides us an upper bound for the ℓ -norm of the solution, which is crucial in the proof of the spectral singularity.

The next proposition relates the whole line m-function M and half line m-function m_φ , which can be found in [11].

Proposition 3.4 (Corollary 21 in [11]). *Fix $E \in \mathbb{R}$ and $\varepsilon > 0$,*

$$(3.25) \quad |M(E + i\varepsilon)| \leq \sup_{\varphi} |m_\varphi(E + i\varepsilon)|.$$

By this proposition, to bound M from above and get spectral continuity as in (2.4), it is enough to obtain uniform upper bounds of m_φ in boundary condition φ for the right half line problem.

For spectral singularity, we need to consider both $m_\varphi(z)$ and $m_\varphi^-(z)$. Let $(U\psi)_n = \psi_{-n+1}$, $n \in \mathbb{Z}$ be a unitary operator on $\ell^2(\mathbb{Z})$. Let $\tilde{H} = UH U^{-1}$. Denote by $\tilde{m}, \tilde{m}_\varphi, \tilde{u}^\varphi$ and $\tilde{\ell}(\varphi)$, correspondingly, m, m_φ, u^φ and $\ell(\varphi)$ of the operator $\tilde{H}$. The following facts are well known in the past literatures (see e.g. section 3, [25]). For any $\varphi \in (-\pi/2, \pi/2]$,

$$(3.26) \quad M(z) = \frac{m_\varphi(z) \tilde{m}_{\pi/2-\varphi} - 1}{m_\varphi(z) + \tilde{m}_{\pi/2-\varphi}}$$

and

$$(3.27) \quad \tilde{\ell}(\pi/2 - \varphi) = \ell^-(\varphi), \quad \|u\|_\ell = \|Uu\|_\ell.$$

In view of (2.10), a direct consequence of (3.26) is (e.g. Lemma 5 in [25]):

Lemma 3.5. *For any $0 < \gamma < 1$, suppose that there exists a $\varphi \in (-\pi/2, \pi/2]$ such that for μ -a.e. E in some Borel set S , we have that $\liminf_{\varepsilon \rightarrow 0} \varepsilon^{1-\gamma} |m_\varphi(E + i\varepsilon)| = \infty$ and $\liminf_{\varepsilon \rightarrow 0} \varepsilon^{1-\gamma} |\tilde{m}_{\pi/2-\varphi}(E + i\varepsilon)| = \infty$. Then for μ -a.e. E in S , $\liminf_{\varepsilon \rightarrow 0} \varepsilon^{1-\gamma} |M(E + i\varepsilon)| = \infty$, namely, the restriction $\mu(S \cap \cdot)$ is γ -spectral singular.*

3.3. Continued fraction. An important tool in the study of quasiperiodic sequence is the continued fraction expansion of irrational numbers. Let $\alpha \in \mathbb{T} \setminus \mathbb{Q}$, α has the following unique expression with $a_n \in \mathbb{N}$:

$$(3.28) \quad \alpha = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}.$$

Let

$$(3.29) \quad \frac{p_n}{q_n} = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\dots + \frac{1}{a_n}}}}.$$

be the continued fraction approximants of α . Let

$$\beta(\alpha) = \limsup_{n \rightarrow \infty} \frac{\ln q_{n+1}}{q_n}.$$

$\beta(\alpha)$ being large means α can be approximated very well by a sequence of rational numbers. Let us mention that $\{\alpha : \beta(\alpha) = 0\}$ is a full measure set.

The following properties about continued fraction expansion are well-known:

$$(3.30) \quad \frac{1}{2q_{n+1}} \leq \|q_n \alpha\|_{\mathbb{T}} \leq \frac{1}{q_{n+1}}.$$

For any $q_n \leq |k| < q_{n+1}$,

$$(3.31) \quad \|q_n \alpha\|_{\mathbb{T}} \leq \|k \alpha\|_{\mathbb{T}}.$$

Combining definition of $\beta(\alpha)$ (2.8) with (3.31), we have: If $\beta(\alpha) = 0$, then for any $\delta > 0$, for $|k|$ large, the following inequality holds:

$$(3.32) \quad \|k \alpha\|_{\mathbb{T}} > e^{-\delta|k|}.$$

3.4. More about β -almost periodicity and the (Λ, β) bound. In this paper, we consider bounded sequences v_n, w_n (for example, v_n, w_n are both generated by some smooth sampling functions). Let $D(n, m)$ and $w(n, m)$ be defined as in (3.6) and (3.7). The mild assumption on v_n, w_n yields the following trivial upper bound for $D(n, m)$ and $w(n, m)$: there is $\Lambda_0 = \Lambda_0(\|v\|_{\infty}, \|w\|_{\infty}) > 0$ such that for any $n \in \mathbb{N}$, and any $E \in \mathcal{N} := \mathcal{N}(H)$,

$$(3.33) \quad \sup_{m \in \mathbb{Z}} \|D(n, m; E)\| \leq e^{\Lambda_0 n},$$

$$(3.34) \quad \sup_{m \in \mathbb{Z}} |w(n, m)| \leq e^{\Lambda_0 n}.$$

Suppose w_n has (Λ, β) - q bound as in (2.2). Without of generality, we assume $\Lambda_0 = \Lambda$ for simplicity. For $1 \leq r \leq q$ and $m \in \mathbb{Z}$, write $w(q, m) = w(q - r, m + r)w(r, m)$. Combine (2.2) with the upper bound (3.34), we have

$$(3.35) \quad \min_{|m| \leq e^{\delta \beta q}} |w(r, m)| \geq e^{-2\Lambda q}, \quad 1 \leq r < q.$$

In particular, $r = 1$ gives

$$(3.36) \quad \min_{|m| \leq e^{\delta \beta q}} |w_m| \geq e^{-2\Lambda q}.$$

Assume further w_n has β - q almost periodicity as in (2.1), by (3.36), β - q periodicity can be strengthened as,

$$(3.37) \quad \max_{|m| \leq e^{\delta \beta q}} \left| \frac{w_{m \pm q}}{w_m} - 1 \right| < e^{-(\beta - 2\Lambda)q}$$

We also abuse the notation frequently by saying the operator H or the transfer matrix $A(n, E)$ has β -almost periodicity and (Λ, β) boundedness if the corresponding v_n, w_n has β almost periodicity and (Λ, β) boundedness.

The lower bound on $w(n, m)$ and upper bound on $D(n, m)$ also imply that for any $E \in \mathcal{N}$, and $|m| \leq e^{\delta \beta q}$

$$(3.38) \quad \|A(q, m)\| < e^{2\Lambda q}, \quad \max_{0 \leq r < q} \|A(r, m)\| < e^{3\Lambda q}.$$

Assume now β -almost periodicity and (Λ, β) bound hold true for the sequence $q_n \rightarrow \infty$, we will use these induced bounds (3.35)-(3.38) for the sequence q_n frequently.

4. (Λ, β) BOUND FOR QUASIPERIODIC SMOOTH SEQUENCE AND THE PROOF OF COROLLARY 2.2

Assume we have a the quasiperiodic sequence $v(\theta + n\alpha)$ generated by a Lipschitz sampling function v . Let q_n be given as in (2.8). By (2.8), for any $0 < \beta < \beta(\alpha)/2$, there is a subsequence q_{n_k} such that $\ln q_{n_k+1} > 2\beta q_{n_k}$. Then for any θ, j and $1 \leq n \leq q_{n_k}$,

$$(4.1) \quad |v(\theta + m\alpha) - v(\theta + (m \pm q_{n_k})\alpha)| \leq \|v\|_{\text{Lip}} \cdot \|q_{n_k}\alpha\| \leq \|v\|_{\text{Lip}} \cdot \frac{1}{q_{n_k+1}} \leq \|v\|_{\text{Lip}} \cdot e^{-2\beta q_{n_k}} \leq e^{-\beta q_{n_k}},$$

provided q_{n_k} large. Same computation works for c . Therefore, $v(\theta + n\alpha)$ and $c(\theta + n\alpha)$ are β -almost periodic for Lipschitz continuous v, c .

The more challenging part is the (Λ, β) bound on $c(\theta + n\alpha)$, where we need some further assumption on c . We will focus on this throughout the rest of this section.

The key ingredient for the proof of the (Λ, β) bound is the following lemma in [1]:

Lemma 4.1. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $\theta \in \mathbb{R}$ and $0 \leq j_0 \leq q_n - 1$ be such that*

$$|\sin \pi(\theta + j_0\alpha)| = \inf_{0 \leq j \leq q_n - 1} |\sin \pi(\theta + j\alpha)|,$$

then for some absolute constant $C > 0$,

$$-C \ln q_n \leq \sum_{j=0, j \neq j_0}^{q_n-1} \ln |\sin \pi(\theta + j\alpha)| + (q_n - 1) \ln 2 \leq C \ln q_n.$$

This lemma was used in [31] to prove some optimal singular continuous spectrum results. By extending the argument in [31] to exponentially many periods, we are able to prove (Λ, β) bound for any analytic sampling function. Actually, we can deal with more general sampling functions with much weaker regularities. Define

$$(4.2) \quad \mathcal{F}(\mathbb{T}, \mathbb{C}) := \left\{ c \in L^\infty(\mathbb{T}, \mathbb{C}) : \exists m \in \mathbb{N}^+, \theta_\ell \in \mathbb{T}, \tau_\ell \in (0, 1], \ell = 1, \dots, m \right. \\ \left. \text{such that } g(\theta) := \frac{c(\theta)}{\prod_{\ell=1}^m |\sin \pi(\theta - \theta_\ell)|^{\tau_\ell}} \in L^\infty(\mathbb{T}, \mathbb{C}) \text{ and } \inf_{\mathbb{T}} |g(\theta)| > 0. \right\}$$

Suppose $c(\theta) \in \mathcal{F}(\mathbb{T}, \mathbb{C})$ with θ_ℓ and $g(\theta)$ given as in (4.2) such that

$$(4.3) \quad c(\theta) = g(\theta) \prod_{\ell=1}^m |\sin \pi(\theta - \theta_\ell)|^{\tau_\ell}.$$

Clearly, $\ln |g(\theta)| \in L^1(\mathbb{T})$. By the well known integral $\int_{\mathbb{T}} \ln |\sin \pi \theta| d\theta = -\ln 2$, it is easy to check that $\ln |c(\theta)| \in L^1(\mathbb{T})$ and is linked to $\ln |g|$ by:

$$(4.4) \quad \int_{\mathbb{T}} \ln |c(\theta)| d\theta = \int_{\mathbb{T}} \ln |g(\theta)| d\theta - \ln 2 \sum_{\ell=1}^m \tau_\ell.$$

The following technique lemma shows that any sampling function in $\mathcal{F}(\mathbb{T}, \mathbb{C})$ with an irrational force can generate a (Λ, β) bounded sequence.

Lemma 4.2. *Assume that there exists $m \in \mathbb{N}^+$, $\theta_\ell \in \mathbb{T}$, $\tau_\ell \in (0, 1]$, $\ell = 1, \dots, m$, $g(\theta) \in L^\infty(\mathbb{T}, \mathbb{C})$ such that $\inf_{\mathbb{T}} |g(\theta)| > 0$ and*

$$(4.5) \quad c(\theta) = g(\theta) \prod_{\ell=1}^m |\sin \pi(\theta - \theta_\ell)|^{\tau_\ell}.$$

Then for any α with $0 < 2\beta < \beta(\alpha)$ and $0 < \delta < \frac{2\sum_{\ell=1}^m \tau_\ell}{1+\sum_{\ell=1}^m \tau_\ell}$, there is a sequence $q_n \rightarrow \infty$ and a full Lebesgue measure set $\Theta = \Theta(\alpha, \theta_1, \dots, \theta_m)$ such that for any $\theta \in \Theta$ and q_n large enough³, $c(\theta + n\alpha)$ satisfies:

$$(4.6) \quad \min_{|k| \leq q_n^{-1} e^{\delta\beta q_n}} \prod_{j=kq_n}^{(k+1)q_n-1} |c(\theta + j\alpha)| > e^{-\Lambda_1 q_n},$$

where

$$(4.7) \quad \Lambda_1 := \Lambda_1(\tau, g, \delta\beta) = \ln 2 \sum_{\ell=1}^m \tau_\ell - \ln \left(\inf_{\mathbb{T}} |g(\theta)| \right) + \delta^2 \min\{\beta, 1\}.$$

Assume further $g(\theta) \in C^0(\mathbb{T}, \mathbb{C})$ and $\ln |g(\theta)| \in L^1(\mathbb{T})$, Λ_1 in (4.7) can be replaced by

$$(4.8) \quad \Lambda_1 = - \int_{\mathbb{T}} \ln |c(\theta)| d\theta + 2\delta^2 \min\{\beta, 1\}.$$

Moreover, $c(\theta + n\alpha)$ is (Λ, β) bounded as defined in (2.2) such that

$$(4.9) \quad \min_{|m| \leq e^{\delta\beta q_n}} \prod_{j=m}^{m+q_n-1} |c(\theta + j\alpha)| > e^{-\Lambda q_n}, \quad \theta \in \Theta$$

where

$$(4.10) \quad \Lambda = - \int_{\mathbb{T}} \ln |c(\theta)| d\theta + 6\delta^2 \min\{\beta, 1\}.$$

Remark 4.1. The above Λ_1 and Λ can be negative in general, which makes (4.6 and (4.9) actually exponentially grow (instead of decay). This is natural since there is actually a ‘large’ scaling of size $\int \ln |g| \sim \int \ln |c|$ for the product in these cases. We are more interested in the case where $\int \ln |c| \leq 0$ where Λ_1 and Λ are indeed positive. In particular, it is always possible to re-scale $c(\theta)$ to make the logarithm average zero. This will lead to an arbitrarily small Λ (positive) in (4.10). Combine this with the uniform Lyapunov upper bound for (3.33) ([15]), we can get some refined results in the zero Lyapunov regime, e.g., the critical EHM model, about the spectral continuity and quasi-ballistic motion. See more discussion in the next Corollary 4.3 and section 7 about Corollary 2.5.

Remark 4.2. It is an easy exercise that if $c(\theta)$ is C^k continuous on $\mathbb{T}$ with finitely many zeros with non-degenerate k -th order derivatives, then $c(\theta) \in \mathcal{F}(\mathbb{T}, \mathbb{C}) \cap \text{Lip}(\mathbb{T}, \mathbb{C})$ with a continuous $g(\theta)$ for all $k \geq 1$. By Lemma 4.2, there is $\Lambda = \Lambda(c)$ such that $c(\theta + n\alpha)$ is (Λ, β) bounded for any $0 < 2\beta < \beta(\alpha)$ and a.e. $\theta \in \mathbb{T}$. From the proof of Lemma 4.2, see (4.11), the subsequence q_{n_k} can be taken to be same as in the β -almost periodicity (4.1). Therefore, Corollary 2.2 follows from Theorem 2.1. We omit the details here. An interesting question is whether the non-degenerate condition on c can be weakened and what is the appropriate ‘non-degenerate’ condition on any Lipschitz function such that (4.2) holds.

Proof. Let $0 < 2\beta < \beta(\alpha)$ and q_n be defined as in (2.8). For any $\delta > 0$, let q_{n_k} be the subsequence such that $\ln q_{n_k+1} \geq 2\beta q_{n_k}$. For simplicity, drop the subindex n_k and denote the subsequence still by q_n , i.e.,

$$(4.11) \quad q_{n+1} > e^{2\beta q_n}$$

³The sequence itself only depends $\beta(\alpha)$, while the largeness depends on $\theta, \alpha, \beta, \delta, \tau$.

We also write $\tilde{\beta} = \min\{\beta, 1\}$. It is obvious that for any $m \in \mathbb{Z}$ and $\theta \in \mathbb{T}$,

$$(4.12) \quad \prod_{j=kq_n}^{(k+1)q_n-1} |g(\theta + j\alpha)| > \left(\inf_{\mathbb{T}} |g(\theta)|\right)^{q_n} = e^{q_n \ln(\inf_{\mathbb{T}} |g(\theta)|)}.$$

In view of (4.5), it is enough to study the lower bound for each $\|\theta - \theta_\ell\|_{\mathbb{T}}$. For any $\alpha \in [0, 1] \setminus \mathbb{Q}$ and any $\theta_\ell, \ell = 1, \dots, m$, let

$$(4.13) \quad \Theta_\ell := \bigcup_{\gamma > 0} \left\{ \theta \in \mathbb{T} : \|\theta - \theta_\ell + n\alpha\|_{\mathbb{T}} \geq \gamma |n|^{-2}, \forall n \in \mathbb{Z} \setminus \{0\} \right\}$$

It is well known that Θ_ℓ is a full measure set. Let

$$(4.14) \quad \Theta := \bigcap_{\ell=1}^m \Theta_\ell.$$

For any $\theta \in \Theta$ and $1 \leq \ell \leq m$, there is $\gamma_\ell = \gamma(\theta, \theta_\ell, \alpha) > 0$ such that

$$(4.15) \quad \|\theta - \theta_\ell + n\alpha\|_{\mathbb{T}} \geq \frac{\gamma_\ell}{|n|^2}, \quad \forall n \in \mathbb{Z} \setminus \{0\}.$$

For all $1 \leq \ell \leq m$ and $|k| < q_n^{-1} e^{\delta \beta q_n}$, let $j_{\ell,k} \in [0, q_n)$ be such that the following holds:

$$|\sin \pi(\theta - \theta_\ell + kq_n\alpha + j_{\ell,k}\alpha)| = \inf_{0 \leq j < q_n} |\sin \pi(\theta - \theta_\ell + kq_n\alpha + j\alpha)|.$$

By (4.15), for all $j_{\ell,k}$,

$$(4.16) \quad \|\theta - \theta_\ell + j_{\ell,k}\alpha\|_{\mathbb{T}} \geq \frac{\gamma_\ell}{|j_{\ell,k}|^2} \geq \frac{\gamma_\ell}{q_n^2}.$$

Let $\tau = \sum_{\ell=1}^m \tau_\ell$. For all $|k| < q_n^{-1} e^{\delta \beta q_n} < e^{\delta \beta q_n}$, we have that

$$(4.17) \quad \begin{aligned} |\sin \pi(\theta - \theta_\ell + kq_n\alpha + j_{\ell,k}\alpha)| &\geq \|\theta - \theta_\ell + kq_n\alpha + j_{\ell,k}\alpha\|_{\mathbb{T}} \geq \|\theta - \theta_\ell + j_{\ell,k}\alpha\|_{\mathbb{T}} - \|kq_n\alpha\|_{\mathbb{T}} \\ &\geq \frac{\gamma_\ell}{q_n^2} - |k| \frac{1}{q_{n+1}} \\ &\geq \frac{\gamma_\ell}{q_n^2} - e^{\delta \beta q_n} e^{-2\beta q_n} \\ &\geq 2e^{-\tau^{-1} \delta^2 \tilde{\beta} q_n} - e^{-(2-\delta) \tilde{\beta} q_n} \\ &\geq e^{-\tau^{-1} \delta^2 \tilde{\beta} q_n} \end{aligned}$$

provided $q_n^{-2} e^{\tau^{-1} \delta^2 \tilde{\beta} q_n} \geq 2\gamma_\ell^{-1}$ and $2 - \delta > \tau^{-1} \delta > \tau^{-1} \delta^2$. The latter gives the restriction on δ such that $\delta < \frac{2}{1+\tau^{-1}}$.

By Lemma 4.1,

$$(4.18) \quad \prod_{j=0, j \neq j_{\ell,k}}^{q_n-1} |\sin \pi(\theta - \theta_\ell + kq_n\alpha + j\alpha)| \geq e^{-(q_n-1) \ln 2 - C \ln q_n} \geq e^{-q_n \ln 2 - \tau^{-1} \delta^2 \tilde{\beta} q_n}$$

provided $C \ln q_n < \tau^{-1} \delta^2 \tilde{\beta} q_n$ where C is the absolute constant in Lemma 4.1 and τ is the same in (4.17).

Now putting (4.17) and (4.18) together, we have

$$\begin{aligned}
& \prod_{j=kq_n}^{(k+1)q_n-1} \left(\prod_{\ell=1}^m |\sin \pi(\theta + j\alpha - \theta_\ell)|^{\tau_\ell} \right) \\
&= \prod_{\ell=1}^m \left(\prod_{j=0}^{q_n-1} |\sin \pi(\theta - \theta_\ell + kq_n\alpha + j\alpha)|^{\tau_\ell} \right) \\
&= \left(\prod_{\ell=1}^m \left(\prod_{j=0, j \neq j_{l,k}}^{q_n-1} |\sin \pi(\theta - \theta_\ell + kq_n\alpha + j\alpha)|^{\tau_\ell} \right) \right) \cdot \left(\prod_{\ell=1}^m |\sin \pi(\theta - \theta_\ell + kq_n\alpha + j_{l,k}\alpha)|^{\tau_\ell} \right) \\
&\geq \left(\prod_{\ell=1}^m (e^{-q_n \ln 2 - \tau^{-1} \delta^2 \tilde{\beta} q_n})^{\tau_\ell} \right) \cdot \left(\prod_{\ell=1}^m (e^{-\tau^{-1} \delta^2 \tilde{\beta} q_n})^{\tau_\ell} \right) \\
&= e^{-q_n (\ln 2 \sum_{\ell=1}^m \tau_\ell) - \delta^2 \tilde{\beta} q_n}.
\end{aligned}$$

Combined with (4.12), we have that for all $|k| \leq q_n^{-1} e^{\delta \beta q_n}$,

$$(4.19) \quad \prod_{j=kq_n}^{(k+1)q_n-1} |c(\theta + j\alpha)| > e^{-\left(\ln 2 \sum_{\ell=1}^m \tau_\ell - \ln \inf_{\mathbb{T}} |g(\theta)| + \delta^2 \tilde{\beta} \right) q_n}$$

provided $q_n > \tilde{q} = \tilde{q}(\max_{\ell} \gamma_{\ell}^{-1}, \delta, \alpha, \sum_{\ell=1}^m \tau_{\ell})$.

Assume further $g(\theta), c(\theta) \in C^0(\mathbb{T}, \mathbb{C})$. Since $\inf |g(\theta)| > 0$, $\ln |g(\theta)|^{-1}$ is also continuous. By Lemma 3.1, there is $n_0 = n_0(\delta^2 \tilde{\beta})$ such that the following upper bound holds uniform in $\theta \in \mathbb{T}$ for $n > n_0$:

$$(4.20) \quad \frac{1}{n} \sum_{j=1}^n \ln |g(\theta + j\alpha)|^{-1} \leq \int_{\mathbb{T}} \ln |g(\theta)|^{-1} d\theta + \delta^2 \tilde{\beta}.$$

In particular, for all $q_n \geq n_0$ and any $k \in \mathbb{Z}$ we have

$$\left(\prod_{j=0}^{q_n-1} |g(\theta + kq_n\alpha + j\alpha)|^{-1} \right)^{\frac{1}{q_n}} \leq e^{-\int_{\mathbb{T}} \ln |g(\theta)| d\theta + \delta^2 \tilde{\beta}} \implies \prod_{j=kq_n}^{(k+1)q_n-1} |g(\theta + j\alpha)| \geq e^{q_n \left(\int_{\mathbb{T}} \ln |g(\theta)| d\theta - \delta^2 \tilde{\beta} \right)}.$$

Therefore, we can replace $\ln(\inf_{\mathbb{T}} |g(\theta)|)$ in (4.19) by $\int_{\mathbb{T}} \ln |g(\theta)| d\theta - \delta^2 \min\{\beta, 1\}$. In view of (4.4), we have

$$(4.21) \quad \Lambda_1 = \ln 2 \sum_{\ell=1}^m \tau_{\ell} - \int_{\mathbb{T}} \ln |g(\theta)| d\theta + 2\delta^2 \min\{\beta, 1\} = - \int_{\mathbb{T}} \ln |c(\theta)| d\theta + 2\delta^2 \min\{\beta, 1\},$$

which gives the desired expression of Λ_1 in (4.8).

Let $\tilde{c}(\theta) = c(\theta) e^{-\int_{\mathbb{T}} \ln |c(\theta)| d\theta}$. It is easy to check that $\int_{\mathbb{T}} \ln |\tilde{c}(\theta)| d\theta = 0$. By (4.21), we have for all $|k| \leq q_n^{-1} e^{\delta \beta q_n}$,

$$(4.22) \quad \prod_{j=kq_n}^{(k+1)q_n-1} |\tilde{c}(\theta + j\alpha)| > e^{-2\delta^2 \tilde{\beta} q_n}.$$

By Lemma 3.1, there is $r_0 = r_0(\delta^2 \tilde{\beta}) \in \mathbb{N}$ such that for any $m \in \mathbb{Z}$ and $r \geq r_0$,

$$(4.23) \quad \prod_{j=m}^{m+r-1} |\tilde{c}(\theta + j\alpha)| \leq e^{r \left(\int_{\mathbb{T}} \ln |\tilde{c}(\theta)| d\theta + \delta^2 \tilde{\beta} \right)} = e^{\delta^2 \tilde{\beta} r}.$$

For $0 \leq r < r_0$, we have the trivial upper bound $\prod_{j=m}^{m+r-1} |\tilde{c}(\theta + j\alpha)| \leq e^{r \ln(\|\tilde{c}\|_{\infty})} \leq e^{r_0 \ln(\|\tilde{c}\|_{\infty} + 1)}$. Therefore, for any $m \in \mathbb{Z}$ and $1 \leq r \leq q_n$,

$$(4.24) \quad \prod_{j=m}^{m+r-1} |\tilde{c}(\theta + j\alpha)| \leq e^{\delta^2 \tilde{\beta} q_n}$$

provided $q_n \geq \delta^{-1} \tilde{\beta}^{-1} r_0 \ln(\|\tilde{c}\|_{\infty} + 1)$.

Then for any $|m| < e^{\delta \tilde{\beta} q_n}$, there is k such that $kq_n \in (m, m + q_n]$, therefore,

$$(4.25) \quad \prod_{j=m}^{m+q_n-1} |\tilde{c}(\theta + j\alpha)| = \prod_{j=m}^{kq_n-1} |\tilde{c}(\theta + j\alpha)| \cdot \prod_{j=kq_n}^{m+q_n-1} |\tilde{c}(\theta + j\alpha)|$$

$$(4.26) \quad = \frac{\prod_{j=(k-1)q_n}^{kq_n-1} |\tilde{c}(\theta + j\alpha)|}{\prod_{j=(k-1)q_n}^{m-1} |\tilde{c}(\theta + j\alpha)|} \cdot \frac{\prod_{j=kq_n}^{(k+1)q_n-1} |\tilde{c}(\theta + j\alpha)|}{\prod_{j=m+q_n}^{(k+1)q_n-1} |\tilde{c}(\theta + j\alpha)|}$$

$$(4.27) \quad > \frac{e^{-2\delta^2 \tilde{\beta} q_n}}{e^{\delta^2 \tilde{\beta} q_n}} \cdot \frac{e^{-2\delta^2 \tilde{\beta} q_n}}{e^{\delta^2 \tilde{\beta} q_n}}$$

$$(4.28) \quad = e^{-6\delta^2 \tilde{\beta} q_n}.$$

Therefore,

$$(4.29) \quad \prod_{j=m}^{m+q_n-1} |c(\theta + j\alpha)| = e^{q_n \int_{\mathbb{T}} \ln |c(\theta)| d\theta} \prod_{j=m}^{m+q_n-1} |\tilde{c}(\theta + j\alpha)| \geq e^{-(-\int_{\mathbb{T}} \ln |c(\theta)| d\theta + 6\delta^2 \tilde{\beta}) q_n} =: e^{-\Lambda q_n},$$

as claimed. This completes the proof of Lemma 4.2. $\square$

As an explicit example, we have the following arbitrarily slow lower bound for the analytic case with zero $\ln$ mean.

Corollary 4.3. *Assume that $c(\theta) \in C^{\omega}(\mathbb{T}, \mathbb{C})$ and $\int_{\mathbb{T}} \ln |c(\theta)| d\theta = 0$. Denote the all zeros⁴ of $c(\theta)$ on $\mathbb{T}$ by $c^{-1}(0) = \{\theta_1, \dots, \theta_m\}$. For any β with $0 < 2\beta < \beta(\alpha)$ and $0 < \delta < 1$, there is there is a sequence $q_n \rightarrow \infty$ and a full Lebesgue measure set $\Theta = \Theta(\alpha, c^{-1}(0))$ such that for any $\theta \in \Theta$, $c(\theta + n\alpha)$ satisfies:*

$$(4.30) \quad \min_{|k| \leq q_n^{-1} e^{\delta \beta q_n}} \prod_{j=kq_n}^{(k+1)q_n-1} |c(\theta + j\alpha)| > e^{-2\delta^2 \min\{\beta, 1\} q_n},$$

$$(4.31) \quad \min_{|m| \leq e^{\delta \beta q_n}} \prod_{j=m}^{m+q_n-1} |c(\theta + j\alpha)| > e^{-6\delta^2 \min\{\beta, 1\} q_n}.$$

⁴Clearly, analytic function $c(\theta)$ only has finitely many zeros on $\mathbb{T}$.

Proof. Clearly, there is an analytic function $\tilde{g}(\theta)$ such that:

$$(4.32) \quad c(\theta) = \tilde{g}(\theta) \prod_{\ell=1}^m (e^{2\pi i \theta} - e^{2\pi i \theta_\ell}), \quad \inf_{\mathbb{T}} |\tilde{g}(\theta)| > 0.$$

Direct computation shows $\int_{\mathbb{T}} \ln |\tilde{g}(\theta)| d\theta = \int_{\mathbb{T}} \ln |c(\theta)| d\theta = 0$ and

$$(4.33) \quad c(\theta) = \tilde{g}(\theta) \prod_{\ell=1}^m (e^{2\pi i \theta} - e^{2\pi i \theta_\ell}) = \tilde{g}(\theta) (2i)^m \prod_{\ell=1}^m e^{i\pi(\theta+\theta_\ell)} \sin \pi(\theta - \theta_\ell).$$

Therefore,

$$(4.34) \quad \frac{|c(\theta)|}{\prod_{\ell=1}^m |\sin \pi(\theta - \theta_\ell)|} = 2^m |\tilde{g}(\theta)|.$$

Apply Lemma 4.2 to (4.33) where $\tau_1 = \dots = \tau_m = 1$ and $|g(\theta)| \equiv 2^m |\tilde{g}(\theta)|$, we have (4.6) and (4.9) hold with $\Lambda_1 = 2\delta^2 \min\{\beta, 1\}$ and $\Lambda = 6\delta^2 \min\{\beta, 1\}$. $\square$

5. SPECTRAL CONTINUITY: PROOF OF THEOREM 2.1

Following the notations and assumptions in Theorem 2.1, consider

$$(5.1) \quad (Hu)_n = w_n u_{n+1} + \overline{w}_{n-1} u_{n-1} + v_n u_n, \quad n \in \mathbb{Z}.$$

Assume that there are positive constants $\beta, \delta, \Lambda > 0$ and a sequence of positive integers $q_n \rightarrow \infty$ such that w_n, v_n has β - q_n almost periodicity and w_n has (Λ, β) - q_n bound.

The key observation is: if H has β - q_n almost periodicity, then it can be approximated by a q_n periodic operator exponentially fast in a finite (exponentially large) lattice. The estimates on the q_n periodic operator eventually lead to the quantitative upper bound for the m -function as in (2.4) through the subordinacy theory Lemma 3.2.

In view of Lemma 3.2, let v^φ be the right half line solution to $Hu = Eu$ with initial condition φ and $\ell = \ell(\varphi, \varepsilon, E)$ is defined as in (3.21). As a direct consequence of Lemma 3.2 and Proposition 3.4, the following relation between the power law of $\|v^\varphi\|_\ell$ and the spectral continuity was proved in [32] (see Lemma 2.1 and the proof of Theorem 6 there).

Lemma 5.1. *Fix $0 < \gamma < 1$. Suppose for μ -a.e. E , there is a sequence of positive numbers $\eta_k \rightarrow 0$ and $L_k = \ell_k(\varphi, \eta_k, E) \rightarrow \infty$ such that for any φ*

$$(5.2) \quad 1/16 (L_k)^\gamma \leq \|v^\varphi\|_{L_k}^2 \leq (L_k)^{2-\gamma}.$$

Then the spectral measure μ is γ -spectral continuous.

Let $A(n; E)$ be defined as in (3.8). Denote by $\text{Tr } A$ the trace of any matrix $A \in \text{GL}(2, \mathbb{C})$. The following estimate on $\text{Tr } A(q_n; E)$ is the key to prove the above power law and spectral continuity.

Theorem 5.2. *Let $H, \beta, \delta, \Lambda$ and q_n be given as in (5.1). Suppose $\beta > 260(1 + \frac{1}{\delta})\Lambda$, then for μ a.e. E , there exists $K(E) \in \mathbb{N}$, for $k \geq K(E)$, we have*

$$(5.3) \quad |\text{Tr } A(q_k; E)| < 2 - 2e^{-60\Lambda q_k}.$$

For any $0 < \gamma < 1$, assume further that

$$(5.4) \quad \beta > 300(1 + \frac{1}{\delta}) \frac{\Lambda}{1-\gamma},$$

we have the power law required by (5.2).

Let

$$(5.5) \quad C = C(\delta, \Lambda) = 300(1 + \frac{1}{\delta})\Lambda.$$

Combining Lemma 5.1 and (5.4) in Theorem 5.2, if $\beta > C$, then μ is γ -spectral continuous for any $\gamma < 1 - \frac{C}{\beta} < 1$ and therefore $\dim_{\text{spe}}(\mu) \geq 1 - \frac{C}{\beta}$. This proves Theorem 2.1.

The trace estimate (5.3) shows that spectrally almost everywhere, $A(q_k; E)$ is strictly elliptic eventually. The quantitative estimate (5.3) allows us to iterate the transfer matrix up to the length scale $e^{\Lambda q_k}$, which gives a well control on the norm of $A(q_k; E)$. The norm estimate eventually leads to the power law as required in (5.2) through (3.3).

The proof of (5.4) and the required power law follows the outline of the Schrödinger case (see [32], Lemma 2.1). The main difference is now the transfer matrix $A(n; E)$ is in $GL(2, \mathbb{C})$. We need to consider some transformations introduced in [22] which conjugate $A(n; E)$ to some $SL(2, \mathbb{R})$ matrix. Then many important techniques developed in [32] for $SL(2, \mathbb{R})$ cocycles are now applicable. The trace estimate (5.3) leads to a norm estimate of $A(q_k; E)$ and eventually leads to the estimate (5.2) for the truncated ℓ^2 norm of the eigenfunction v^φ by (3.3). We will omit the details here and focus on the proof of the trace estimate (5.3). For the sake of completeness, we sketch the proof of (5.4) and the power law (5.2) in the Appendix A.1 for reader's convenience.

The rest of the section is organized as follows: In section 5.1, we introduce the transformation we will use to conjugate $GL(2, \mathbb{C})$ to $SL(2, \mathbb{R})$ and develop all the useful lemmas about the conjugate. In section 5.2, we study the case where the trace of the transfer matrix is greater than 2. In section 5.3, we study the case where the trace of the transfer matrix is close to 2.

Throughout this section, we assume v_n, w_n have β - q almost periodicity and w_n has (Λ, β) - q bound for some q large enough such that $e^{-(\beta-2\Lambda)q} < 1/10$. We also use the induced estimates (3.35)-(3.38) discussed in section 3.4 directly, referred also as β - q almost periodicity and (Λ, β) - q bound.

5.1. Conjugate between $SL(2, \mathbb{R})$ and $GL(2, \mathbb{C})$ matrices. The trace estimate (5.3) was first proved in [32] for $SL(2, \mathbb{R})$ cocycles. The generalization to $GL(2, \mathbb{C})$ case is very delicate. We need to consider the following transformation: let

$$(5.6) \quad T_n = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\frac{w_n}{w_{n-1}}} \end{pmatrix}$$

and

$$(5.7) \quad r_n = \frac{w_{n+1}}{\sqrt{|w_{n+1} w_n|}}.$$

Let $A_n(E)$ be given as in (3.4). Define

$$(5.8) \quad \tilde{A}_n(E) := r_{n-1} T_n^{-1} A_n T_{n-1} = \frac{1}{\sqrt{|w_n w_{n-1}|}} \begin{pmatrix} E - v_n & -|w_{n-1}| \\ |w_n| & 0 \end{pmatrix}.$$

The n -transfer matrix $\tilde{A}(n, m; E)$ and $\tilde{A}(n; E)$ for $\tilde{A}_n$ will be defined in the same way as in (3.5):

$$(5.9) \quad \tilde{A}(n, m; E) = \prod_{j=m}^{n+m-1} \tilde{A}_j(E), \quad n \in \mathbb{N}^+, \quad m \in \mathbb{Z}$$

and

$$(5.10) \quad \tilde{A}(n; E) = \tilde{A}(n, 1; E), \quad n > 0; \quad \tilde{A}(0; E) = Id; \quad \tilde{A}(n; E) = \tilde{A}^{-1}(-n, n+1; E), \quad n < 0.$$

We also denote the scalar product of r_n in the same way as $w(n, m)$ in (3.10) for $n \in \mathbb{N}^+$,

$$(5.11) \quad r(n, m) = \prod_{j=m}^{n+m-1} r_j, \quad m \in \mathbb{Z}.$$

Direct computation shows

$$(5.12) \quad \tilde{A}(n, m; E) = r(n, m-1) T_{n+m-1}^{-1} A(n, m; E) T_{m-1}.$$

In view of (5.6) and (5.8), it is easy to check that $\|T_n\| = 1$ and $\tilde{A}_n, \tilde{A}(n, m) \in \text{SL}(2, \mathbb{R})$ for any n, m . By (5.8) and (5.12), we are able to apply the techniques developed in [32] for $\text{SL}(2, \mathbb{R})$ matrix and then switch between the singular $\text{GL}(2, \mathbb{C})$ case and the $\text{SL}(2, \mathbb{R})$ case.

The β -almost periodicity and the (Λ, β) boundedness of w_n imply the β -almost periodicity of r and T in the following sense:

Lemma 5.3. *If $\beta > 2\Lambda$, then for all $m \in \mathbb{Z}$ such that $|m| < e^{\delta\beta q}$,*

$$(5.13) \quad \left| |r^\pm(q, m)| - 1 \right| < e^{-(\beta-2\Lambda)q}$$

$$(5.14) \quad \|T_{m+q}^{-1} \cdot T_m - I\| = \|T_m \cdot T_{m+q}^{-1} - I\| < 4e^{-(\beta-2\Lambda)q}.$$

Assume further that $N \in \mathbb{N}^+$, $Nq \leq e^{\delta\beta q}$, then

$$(5.15) \quad \left| |r^\pm(Nq, 0)| - 1 \right| < Ne^{-(\beta-2\Lambda)q}$$

$$(5.16) \quad \|T_0 \cdot T_{Nq}^{-1} - I\| = \|T_{Nq}^{-1} \cdot T_0 - I\| < 4Ne^{-(\beta-2\Lambda)q}.$$

Note that $r(n, m)$ and T_n are essentially scalar products, the proof is based on the following direct computation:

Proof. Set $z_m = \frac{w_m}{w_{m+q}}$. By (3.37), for $|m| \leq e^{\delta\beta q}$ and q large,

$$\left| |z_m|^\pm - 1 \right| \leq |z_m^\pm - 1| < e^{-(\beta-2\Lambda)q} < \frac{1}{2}.$$

Clearly, $|r_n| = \sqrt{\frac{|w_{n+1}|}{|w_n|}}$. In view of (5.11) and (5.6), we have

$$(5.17) \quad \left| |r(q, m)| - 1 \right| = \left| \sqrt{\frac{|c_{m+1}|}{|c_m|} \cdots \frac{|c_{m+q}|}{|c_{m+q-1}|}} - 1 \right| = \left| \sqrt{|z_m|^{-1}} - 1 \right| < e^{-(\beta-2\Lambda)q}$$

and

$$T_{m+q}^{-1} T_m = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\frac{w_m w_{m+q}^{-1}}{w_{m+q} w_m^{-1}}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{w_m w_m^{-1}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{w_m w_{m+q}^{-1}}{w_{m+q} w_m^{-1}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{z_m}{|z_m|} \end{pmatrix}$$

Therefore,

$$\|T_{m+q}^{-1} T_m - I\| \leq \left| \frac{z_m}{|z_m|} - 1 \right| = \frac{|z_m - |z_m||}{|z_m|} \leq \frac{2|z_m - 1|}{|z_m|} \leq 4e^{-(\beta-2\Lambda)q}.$$

In particular, in (5.17), let $m = 0, q, 2q, \dots, (N-1)q$ for $Nq < e^{\delta\beta q}$. Direct computation shows that

$$\left| |r(Nq, 0)| - 1 \right| = \left| \prod_{k=0}^{N-1} |r(q, kq)| - 1 \right| < Ne^{-(\beta-2\Lambda)q},$$

$$T_{Nq}^{-1} T_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\overline{w}_{Nq} w_{Nq}^{-1}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{w_0 \overline{w}_0^{-1}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{w_0 w_{Nq}^{-1}}{|w_0 w_{Nq}^{-1}|} \end{pmatrix},$$

and

$$\begin{aligned} \|T_{Nq}^{-1} T_0 - I\| &\leq \left| \frac{w_0 w_{Nq}^{-1}}{|w_0 w_{Nq}^{-1}|} - 1 \right| \leq \left| \prod_{k=0}^{N-1} \frac{z_{kq}}{|z_{kq}|} - 1 \right| \\ &\leq \sum_{k=0}^{N-1} \left| \frac{z_{kq}}{|z_{kq}|} - 1 \right| \leq 4N e^{-(\beta-2\Lambda)q}. \end{aligned}$$

□

(5.12) only implies $\|\tilde{A}(q; E)\| \approx \|A(q; E)\|$, while $\|A(q; E) - \tilde{A}(q; E)\|$ is not necessarily small. Lemma 5.3 actually shows $\tilde{A}(q; E)$ and $A(q; E)$ are close to each other up to a conjugate. This will be enough to control the difference between their traces. Fix E , we write $A(n, m) = A(n, m; E)$ for short.

Lemma 5.4. *For all $m \in \mathbb{Z}$ such that $|m| < e^{\delta\beta q}$, let $\Phi = \text{Arg } r(q, m)$ be the Principal value of $r(q, m) \in \mathbb{C}$. For $\beta > 4\Lambda$,*

$$(5.18) \quad \frac{1}{2} \|A(q, m)\| \leq \|\tilde{A}(q, m)\| \leq 2 \|A(q, m)\| < 2e^{2\Lambda q}$$

$$(5.19) \quad \|\tilde{A}(q, m+1) - e^{i\Phi} T_m^{-1} A(q, m+1) T_m\| < 12e^{-(\beta-4\Lambda)q}$$

and consequently,

$$(5.20) \quad \left| |\text{Tr } \tilde{A}(q, m)| - |\text{Tr } A(q, m)| \right| < 12e^{-(\beta-4\Lambda)q}.$$

Proof. By (5.13), we have $|r^\pm(q, m)| \leq 2$. (5.18) follows from (3.38) and (5.12) since $\|T_m^\pm\| = 1$. By (5.12), we have

$$(5.21) \quad \tilde{A}(q, m+1) = r(q, m) T_{m+q}^{-1} A(q, m+1) T_m = (|r(q, m)| T_{m+q}^{-1} T_m) e^{i\Phi} T_m^{-1} A(q, m+1) T_m$$

Therefore,

$$(5.22) \quad \|\tilde{A}(q, m+1) - e^{i\Phi} T_m^{-1} A(q, m+1) T_m\| = \|(|r(q, m)| T_{m+q}^{-1} T_m - I) e^{i\Phi} T_m^{-1} A(q, m+1) T_m\|$$

$$(5.23) \quad \leq \| (|r(q, m)| T_{m+q}^{-1} T_m - I) \| \cdot \| e^{i\Phi} T_m^{-1} A(q, m+1) T_m \|$$

$$(5.24) \quad \leq 6e^{-(\beta-2\Lambda)q} \|A(q, m+1)\|$$

$$(5.25) \quad \leq 12e^{-(\beta-4\Lambda)q}.$$

The last inequality follows from (5.13) and (5.14) since

$$\| (|r(q, m)| T_{m+q}^{-1} T_m - I) \| \leq \| |r(q, m)| - 1 \| \cdot \| T_{m+q}^{-1} T_m - I \| + \| |r(q, m)| - 1 \| + \| T_{m+q}^{-1} T_m - I \|.$$

$$(5.20) \text{ follows directly from (5.19) since } |\text{Tr } A(q, m+1)| = \left| \text{Tr} (e^{i\Phi} T_m^{-1} A(q, m+1) T_m) \right|.$$

□

Standard telescoping argument allows us to pass the β -almost periodicity from the sequences w_n, v_n to the matrices $A(n, m), \tilde{A}(n, m)$, up to product length q ,

Lemma 5.5. *For all $m \in \mathbb{Z}$ such that $|m| < e^{\delta\beta q}$, $\beta > 6\Lambda$,*

$$(5.26) \quad \|A(q, m; E) - A(q, m + q; E)\| \leq e^{(-\beta+6\Lambda)q}$$

and

$$(5.27) \quad \|\tilde{A}(q, m; E) - \tilde{A}(q, m + q; E)\| \leq e^{(-\beta+6\Lambda)q}.$$

Proof. Write $m' = m + q$ for short.

$$\begin{aligned} A(q, m) - A(q, m') &= \sum_{j=0}^{q-1} A(q-j-1, m+j+1) (A_{m+j} - A_{m'+j}) A(j, m') \\ &= \sum_{j=0}^{q-1} \frac{D(q-j-1, m+j+1)}{w(q-j-1, m+j+1)} \left(\frac{D_{m+j}}{w_{m+j}} - \frac{D_{m'+j}}{w_{m'+j}} \right) \frac{D(j, m')}{w(j, m')}. \end{aligned}$$

By the trivial upper bound (3.33) for $D(n, m)$ and the lower bound (3.35) for $w(n, m)$, we have

$$\begin{aligned} \|A(q, m) - A(q, m')\| &\leq \sum_{j=0}^{q-1} \frac{e^{(q-j-1)\Lambda}}{|w(q-j-1, m+j)|} |w_{m'+j} D_{m+j} - w_{m+j} D_{m'+j}| \frac{e^{\Lambda j}}{|w(j+1, m')|} \\ &\leq \sum_{j=0}^{q-1} \frac{e^{(q-j-1)\Lambda}}{e^{-2\Lambda q}} |w_{m'+j} D_{m+j} - w_{m+j} D_{m'+j}| \frac{e^{j\Lambda}}{e^{-2\Lambda q}} \\ &\leq q e^{5\Lambda q} \max_{|m| \leq e^{\delta\beta q}} |w_{m+q} D_m - w_m D_{m+q}| \\ &\leq q e^{5\Lambda q} \max_{|m| \leq e^{\delta\beta q}} (|(w_{m+q} - w_m) D_m| + |w_m (D_{m+q} - D_m)|) \\ &\leq 2q e^{5\Lambda q} e^{\Lambda} e^{-\beta q} \\ &\leq e^{-(\beta-6\Lambda)q} \end{aligned}$$

provided $\sup_n |w_n|, \sup_{n,E} \|D_n\| \leq e^{\Lambda}$ and q large such that $2qe^{\Lambda} \leq e^{\Lambda q}$.

Let $\tilde{w}_n = \sqrt{|w_n w_{n-1}|}$, $\tilde{D}_n = \begin{pmatrix} E - v_n & -|w_{n-1}| \\ |w_n| & 0 \end{pmatrix}$. Define $\tilde{w}(n, m), \tilde{D}(n, m; E)$ exact in the same as for w, D . It is easy to check that the Λ bounds of w_n and D_n hold true for $\tilde{w}_n, \tilde{D}_n$:

$$(5.28) \quad |\tilde{w}(n, m)| \leq e^{\Lambda n}, \quad \sup_E \|\tilde{D}(n, m; E)\| \leq e^{\Lambda n}, \quad \forall n \geq 0, m \in \mathbb{Z}$$

and

$$(5.29) \quad |\tilde{w}(r, m)| \geq e^{-2\Lambda q}, \quad 0 \leq r \leq q, \quad |m| \leq e^{\delta\beta q}.$$

The β -almost periodicity of w_n, v_n are also passed directly to $\tilde{w}_n, \tilde{D}_n$:

$$(5.30) \quad \max_{|m| \leq e^{\delta\beta q}} |\tilde{w}_m - \tilde{w}_{m \pm q}| \leq 2e^{\Lambda} e^{-\beta q}, \quad \max_{|m| \leq e^{\delta\beta q}} \|\tilde{D}_m - \tilde{D}_{m \pm q}\| \leq e^{-\beta q}.$$

By the definition of $\tilde{A}_n$ in (5.8), we have $\tilde{A}_n = \frac{1}{\tilde{w}_n} \tilde{D}_n$. Exact the same computation proves that

$$(5.31) \quad \|\tilde{A}(q, m) - \tilde{A}(q, m')\| \leq q(2e^{2\Lambda} + e^{\Lambda}) e^{5\Lambda q} e^{-\beta q} \leq e^{-(\beta-6\Lambda)q}.$$

□

The above telescoping argument can not be extended to exponential scale $e^{\Lambda q}$ as for r and T in (5.15), (5.16) directly. One main reason is we lose control of the matrix norm super-exponentially as $\|D(e^{\Lambda q})\| \lesssim e^{\Lambda e^{\Lambda q}}$. Such growth can not be controlled by condition such as $\beta \gtrsim \Lambda$. The key to prove the trace estimate (5.3) is to avoid using such rough bound for the matrix norm at an exponential scale. This is one breakthrough in [32]. By all the above estimates of the conjugate r, T and some simple linear algebra facts of $\text{SL}(2, \mathbb{R})$ matrix found in [32], we are able to prove this extension for the $\text{GL}(2, \mathbb{C})$ case. We will see more details in the next two subsections.

Similar to [32], we consider the following two cases where $|\text{Tr} A(q)|$ is away from 2 and close to 2.

5.2. The case where the trace is away from 2. We start with the hyperbolic case in the following sense: let

$$(5.32) \quad S_q^1 = \{E : |\text{Tr} A(q; E)| > 2 + 2e^{-60\Lambda q}\}$$

We may fix E and write $A(q) = A(q; E)$ for simplicity whenever it is clear.

Lemma 5.6. *Let q_n be given as in Theorem 5.2. If $\beta > (260 + \frac{61}{8})\Lambda$, then the set*

$$(5.33) \quad \limsup_{n \rightarrow \infty} S_{q_n}^1 = \left\{ E : E \text{ belongs to infinitely many } S_{q_n}^1 \right\}$$

has spectral measure zero.

Lemma 5.4 implies for large β , $\text{Tr} \tilde{A}(q; E)$ and $\text{Tr} A(q; E)$ lie in the same region, i.e., if $E \in S_q^1$, then

$$(5.34) \quad |\text{Tr} \tilde{A}(q; E)| > 2 + 2e^{-60\Lambda q} - 12e^{-(\beta-4\Lambda)q} > 2 + e^{-60\Lambda q},$$

provided $e^{(\beta-64\Lambda)q} > 12$.

The following linear algebra facts were proved in [32]

Lemma 5.7. *Suppose $G \in \text{SL}(2, \mathbb{R})$ with $2 < |\text{Tr} G| \leq 6$. The invertible matrix B such that*

$$(5.35) \quad G = B \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix} B^{-1}$$

where $\rho^{\pm 1}$ are the two conjugate real eigenvalues of G with $|\det B| = 1$ satisfies

$$(5.36) \quad \|B\| = \|B^{-1}\| < \frac{\sqrt{\|G\|}}{\sqrt{|\text{Tr} G| - 2}}$$

If $|\text{Tr} G| > 6$, then $\|B\| \leq \frac{2\sqrt{\|G\|}}{\sqrt{|\text{Tr} G| - 2}}$.

Apply the above lemma to $\tilde{A}(q; E) \in \text{SL}(2, \mathbb{R})$ satisfying (5.18) and (5.34), we have the following decomposition

$$(5.37) \quad \tilde{A}(q) = B \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix} B^{-1}$$

where $\rho^{\pm 1}$ are the two conjugate real eigenvalues of $\tilde{A}(q)$ with $|\rho| > |\text{Tr} \tilde{A}(q)| - 1 > 1 + e^{-60\Lambda q}$ and B satisfies $|\det B| = 1$ and

$$(5.38) \quad \|B\| = \|B^{-1}\| < e^{32\Lambda q}.$$

By (5.37) and (5.38), we have that for any $N \in \mathbb{N}^+$,

$$(5.39) \quad \tilde{A}^N(q) := [\tilde{A}(q)]^N = B \begin{pmatrix} \rho^N & 0 \\ 0 & \rho^{-N} \end{pmatrix} B^{-1}, \quad \|\tilde{A}^N(q)\| \leq e^{64\Lambda q} |\rho|^N$$

In the rest of this section, consider $\beta > \frac{61}{\delta}\Lambda$ and set

$$(5.40) \quad N = [e^{61\Lambda q}] < e^{\delta\beta q}.$$

The above decomposition now turns the matrix product $[\tilde{A}(q)]^N$ into a scalar product of ρ^N with a uniformly controlled conjugate B (independent of N). This is one key algebra ingredient observed in [32]. This technique now allows us to extend the orbit of $\tilde{A}(q)$ to the exponentially long scale $N = e^{61\Lambda q}$.

The following technique lemma was proved in [32] (see Lemma A.1 there):

Lemma 5.8. *Suppose G is a two by two matrix satisfying*

$$(5.41) \quad \|G^j\| \leq M < \infty, \quad \text{for all } 0 < j \leq N \in \mathbb{N}^+,$$

where $M \geq 1$ only depends on N . Let $G_j = G + \Delta_j$, $j = 1, \dots, N$, be a sequence of two by two matrices with

$$(5.42) \quad \delta = \max_{1 \leq j \leq N} \|\Delta_j\|.$$

If

$$(5.43) \quad NM\delta < 1/2,$$

then for any $1 \leq n \leq N$

$$(5.44) \quad \left\| \prod_{j=1}^n G_j - G^n \right\| \leq 2NM^2\delta.$$

Let $N = [e^{61\Lambda q}]$, $G = \frac{1}{\rho}\tilde{A}(q)$ and $G_j = \frac{1}{\rho}\tilde{A}(q, jq + 1)$, $|j| = 0, 1, \dots, N$. By (5.18) and (5.27), it is easy to check that $\|G^j\| \leq e^{64\Lambda q}$ and $\|G_j - G\| \leq Ne^{(-\beta+6\Lambda)q} \leq e^{(-\beta+67\Lambda)q}$. The above lemma is applicable provided $\beta > (260 + \frac{61}{\delta})\Lambda$. One can prove that

$$(5.45) \quad \|\tilde{A}(Nq) - \tilde{A}^N(q)\| \leq |\rho|^N e^{(-\beta+260\Lambda)q}$$

$$(5.46) \quad \|\tilde{A}(-Nq) - \tilde{A}^{-1}(Nq)\| \leq 2|\rho|^N e^{(-\beta+260\Lambda)q}$$

The proof (5.45) and (5.46) is a direct application of Lemma 5.8 and resembles the proof of Claim 3, [32]. We omit the details here.

Similar to (5.19), we can prove $A(\pm Nq)$ and $\tilde{A}(\pm Nq)$ are close to each other up the size $|\rho|^N$.

Lemma 5.9. *Let $\eta = r^{-1}(Nq, 0)$, $\zeta = r(Nq, -Nq)$ and $\phi = \text{Arg } \eta$, $\psi = \text{Arg } \zeta$ be the Principal values of η and ζ accordingly. For $\beta > (260 + \frac{61}{\delta})\Lambda$,*

$$(5.47) \quad \|A^\pm(Nq) - e^{\pm i\phi} T_0 \tilde{A}^\pm(Nq) T_0^{-1}\| < e^{(-\beta+127\Lambda)q} |\rho|^N,$$

$$(5.48) \quad \|A(-Nq) - e^{i\psi} T_0 \tilde{A}(-Nq) T_0^{-1}\| < e^{(-\beta+127\Lambda)q} |\rho|^N,$$

and consequently,

$$(5.49) \quad \|A^{-1}(Nq) - e^{-i(\phi+\psi)} A(-Nq)\| < 4e^{(-\beta+260\Lambda)q} |\rho|^N.$$

Proof. By (5.8),

$$(5.50) \quad A(Nq) = \eta T_{Nq} \tilde{A}(Nq) T_0^{-1} = (|\eta| T_{Nq} T_0^{-1}) e^{i\phi} T_0 \tilde{A}(Nq) T_0^{-1}.$$

Therefore, by (5.15) and (5.16),

$$\begin{aligned}\|A(Nq) - e^{i\phi} T_0 \tilde{A}(Nq) T_0^{-1}\| &\leq \| |\eta| T_{Nq} T_0^{-1} - I \| \cdot \| e^{i\phi} T_0 \tilde{A}(Nq) T_0^{-1} \| \\ &\leq 6e^{(-\beta+63\Lambda)q} \|\tilde{A}(Nq)\| \\ &\leq e^{(-\beta+127\Lambda)q} |\rho|^N,\end{aligned}$$

provided $e^{(\beta-260\Lambda)q} > 1$ and $e^{\Lambda q} > 12$. The last inequality follows from (5.39) and (5.45):

$$\|\tilde{A}(Nq)\| \leq \|\tilde{A}^N(q)\| + e^{(-\beta+260\Lambda)q} |\rho|^N \leq e^{64\Lambda q} |\rho|^N + e^{(-\beta+260\Lambda)q} |\rho|^N \leq 2e^{64\Lambda q} |\rho|^N.$$

Note that $\tilde{A}(Nq) \in \text{SL}(2, \mathbb{R})$, then $\|\tilde{A}^{-1}(Nq)\| = \|\tilde{A}(Nq)\| \leq 2e^{64\Lambda q} |\rho|^N$. The proof for $A^{-1}(Nq)$ is exactly the same since

$$(5.51) \quad A^{-1}(Nq) = \eta^{-1} T_0 \tilde{A}^{-1}(Nq) T_{Nq}^{-1} = e^{-i\phi} T_0 \tilde{A}^{-1}(Nq) T_0^{-1} (|\eta|^{-1} T_0 T_{Nq}^{-1}).$$

(5.8) and (5.10) imply that

$$\begin{aligned}A(-Nq) &= A^{-1}(Nq, -Nq+1) = \left[r^{-1}(Nq, -Nq) T_0 \tilde{A}(Nq, -Nq+1) T_{-Nq}^{-1} \right]^{-1} \\ &= r(Nq, -Nq) T_{-Nq} \tilde{A}(-Nq) T_0^{-1}.\end{aligned}$$

(5.46) implies that $\|\tilde{A}(-Nq)\| \leq \|\tilde{A}^{-1}(Nq)\| + 2|\rho|^N e^{(-\beta+260\Lambda)q} \leq 3e^{64\Lambda q} |\rho|^N$. Now by (5.15) and (5.16), exact the same argument for (5.47) proves (5.48) provided $e^{(\beta-260\Lambda)q} > 2$ and $e^{\Lambda q} > 18$.

The proof of (5.49) follows directly from (5.46), (5.47) and (5.48) since

$$\begin{aligned}\|A^{-1}(Nq) - e^{-i(\phi+\psi)} A(-Nq)\| &\leq \|A^{-1}(Nq) - e^{-i\phi} T_0 \tilde{A}^{-1}(Nq) T_0^{-1}\| \\ &\quad + \|e^{-i\phi} T_0 \tilde{A}^{-1}(Nq) T_0^{-1} - e^{-i\phi} T_0 \tilde{A}(-Nq) T_0^{-1}\| \\ &\quad + \|e^{-i\phi} T_0 \tilde{A}(-Nq) T_0^{-1} - e^{-i(\phi+\psi)} A(-Nq)\| \\ &\leq \|A^{-1}(Nq) - e^{-i\phi} T_0 \tilde{A}^{-1}(Nq) T_0^{-1}\| \\ &\quad + \|\tilde{A}^{-1}(Nq) - \tilde{A}(-Nq)\| \\ &\quad + \|e^{i\psi} T_0 \tilde{A}(-Nq) T_0^{-1} - A(-Nq)\| \\ &\leq 2e^{(-\beta+127\Lambda)q} |\rho|^N + 2e^{(-\beta+260\Lambda)q} |\rho|^N \\ &\leq 4e^{(-\beta+260\Lambda)q} |\rho|^N.\end{aligned}$$

□

With the above preparation, we are in the place to prove Lemma 5.6. It is easy to see that all the estimates from (5.45) to (5.49) preserve errors between the traces. Now combine (5.45) with (5.47), we have

$$(5.52) \quad \left| |\text{Tr } A(Nq)| - |\text{Tr } \tilde{A}^N(q)| \right| \leq 2e^{(-\beta+260\Lambda)q} |\rho|^N \leq \frac{1}{2} |\rho|^N,$$

provided $e^{(\beta-260\Lambda)q} > 4$. Therefore, by (5.39),

$$(5.53) \quad |\text{Tr } A(Nq)| \geq |\text{Tr } \tilde{A}^N(q)| - \frac{1}{2} |\rho|^N \geq \frac{1}{2} |\rho|^N.$$

(5.49) implies that for any vector $X \in \mathbb{C}^2$,

$$(5.54) \quad \|A^{-1}(Nq)X\| \leq \|A(-Nq)X\| + 4e^{(-\beta+260\Lambda)q} |\rho|^N \|X\| \leq \|A(-Nq)X\| + \frac{1}{8} |\rho|^N \|X\|,$$

provided $e^{(\beta-260\Lambda)q} > 32$.

By (5.8) and (5.12), it is easy to check that $|\det A(Nq)| = |r^{-1}(Nq, 0)|$. Therefore, (5.15) implies that

$$(5.55) \quad |\det A(Nq)| < 1 + e^{-(\beta-63\Lambda)q} < 2$$

Consider the generalized eigenequation $Hu = Eu$ with normalized initial value $X = \begin{pmatrix} u_1 \\ u_0 \end{pmatrix}$, $\|X\| = 1$. By (3.3) and the Cayley-Hamilton theorem for $\text{GL}(2, \mathbb{C})$ matrix $A(Nq)$, we have:

$$(5.56) \quad A(Nq)X = \begin{pmatrix} u_{Nq+1} \\ u_{Nq} \end{pmatrix}, \quad A(-Nq)X = \begin{pmatrix} u_{-Nq+1} \\ u_{-Nq} \end{pmatrix}.$$

and

$$(5.57) \quad A(Nq)X + (\det A) \cdot A^{-1}(Nq)X = -(\text{Tr } A(Nq))X$$

Combine (5.53), (5.54), (5.55) with (5.57), we have

$$(5.58) \quad \|A(Nq)X\| + \|A(-Nq)X\| \geq \frac{1}{8}|\rho|^N.$$

Now by the choice of ρ and N , for q large, we have

$$(5.59) \quad \|A(Nq)X\| + \|A(-Nq)X\| \geq \frac{1}{8}(1 + e^{-60\Lambda q})[e^{61\Lambda q}] \geq 4e^q$$

which implies

$$(5.60) \quad \max\{|u_{Nq+1}|, |u_{Nq}|, |u_{-Nq+1}|, |u_{-Nq}|\} \geq e^q.$$

In conclusion, we can claim the existence of a subsequence of u_n at energy E with following exponential growth:

Claim 5.10. *Assume v_n, w_n have β - q almost periodicity as in (2.1) and w_n has (Λ, β) - q bound (2.2), (2.2) for $q > q_0(\Lambda, \delta, \beta)$. Suppose $E \in S_q^1$ and $\beta > (260 + \frac{61}{\delta})\Lambda$, then there are integer sequences $x_q^1, x_q^2, x_q^3, x_q^4 \in \mathbb{Z}$ independent of E , such that $\min_i |x_q^i| \rightarrow \infty$ as $q \rightarrow \infty$ and*

$$(5.61) \quad \max_i |u_{x_q^i}^E| > e^q,$$

where u_n^E solves the half-line problem $Hu = Eu$ with normalized boundary condition $|u_0|^2 + |u_1|^2 = 1$.

Now Lemma 5.6 follows directly from Claim (5.10) and the following lemma:

Theorem 5.11 (Extended Schnol's Theorem, Lemma 2.4, [32]). *Fix any $y > 1/2$. For any sequence $|x_k| \rightarrow \infty$ (where the sequence is independent of E), for spectrally a.e. E , there is a generalized eigenvector u^E of $Hu = Eu$, such that*

$$|u_{x_k}^E| < C(1 + |k|)^y.$$

5.3. The case where the trace is close to 2. In this part, we consider those energy E where the trace of $A(q; E)$ is close to 2. Let

$$(5.62) \quad S_q^2 = \{E : |\text{Tr } A(q; E)| - 2| < 2e^{-60\Lambda q}\}$$

Again we assume that q is large and v_n, w_n satisfy β - q almost periodicity (2.1) and Λ - q bound in (2.2) with positive finite parameters β, Λ, δ . We can prove that

Lemma 5.12. *If $\beta > (130 + \frac{29}{\delta})\Lambda$, then*

$$(5.63) \quad \mu(S_q) < e^{-\frac{1}{10}\Lambda q},$$

where μ is the spectral measure of H .

Proof of Theorem 5.2: Assume now $\beta > 260(1 + \frac{1}{\delta})\Lambda$. Let q_n be given as in Theorem 5.2. Lemma 5.6 implies that for spectrally a.e. E , there is $K_1(E)$ such that,

$$(5.64) \quad |\mathrm{Tr} A(q_k; E)| < 2 + 2e^{-60\Lambda q_k}, \quad \forall k \geq K_1(E)$$

Combine Lemma 5.12 with the Borel-Cantelli lemma, we have $\mu\left(\limsup_n S_{q_n}^2\right) = 0$, i.e., for spectrally a.e. E , there is $K_2(E)$ such that

$$(5.65) \quad \left| |\mathrm{Tr} A(q_k; E)| - 2 \right| > 2e^{-60\Lambda q_k}, \quad \forall k \geq K_2(E).$$

Clearly, (5.64) and (5.65) complete the proof of Theorem 5.2 by taking $K = \max\{K_1, K_2\}$. $\square$

In the rest of the section, we focus on proving (5.63). Similar to the hyperbolic case, $\mathrm{Tr} \tilde{A}(q; E)$ and $\mathrm{Tr} A(q; E)$ are close up to exponential error by Lemma 5.4. More precisely, let

$$(5.66) \quad \tilde{S}_q^2 := \left\{ E : \left| |\mathrm{Tr} \tilde{A}(q; E)| - 2 \right| < 3e^{-60\Lambda q} \right\}$$

Clearly, Lemma 5.4 implies that for $\beta > 6\Lambda$, $S_q^2 \subset \tilde{S}_q^2$.

The following elementary linear algebra facts were proved in [32]

Lemma 5.13 (Lemma 2.9, Lemma 2.10 [32]). *Suppose $A \in \mathrm{SL}(2, \mathbb{R})$ has eigenvalues $\rho^{\pm 1}$, $\rho > 1$. For any $k \in \mathbb{N}$, if $\mathrm{Tr} A \neq 2$, then*

$$(5.67) \quad A^k = \frac{\rho^k - \rho^{-k}}{\rho - \rho^{-1}} \cdot \left(A - \frac{\mathrm{Tr} A}{2} \cdot I \right) + \frac{\rho^k + \rho^{-k}}{2} \cdot I$$

Otherwise, $A^k = k(A - I) + I$.

Assume further that $||\mathrm{Tr} A| - 2| < \tau < 1$, then there are universal constants $1 < C_1 < \infty, c_1 > 1/3$ such that for $1 \leq k \leq \tau^{-1}$, we have

$$(5.68) \quad c_1 < \frac{\rho^k + \rho^{-k}}{2} < C_1, \quad c_1 k < \frac{\rho^k - \rho^{-k}}{\rho - \rho^{-1}} < C_1 k.$$

Now fix $E \in \tilde{S}_q^2$, the above lemma actually shows that the k -th power of $\tilde{A}(q; E)$ grows almost linearly with respect to k as :

$$(5.69) \quad \tilde{A}^k(q) \sim k(\tilde{A}(q) - \frac{1}{2}\mathrm{Tr} \tilde{A}(q) \cdot I) + I, \quad 1 \leq k \leq N.$$

This simple observation will be an important part of our quantitative estimates in the near parabolic case. The arguments to derive (5.63) from (5.69) follow the outline of the near parabolic case in [32] with slight modification concerning all the estimates of the conjugacy in section 5.1. We sketch the proof below for reader's convenience.

Proof of Lemma 5.12: First, Lemma 5.13 provides the following norm estimates: there is absolute constant $C_1 > 0$ such that for any $1 \leq j < N = \lfloor e^{29\Lambda q} \rfloor < e^{29\Lambda q} < e^{\delta\beta q}$,

$$(5.70) \quad \|\tilde{A}^j(q; E)\| < 3C_1 j \cdot \|\tilde{A}(q)\|.$$

By (5.18) and the choice of N , we have

$$(5.71) \quad \|\tilde{A}^j(q; E)\| < 6C_1 j e^{2\Lambda q} < j e^{3\Lambda q} < e^{32\Lambda q}.$$

In the same way as the proof of (5.45) and (5.46), for any $1 \leq k \leq N$, combine (5.71) with (5.27), we can apply Lemma 5.8 to obtain

$$(5.72) \quad \|\tilde{A}(kq) - \tilde{A}^k(q)\| \leq e^{(-\beta + 130\Lambda)q} < 1,$$

provided $\beta > (130 + \frac{29}{\delta})\Lambda$ and q large.

In view of (5.67), (5.69) and (5.72), it is clear that $\tilde{A}(kq)$ has the same linear expansion as in (5.69). Combine (5.67), (5.72) with the conjugate relation:

$$(5.73) \quad A(kq)X = r^{-1}(kq, 0)T(kq)\tilde{A}(kq)T^{-1}(0)X, \quad X \in \mathbb{C}^2,$$

we can prove that:

Claim 5.14. *For any $\varepsilon > 0, E$ and $\varphi \in [0, 2\pi)$, let $\ell = \ell(\varphi, \varepsilon, E), u^\varphi, v^\varphi$ be given as in (3.21). Suppose $E \in S_q^2, \varepsilon < e^{-29\Lambda q}$ and $\beta > (130 + \frac{29}{\delta})\Lambda$, then*

$$(5.74) \quad \|u^\varphi\|_\ell^2 > e^{\frac{1}{6}\Lambda q}.$$

The proof of Claim 5.14 follows the outline of the proof of Claim 5 in [32]. The key is to use the linear expression (5.69) to control both the upper and lower bound of bound of $\|A(n)X\|$. The main difference is we need to consider the conjugacy (5.73) and switch between the orbits of $A(n)X$ and $\tilde{A}(n)\tilde{X}$. We omit the details here. For sake of completeness, we include the proof in Appendix A.2.

We proceed to prove Lemma 5.12 by Claim 5.14. $\text{Tr } A(q; E)$ is a polynomial in E with degree q . S_q^2 can be written as a union of at most q band: $S_q^2 = \bigcup_{j=1}^q I_j$. Note $\text{Tr } \tilde{A}(q; E)$ is also a polynomial in E with degree q with real coefficients, by Proposition A.3, we have $|\tilde{S}_q^2| \leq C_2\sqrt{6e^{-60\Lambda q}}$, where C_2 only depends on $\|w\|_\infty, \|v\|_\infty$. Then this gives us a uniform control on the width of each band I_j :

$$(5.75) \quad S_q^2 = \bigcup_{j=1}^q I_j, \quad \varepsilon_q^j := |I_j| \leq |S_q^2| \leq |\tilde{S}_q^2| \leq e^{-29\Lambda q}.$$

Now pick $E_j \in I_j \cap \sigma(H) \neq \emptyset$ to be the center in the sense that $I_j \subset (E_j - \varepsilon_q^j, E_j + \varepsilon_q^j)$. For any φ , let $u^\varphi(E_j)$ be the right half line solution associated with the energy E_j . By Claim 5.14, we have

$$(5.76) \quad \|u^\varphi(E_j)\|_{\ell_q(j)}^2 \geq e^{\frac{1}{6}\Lambda q}, \quad j = 1, \dots, q$$

where $\ell_q(j) = \ell(\varphi, E_j, \varepsilon_q^j)$ is given as in (3.21).

A direct consequence of (5.76) and the subordination theory Lemma 3.2 is

$$(5.77) \quad \varepsilon_q^j \cdot |m_\varphi(E_j + i\varepsilon_q^j)| < \frac{5 + \sqrt{24}}{2} \cdot e^{-\frac{1}{6}\Lambda q}, \quad j = 1, \dots, q$$

Then by (2.3) and (3.25), we have

$$(5.78) \quad \mu(I_j) \leq \sup_\varphi 2\varepsilon_q^j |m_\varphi(E_j + i\varepsilon_q^j)| < (5 + \sqrt{24})e^{-\frac{1}{6}\Lambda q}, \quad j = 1, \dots, q.$$

Clearly, (5.78) completes the proof of Lemma 5.12 provided $q(5 + \sqrt{24})e^{-\frac{1}{6}\Lambda q} \leq e^{-\frac{1}{10}\Lambda q}$. $\square$

6. SPECTRAL SINGULARITY FOR ANALYTIC QUASIPERIODIC JACOBI OPERATOR

In this section, we focus on analytic quasiperiodic potential given by $v_n = v(\theta + n\alpha), w_n = c(\theta + n\alpha), n \in \mathbb{Z}, \theta \in \mathbb{T}$ where $v \in C^\omega(\mathbb{T}, \mathbb{R})$ and $c \in C^\omega(\mathbb{T}, \mathbb{C})$ are analytic functions on $\mathbb{T}$ taken values in $\mathbb{R}$ and $\mathbb{C}$ respectively. Both $v(\theta)$ and $c(\theta)$ have bounded analytic extensions to the strip $\{z : |\text{Im}z| < \rho\}$.

Follow the notations in section 3.1. We list the corresponding quasiperiodic versions here again for reader's convenience. The analytic quasiperiodic Jacobi operator on $\ell^2(\mathbb{Z})$ is given by:

$$(6.1) \quad (H_{v,c}u)_n = c(\theta + n\alpha)u_{n+1} + \bar{c}(\theta + (n-1)\alpha)u_{n-1} + v(\theta + n\alpha)u_n, \quad n \in \mathbb{Z}.$$

The transfer matrix is given by:

$$A(\theta, E, \alpha) = \frac{1}{c(\theta)} \begin{pmatrix} E - v(\theta) & -\bar{c}(\theta - \alpha) \\ c(\theta) & 0 \end{pmatrix}$$

and

$$A(n; \theta, E, \alpha) = \prod_{j=1}^n A(\theta + (j-1)\alpha, E, \alpha), \quad n > 0.$$

The spectral singularity in Theorem 2.4 is reduced to the following lemma about the norm of the transfer matrices, which was proved in [32]:

Lemma 6.1 ([32], Lemma 3.1). *Fix $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ with $\beta = \beta(\alpha) < \infty$ and $\theta \in \mathbb{T}$. Suppose there is a constant $c > 0$ such that for any E , there is $\ell_0 = \ell(E, \beta, \rho, \theta)$ such that for any $\ell > \ell_0$, the following two estimates hold:*

$$(6.2) \quad \sum_{k=1}^{\ell} \|A(k; \theta, E, \alpha)\|^2 \geq \ell^{1+\frac{2c}{\beta}},$$

and

$$(6.3) \quad \sum_{k=1}^{\ell} \|A(k; \theta - \alpha, E, -\alpha)\|^2 \geq \ell^{1+\frac{2c}{\beta}},$$

then we have the following upper bound for the spectral dimension defined in (2.3) of the spectral measure $\mu = \mu_{\alpha, \theta}$:

$$(6.4) \quad \dim_{\text{spe}}(\mu) \leq \gamma_0 := \frac{1}{1 + c/\beta} < 1.$$

This is a direct consequence of the subordinate theory (3.23) and Last-Simon upper bound on the generalized eigenfunction (3.24). Actually, in view of Lemma 3.5, it is enough to find a φ such that both m_φ and $\tilde{m}_{\pi/2-\varphi}$ are γ -spectral singular, where m_φ and $\tilde{m}_{\pi/2-\varphi}$ are half line m -function defined in section 3.2. The estimate on the half line m -function relies on the subordinacy theory Lemma 3.2. The quantitative estimates need both an upper bound and a lower bound on the ℓ -norm of u^φ, v^φ . Lemma 3.3 provides two eigen functions u^φ and $u^{\varphi, -}$, both obeying the sub-linear growth as in (3.24). (6.2) and (6.3) provide the lower bound as required in the subordinacy theory for m_φ and $\tilde{m}_{\pi/2-\varphi}$ respectively, which eventually lead to the spectral singularity. In the rest of this section, we will focus on the proof of (6.2) and (6.3). We refer readers to [32], section 3 for more details about this lemma and spectral singularity.

For a $\text{GL}(2, \mathbb{C})$ matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we denote by $\|\cdot\|_{HS}$ the Hilbert-Smith norm of A :

$$(6.5) \quad \|A\|_{HS} = \sqrt{|a|^2 + |b|^2 + |c|^2 + |d|^2}.$$

In the rest of this section, we write $\|\cdot\| = \|\cdot\|_{HS}$ for simplicity whenever it is clear.

The key to prove (6.2) and (6.3) is the following lemma:

Lemma 6.2. *Assume that $L(E) \geq a > 0$. There are $c_2 = c_2(a, S, \rho) > 0$, $n_0 = n_0(a, \rho) > 0$ and a positive integer $d = d(S, \rho, \|v\|_\rho, \|c\|_\rho) \in \mathbb{N}^+$ such that for $E \in S$ and $n > n_0$, there exists an interval $\Delta_n \subset \mathbb{T}$ satisfying the following properties:*

$$(6.6) \quad \text{Leb}(\Delta_n) \geq \frac{c_2}{4dn}$$

and for any $\theta \in \Delta_n$,

$$(6.7) \quad \|A(n; \theta, E, \alpha)\|_{HS}^2 > e^{nL(E)/8}.$$

Lemma 6.2 will be the key ingredient to the proof of spectral singularity, we will return to its proof in the end of this section. We will derive (6.2) and (6.3) from Lemma 6.2 and finish the proof of Theorem 2.4 first.

Let q_n be given as in the continued fraction approximants to α , see (2.8). The following lemma about the ergodicity of an irrational rotation can be found e.g. in [25].

Lemma 6.3 (Lemma 9, [25]). *Let $\Delta \subset [0, 1]$ be an arbitrary segment. If $|\Delta| > \frac{1}{q_n}$. Then, for any θ ; there exists a j in $\{0, 1, \dots, q_n + q_{n-1} - 1\}$ such that $\theta + j\alpha \in \Delta$.*

Combine Lemma 6.2 with Lemma 6.3, we immediately have the following localization density result:

Lemma 6.4. *Fix $E \in S, \theta \in \Theta$ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. There is $n_1 = n_1(E, \rho, \alpha, \theta)$ such that for any $q_n \geq n_1$ and any $m \in \mathbb{N}$, there is $j_m = j_m(\theta) \in [2m q_n, (2m + 2) q_n)$ such that*

$$(6.8) \quad \|A(j_N; \theta, E, \alpha)\| > e^{c_0 q_n L(E)}$$

where $c_0 = c_0(a, \rho)$ explicitly depends on c_2 and d given in Lemma 6.2.

Proof. We fix E, α and write $A(n; \theta) = A(n; \theta, E, \alpha)$ for simplicity. Let n_0 be given as in Lemma 6.2. Given q_n , let

$$(6.9) \quad k_n = \left\lfloor \frac{c_2 q_n}{4d} \right\rfloor - 1 \geq \frac{c_2}{5d} q_n \geq n_0,$$

provided q_n large, where c_2 and d are given as in Lemma 6.2. By Lemma 6.2, there is an interval $\Delta_{k_n} \subset \mathbb{T}$ such that the following hold:

$$(6.10) \quad \text{Leb}(\Delta_{k_n}) \geq \frac{c_2}{4dk_n} > \frac{1}{q_n}$$

and

$$(6.11) \quad \|A(k_n; \theta)\|^2 > e^{k_n L(E)/8} > e^{\frac{c_2}{40d} q_n L(E)}, \quad \forall \theta \in \Delta_{k_n}.$$

Fix θ and $m \in \mathbb{N}$, apply Lemma 6.3 to Δ_{k_n} and $\theta + 2m q_n$, we have that there exists a j in $\{0, 1, \dots, q_n + q_{n-1} - 1\}$ such that $(\theta + 2m q_n \alpha) + j\alpha \in \Delta_{k_n}$. By (6.11), we have

$$(6.12) \quad \|A(k_n; \theta + 2m q_n \alpha + j\alpha)\| > e^{4c_0 q_n L(E)},$$

where $c_0 = \frac{c_2}{320d}$.

It is easy to check that

$$(6.13) \quad A(2m q_n + j + k_n; \theta) = A(k_n; \theta + 2m q_n \alpha + j\alpha) A(2m q_n + j; \theta).$$

By (6.12), we have that either

$$(6.14) \quad \|A^{-1}(2m q_n + j; \theta)\| \geq e^{2c_0 q_n L(E)}$$

$$(6.15) \quad \text{or } \|A(2m q_n + j + k_n; \theta)\| \geq e^{2c_0 q_n L(E)}.$$

Direct computation shows that

$$(6.16) \quad \|A^{-1}(2m q_n + j; \theta)\| = \frac{1}{|\det A(2m q_n + j; \theta)|} \|A(2m q_n + j; \theta)\|$$

$$(6.17) \quad = \frac{|c(\theta + (2m q_n + j)\alpha)|}{|c(\theta)|} \|A(2m q_n + j; \theta)\|$$

$$(6.18) \quad \leq \frac{\|c\|_\infty}{|c(\theta)|} \|A(2m q_n + j; \theta)\|$$

Suppose (6.14) holds, then

$$(6.19) \quad \|A(2m q_n + j; \theta)\| \geq \frac{|c(\theta)|}{\|c\|_\infty} e^{2c_0 q_n L(E)} \geq e^{c_0 q_n L(E)}$$

provided

$$(6.20) \quad e^{c_0 q_n L(E)} \geq \frac{\|c\|_\infty}{|c(\theta)|}.$$

Let j_m be $2m q_n + j$ or $2m q_n + j + k_n$, for which j_N satisfies (6.8). Clearly, by the choice of j, k_n , $j_m(\theta) \in [2m q_n, (2m+2) q_n)$ for all $m \in \mathbb{N}$ and

$$(6.21) \quad q_n \geq n_1 := \max \left\{ \frac{5dn_0}{c_2}, \frac{\ln \frac{\|c\|_\infty}{|c(\theta)|}}{c_0 L(E)} \right\}.$$

Note that if $m = j = 0$ in (6.12), we pick $j_0 = k_n \geq 1$. So $j_0 \in [1, 2q_n)$. $\square$

With the above localization density lemma, we can complete the proof of Theorem 2.4 by checking (6.2) and (6.3) in Lemma 6.1 for a.e. $\theta \in \mathbb{T}$.

Proof of Theorem 2.4.

For any $\ell \in \mathbb{N}$, there is q_n such that, $\ell \in [2q_n, 2q_{n+1})$. Let $\ell = 2Nq_n + r$, where $0 \leq r < 2q_n$, $1 \leq N < \frac{q_{n+1}}{q_n}$. Let n_1 be given as in (6.21). It is easy to check that $q_n \geq n_1$ provided

$$(6.22) \quad \ell \geq 2e^{2n_1\beta(\alpha)}.$$

Now apply Lemma 6.4 to q_n and $0 \leq m \leq N-1$. There are $j_m \in [2m q_n, (2m+2) q_n) \subset [0, \ell]$ such that $\|A(j_m; \theta, E, \alpha)\| > e^{c_0 q_n L(E)}$. Therefore,

$$(6.23) \quad \begin{aligned} \sum_{k=1}^{\ell} \|A(k; \theta, E, \alpha)\|^2 &\geq \sum_{m=0}^{N-1} \|A(j_m; \theta, E, \alpha)\|^2 \\ &\geq N e^{2c_0 q_n L(E)}. \end{aligned}$$

Clearly, $\ell = 2Nq_n + r < 4Nq_n$. By (6.23), we have

$$\sum_{k=1}^{\ell} \|A_k(\theta)\|^2 \geq \frac{\ell}{4q_n} e^{2c_0 q_n L(E)} \geq \ell e^{c_0 q_n L(E)} \geq \ell e^{c_0 a q_n}$$

provided $e^{c_0 q_n L(E)} \geq 4q_n$. Then for sufficiently large ℓ such that $\frac{\ln q_{n+1}}{q_n} < 2\beta$, we have

$$(6.24) \quad \sum_{k=1}^{\ell} \|A_k(\theta)\|^2 \geq \ell q_{n+1}^{\frac{c_0 a}{2\beta}} \geq \ell \cdot \left(\frac{\ell}{2}\right)^{\frac{c_0 a}{2\beta}} \geq \ell \cdot \ell^{\frac{c_0 a}{4\beta}} =: \ell^{1+\frac{2c}{\beta}},$$

provided $\ell \geq 4$, where $c = \frac{1}{8}c_0 a$. This proved (6.2).

For the same θ and E , repeat the above procedure for $A(n; \theta - \alpha, E, -\alpha)$. We have a sequence of positive integers $\tilde{j}_m = \tilde{j}_m(\theta - \alpha) \in [2mq_n, 2(m+1)q_n)$ for any $N \in \mathbb{N}$ and $q_n \geq n_1(E, \rho, -\alpha, \theta - \alpha)$ such that

$$(6.25) \quad \|A(\tilde{j}_m; \theta - \alpha, E - \alpha, E)\| > e^{c_0 q_n L(E)}.$$

Note that $c_0 = c_0(a, \rho)$ does not depend on $\theta - \alpha$ and is the same as in (6.8) and (6.24). The same reasoning proves (6.3).

Then by Lemma 6.1, we have for all $\theta \in \Theta$ and $\beta(\alpha) < \infty$, $\dim_{\text{spe}}(\mu_{\alpha, \theta}) < \frac{1}{1+c/\beta} < 1$, which completes the proof of Theorem 2.4. $\square$

In the rest of the section, we focus on the proof of Lemma 6.2. In [32], the authors proved the analytic $SL(2, \mathbb{R})$ version of this lemma. One advantage for Shrödinger case is the H-S norm $\|A(n; \theta)\|_{HS}^2$ is a real analytic function which can be approximated by trigonometric functions in some uniform sense. For $GL(2, \mathbb{C})$ case, the HS norm of the transfer matrices are meromorphic functions. We need finer decomposition to deal with the poles.

Fix E, α , for $n \in \mathbb{N}^+, \theta \in \mathbb{T}$, let

$$(6.26) \quad F_n(\theta) = \|A(n; \theta, E, \alpha)\|_{HS}^2$$

be defined as in (6.5). We have the following decomposition of $F_n(\theta)$:

Lemma 6.5. *For any E and $n \in \mathbb{N}^+$, there are positive functions $f_n(\theta)$ and $g_n(\theta)$ such that*

$$(6.27) \quad F_n(\theta) = \frac{f_n(\theta)}{g_n(\theta)},$$

$$(6.28) \quad \inf_n \frac{1}{n} \int \ln g_n(\theta) d\theta = 0, \quad \inf_n \frac{1}{n} \int \ln f_n(\theta) d\theta = 2L(E).$$

For any $\varepsilon > 0$, there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that for any $n > n_0$ and any $\theta \in \mathbb{T}$,

$$(6.29) \quad 0 < g_n(\theta) < e^{\varepsilon n}.$$

Furthermore, for E in a compact set S , there are $n_1 = n_1(\rho) > 0$ and $d = d(S, \rho, \|v\|_\rho, \|c\|_\rho) > 0$ such that for any $n > n_1$, there are two functions $P_n(\theta), R_n(\theta)$ satisfying the following decomposition:

$$(6.30) \quad f_n(\theta) = P_n(\theta) + R_n(\theta),$$

$$(6.31) \quad |R_n(\theta)| < 1,$$

$$(6.32) \quad P_n(\theta) = \sum_{|k| \leq d \cdot n} \hat{f}_n(k) e^{2\pi i k \theta},$$

where $\hat{f}_n(k)$ is the k -th Fourier coefficient of $f_n(\theta)$.

Proof. Follow the notations in (3.4), let

$$A(\theta, E) = \frac{1}{c(\theta)} D(\theta, E), \quad D(\theta, E) = \begin{pmatrix} E - v(\theta) & -\bar{c}(\theta - \alpha) \\ c(\theta) & 0 \end{pmatrix}$$

and

$$(6.33) \quad A(n; \theta, E) = \frac{1}{c(n; \theta)} D(n; \theta, E), \quad \text{where}$$

$$(6.34) \quad c(n; \theta) = \prod_{j=1}^n c(\theta + (j-1)\alpha), \quad D(n; \theta, E) = \prod_{j=1}^n D(\theta + (j-1)\alpha, E) = \begin{pmatrix} D^1(\theta) & D^2(\theta) \\ D^3(\theta) & D^4(\theta) \end{pmatrix}$$

Without loss of generality, we assume $\int_{\mathbb{T}} \ln |c(\theta)| d\theta = 0$. Otherwise, the argument simply differs by a constant factor. See Remark 6.1 after the proof.

Let $g_n(\theta) := |c(n; \theta)|^2$ and $f_n(\theta) := \|D(n; \theta, E)\|_{HS}^2$. Clearly,

$$(6.35) \quad F_n(\theta) = \|A(n; \theta, E, \alpha)\|_{HS}^2 = \frac{f_n(\theta)}{g_n(\theta)}$$

$$(6.36) \quad f_n(\theta) = \|D(n; \theta, E)\|_{HS}^2 = |D^1(\theta)|^2 + |D^2(\theta)|^2 + |D^3(\theta)|^2 + |D^4(\theta)|^2.$$

Birkhoff Ergodic Theory implies that for any irrational α ,

$$(6.37) \quad \begin{aligned} \lim_n \int_{\mathbb{T}} \frac{1}{n} \ln g_n(\theta) d\theta &= \inf_n \frac{1}{n} \int \ln g_n(\theta) d\theta \\ &= \lim_n \int_{\mathbb{T}} \frac{1}{n} \sum_{j=1}^n \ln |c(\theta + (j-1)\alpha)|^2 d\theta = \int_{\mathbb{T}} \ln |c(\theta)|^2 d\theta = 0 \end{aligned}$$

In view of (6.35) and the definition of Lyapunov exponent (3.12), we have

$$(6.38) \quad \begin{aligned} \inf_n \int_{\mathbb{T}} \frac{1}{n} \ln f_n(\theta) d\theta &= \inf_n \int_{\mathbb{T}} \frac{1}{n} \ln \left(g_n(\theta) \|A(n; \theta)\|_{HS}^2 \right) d\theta \\ &= \inf_n \int_{\mathbb{T}} \frac{1}{n} \ln g_n(\theta) d\theta + \inf_n \int_{\mathbb{T}} \frac{1}{n} \ln \|A(n; \theta)\|_{HS}^2 d\theta \\ &= 2L(E). \end{aligned}$$

Note $c(\theta)$ is continuous in θ , by (3.14), for any $\varepsilon > 0$, there is $n_1 = n_1(\varepsilon)$ such that for any $n > n_1$ and any $\theta \in \mathbb{T}$, we have the following upper semicontinuity (uniform in θ):

$$(6.39) \quad \frac{1}{n} \ln g_n(\theta) \leq \int_{\mathbb{T}} \ln |c(\theta)|^2 d\theta + \varepsilon = \varepsilon.$$

This gives $g_n(\theta) \leq e^{\varepsilon n}$ and finishes the proof of (6.27)-(6.29).

The further decomposition of $f_n(\theta)$ into P_n and R_n follows the strategy in [32]. Note that $v(\theta)$ and $c(\theta)$ are both analytic with bounded extension to the strip $\{z : |\operatorname{Im} z| < \rho\}$. In view of (6.34), all $D^i(\theta)$, $i = 1, 2, 3, 4$ have analytic extension to the strip $\{z : |\operatorname{Im} z| < \rho\}$. For compact S , there is $C_1 = C_1(S, \rho, \|v\|_{\rho}, \|c\|_{\rho})$ such that

$$(6.40) \quad \|D^i\|_{\rho} := \sup_{|\operatorname{Im} z| < \rho} |D^i(z)| < \sup_{|\operatorname{Im} z| < \rho} \|D_n(z)\|_{HS}^2 < e^{C_1 n}, \quad E \in S, \quad i = 1, 2, 3, 4.$$

Consider the Fourier expansion of the periodic-1 functions $D^i(\theta)$:

$$(6.41) \quad D^i(\theta) = \sum_{k \in \mathbb{Z}} \widehat{D}^i(k) e^{2\pi i k \theta}, \quad i = 1, 2, 3, 4$$

The Fourier coefficients of $D^i(\theta)$ has exponential decay as

$$(6.42) \quad |\widehat{D}^i(k)| < \|D^i\|_{\rho} \cdot e^{-2\pi \rho |k|} < e^{C_1 n} \cdot e^{-2\pi \rho |k|}, \quad \forall k \in \mathbb{Z}, \quad i = 1, 2, 3, 4.$$

Combine

$$(6.43) \quad |D^i(\theta)|^2 = \left(\sum_{k \in \mathbb{Z}} \widehat{D}^i(k) e^{2\pi i k \theta} \right) \left(\sum_{k \in \mathbb{Z}} \overline{\widehat{D}^i(k)} e^{-2\pi i k \theta} \right)$$

with (6.42), it is easy to check that the Fourier coefficients of $|D^i(\theta)|^2$ has exponential decay as:

$$(6.44) \quad |\widehat{|D^i|^2}(\cdot)(k)| < e^{C_1 n} \cdot e^{-\pi \rho |k|}, \quad \forall k \in \mathbb{Z}, \quad i = 1, 2, 3, 4.$$

Let $f_n(\theta)$ be given as in (6.36). Consider the Fourier expansion of $f_n(\theta)$:

$$(6.45) \quad f_n(\theta) = \sum_{k \in \mathbb{Z}} \widehat{f}_n(k) e^{2\pi i k \theta}.$$

By (6.36) and (6.44), clearly, $\widehat{f}_n(k)$ has the same exponential decay in $|k|$:

$$(6.46) \quad |\widehat{f}_n(k)| < 4e^{C_1 n} \cdot e^{-\pi \rho |k|}, \quad \forall k \in \mathbb{Z}.$$

Pick

$$(6.47) \quad d = \left\lceil \frac{C_1}{\pi \rho} \right\rceil + 2.$$

We split $f_n(\theta)$ into two parts:

$$f_n(\theta) = P_n(\theta) + R_n(\theta), \quad P_n(\theta) = \sum_{|k| \leq d \cdot n} \widehat{f}_n(k) e^{2\pi i k \theta}, \quad R_n(\theta) = \sum_{|k| > d \cdot n} \widehat{f}_n(k) e^{2\pi i k \theta}.$$

For any $\theta \in \mathbb{T}$,

$$\begin{aligned} |R_n(\theta)| &\leq \sum_{|k| > d \cdot n} |\widehat{f}_n(k)| \leq \sum_{|k| > d \cdot n} 4e^{C_1 n} \cdot e^{-\pi \rho |k|} \\ &\leq \frac{8}{1 - e^{-\pi \rho}} e^{C_1 n} e^{-\pi \rho d n} \\ &\leq \frac{8}{1 - e^{-\pi \rho}} e^{-(\pi \rho d - C_1) n}. \end{aligned}$$

By the choice of d in (6.47), we have $\pi \rho d > C_1 + \pi \rho$. Then for any $\theta \in \mathbb{T}$,

$$(6.48) \quad |R_n(\theta)| \leq \frac{8}{1 - e^{-\pi \rho}} e^{-\pi \rho n} < 1,$$

provided $n > n_2(\rho) := (\pi \rho)^{-1} \ln(\frac{8}{1 - e^{-\pi \rho}})$. This finishes the proof of (6.30)-(6.32). $\square$

Remark 6.1. Suppose $b = \int_{\mathbb{T}} \ln |c(\theta)| d\theta \neq 0$. In (6.33), we set

$$(6.49) \quad A(\theta, E) = \frac{1}{\tilde{c}(\theta)} \tilde{D}(\theta, E), \quad \text{where } \tilde{c}(\theta) = e^{-b} c(\theta), \quad \tilde{D}(\theta, E) = e^{-b} D(\theta, E).$$

Clearly,

$$(6.50) \quad \int_{\mathbb{T}} \ln |\tilde{c}(\theta)| d\theta = 0, \quad \lim_n \int_{\mathbb{T}} \frac{1}{n} \ln \|\tilde{D}(n; \theta, E)\| d\theta = L(E).$$

Let $g_n(\theta) := |\tilde{c}(n; \theta)|^2$ and $f_n(\theta) := \|\tilde{D}(n; \theta, E)\|_{HS}^2$. The rest of the decomposition are exactly the same.

Combine Lemma 6.5 with the positive assumption on Lyapunov exponent, we can now finish **The proof of Lemma 6.2:** Assume that the Lyapunov exponent $L(E) \geq a > 0$ for $E \in S$. Pick $\varepsilon = a/8$. Let $n_1 = n_1(\varepsilon)$ and $n_2 = n_2(\rho)$ be given as in Lemma 6.5. Then for all $n > \max\{n_1, n_2\}$, we have $g_n(\theta), f_n(\theta), P_n(\theta)$ and $R_n(\theta)$ as in Lemma 6.5, satisfying (6.27)-(6.32). Denote

$$\begin{aligned} \Theta_n^1 &= \{\theta : F_n(\theta) > e^{nL(E)/8}\}, \\ \Theta_n^2 &= \{\theta : P_n(\theta) > e^{nL(E)/3}\}, \\ \Theta_n^3 &= \{\theta : f_n(\theta) > e^{nL(E)/2}\}. \end{aligned}$$

Let $n_3 := 4a^{-1}$. Then for all $n > n_3$, we have $e^{nL(E)} > e^{na} > e^4 > 50$. By using the fact $x^{1/2} - x^{1/3} > x^{1/3} - x^{1/4} > 1$ for all $x > 50$, it is easy to check that for $n > n_3$,

$$(6.51) \quad e^{nL(E)/2} - e^{nL(E)/3} > e^{nL(E)/3} - e^{nL(E)/4} > 1.$$

Assume that $f_n(\theta) > e^{nL(E)/2}$. By (6.30) and (6.51), we have for $n > n_3$,

$$P_n(\theta) > f_n(\theta) - |R_n(\theta)| > e^{nL(E)/2} - 1 > e^{nL(E)/3}.$$

Then

$$f_n(\theta) > P_n(\theta) - |R_n(\theta)| > e^{nL(E)/3} - 1 > e^{nL(E)/4}.$$

In view of (6.27) and (6.29), we have then for $n > \max\{n_1, n_3\}$,

$$F_n(\theta) = \frac{f_n(\theta)}{g_n(\theta)} > \frac{e^{nL(E)/4}}{e^{n\varepsilon}} > \frac{e^{nL(E)/4}}{e^{nL(E)/8}} = e^{nL(E)/8}.$$

Therefore, we have for $n > n_0 := \max\{n_1, n_2, n_3\}$,

$$(6.52) \quad \Theta_n^3 \subseteq \Theta_n^2 \subseteq \Theta_n^1.$$

Meanwhile, by (6.28),

$$\begin{aligned} 2nL(E) &\leq \int_{\mathbb{T}} \ln f_n(\theta) d\theta \\ &\leq \text{Leb}(\Theta_n^3) \ln \|f_n\|_\rho + (1 - \text{Leb}(\Theta_n^3)) \ln e^{nL(E)/2} \\ &\leq \text{Leb}(\Theta_n^3) \cdot C_1 n + (1 - \text{Leb}(\Theta_n^3)) \cdot nL(E)/2. \end{aligned}$$

This implies $\text{Leb}(\Theta_n^3) \geq \frac{3L(E)}{2C_1 - L(E)}$. Note that $L(E) \geq a > 0, E \in S$, we have

$$(6.53) \quad \text{Leb}(\Theta_n^3) \geq \frac{3a}{2C_1 - a} =: c_2(a, S, \rho) > 0.$$

In view of (6.52), we have for $n > n_0$,

$$(6.54) \quad \text{Leb}(\Theta_n^2) \geq c_2(a, S, \rho) > 0.$$

By (6.32), $P_n(\theta)$ is a trigonometric polynomial of degree (at most) $2dn$, where d is given by (6.47) in Lemma 6.5. The set Θ_n^2 consists of no more than $4dn$ intervals. Therefore, there exists a segment, $\Delta_n \subset \Theta_n^2 \subset \Theta_n^1$, with $\text{Leb}(\Delta_n) > \frac{c_2}{4dn}$. For any $n > n_0$ and $\theta \in \Delta_n \subset \Theta_n^1$,

$$\|A_n(\theta)\|_{HS}^2 = F_n(\theta) > e^{nL(E)/8}$$

and

$$\text{Leb}(\Delta_n) > \frac{c_2}{4dn},$$

as claimed. $\square$

7. THE EXTENDED HARPER'S MODEL: PROOF OF COROLLARY 2.5

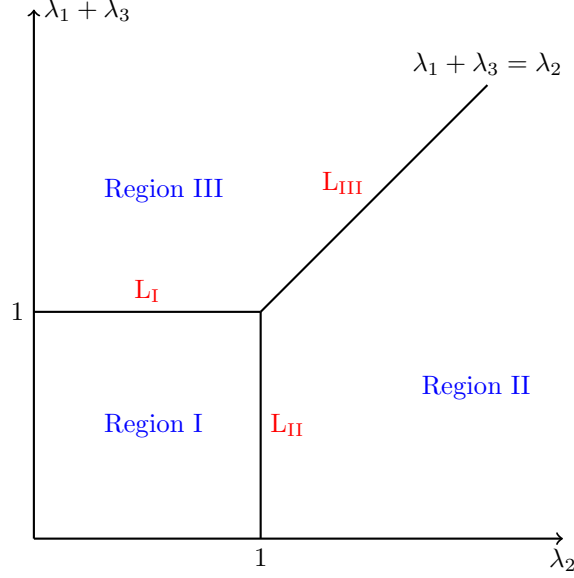
Recall the extended Harper's model (EHM) defined in (2.13) as:

$$(7.1) \quad (H_{\lambda, \alpha, \theta} u)_n = c_\lambda(\theta + n\alpha)u_{n+1} + \bar{c}_\lambda(\theta + (n-1)\alpha)u_{n-1} + 2 \cos 2\pi(\theta + n\alpha)u_n,$$

where

$$(7.2) \quad c_\lambda(\theta) = \lambda_1 e^{-2\pi i(\theta + \frac{\alpha}{2})} + \lambda_2 + \lambda_3 e^{2\pi i(\theta + \frac{\alpha}{2})}.$$

By some earlier work [28], we consider the following partitioning of the parameter space into the following three regions:



Region I: $0 \leq \lambda_1 + \lambda_3 \leq 1, 0 < \lambda_2 \leq 1$,

Region II: $\max\{\lambda_1 + \lambda_3, 1\} \leq \lambda_2, \lambda_1 + \lambda_3 > 0$,

Region III: $\max\{1, \lambda_2\} \leq \lambda_1 + \lambda_3, \lambda_2 > 0$.

Let $L(E, \lambda)$ be the *Lyapunov exponent* of the extended Harper's model, defined as in (3.12). The main achievement of [28] is to prove the following explicit formula of $L(E, \lambda)$, valid for all λ and all irrational α :

Theorem 7.1 ([28]). *Fix an irrational frequency α . Then $L(E, \lambda)$ restricted to the spectrum is zero within both region II and III. In region I it is given by the formula on the spectrum,*

$$(7.3) \quad L(E, \lambda) = \begin{cases} \ln \left(\frac{1 + \sqrt{1 - 4\lambda_1\lambda_3}}{2\lambda_1} \right) & , \text{ if } \lambda_1 \geq \lambda_3, \lambda_2 \leq \lambda_3 + \lambda_1, \\ \ln \left(\frac{1 + \sqrt{1 - 4\lambda_1\lambda_3}}{2\lambda_3} \right) & , \text{ if } \lambda_3 \geq \lambda_1, \lambda_2 \leq \lambda_3 + \lambda_1, \\ \ln \left(\frac{1 + \sqrt{1 - 4\lambda_1\lambda_3}}{\lambda_2 + \sqrt{\lambda_2^2 - 4\lambda_1\lambda_3}} \right) & , \text{ if } \lambda_2 \geq \lambda_3 + \lambda_1. \end{cases}$$

Denote by $\text{Region I}^\circ, \text{Region II}^\circ, \text{Region III}^\circ$ the interior of Region I, II, III respectively. A complete understanding of the spectral properties of the extended Harper's model for a.e. θ has been established in [23, 22, 3, 20]. We collect the spectral decomposition results in these papers as the follow theorem for reader's convenience. Follow the notations in Corollary 2.5, denote the three parameter regions of $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$ by:

$$\mathcal{R}_1 = \{\lambda \in \mathbb{R}^3 : 0 < \lambda_1 + \lambda_3 < 1, 0 < \lambda_2 < 1\}.$$

$$\mathcal{R}_2 = \{\lambda \in \mathbb{R}^3 : \lambda_2 > \max\{\lambda_1 + \lambda_3, 1\}, \lambda_1 + \lambda_3 \geq 0 \text{ or } \lambda_1 + \lambda_3 > \max\{\lambda_2, 1\}, \lambda_1 \neq \lambda_3, \lambda_2 > 0\}.$$

$$\mathcal{R}_3 = \{\lambda \in \mathbb{R}^3 : 0 \leq \lambda_1 + \lambda_3 \leq 1, \lambda_2 = 1 \text{ or } \lambda_1 + \lambda_3 \geq \max\{\lambda_2, 1\}, \lambda_1 = \lambda_3, \lambda_2 > 0\}.$$

Theorem 7.2 ([23, 22, 3, 20]). *The following Lebesgue decomposition of the spectrum of $H_{\lambda, \alpha, \theta}$ holds for a.e. θ .*

- For $\lambda \in \mathcal{R}_1$, if $\beta(\alpha) < L(E, \lambda)$, then $H_{\lambda, \alpha, \theta}$ has pure point spectrum. If $\beta(\alpha) > L(E, \lambda)$, then $H_{\lambda, \alpha, \theta}$ has purely singular continuous spectrum.

- For $\lambda \in \mathcal{R}_2$ and all irrational α , $H_{\lambda,\alpha,\theta}$ has purely absolutely continuous spectrum.
- For $\lambda \in \mathcal{R}_3$ and all irrational α , $H_{\lambda,\alpha,\theta}$ has purely singular continuous spectrum.

Now we are in the place to analyze the spectral dimension of EHM in each region.

Clearly, Region $\text{I}^\circ = \mathcal{R}_1$. In view of (7.3), it is easy to check that $L(E, \lambda) > 0$ on $\mathcal{R}_1$ for all α and E . Therefore, by Theorem 2.3, we have part (1) of Corollary 2.5.

Next, consider Region $\mathcal{R}_2$. Theorem 7.2 shows that $H_{\lambda,\alpha,\theta}$ has purely a.c. spectrum in region $\mathcal{R}_2$ for all α and a.e. θ . In view of Definition 2.3, absolutely continuous measure has full spectral dimension⁵. This gives part (2) of Corollary 2.5.

Part (3) (region $\mathcal{R}_3$) is the only place requires extra work. By Theorem 7.1 and Theorem 7.2, in region $\mathcal{R}_3$, $L(E, \lambda) = 0$ on the spectrum and $H_{\lambda,\alpha,\theta}$ does not have a.c. spectrum. Lack of positivity of Lyapunov exponent, we do not have the spectral singularity and the upper bound provided by Theorem 2.4. While the lower bound from Theorem 2.1 still holds. Moreover, in view of Lemma 3.1 and Corollary 4.3, we can obtain arbitrarily small exponential growth of the transfer matrix. This allows us to obtain the increased range of $\beta(\alpha)$ in the critical region in part (3).

Recall the notations of the transfer matrix in (3.6) and (3.7) for EHM: let

$$A^\lambda(\theta, E, \alpha) = \frac{1}{c_\lambda(\theta)} D^\lambda(\theta, E, \alpha), \quad D^\lambda(\theta, E, \alpha) = \begin{pmatrix} E - v(\theta) & -\bar{c}_\lambda(\theta - \alpha) \\ c_\lambda(\theta) & 0 \end{pmatrix}.$$

For $n > 0, m \in \mathbb{Z}$,

$$(7.4) \quad A^\lambda(n, m; \theta) = \prod_{j=m}^{m+n-1} A^\lambda(\theta + j\alpha),$$

$$(7.5) \quad D^\lambda(n, m; \theta) = \prod_{j=m}^{m+n-1} D^\lambda(\theta + j\alpha), \quad c_\lambda(n, m; \theta) = \prod_{j=m}^{m+n-1} c_\lambda(\theta + j\alpha).$$

It is easy to check that

$$(7.6) \quad L(E, \lambda) = L(D^\lambda) - \int_{\mathbb{T}} \ln |c_\lambda(\theta)| d\theta.$$

Note that

$$(7.7) \quad b_\lambda := \int_{\mathbb{T}} \ln |c_\lambda(\theta)| d\theta$$

is not necessarily zero in region $\mathcal{R}_3$. Suppose not, consider the rescaling trick in Remark 6.1. Set

$$(7.8) \quad \tilde{c}_\lambda(\theta) = e^{-b_\lambda} c_\lambda(\theta), \quad \tilde{D}^\lambda(\theta, E) = e^{-b_\lambda} D^\lambda(\theta, E).$$

Clearly, in Region X,

$$(7.9) \quad \int_{\mathbb{T}} \ln |\tilde{c}(\theta)| = 0, \quad L(\tilde{D}^\lambda) = L(D^\lambda) - b_\lambda = L(E, \lambda) = 0.$$

Let $\tilde{D}^\lambda(n, m; \theta), \tilde{c}_\lambda(n, m; \theta)$ be defined the same way as in (7.5). For irrational α , let $\beta(\alpha)$ and q_n ⁶ be defined as in (2.8). Now assume $\beta(\alpha) > 0$, let $\tilde{\beta} = \min\{\beta(\alpha)/3, 1\}$. It was proved in [28] that $L(E, \alpha)$ is continuous in E for irrational α . In view of Lemma 3.1, the $\limsup$ is uniform in both θ

⁵Actually, it is well known that a.c. measure has full dimension for most commonly used fractal dimensions, e.g. Hausdorff/packing dimension etc. See more background knowledge about fractal dimension in e.g. [14]

⁶We still denote the subsequence reaching the $\limsup$ by q_n .

and E . Therefore, for any $\delta > 0$, there is $n_0 = n_0(\delta, \tilde{\beta})$ such that for any $n > n_0$, $m \in \mathbb{Z}$, $\theta \in \mathbb{T}$ and $E \in \sigma(H_{\lambda, \alpha, \theta})$,

$$(7.10) \quad \|\tilde{D}^\lambda(n, m; \theta)\| \leq e^{\delta^2 \tilde{\beta} n},$$

$$(7.11) \quad |\tilde{c}_\lambda(n, m; \theta)| \leq e^{\delta^2 \tilde{\beta} n}.$$

Note that in the proof Theorem 5.2, we only need to consider the above upper bound for E restricted in the spectrum. By Corollary 4.3, for a.e. θ and q_n large,

$$(7.12) \quad \min_{|m| \leq e^{\delta \tilde{\beta} q_n}} |\tilde{c}_\lambda(q_n, m; \theta)| > e^{-6\delta^2 \tilde{\beta} q_n}.$$

Combing (7.10), (7.11) and (7.12), exact the same computation in section 3.4 shows that for a.e. θ , $0 < \delta < \frac{1}{\sqrt{7}}$ and q_n large,

$$(7.13) \quad \min_{|m| \leq e^{\delta \tilde{\beta} q_n}} |\tilde{c}_\lambda(r, m; \theta)| \geq e^{-7\delta^2 \tilde{\beta} q_n}, \quad 1 \leq r \leq q_n,$$

$$(7.14) \quad \max_{|m| \leq e^{\delta \tilde{\beta} q_n}} \left| \frac{\tilde{c}_\lambda(\theta + (m \pm q_n)\alpha)}{\tilde{c}_\lambda(\theta + m\alpha)} - 1 \right| < e^{-(\beta - 7\delta^2 \tilde{\beta})q_n},$$

$$(7.15) \quad \sup_{E \in \sigma(H_{\lambda, \alpha, \theta})} \|A^\lambda(r, m; \theta)\| < e^{8\delta^2 \tilde{\beta} q_n}, \quad 0 \leq r \leq q_n, \quad |m| \leq e^{\delta \tilde{\beta} q_n}.$$

Therefore, we can replace all the Λ in the proof Theorem 5.2 by $10\delta^2 \tilde{\beta}$. Then for any $\beta(\alpha) > 0$ and $0 < \gamma < 1$, (5.4) holds true provided

$$(7.16) \quad \delta < \frac{1}{6000}(1 - \gamma).$$

Therefore, by Lemma 5.1 and Theorem 5.2, for any $\beta(\alpha) > 0$, $\gamma < 1$ and a.e. θ , $\mu_{\lambda, \alpha, \theta}$ is γ -spectral continuous. By (2.5), $\dim_{\text{spe}}(\mu_{\lambda, \alpha, \theta})=1$, which completes the proof of part (3) of Corollary 2.5. $\square$

APPENDIX A. APPENDIX

A.1. Proof of (5.4) in Theorem 5.2. We have showed in the first part of Theorem 5.2 that if $\beta > 260(1 + \frac{1}{\delta})\Lambda$, then for μ a.e. E , there exists $K(E) \in \mathbb{N}$, for $k \geq K(E)$, we have

$$(A.1) \quad |\text{Tr} A(q_k; E)| < 2 - 2e^{-60\Lambda q_k}$$

Now by (5.20), we have

$$(A.2) \quad |\text{Tr} \tilde{A}(q_k; E)| < 2 - 2e^{-60\Lambda q_k} + 12e^{(-\beta + 4\Lambda)q_k} < 2 - e^{-60\Lambda q_k},$$

provided $e^{(\beta - 64\Lambda)q_k} > 12$. Fix E and $q = q_k$ and write $\tilde{A}(q_k; E) = \tilde{A}(q)$. Now apply Lemma (5.13) to these $\tilde{A}(q)$ satisfying A.2. Note $\tilde{A}(q) \in \text{SL}(2, \mathbb{R})$, and $|\text{Tr} \tilde{A}(q)| < 2$, the eigenvalue ρ of $\tilde{A}(q)$ is purely imaginary with modulus 1, i.e., $\rho = e^{i\psi}$, for some $\psi \in (-\pi, \pi)$. By (5.67), we have for any j ,

$$(A.3) \quad \tilde{A}^j(q) = \frac{\sin j\psi}{\sin \psi} \cdot \left(\tilde{A}(q) - \frac{\text{Tr} \tilde{A}(q)}{2} \cdot I \right) + \frac{\cos j\psi}{2} \cdot I, \quad \psi \in (-\pi, \pi)$$

Then $|2 \cos \psi| = |\text{Tr} \tilde{A}(q)| < 2 - e^{-60\Lambda q}$ implies $|\sin \psi| > \sqrt{1 - (1 - \frac{1}{2}e^{-60\Lambda q})^2} > e^{-40\Lambda q}$. By (A.3) and (5.18),

$$(A.4) \quad \|\tilde{A}^j(q)\| \leq 2e^{40\Lambda q} \|\tilde{A}(q)\| + 1 \leq e^{43\Lambda q},$$

provided $q > q(\Lambda)$.

Now for any $0 < \gamma < 1$, let $\xi = \frac{95}{1-\gamma} < e^{\delta\beta q}$ and

$$(A.5) \quad N = [e^{\xi\Lambda q}].$$

Apply Lemma 5.8 to $G = \tilde{A}(q)$, $G_j = \tilde{A}(q, jq + 1)$, $j = 0, \dots, N$, by (5.27) and (A.4), for all $j \leq N$ we have

$$(A.6) \quad \|\tilde{A}(jq) - \tilde{A}^j(q)\| < e^{(-\beta+93\Lambda+2\xi\Lambda)q} < e^{-\Lambda q} < 1.$$

provided $\beta > (94 + 2\xi)\Lambda$. Therefore, by (A.4),

$$\|\tilde{A}(jq)\| \leq \|\tilde{A}^j(q)\| + 1 \leq 2e^{40\Lambda q} \|\tilde{A}(q)\| + 2 \leq e^{43\Lambda q}$$

By (5.15), $|r^{-1}(jq, 0)| \leq 1 + Ne^{(-\beta+2\Lambda)q} \leq 1 + e^{(-\beta+2\Lambda+\xi\Lambda)q} < 2$ provided $\beta > 3\Lambda + \xi\Lambda$. Then by (5.12) and (5.18), for all $0 \leq j \leq N$ and $1 \leq r \leq q$,

$$(A.7) \quad \|A(jq)\| \leq |r^{-1}(jq, 0)| \cdot \|T_{jq}\| \cdot \|\tilde{A}(jq)\| \cdot \|T_0^{-1}\| \leq 2e^{43\Lambda q}$$

$$(A.8) \quad \|A(jq+r)\| \leq \|A(r, jq+1)\| \cdot \|A(jq)\| \leq e^{46\Lambda q}$$

Therefore,

$$(A.9) \quad \sum_{n=1}^{Nq} \|A(n; E)\|^2 \leq \sum_{k=0}^{N-1} \sum_{r=1}^q \|A(kq+r; E)\|^2 \leq Nq e^{92\Lambda q} \leq e^{(\xi+93)\Lambda q}$$

$$(A.10) \quad \frac{1}{(Nq)^{2-\gamma}} \sum_{n=1}^{Nq} \|A(n; E)\|^2 \leq e^{(-(1-\gamma)\xi+94)\Lambda q} = e^{-\Lambda q} < 1$$

In conclusion, for any $0 < \gamma < 1$ and μ a.e. E , we have a sequence $q_k \rightarrow \infty$ and $\ell_k = [e^{95(1-\gamma)^{-1}\Lambda q_k}]q_k$ such that

$$(A.11) \quad \sum_{n=1}^{\ell_k} \|A(n; E)\|^2 \leq \ell_k^{2-\gamma}$$

provided

$$(A.12) \quad \beta > (3\xi + \xi/\delta)\Lambda = (285 + \frac{95}{\delta})\frac{\Lambda}{1-\gamma} > (94 + 2\xi + \xi/\delta)\Lambda.$$

It was proved in [32] that (A.11) implies (5.2) directly from the relation (3.3) and (3.21) and . We omit the proof for this part here. See more details about this direct computation in the proof Lemma 2.1 in [32]. $\square$

A.2. Proof of Claim 5.14. For any $0 < \varepsilon < e^{-29\Lambda q}$, let $\ell = \ell(\varphi, \varepsilon, E)$, u^φ, v^φ be given as in (3.21). Write $\ell(\varepsilon) = [\ell] + \ell - [\ell]$, and $[\ell] = K(\varepsilon) \cdot q + r(\varepsilon)$, where $0 \leq r = [\ell] \bmod q < q$ and $0 \leq \ell - [\ell] < 1$. Let $X = \begin{pmatrix} \cos \varphi \\ -\sin \varphi \end{pmatrix}$ and $\tilde{X} = T_0^{-1}X$. Clearly, $\|X\| = \|\tilde{X}\| = 1$.

We need to show first suppose $K < N_q = [e^{29\Lambda q}]$, then for any $\varepsilon < e^{-29\Lambda q}$:

$$(A.13) \quad K > e^{\Lambda q}$$

For any $n \leq [\ell] + 1$, write $n = kq + r$, where $0 \leq k \leq K$, $0 \leq r \leq q$. By (5.15), (5.71) and (5.72), we have

$$\|A(kq)\| \leq |r^{-1}(kq, 0)| \cdot (\|\tilde{A}^k(q)\| + 1) \leq 2(6C_1 k e^{2\Lambda q} + 1) < k e^{3\Lambda q}$$

Then by (5.18),

$$\|A(kq+r)X\| \leq \|A(r, kq+1)\| \cdot \|A(kq)\| \cdot \|X\| \leq k e^{5\Lambda q}$$

Direct computation shows

$$\begin{aligned}
\|u^\varphi\|_\ell^2 &\leq \sum_{n=1}^{[\ell]+1} \|A(n) \cdot X\|^2 \leq \sum_{r=1}^q \|A(r) \cdot X\|^2 + \sum_{k=1}^K \sum_{r=1}^q \|A(kq+r) \cdot X\|^2 \\
&\leq q \cdot e^{4\Lambda q} + \sum_{k=1}^K \sum_{r=1}^q k^2 e^{10\Lambda q} \\
&\leq q \cdot e^{4\Lambda q} + K^3 q e^{10\Lambda q} \\
&\leq K^3 e^{11\Lambda q}
\end{aligned}$$

Since φ is arbitrary, we have $\|v^\varphi\|_\ell^2 \leq K^3 e^{11\Lambda q}$ in the same way. By the definition of ℓ in (3.21), we have

$$(A.14) \quad K^6 e^{22\Lambda q} \geq \|u^\varphi\|_{\ell(\varepsilon)} \|v^{\varphi(\varepsilon)}\|_\ell = \frac{1}{2\varepsilon} \geq e^{28\Lambda q}$$

Therefore, $K > e^{\Lambda q}$ as claim in (A.13).

To bound $\|u^\varphi\|_\ell^2$ from below, we need to consider two cases of initial value φ .

Case I: Assume φ satisfies

$$(A.15) \quad \left\| \left(\tilde{A}(q) - \frac{\text{Tr } \tilde{A}(q)}{2} \cdot I \right) \cdot \tilde{X} \right\| \geq e^{-\frac{1}{4}\Lambda q}.$$

By (5.67), for any $e^{\frac{1}{2}\Lambda q} \leq k \leq K \leq N_q$, we have

$$\begin{aligned}
\|\tilde{A}^k(q) \cdot \tilde{X}\| &\geq \frac{\rho^k - \rho^{-k}}{\rho - \rho^{-1}} \cdot \left\| \left(\tilde{A}(q) - \frac{\text{Tr } \tilde{A}(q)}{2} \cdot I \right) \tilde{X} \right\| - \frac{\rho^k + \rho^{-k}}{2} \cdot \|\tilde{X}\| \\
&\geq \frac{1}{3} k \cdot e^{-\frac{1}{4}\Lambda q} - C_1 \\
&\geq 3,
\end{aligned}$$

provided $e^{\frac{1}{4}\Lambda q} > 3(C_1 + 3)$.

By (5.72), we have then

$$\|\tilde{A}(kq) \cdot \tilde{X}\| \geq \|\tilde{A}^k(q) \cdot \tilde{X}\| - \left\| \left(\tilde{A}(kq) - \tilde{A}^k(q) \right) \cdot \tilde{X} \right\| \geq 2.$$

By (5.12) and (5.15), for $e^{\frac{1}{4}\Lambda q} \leq k \leq K$, we have

$$(A.16) \quad \|A(kq)X\| = |r^{-1}(kq, 0)| \cdot \|T_{kq} \tilde{A}(kq) T_0^{-1} X\| = |r^{-1}(kq, 0)| \cdot \|\tilde{A}(kq) \tilde{X}\| \geq 1$$

Therefore,

$$\|u^\varphi\|_\ell^2 \geq \frac{1}{2} \sum_{n=1}^{[\ell]-1} \|A_n \cdot X\|^2 \geq \frac{1}{2} \sum_{e^{\frac{1}{4}\Lambda q} \leq k \leq K} \|A(kq) \cdot X\|^2 \geq \frac{1}{2} (K - e^{\frac{1}{4}\Lambda q}) > e^{\frac{1}{2}\Lambda q}$$

Case II: Assume φ satisfies

$$(A.17) \quad \left\| \left(\tilde{A}(q) - \frac{\text{Tr } \tilde{A}(q)}{2} I \right) \cdot \tilde{X} \right\| < e^{-\frac{1}{4}\Lambda q},$$

By (5.67), for any $1 \leq k \leq e^{\frac{1}{5}\Lambda q} < N_q$ we get

$$\begin{aligned} \|\tilde{A}^k(q) \cdot \tilde{X}\| &\geq \frac{\rho^k + \rho^{-k}}{2} \cdot \|\tilde{X}\| - \frac{\rho^k - \rho^{-k}}{\rho - \rho^{-1}} \cdot \left\| \left(\tilde{A}(q) - \frac{\text{Tr } \tilde{A}(q)}{2} I \right) \tilde{X} \right\| \\ &\geq \frac{1}{2} - C_1 k \cdot e^{-\frac{1}{4}\Lambda q} \\ &\geq \frac{1}{3}, \end{aligned}$$

provided $e^{\frac{1}{20}\Lambda q} > 6C_1$.

By (5.72), we have

$$\|\tilde{A}(kq) \cdot \tilde{X}\| \geq \|\tilde{A}^k(q) \cdot \tilde{X}\| - \|(\tilde{A}(kq) - \tilde{A}^k(q)) \cdot \tilde{X}\| \geq \frac{1}{4}.$$

By (5.12) and (5.15), for any $1 \leq k \leq e^{\frac{1}{5}\Lambda q} < N_q$, we have

$$(A.18) \quad \|A(kq)X\| = |r^{-1}(kq, 0)| \cdot \|T_{kq}\tilde{A}(kq)T_0^{-1}X\| = |r^{-1}(kq, 0)| \cdot \|\tilde{A}(kq)\tilde{X}\| \geq \frac{1}{5}$$

Therefore,

$$\|u^\varphi\|_\ell^2 \geq \frac{1}{2} \sum_{n=1}^{[\ell]-1} \|A_n \cdot X\|^2 \geq \frac{1}{2} \sum_{1 \leq k \leq e^{\frac{1}{5}\Lambda q}} \|A(kq) \cdot X\|^2 \geq \frac{1}{50} e^{\frac{1}{5}\Lambda q} \geq e^{\frac{1}{6}\Lambda q}.$$

□

A.3. The refined estimate on the preimage of $\mathcal{P}_n(\mathbb{R})$. Let $\mathcal{P}_n(\mathbb{R})$ denote the polynomials over $\mathbb{R}$ of exact degree n . Let the class $\mathcal{P}_{n;n}(\mathbb{R})$ be elements in $\mathcal{P}_n(\mathbb{R})$ with n distinct real zeros. The following proposition was proved in Theorem 6.1,[29]:

Proposition A.1. *Let $p \in \mathcal{P}_{n;n}(\mathbb{R})$ with $y_1 < \dots < y_{n-1}$ the local extrema of p . Let*

$$(A.19) \quad \zeta(p) := \min_{1 \leq j \leq n-1} |p(y_j)|$$

and $0 \leq a < b$. Then,

$$(A.20) \quad |p^{-1}(a, b)| \leq 2 \text{diam}(z(p-a)) \max \left\{ \frac{b-a}{\zeta(p)+a}, \left(\frac{b-a}{\zeta(p)+a} \right)^{\frac{1}{2}} \right\}$$

where $z(p)$ is the zero set of p and $|\cdot|$ denotes the Lebesgue measure.

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REFERENCES

1. A. Avila and S. Jitomirskaya, *The ten martini problem*, Annals of Mathematics 170, 303-342 (2009).
2. A. Avila and S. Jitomirskaya, *Hölder continuity of absolutely continuous spectral measures for one-frequency Schrödinger operators*. Commun. Math. Phys. 301, 563-581 (2011).
3. A. Avila, S. Jitomirskaya and C. Marx, *Spectral theory of Extended Harpers Model and a question by Erds and Szekeres*, Inventiones mathematicae 210.1, 283-339 (2017).
4. A. Avila, J. You and Q. Zhou, *Sharp phase transitions for the almost Mathieu operator*. Duke Math. J, 166.14, 2697-2718 (2017).

5. J. Barbaroux, F. Germinet and S. Tcheremchantsev, *dimensions and the phenomenon of intermittency in quantum dynamics*. Duke Math. J. 110, 161-193 (2001).
6. S. Becker, R. Han and S. Jitomirskaya, Cantor spectrum of graphene in magnetic fields, preprint arXiv:1803.00988.
7. R. Carmona and J. Lacroix, *Spectral theory of random Schrödinger operators*. Springer Science & Business Media, (2012).
8. D. Damanik, Lyapunov exponents and spectral analysis of ergodic Schrödinger operators: a survey of Kotani theory and its applications. *Proc. Sympos. Pure Math.* 76, Part 2, Providence, RI: Amer. Math. Soc., 2007, pp. 539-563
9. D. Damanik, *Schrödinger operators with dynamically defined potentials*. Ergodic Theory and Dynamical Systems 37(6), 1681-1764 (2017).
10. D. Damanik, A. Grodetski and W. Yessen. *The Fibonacci Hamiltonian*. Inventiones mathematicae 206.3 629-692 (2016).
11. D. Damanik, R. Killip and D. Lenz, *Uniform spectral properties of one-dimensional quasicrystals. iii. α -continuity*. Commun. Math. Phys. 212, 191-204 (2000).
12. D. Damanik and S. Tcheremchantsev, *Upper bounds in quantum dynamics*. J. Amer. Math. Soc. 20, 799-827 (2007).
13. R. del Rio, S. Jitomirskaya, Y. Last and B. Simon, *Operators with singular continuous spectrum, IV. Hausdorff dimensions, rank one perturbations, and localization*. J. Anal. Math. 69, 153-200 (1996).
14. K. Falconer, *Techniques in Fractal Geometry*, John Wiley & Sons, Ltd., Chichester, (1997).
15. A. Furman, *On the multiplicative ergodic theorem for uniquely ergodic systems* Annales de l'Institut Henri Poincaré (B) Probability and Statistics. No longer published by Elsevier, 33(6): 797-815 (1997).
16. D.J. Gilbert, *On subordinacy and analysis of the spectrum of Schrödinger operators with two singular endpoints*. Proc. Roy. Soc. Edinburgh A 112, 213-229 (1989).
17. F. Germinet, A. Kiselev and S. Tcheremchantsev, *Transfer matrices and transport for 1D Schrödinger operators with singular spectrum*. Ann. Inst. Fourier 54, 787-830 (2004)
18. D.J. Gilbert and D. Pearson, *On subordinacy and analysis of the spectrum of one-dimensional Schrödinger operators*. J. Math. Anal. Appl. 128, 30-56 (1987).
19. I. Guarneri and H. Schulz-Baldes, *Lower bounds on wave packet propagation by packing dimensions of spectral measures*. Math. Phys. Electron. J. 5(1), 16 (1999).
20. R. Han. *Absence of point spectrum for the self-dual extended Harpers model*. Int. Math. Res. Not. rrw279 (2017).
21. R. Han. *Dry ten martini problem for the non-self-dual extended Harpers model*. Transactions of the American Mathematical Society, 370.1, 197-217 (2018).
22. R. Han and S. Jitomirskaya, *Full measure reducibility and localization for quasi-periodic Jacobi operators: a topological criterion*. Advances in Mathematics 319, 224-250 (2017).
23. S. Jitomirskaya, D.A. Koslover and M.S. Schulteis, *Localization for a Family of One-dimensional Quasi-periodic Operators of Magnetic Origin*, Ann. Henri Poincaré 6, 103-124 (2005).
24. S. Jitomirskaya and Y. Last, *Power-law subordinacy and singular spectra. I. Half-line operators*. Acta Math. 183, 171-189 (1999).
25. S. Jitomirskaya and Y. Last. *Power-law subordinacy and singular spectra. II. Line operators*. Commun. Math. Phys. 211, 643-658 (2000).
26. S. Jitomirskaya and W. Liu, *Universal hierarchical structure of quasiperiodic eigenfunctions*. Annals of Math, to appear.
27. S. Jitomirskaya and W. Liu, *Arithmetic Spectral Transitions for the Maryland Model*. Comm. Pure Appl. Math., 70: 1025-1051 (2017).
28. S. Jitomirskaya and C. A. Marx, *Analytic quasi-periodic cocycles with singularities and the Lyapunov Exponent of Extended Harper's Model*, Commun. Math. Phys. 316, 237-267 (2012).
29. S. Jitomirskaya and C.A. Marx, *Analytic quasi-periodic Schrödinger operators and rational frequency approximations*, Geom. Funct. Anal. 22, 1407-1443 (2012).
30. S. Jitomirskaya and R. Mavi, *Dynamical bounds for quasiperiodic Schrödinger operators with rough potentials*. Int. Math. Res. Not. 1, 96-120 (2017).
31. S. Jitomirskaya and F. Yang, *Singular continuous spectrum for singular potentials*. Communications in Mathematical Physics 351.3 1127-1135 (2017).
32. S. Jitomirskaya and S. Zhang, *Quantitative continuity of singular continuous spectral measures and arithmetic criteria for quasiperiodic Schrödinger operators* arXiv:1510.07086 (2015).
33. S. Kotani, *Lyapunov indices determine absolutely continuous spectra of stationary random one-dimensional Schrödinger operators*, Stochastic Analysis (Katata/Kyoto, 1982), 225-247, North-Holland Math. Library 32, North-Holland, Amsterdam, 1984

- 34. R. Killip, A. Kiselev and Y. Last, *Dynamical upper bounds on wavepacket spreading*. Amer. J. Math. 125, 1165-1198 (2003).
- 35. Y. Last, *Quantum dynamics and decompositions of singular continuous spectra*. J. Funct. Anal. 142, 406-445 (1996).
- 36. Y. Last and B. Simon, *Eigenfunctions, transfer matrices, and absolutely continuous spectrum of one-dimensional Schrödinger operators*. Inventiones mathematicae 135.2, 329-367 (1999).
- 37. C. A. Marx, *Dominated splittings and the spectrum of quasi-periodic Jacobi operators*. Nonlinearity 27 (2014) 3059-3072
- 38. B. Simon, *Operators with singular continuous spectrum: I. General operators*. Ann. of Math. 141, 131-145 (1995)
- 39. B. Simon, *Equilibrium measures and capacities in spectral theory*. Inverse Problems and Imaging 1, 376-382 (2007).
- 40. D. J. Thouless, *Bandwidth for a quasiperiodic tight binding model*. Phys. Rev. B 28, 4272-4276 (1983).
- 41. F. Yang, *Spectral transition line for the extended Harper's model in the positive Lyapunov exponent regime*. J. Funct. Anal., to appear.

Rui Han,
School of Math, Institute for Advanced Study.
E-mail address: rhan@ias.edu

Fan Yang,
School of Math, Institute for Advanced Study.
E-mail address: yangf@ias.edu

Shiwen Zhang,
Dept. of Math., Michigan State University.
E-mail address: zhangshiwen@math.msu.edu