A HIGHER DIMENSIONAL BOURGAIN-DYATLOV FRACTAL UNCERTAINTY PRINCIPLE

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Abstract. We establish a version of the fractal uncertainty principle, obtained by Bourgain and Dyatlov in 2016, in higher dimensions. The Fourier support is limited to sets $Y \subset \mathbb{R}^d$ which can be covered by finitely many products of δ-regular sets in one dimension, but relative to arbitrary axes. Our results remain true if Y is distorted by diffeomorphisms. Our method combines the original approach by Bourgain and Dyatlov, in the more quantitative 2017 rendition by Jin and Zhang, with Cartan set techniques.

1. Introduction

Bourgain-Dyatlov [BouDya] proved the following result.

Theorem 1.1. *Let $X, Y \subset \mathbb{R}$ and $N \geqslant 1$ be such that (i) $X \subset [-1, 1]$ is δ-regular with constant C_R on scales N^{-1} to 1, (ii) $Y \subset [-N, N]$, is δ-regular with constant C_R on scales 1 to N. Then there exist constants $\beta > 0$, and C depending on δ, C_R so that*

$$\|f\|_{L^2(X)} \leqslant CN^{-\beta}\|f\|_{L^2(\mathbb{R})}$$

for all $f \in L^2(\mathbb{R})$ with $\mathrm{supp}(\hat{f}) \subset Y$.

The δ-regularity condition is akin to asking for a Frostman measure at dimension δ, see Definition 6.1 below for the precise statement. Theorem 1.1 is most interesting for δ close to 1. For $\delta < \frac{1}{2}$, Cauchy-Schwarz and measure estimates in phase space suffice. The β was made effective later by Jin and Zhang [JinZha]. Combining this fractal uncertainty principle with earlier results by Dyatlov and Zahl [DyaZah] led to a breakthrough on the existence for an essential spectral gap for convex co-compact hyperbolic surfaces. This refers to a strip to the left of the 1/2 line in the complex plane in which the Selberg zeta function has only finitely many zeros. This result can be reformulated in terms of strips below the real axis in which the meromorphic continuation of the resolvent of the Laplacian of the hyperbolic surface exhibits only finitely many resonances. This in turn can be rephrased as a decay rate of the resolvent for large energies within such a strip.

For other applications see [BouDya2, DyaJin, DyaJin2], and for a survey [Dya].

It remained an open problem to establish an analogue of Theorem 1.1 in higher dimensions. This is the main goal of this paper.

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We now present our main results. Let $X \subset [-1,1]^d$ be a δ-regular set in the sense of Bourgain-Dyatlov with $\delta \in (0,d)$ and constant C_R, on scales N^{-1} to 1. In [BouDya] this concept is defined only on the line, but the definition, together with its main properties, carries over to higher dimensions. Strictly speaking, we do not need the regularity condition per se, but rather the porosity property of such sets as stated precisely in Definition 5.1 below. Second, let $Y \subset [-N,N]^d$ be of the form

$$Y = \Big\{ \sum_{i=1}^d \xi_i \vec{e}_i \mid \xi_i \in Y_i \Big\}, \tag{1.1}$$

where $\vec{e}_i$ are unit vectors with $|\det(\vec{e}_1, \ldots, \vec{e}_d)| \geqslant \varepsilon_0$, a positive constant (possibly small), and $Y_i \subset [-2N, 2N]$ is a δ_1-regular set with $\delta_1 \in (0,1)$ and constant C_R, on scales 1 to N.

Theorem 1.2. *Let X, Y be as in the previous paragraph in dimension $d \geqslant 2$. Then there exists constant $C = C(d, \varepsilon_0, \delta, \delta_1, C_R) > 0$ such that for*

$$\beta = \exp\Big\{ -\exp\Big[\Big(\frac{(C_R^2/\iota)^{\frac{2d-2\delta+2}{d-\delta}}}{\delta_1(1-\delta_1)}\Big)^{\frac{2}{1-\delta_1}}\Big]\Big\},$$

where $\iota > 0$ is a small constant depending on d and ε_0, and for any $f \in L^2(\mathbb{R}^d)$ with $\mathrm{supp}(\widehat{f}) \subset Y$ one has

$$\|f\|_{L^2(X)} \leqslant C N^{-\beta} \|f\|_{L^2(\mathbb{R}^d)} \tag{1.2}$$

for sufficiently large $N \geqslant N_0(d, \varepsilon_0, \delta, \delta_1, C_R)$.

As a corollary of our main theorem, we allow Y to be covered by the union of a finite number of Y_j's, each satisfying (1.1) but with a uniform ε_0.

$$\begin{aligned} Y &\subset \bigcup_{j=1}^m Y_j \\ Y_j &= \Big\{ \sum_{i=1}^d \xi_{j,i} \vec{e}_{j,i} \mid \xi_{j,i} \in Y_{j,i} \Big\}. \end{aligned} \tag{1.3}$$

Furthermore, the number m of covers can grow in N. To be specific, we prove

Corollary 1.3. *Let X be as above and Y be as in (1.3). Suppose m grows with N as follows*

$$m = \lfloor N^\gamma \rfloor,$$

in which $0 \leqslant \gamma < \beta$. Then for any $f \in L^2(\mathbb{R}^d)$ with $\mathrm{supp}(\widehat{f}) \subset Y$, and constants C, β in Theorem 1.2, one has

$$\|f\|_{L^2(X)} \leqslant C N^{\gamma-\beta} \|f\|_2, \tag{1.4}$$

for sufficiently large $N \geqslant N_0(d, \varepsilon_0, \delta, \delta_1, C_R)$.

Theorem 1.2 and Corollary 1.3 require that the Fourier support Y may be covered by products of regular sets in one dimension *along lines*, cf. (1.3). Our third result asserts that one may distort these lines by means of diffeomorphisms which are obtained as follows. Let $\Psi_N : [-N,N]^d \to [-N,N]^d$ be a diffeomorphism such that

$$\|D\Phi_N\|_\infty + \|D\Phi_N^{-1}\|_\infty + N\|D^2\Phi_N\|_\infty \leqslant C(d, D_0), \tag{1.5}$$

where the supremum norm is taken over the cube $[-N,N]^d$. One example of a diffeomorphism satisfying (1.5) is $\Psi_N(x) = N\Psi_0(x/N)$, where Ψ_0 is a diffeomorphism from $[-1,1]^d$ to $[-1,1]^d$ such that

$$\|D\Psi_0\|_\infty + \|D\Psi_0^{-1}\|_\infty + \|D^2\Psi_0\|_\infty \leqslant D_0. \tag{1.6}$$

where the supremum norm is taken over the cube.

Theorem 1.4. *Theorem 1.2 remains correct with $\Phi_N(Y)$ in place of Y. Constants depend on D_0, but not on Ψ_0.*

In the following section we demonstrate the Cartan techniques by reproving a certain step in [BouDya] which was proved there by means of harmonic measure of the strip with a real line-segment removed. In Section 3 we go beyond the one-dimensional setting via these Cartan methods. The subsequent sections implement the argument in analogy with [BouDya] albeit in dimensions two and higher. We haven striven to present the argument in a modular fashion. In particular, the delicate Beurling-Malliavin step appears only in Section 6 in order to prove the existence of *damping functions*. We do not use a higher-dimensional version of the Beurling-Malliavin theorem which appears to be unknown. Rather, we reduce ourselves in that step to the aforementioned product structure of Y (or covers of finitely many of such products) precisely so as to be able to still use the one-dimensional construction of such damping functions. Moreover, as in [JinZha] it is important for us to use the weaker form of the Beurling-Malliavin theorem obtained via outer functions, see [KhaMasNaz]. Any other construction of damping functions in Section 6 would lead to different formulations of our main theorems in terms of the conditions on Y without needing to change anything in the other sections. Theorem 1.4 is proved in Section 6.4. A FUP for Fourier integral operators is presented in Section 6.5.

2. L^2 localization in one dimension

Let us first introduce notations. For $\xi = (\xi_1, \xi_2, ..., \xi_d) \in \mathbb{R}^d$, let $|\xi|_1 := \sum_{j=1}^d |\xi_k|$, $|\xi|_2 := \sum_{j=1}^d |\xi_k|^2$ and $\langle\xi\rangle := (1+|\xi|_2^2)^{\frac{1}{2}}$. Let $e(\theta) := e^{2\pi i\theta}$. For $x \in \mathbb{R}$, let $\lceil x\rceil := \min\{n \in \mathbb{N} : n \geqslant x\}$, and $\lfloor x\rfloor := \max\{n \in \mathbb{N} : n \leqslant x\}$.

Throughout, we let $\mathcal{R}(q)$ be the rectangle with vertices $\pm iq$, $1\pm iq$. We begin with quantitative bounds on the Schwarz-Christoffel map from the disk onto a rectangle. The goal is to control this conformal mapping as the eccentricity of $\mathcal{R}(q)$ tends to 0.

Lemma 2.1. *Let $0 < q \leqslant 1$ and define Φ_q to be the unique conformal map, continuous up to the boundary, which takes the unit disk $\mathbb{D}$ onto the rectangle $\mathcal{R}(q)$ and so that $\Phi_q(-1) = 0$, $\Phi_q(\pm i) = \pm iq$. Then $\Phi_q(1) = 1$, $\Phi_q(e^{\pm i\theta(q)}) = 1 \pm iq$ where*

$$\theta(q) = 8\exp\big(-\frac{\pi}{2q}\big)(1+O(q)), \quad q \to 0$$

Moreover,

$$\Phi_q([a_1(q), a_2(q)]) = \big[\frac{1}{4}, \frac{3}{4}\big], \quad a_j(q) = 1 - \delta_j(q)$$

with

$$\delta_1(q) = 4\exp\big(-\frac{\pi}{8q}\big)(1+O(q)), \quad \delta_2(q) = 4\exp\big(-\frac{3\pi}{8q}\big)(1+O(q))$$

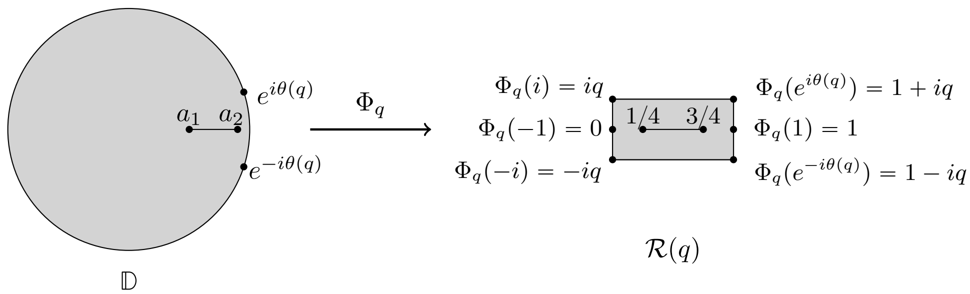

FIGURE 1. Conformal map Φ_q

as $q \to 0$. Let $E \subset [a_1(q), a_2(q)]$ be a measurable set. Then for sufficiently small q one has $|\Phi_q(E)| \leqslant 2\delta_2(q)^{-2}|E|$, where $|\cdot|$ denotes Lebesgue measure.

Proof. Let $0 < k < 1$ and consider the elliptic integral of the first kind

$$\operatorname{arcsn}(z, k) = \int_0^z \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \qquad \operatorname{Im} z > 0$$

which maps the upper half-plane onto the rectangle with vertices $\pm L(k), \pm L(k) + iH(k)$. Here $2L(k)$ and $iH(k)$ are the periods of the elliptic function $\operatorname{sn}(z, k)$ and satisfy, as $k \to 0$,

$$\begin{aligned}
L(k) &= \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \frac{\pi}{2} + O(k^2), \\
H(k) &= \int_1^{k^{-1}} \frac{dt}{\sqrt{(t^2-1)(1-k^2t^2)}} = \int_0^\infty \frac{ds}{\sqrt{(1+s^2)(1+k^2s^2)}} \\
&= \log 4 - \log k + O(k)
\end{aligned}$$

The latter expansion is a standard fact, see for example [AbrSte, Section 17.3.26]. Let $q := \frac{L(k)}{H(k)}$ and set

$$F_q(z) = -\frac{i}{H(k)} \operatorname{arcsn}(z, k) \tag{2.1}$$

which maps the upper half-plane onto the rectangle with vertices $\pm iq, 1 \pm iq$. With $k = e^{-\frac{\pi}{2}\ell}$,

$$\begin{aligned}
q &= \frac{\frac{\pi}{2} + O(k^2)}{\log 4 + \frac{\pi}{2}\ell + O(k)} \\
&= \ell^{-1}\big(1 - \frac{\log 16}{\pi \ell} + O(k)\big)
\end{aligned}$$

and thus

$$\begin{aligned}
\ell &= q^{-1}\big(1 - \frac{2\log 4}{\pi} q + O(q^2)\big) \\
k &= 4\exp\big(-\frac{\pi}{2q}\big)(1 + O(q))
\end{aligned}$$

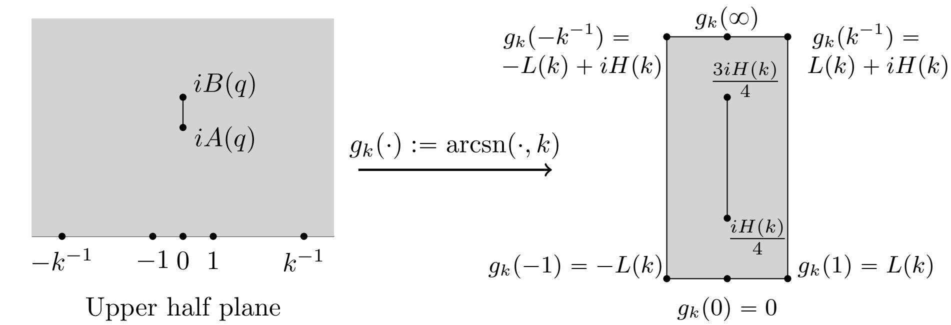

FIGURE 2. Elliptic integral $\mathrm{arcsn}(z, k)$

Define $A(q), B(q)$ by $F_q(iA(q)) = \frac{1}{4}$, $F_q(iB(q)) = \frac{3}{4}$. Thus,

$$\int_0^{A(q)} \frac{ds}{\sqrt{(1+s^2)(1+k^2s^2)}} = \frac{1}{4}H(k)$$
$$\int_0^{B(q)} \frac{ds}{\sqrt{(1+s^2)(1+k^2s^2)}} = \frac{3}{4}H(k)$$

We make the ansatz $A(q) = ck^{-\frac{1}{4}}(1+\varepsilon(q))$. Then

$$\begin{aligned}\int_0^{A(q)} \frac{ds}{\sqrt{(1+s^2)(1+k^2s^2)}} &= (1+O(k^{\frac{3}{2}}))\int_0^{A(q)} \frac{ds}{\sqrt{1+s^2}} \\ &= \mathrm{arcsinh}(ck^{-\frac{1}{4}}(1+\varepsilon(q))(1+O(k^{\frac{3}{2}})) \\ &= \log(2ck^{-\frac{1}{4}}(1+\varepsilon(q))(1+O(k^{\frac{3}{2}})) \\ &= \frac{1}{4}(\log 4 - \log k + O(k))\end{aligned}$$

Hence,

$$\begin{aligned}\log(2c) - \frac{1}{4}\log k + \log(1+\varepsilon(q)) &= \frac{1}{4}(\log 4 - \log k + O(k)) \\ c &= \frac{1}{2}\sqrt{2}, \qquad \varepsilon(q) = O(k) \\ A(q) &= \frac{1}{2}\sqrt{2}\,k^{-\frac{1}{4}}(1+O(k))\end{aligned}$$

Similarly, with $B(q) = \tilde{c}k^{-\frac{3}{4}}(1+\tilde{\varepsilon}(q))$

$$\log(2\tilde{c}) - \frac{3}{4}\log k + \log(1+\tilde{\varepsilon}(q)) = \frac{3}{4}(\log 4 - \log k + O(k))(1+O(k^{\frac{1}{2}}))$$
$$\tilde{c} = \sqrt{2}, \quad \tilde{\varepsilon}(q) = O(k^{\frac{1}{2}}\log k)$$

and so

$$B(q) = \sqrt{2}\,k^{-\frac{3}{4}}(1+O(k^{\frac{1}{2}}\log k))$$

Expressing k in terms of q we obtain

$$A(q) = \frac{1}{2}\exp\big(\frac{\pi}{8q}\big)(1+O(q)), \quad B(q) = \frac{1}{2}\exp\big(\frac{3\pi}{8q}\big)(1+O(q))$$

Next, we conformally map the upper half plane $\operatorname{Im} z > 0$ onto the unit disk $|w| < 1$ via $z = \varphi(w) = i\frac{w+1}{1-w}$, $w = \frac{z-i}{z+i}$. One has $\varphi(-1) = 0$, $\varphi(\pm i) = \mp 1$, $\varphi(e^{i\theta}) = -k^{-1}$ with $\theta = 2k + O(k^3)$. Furthermore, $\varphi([a_1(q), a_2(q)]) = i[A(q), B(q)]$ where

$$\begin{aligned} a_1(q) &= \frac{A(q)-1}{A(q)+1} = 1 - 2A(q)^{-1} + O(A(q)^{-2}), \\ a_2(q) &= \frac{B(q)-1}{B(q)+1} = 1 - 2B(q)^{-1} + O(B(q)^{-2}) \end{aligned}$$

Setting $a_j(q) = 1 - \delta_j(q)$ we have

$$\delta_1(q) = 4\exp\big(-\frac{\pi}{8q}\big)(1 + O(q)), \quad \delta_2(q) = 4\exp\big(-\frac{3\pi}{8q}\big)(1 + O(q))$$

as claimed. The final claim of the lemma follows from

$$|(F_q \circ \varphi)'(w)| \leqslant |F_q'(z)||\varphi'(w)| \leqslant 2(1 - |w|)^{-2}$$

where $\varphi(w) = z$, $w \in (0, 1)$. We used here that for $z = is$, $s > 0$,

$$\begin{aligned} |F_q'(z)| &= H(k)^{-1}(1 + |z|^2)^{-\frac{1}{2}}(1 + k^2|z|^2)^{-\frac{1}{2}} \\ &\leqslant H(k)^{-1}(1 + |z|^2)^{-\frac{1}{2}} \leqslant 1 \end{aligned}$$

for small q. $\square$

By a subharmonic function v on a domain $\Omega \subset \mathbb{C}$ we mean a function $v : \Omega \to [-\infty, \infty)$, which is upper semi-continuous and satisfies the sub mean-value property. We recall the basic Riesz representation of subharmonic function on the disk, albeit with precise quantitative control on the Riesz mass and the harmonic part. In view of Lemma 2.1 we need to consider the case where the lower bound on the subharmonic function is attained arbitrarily close to the boundary of the unit disk.

Lemma 2.2. *Let v be subharmonic on a neighborhood of $\mathbb{D}$, with $v \leqslant M$ on $\mathbb{D}$, and assume $\sup_{\rho\mathbb{D}} v \geqslant m$ for some $0 < \rho < 1$. Let $\rho < r_1 < r < 1$. Then there exist a nonnegative measure μ on $\mathbb{D}$, called the Riesz measure, with the property that for all $w \in r\mathbb{D}$*

$$v(w) = \int_{r\mathbb{D}} \log|z - w|\, \mu(dz) + h(w) \tag{2.2}$$

with h harmonic on $r\mathbb{D}$. We have the quantitative bounds on the Riesz mass

$$\mu(r\mathbb{D}) \leqslant \frac{M - m}{\log\left(\frac{1+\rho r}{\rho + r}\right)} \tag{2.3}$$

and on the deviations of the harmonic function

$$\min_{c\in\mathbb{R}} \max_{|w|\leqslant r_1} |h(w) - c| \leqslant \frac{1}{2}(M - m)\frac{r + r_1}{r - r_1}\frac{\log\left(\frac{1+\rho r}{1-r^2}\right)}{\log\left(\frac{1+\rho r}{\rho+r}\right)} =: \varepsilon \tag{2.4}$$

The constant c which minimizes the left-hand side satisfies

$$c \geqslant m - \varepsilon - \log(r + \rho)\mu(r\mathbb{D}) \tag{2.5}$$

Proof. We will assume that v is smooth, the general case following by approximation. The Green function $G : \mathbb{D} \times \mathbb{D} \to \mathbb{R}$ given by

$$G(z,w) := \frac{1}{2\pi} \log \left| \frac{z-w}{1-z\overline{w}} \right|$$

satisfies $\Delta_z G(z,w) = \delta_w$ and $G(z,w) = 0$ when $|z| = 1$.

Let $w \in \mathbb{D}$. By Green's second identity for the domain $\mathbb{D}$, we have

$$v(w) - \int_{\mathbb{D}} G(z,w) \Delta v(z) \operatorname{Vol}(dz) = \int_{\partial\mathbb{D}} v(z) \frac{\partial G}{\partial n_z}(z,w)\, \sigma(dz),$$

where Vol is the standard volume measure and σ is the (unnormalized) arc length measure on the circle $\partial\mathbb{D}$. Since v is smooth and subharmonic, Δv is a non-negative, continuous function, call it $2\pi\mu$. Therefore

$$v(w) = \int_{\mathbb{D}} 2\pi G(z,w)\, \mu(dz) + h_0(w), \tag{2.6}$$

where

$$h_0(w) := \int_{\partial\mathbb{D}} v(z) \frac{\partial G}{\partial n_z}(z,w)\, \sigma(dz). \tag{2.7}$$

Let $0 < r < 1$. On the disk $r\mathbb{D}$ we have the Riesz representation

$$v(w) = \int_{r\mathbb{D}} \log|z-w|\, \mu(dz) + h(w), \tag{2.8}$$

where

$$h(w) := \int_{\mathbb{D}\setminus r\mathbb{D}} \log \left| \frac{z-w}{1-z\overline{w}} \right| \mu(dz) - \int_{r\mathbb{D}} \log|1-z\overline{w}|\, \mu(dz) + h_0(w) \tag{2.9}$$

is harmonic in $r\mathbb{D}$. Note that $\frac{\partial G}{\partial n_z}(z,w)$ is the Poisson kernel whence

$$h_0(w) = \int_0^1 v(e(\theta)) P_{|w|}(\varphi - \theta)\, d\theta, \qquad w = |w| e(\varphi). \tag{2.10}$$

We now set out to bound the Riesz measure μ. Without loss of generality, assume $m = v(\rho)$. Then setting $w = \rho$ in (2.6) yields

$$\int_{\mathbb{D}} \log \frac{|1-\rho z|}{|z-\rho|}\, \mu(dz) = h_0(\rho) - v(\rho) \leqslant M - m, \tag{2.11}$$

in which we used

$$h_0(\rho) \leqslant M. \tag{2.12}$$

This follows from the maximum principle and the fact that h_0 is the harmonic function on $\mathbb{D}$ with boundary values v by (2.10). By an elementary calculation,

$$\min_{|z| \leqslant r} \frac{|1-\rho z|}{|z-\rho|} = \frac{1+\rho r}{\rho + r} > 1$$

for all $0 < \rho, r < 1$. Inserting this bound into (2.11) implies that

$$\mu(r\,\mathbb{D}) \leqslant \frac{M-m}{\log\left(\frac{1+\rho r}{\rho + r}\right)}. \tag{2.13}$$

Let $\rho < r_1 < r < 1$. For all $w \in r\mathbb{D}$ we have

$$\begin{aligned} h(w) &= \int_{\mathbb{D}\setminus r\mathbb{D}} 2\pi G(w,z)\,\mu(dz) - \int_{r\mathbb{D}} \log|1 - z\bar{w}|\,\mu(dz) + h_0(w) \\ &\leqslant -\log(1-r^2)\mu(r\mathbb{D}) + M =: h^* \end{aligned} \tag{2.14}$$

By Harnack's inequality on $r_1\mathbb{D}$ we conclude from this that for any $w \in r_1\mathbb{D}$,

$$(h^* - h(w)) \leqslant \frac{r + r_1}{r - r_1}(h^* - h(\rho))$$

whence

$$h(w) \geqslant \frac{r + r_1}{r - r_1} h(\rho) - \frac{2r_1}{r - r_1} h^*$$

By (2.8),

$$h(\rho) = v(\rho) - \int_{r\mathbb{D}} \log|z - \rho|\,\mu(dz) \geqslant m - \log(r + \rho)\mu(r\mathbb{D}) \tag{2.15}$$

and thus

$$h(w) \geqslant \frac{r + r_1}{r - r_1}\big(m - \log(r+\rho)\mu(r\mathbb{D})\big) - \frac{2r_1}{r - r_1} h^* =: h_*$$

In summary,

$$\begin{aligned} \min_{c\in\mathbb{R}} \max_{|w|\leqslant r_1} |h(w) - c| &\leqslant \frac{1}{2}(h^* - h_*) \\ &= \frac{1}{2}\frac{r + r_1}{r - r_1}\Big(h^* - m + \log(r+\rho)\mu(r\mathbb{D})\Big) \\ &= \frac{1}{2}\frac{r + r_1}{r - r_1}\Big(M - m + \log\big(\frac{r+\rho}{1-r^2}\big)\mu(r\mathbb{D})\Big) \end{aligned} \tag{2.16}$$

Finally, bounded the μ-mass by (2.13) finally implies that

$$\min_{c\in\mathbb{R}} \max_{|w|\leqslant r_1} |h(w) - c| \leqslant \frac{1}{2}(M - m)\frac{r + r_1}{r - r_1}\frac{\log\left(\frac{1+\rho r}{1-r^2}\right)}{\log\left(\frac{1+\rho r}{\rho + r}\right)} =: \varepsilon$$

as claimed. Finally, to establish (2.5), we return to (2.15) and note that the left-hand side at most $c + \varepsilon$ for c the minimizer in the previous line. Then

$$c \geqslant m - \log(r + \rho)\mu(r\mathbb{D}) - \varepsilon$$

Note that one may insert (2.13) on the right-hand side to control the mass. $\square$

We now apply the Cartan estimate for logarithmic potentials to the Riesz representation (2.2) in order to derive lower bounds on v up to a small measure of exceptions.

Corollary 2.3. *Let v be as in Lemma 2.2 with $\rho = 1 - 3\delta$, $0 < \delta < \frac{1}{3}$. Then for all $0 < H \leqslant 1$ there exist disks $D(z_j, s_j)$ so that*

$$v(z) \geqslant m - (M - m)\big[2\delta^{-3}\log(2/\delta) + \delta^{-2}\log(2e/H)\big]$$

for all $z \in r_1\mathbb{D}\setminus\bigcup_j D(z_j, s_j)$ with $\sum_j s_j \leqslant 5H$ and $r_1 = 1 - 2\delta$.

Proof. By Cartan's estimate for any $H > 0$ there exist disks $D(z_j, s_j)$ such that $\sum_j s_j \leqslant 5H$ and

$$\int_{r\mathbb{D}} \log|w-z|\,\mu(dw) \geqslant \mu(r\mathbb{D})\log(H/e), \qquad \forall\, z \in r_1\mathbb{D}\backslash\bigcup_j D(z_j, s_j) \tag{2.17}$$

See [Lev], Theorem 3, Section 11.2. To invoke the measure bound (2.3) we estimate

$$\begin{aligned}\log\Big(\frac{1+\rho r}{\rho + r}\Big) &= \log\Big(\frac{2-4\delta+3\delta^2}{2-4\delta}\Big)\\ &= \log\Big(1+\frac{3\delta^2}{2-4\delta}\Big) \geqslant \log(1+\frac{3}{2}\delta^2) \geqslant \delta^2\end{aligned}$$

since $\delta^2 \leqslant \frac{1}{2}$ and $\log(1+\frac{3}{2}x) \geqslant x$ for $0 \leqslant x \leqslant \frac{1}{2}$. Consequently,

$$\mu(r\mathbb{D}) \leqslant \delta^{-2}(M-m)$$

Next,

$$\frac{1+\rho r}{1-r^2} \leqslant \frac{2}{2\delta-\delta^2} \leqslant \delta^{-1}(1+\delta)$$

as well as

$$\frac{r+r_1}{r-r_1} = \frac{2-3\delta}{\delta} \leqslant 2\delta^{-1}$$

whence (2.4) implies

$$\min_{c\in\mathbb{R}}\max_{|w|\leqslant r_1} |h(w)-c| \leqslant \varepsilon \leqslant (M-m)\delta^{-3}\log(2/\delta) =: \tilde{\varepsilon}.$$

Finally, by (2.5), one has

$$c \geqslant m - \varepsilon - \log(r+\rho)\mu(r\mathbb{D}) \geqslant m - \varepsilon - \log(2)\mu(r\mathbb{D})$$

In view of (2.2) and the preceding estimates we obtain

$$\begin{aligned} v(z) &\geqslant c + \mu(r\mathbb{D})\log(H/e) - \varepsilon \geqslant m - 2\varepsilon + \log(H/(2e))\mu(r\mathbb{D})\\ &\geqslant m - (M-m)\big[2\delta^{-3}\log(2/\delta) - \delta^{-2}\log(H/(2e))\big]\end{aligned} \tag{2.18}$$

for all z as in (2.17). □

By means of the conformal transformation Φ_q from Lemma 2.1 we can obtain a version of the Riesz representation theorem on thin rectangles $\mathcal{R}(q)$.

Corollary 2.4. *There exists $q_* \in (0,1]$ with the following property: let u be subharmonic on $\mathcal{R}(q)$ for some $0 < q \leqslant q_*$, continuous up to the boundary. Assume that $u \leqslant M$ on $\mathcal{R}(q)$ and $\max_{x\in[1/4,3/4]} u(x) \geqslant m$. Then*

$$u(x) \geqslant m - (M-m)\exp\big(\frac{9\pi}{8q}\big)\big[\log(4) + \frac{9\pi}{4q} + \exp\big(-\frac{3\pi}{8q}\big)\log(2e/H)\big] \tag{2.19}$$

for all $x \in [1/4,3/4]\backslash\bigcup_j I_j$ where $\sum_j |I_j| \leqslant 3H\exp\big(\frac{3\pi}{4q}\big)$.

Proof. Let $v = u\circ\Phi_q$, with Φ_q as in Lemma 2.1. Then v satisfies the assumptions of Corollary 2.3 with $\rho \geqslant 1-\delta_2(q)$, and

$$\begin{aligned}\delta_2(q) &= 4\exp\big(-\frac{3\pi}{8q}\big)(1+O(q)) \geqslant 3\delta\\ \delta &:= \exp\big(-\frac{3\pi}{8q}\big) < \frac{1}{3},\end{aligned} \tag{2.20}$$

provided q_* is small enough. By Corollary 2.3 we have

$$\begin{aligned} v(z) &\geqslant m-(M-m)\exp\big(\frac{9\pi}{8q}\big)\big[2\log(2/\delta)+\delta\log(2e/H)\big] \\ &= m-(M-m)\exp\big(\frac{9\pi}{8q}\big)\big[\log(4)+\frac{9\pi}{4q}+\exp\big(-\frac{3\pi}{8q}\big)\log(2e/H)\big] \end{aligned}$$

for all $z\in r_1\mathbb{D}\backslash\bigcup_j D(z_j,s_j)$, $\sum_j s_j\leqslant 5H$, where $r_1=1-2\delta$. The inverse image of $[1/4,3/4]$ under Φ_q is $[a_1(q),a_2(q)]$. Define $\tilde{I}_j:=\mathbb{R}\cap D(z_j,s_j)$, $I_j=\Phi_q(\tilde{I}_j)$, and $E:=\bigcup_j\tilde{I}_j$ so that $\sum_j|\tilde{I}_j|\leqslant 10H$. By Lemma 2.1 we have

$$|\Phi_q(E)|\leqslant 20H\delta_2(q)^{-2}< 3H\exp\big(\frac{3\pi}{4q}\big)$$

as claimed. □

Next, we apply the previous results on subharmonic functions to $\log|F|$, where F is analytic.

Corollary 2.5. *Let F be an analytic function on a neighborhood of $\mathcal{R}(q)$ with $0<q\leqslant q^*$, and F not identically equal to zero. Denote*

$$B_1:=\|F\|_{L^2([1/4,3/4])},\qquad B_2:=\|F\|_{L^2(\partial\mathcal{R}(q))}.$$

Then for some absolute constant C_0, and all $H>0$,

$$\begin{aligned} &B_1^{K+1}\leqslant e^{\frac{C_0K}{q}}\,B_2^K\,|F(x)|, \\ &\textit{holds for any}\ \ K\geqslant\exp\big(\frac{9\pi}{8q}\big)\big[\log(4)+\frac{9\pi}{4q}+\exp\big(-\frac{3\pi}{8q}\big)\log(2e/H)\big] \end{aligned}\tag{2.21}$$

for all $x\in[1/4,3/4]\backslash\bigcup_j I_j$ where $\sum_j|I_j|\leqslant 3H\exp\big(\frac{3\pi}{4q}\big)$.

Proof. We apply our previous results to $u(z):=\log|F(z)|$, which is subharmonic on a neighborhood of $\mathcal{R}(q)$. However, Corollary 2.4 does not apply directly since we do not have a point wise upper bound on u. Returning to the subharmonic function $v=u\circ\Phi_q$ on the unit disk $\mathbb{D}$, we note that the point wise upper bound M on v only entered through the estimate $h_0\leqslant M$, see (2.12), (2.14). The analytic function $\tilde{F}=F\circ\Phi_q$ satisfies $\log|\tilde{F}|=v$. Denoting by

$$P_w(d\theta)=P_{|w|}(d(\theta-\phi))=\frac{1-|w|^2}{1-2|w|\cos(2\pi(\theta-\phi))+|w|^2}$$

the Poisson kernel centered at $w=|w|e(\phi)$, we estimate h_0 from (2.11) as follows:

$$\begin{aligned} h_0(w) &= \int_0^1 v(e(\theta))\,P_w(d\theta)=\int_0^1\log|\tilde{F}(e(\theta))|\,P_w(d\theta) \\ &\leqslant \log\Big(\int_0^1|\tilde{F}(e(\theta))|\,P_w(d\theta)\Big) \\ &\leqslant \log\Big(\int_0^1|\tilde{F}(e(\theta))|\,d\theta\,\big\|\frac{P_w(d\theta)}{d\theta}\big\|_\infty\Big) \\ &\leqslant \log(B_2)+\log\big(\big\|\frac{d\theta}{d\sigma}\big\|_{L^2(\partial\mathcal{R}(q))}\big)+\log\big\|\frac{P_w(d\theta)}{d\theta}\big\|_\infty \end{aligned}\tag{2.22}$$

where $d\sigma$ denotes arc length measure on $\partial\mathcal{R}(q)$, and the correspondence between $\partial\mathbb{D}$ and $\partial\mathcal{R}(q)$ is given by $\xi \mapsto \Phi_q(e(\xi))$. On the one hand,

$$\Big\|\frac{P_w(d\theta)}{d\theta}\Big\|_\infty \leqslant 2(1-|w|)^{-1}$$

and on the other hand,

$$\Big\|\frac{d\theta}{d\sigma}\Big\|^2_{L^2(\partial\mathcal{R}(q))} = \int_{\partial\mathcal{R}(q)} \Big|\frac{d\theta}{d\sigma}\Big|^2 d\sigma = \int_0^1 \Big|\frac{d\sigma}{d\theta}(\xi)\Big|^{-1} d\xi \tag{2.23}$$

Using the notations of Lemma 2.1, the boundary map $\partial\mathbb{D} \to \partial\mathcal{R}(q)$ induced by Φ_q is

$$\begin{aligned}\xi \mapsto \zeta(\xi) &:= iH(k)^{-1}\mathrm{arcsn}(x(\xi),k),\\ x(\xi) &:= \varphi(e(\xi)) = -\cot(\pi\xi), \quad x'(\xi) = \pi(1+x(\xi)^2)\end{aligned}$$

where $\varphi(w) = i\frac{w+1}{1-w}$ takes the disk to the upper half-plane. If $0 < 2\pi\xi < \theta(q)$, then $\zeta(\xi) = 1 + iy(\xi)$ where

$$\frac{dy}{d\xi} = \frac{\pi}{H(k)}\frac{1+x^2}{\sqrt{(x^2-1)(k^2x^2-1)}} \geqslant \frac{\pi}{kH(k)}, \quad x(\xi) < -k^{-1}.$$

Therefore, this region contributes

$$\leqslant \frac{1}{2}kH(k)\theta(q) \lesssim 1 \ \text{ uniformly in } \ q$$

to the integral in (2.23). Next, if $\theta(q) < 2\pi\xi < \pi/2$, then $\zeta = u + iq$ with

$$\Big|\frac{du}{d\xi}\Big| = \frac{\pi}{H(k)}\frac{1+x^2}{\sqrt{(x^2-1)(1-k^2x^2)}} \geqslant \frac{\pi}{H(k)}, \quad -k^{-1} < x(\xi) < -1$$

and so this case contributes $\lesssim H(k)$ to (2.23). Finally, the region $\pi/2 < 2\pi\xi < 2\pi$ similarly adds at most $\lesssim H(k)$ to (2.23).

Combining these estimates with (2.22) yields

$$\begin{aligned}h_0(w) &\leqslant \log(B_2) + \log\big(CH(k)\big) + \log(2/(\pi(1-r))\\ &\leqslant \log(B_2) + C_0q^{-1} =: M\end{aligned} \tag{2.24}$$

for all $|w| < r = 1-\delta$ with some absolute constant C_0, cf. (2.20). This bound replaces (2.12) and (2.14) above.

As for the lower bound m on u, one has $m \geqslant \log(B_1)$ and thus (2.19) holds with

$$M - m \leqslant \log(B_2/B_1) + C_0q^{-1}$$

Finally, (2.21) follows from (2.19) by exponentiating. □

Integrating the previous result over a small set of x yields the following localization estimate for the L^2 norm of F.

Proposition 2.6. *There exists an absolute constant $C_1 > 0$ with the following property: Let F be an analytic function on a neighborhood of $\mathcal{R}(q)$ with $0 < q \leqslant q^*$, and F not identically equal to zero. Denote*

$$B_1 := \|F\|_{L^2([1/4,3/4])}, \qquad B_2 := \|F\|_{L^2(\partial\mathcal{R}(q))}.$$

For any $J \subset [1/4,3/4]$ some Borel set of positive measure,

$$B_1 \leqslant e^{\frac{C_1}{q}} B_2^{1-\kappa}\|F\|^{\kappa}_{L^2(J)}$$

with $0 < \kappa \leqslant e^{-\frac{C_1}{q}}\big(\log(1/|J|)\big)^{-1}$.

Proof. We apply Corollary 2.5 with $3H\exp\big(\frac{3\pi}{4q}\big) = \frac{1}{2}|J|$. Thus,

$$\begin{aligned} B_1^{K+1}(|J|/2)^{\frac{1}{2}} &\leqslant e^{\frac{C_0 K}{q}}\, B_2^K\, \|F\|_{L^2(J)} \\ K &:= \exp\big(\frac{9\pi}{8q}\big)\Big[\log(4) + \frac{9\pi}{4q} + \exp\big(-\frac{3\pi}{8q}\big)\big(\log(12e/|J|) + \frac{3\pi}{4q}\big)\Big] \end{aligned} \tag{2.25}$$

or

$$B_1 \leqslant e^{\frac{C_0}{q}}(|J|/2)^{-\frac{\kappa}{2}} B_2^{1-\kappa}\|F\|^{\kappa}_{L^2(J)}, \qquad \kappa \leqslant (1+K)^{-1}. \tag{2.26}$$

We write $\kappa \leqslant (1+K)^{-1}$ instead of $\kappa = (1+K)^{-1}$, since we may increase the value of K. One checks that

$$\begin{aligned} \log\big((|J|/2)^{-\frac{\kappa}{2}}\big) \leqslant& \frac{\log(2/|J|)}{\exp\big(\frac{9\pi}{8q}\big)\Big[\log(4) + \frac{9\pi}{4q} + \exp\big(-\frac{3\pi}{8q}\big)\big(\log(12e/|J|) + \frac{3\pi}{4q}\big)\Big]} \\ \leqslant& \exp\big(-\frac{3\pi}{4}\big) < 0.1, \end{aligned} \tag{2.27}$$

uniformly in $0 < q < 1$ and in $|J|$. Note that

$$\begin{aligned} K \leqslant& \begin{cases} \exp\big(\frac{9\pi}{8q}\big)\Big[\log(4) + \frac{9\pi}{4q} + \exp\big(-\frac{3\pi}{8q}\big)\big(\log(12e) + \frac{3\pi}{2q}\big)\Big], & \text{if } \log 2 \leqslant \log(1/|J|) < \frac{3\pi}{4q} \\ 8\exp\big(\frac{9\pi}{8q}\big)\Big[1 + \exp\big(-\frac{3\pi}{8q}\big)\Big]\log(1/|J|), & \text{if } \max\big(\log 2, \frac{3\pi}{4q}\big) \leqslant \log(1/|J|) \end{cases} \\ \leqslant& e^{\frac{C_2}{q}}\log(1/|J|) - 1, \end{aligned}$$

for some absolute constant $C_2 > 0$. Taking $C_1 := \max(2C_0, C_2)$ and

$$K_0 := e^{\frac{C_1}{q}}\log(1/|J|).$$

We conclude from (2.25), (2.26) and (2.27) with the estimate $K \leqslant K_0 - 1$ that

$$B_1 \leqslant e^{\frac{C_0}{q}+0.1}B_2^{1-\kappa}\|F\|^{\kappa}_{L^2(J)} \leqslant e^{\frac{C_1}{q}}B_2^{1-\kappa}\|F\|^{\kappa}_{L^2(J)}, \qquad \kappa \leqslant K_0^{-1}$$

as claimed. □

We next apply Proposition 2.6 to a band limited L^2 function in order to obtain the main result of this section.

Proposition 2.7. *Fix* $\lambda \in (0, \frac{1}{2}]$ *and for each integer* n *let* $I_n \subset [n, n+1]$ *be some Borel set with* $|I_n| = \lambda$. *Let* $f \in L^2(\mathbb{R})$ *be band-limited, i.e.,* $\hat{f}$ *is of compact support. Then for each* $0 < q \leqslant q^*$

$$\|f\|^2_{L^2(\mathbb{R})} \leqslant 12\, e^{\frac{10C_1}{q}}\Big(\sum_n \|f\|^2_{L^2(I_n)}\Big)^{\kappa}\|e^{2\pi q|\xi|}\hat{f}(\xi)\|^{2(1-\kappa)}_{L^2(\mathbb{R})} \tag{2.28}$$

with $0 < \kappa \leqslant e^{-\frac{5C_1}{q}}(-\log\lambda)^{-1}$, *and* C_1, q^* *are as in Proposition 2.6.*

Proof. Let F be the entire function with $F = f$ on the real line. Fix $0 \leqslant t \leqslant 1$ and define $\mathcal{R}_{n,t}(q)$ to be the rectangle with vertices $n - 1 - t \pm iq, n + 2 + t \pm iq$. We claim that by Proposition 2.6 we have

$$\|f\|_{L^2([n,n+1])} \leqslant e^{\frac{5C_1}{q}}\|F\|^{1-\kappa}_{L^2(\partial\mathcal{R}_{n,t}(q))}\|f\|^{\kappa}_{L^2(I_n)} \tag{2.29}$$

with $\kappa \leqslant e^{-\frac{5C_1}{q}}\big(\log((3+2t)/|I_n|)\big)^{-1}$. To see this, we set $n=0$ without loss of generality, translate $\mathcal{R}_{n,t}(q) \to \mathcal{R}_{n,t}(q)+1+t$, and dilate $z \mapsto z/(3+2t)$. After these operations, the transformed interval I_0 lies in

$$[(1+t)/(3+2t),(2+t)/(3+2t)] \subset [1/4,3/4],$$

and the height q becomes $q/(3+2t) \geqslant q/5$, whence the claim.

Squaring, summing, and applying Hölder's inequality yields

$$\|f\|_{L^2(\mathbb{R})}^2 \leqslant e^{\frac{10C_1}{q}}\Big(\sum_n \|F\|_{L^2(\partial\mathcal{R}_{n,t}(q))}^2\Big)^{1-\kappa}\Big(\sum_n \|f\|_{L^2(I_n)}^2\Big)^{\kappa}$$

Let $\mathbb{E}$ denote the expected value with respect to $0 \leqslant t \leqslant 1$, uniformly distributed. On the one hand, taking expectations of the previous line yields

$$\|f\|_{L^2(\mathbb{R})}^2 \leqslant e^{\frac{10C_1}{q}}\Big(\sum_n \mathbb{E}\|F\|_{L^2(\partial\mathcal{R}_{n,t}(q))}^2\Big)^{1-\kappa}\Big(\sum_n \|f\|_{L^2(I_n)}^2\Big)^{\kappa} \tag{2.30}$$

On the other hand, since

$$\sup_{0\leqslant t\leqslant 1}\sum_n \mathbb{1}_{[n-1-t,n+2+t)} \leqslant 5 \tag{2.31}$$

we have

$$\begin{aligned}\sum_n \mathbb{E}\|F\|_{L^2(\partial\mathcal{R}_{n,t}(q))}^2 &\leqslant 5\|F(\cdot+iq)\|_{L^2(\mathbb{R})}^2 + 5\|F(\cdot-iq)\|_{L^2(\mathbb{R})}^2 \\ &\quad + 2\sum_n\int_0^1\int_{-q}^{q}|F(n-t+is)|^2\,dsdt\end{aligned} \tag{2.32}$$

Since $\|F(\cdot \pm iq)\|_{L^2(\mathbb{R})} = \|e^{\pm 2\pi q\xi}\hat{f}(\xi)\|_{L^2(\mathbb{R})}$, and

$$\begin{aligned}\sum_n\int_0^1\int_{-q}^{q}|F(n-t+is)|^2\,dsdt &= \int_{\mathbb{R}}\int_{-q}^{q}|F(x+is)|^2\,dsdx \\ = \int_{-q}^{q}\int_{\mathbb{R}} e^{4\pi s\xi}|\hat{f}(\xi)|^2\,d\xi ds &\leqslant 2q\|e^{2\pi q|\xi|}\hat{f}(\xi)\|_{L^2(\mathbb{R})}^2\end{aligned}$$

Assuming as we may that $q^* \leqslant \frac{1}{2}$ we infer from (2.32) that

$$\sum_n \mathbb{E}\|F\|_{L^2(\partial\mathcal{R}_{n,t}(q))}^2 \leqslant 12\|e^{2\pi q|\xi|}\hat{f}(\xi)\|_{L^2(\mathbb{R})}^2$$

Inserting this into (2.30) concludes the proof. □

3. L^2 localization in higher dimensions

Our goal is to prove a version of Proposition 2.7 for band-limited functions $f \in L^2(\mathbb{R}^d)$, $d \geqslant 2$. For the sake of simplicity, we first limit ourselves to $d=2$ and begin with a Cartan-type estimate for functions on $\mathbb{D}\times\mathbb{D}$ which are subharmonic relative to each variable.

We begin with the definition of a Cartan-2 set, cf. [GolSch1, Definition 8.1] and [GolSch2, Definition 2.12].

Definition 3.1. We say that $\mathcal{B} \subset \mathbb{C}^2$ is a Cartan-2 set with parameter $H>0$ if for all $(z_1,z_2) \in \mathcal{B}$ one has either

- $z_1 \in \bigcup_j D(\zeta_j,s_j)$ with $\sum_j s_j \leqslant 5H$,

- or for all other z_1, one has $z_2 \in \bigcup_k D(w_k, t_k)$ with $\sum_k t_k \leqslant 5H$ and (w_k, t_k) depend on z_1.

Of particular relevance to us with be the fact that a Cartan-2 set has a real "trace" of small measure.

Lemma 3.1. *Let $\mathcal{B} \subset \prod_{j=1}^2 D(z_{j,0}, 1)$ be a Cartan-2 set with parameter $H > 0$. Then*

$$|\mathcal{B} \cap \mathbb{R}^2| \leqslant 40H$$

Proof. Follows from Fubini and $|D(\zeta, s) \cap \mathbb{R}| \leqslant 2s$ for all $\zeta \in \mathbb{C}$. ☐

We can now formulate a Cartan-type bound for pluri-subharmonic functions.

Lemma 3.2. *Let $v : \overline{\mathbb{D} \times \mathbb{D}} \to [-\infty, \infty)$ be continuous so that $v = v(z_1, z_2)$ is separately subharmonic in each variable. Suppose for $0 < \rho < r < 1$*

$$\max_{|z_1| \leqslant r, |z_2| \leqslant r} \int_{\mathbb{S}^1 \times \mathbb{S}^1} v(e(\theta_1), e(\theta_2))\, P_{z_1}(d\theta_1) P_{z_2}(d\theta_2) \leqslant M \tag{3.1}$$

and

$$\max_{|z_1| \leqslant \rho, |z_2| \leqslant \rho} v(z_1, z_2) \geqslant m \tag{3.2}$$

Let $\rho = r(1 - 3\delta)$ with $0 < \delta < \frac{1}{3}$. Then for any $0 < H \leqslant 1$ one has

$$\begin{aligned} v(z_1, z_2) &\geqslant m - (M - m)(L + 1)^2 \\ L &:= 2\delta^{-3} \log(2/\delta) + \delta^{-2} \log(2e/H) \end{aligned} \tag{3.3}$$

for all $(z_1, z_2) \in r_1\mathbb{D} \times r_1\mathbb{D} \backslash \mathcal{B}$ where $\mathcal{B}$ is a Cartan-2 set with parameter rH, and $r_1 = r(1 - 2\delta)$.

Proof. The function

$$h(z_1, z_2) := \int_{\mathbb{S}^1 \times \mathbb{S}^1} v(e(\theta_1), e(\theta_2))\, P_{z_1}(d\theta_1) P_{z_2}(d\theta_2) \tag{3.4}$$

is separately harmonic in each variable, is continuous up to $\partial(\mathbb{D} \times \mathbb{D})$, and satisfies $v \leqslant h$ pointwise. This latter property follows from the pointwise inequalities

$$v(z_1, z_2) \leqslant \int_{\mathbb{S}^1} v(z_1, e(\theta_2))\, P_{z_2}(d\theta_2)$$

which holds due to harmonicity of the right-hand side in z_2, whence

$$\begin{aligned} v(z_1, z_2) &\leqslant \int_{\mathbb{S}^1} v(e(\theta_1), z_2)\, P_{z_1}(d\theta_1) \\ &\leqslant \int_{\mathbb{S}^1 \times \mathbb{S}^1} v(e(\theta_1), e(\theta_2))\, P_{z_1}(d\theta_1) P_{z_2}(d\theta_2) = h(z_1, z_2) \end{aligned} \tag{3.5}$$

as claimed. Define

$$\tilde{v}(z_1) := \max_{|z_2| \leqslant \rho} v(z_1, z_2) \tag{3.6}$$

Then $\tilde{v}$ is continuous (by uniform continuity), and subharmonic (as the supremum of a family of subharmonic functions). It satisfies $\tilde{v}(z_1) \leqslant M$ for all $|z_1| \leqslant r$ by (3.1) and (3.5), and $\max_{|z_1| \leqslant \rho} \tilde{v}(z_1) \geqslant m$. The latter follows from

$$v(z_1, z_2) \leqslant \tilde{v}(z_1) \qquad \forall\ |z_1| \leqslant r,\ |z_2| \leqslant \rho$$

and (3.2).

We apply Corollary 2.3 to $\tilde v$, which requires rescaling from $\mathbb{D}$ to $r\mathbb{D}$. Thus, with $\rho = r(1-3\delta)$, and $r_1 = r(1-2\delta)$,

$$\tilde v(z_1) \geqslant m - (M-m)L =: m^* \tag{3.7}$$

for all $z_1 \in r_1\mathbb{D}\backslash \bigcup_j D(\zeta_j, s_j)$ with $\sum_j s_j \leqslant 5rH$. Fix such a *good* z_1. By definition, there exists z_2^* with $|z_2^*| \leqslant \rho$ and $v(z_1, z_2^*) \geqslant m^*$. On the other hand, $v(z_1, z_2) \leqslant M$ for all $|z_2| \leqslant r$.

Once again, by Corollary 2.3 rescaled from $\mathbb{D}$ to $r\mathbb{D}$, it follows that

$$\begin{aligned} v(z_1, z_2) &\geqslant m^* - (M - m^*)L \\ &\geqslant m - (M-m)L(2+L) \end{aligned} \tag{3.8}$$

for all $z_2 \in r_1\mathbb{D}\backslash \bigcup_j D(w_j, t_j)$ with $\sum_j t_j \leqslant 5rH$. These disks depend on z_1. $\square$

By means of Lemma 3.2 we establish a two-dimensional analogue of Proposition 2.6.

Proposition 3.3. *Let F be an analytic function of two variables on a neighborhood of $\mathcal{R}(q) \times \mathcal{R}(q)$ with $0 < q \leqslant q^*$, and F not identically equal to zero. Denote*

$$B_1 := \|F\|_{L^2([1/4,3/4]\times[1/4,3/4])}, \qquad B_2 := \|F\|_{L^2(\partial\mathcal{R}(q)\times\partial\mathcal{R}(q))}.$$

For any $J \subset [1/4,3/4]\times[1/4,3/4]$ some Borel set of positive measure,

$$B_1 \leqslant e^{\frac{C}{q}} B_2^{1-\kappa} \|F\|_{L^2(J)}^{\kappa}$$

with $0 < \kappa \leqslant e^{-\frac{C}{q}}\big(\log(1/|J|)\big)^{-2}$ with some absolute constant C.

Proof. Set $u(z_1, z_2) := \log|F(z_1, z_2)|$, which is pluri-subharmonic on a neighborhood of $\mathcal{R}(q) \times \mathcal{R}(q)$. We pull u back to the polydisk $\mathbb{D}\times\mathbb{D}$, and define

$$v(z_1, z_2) = u(\Phi_q(z_1), \Phi_q(z_2)) = \log|\tilde F(z_1, z_2)|, \qquad \tilde F(z_1, z_2) = F(\Phi_q(z_1), \Phi_q(z_2)).$$

With h defined as in (3.4), for all $|z_1|, |z_2| \leqslant r$,

$$\begin{aligned} h(z_1, z_2) &= \int_0^1\int_0^1 v(e(\theta_1), e(\theta_2))\, P_{z_1}(d\theta_1)P_{z_2}(d\theta_2) \\ &= \int_0^1\int_0^1 \log|\tilde F(e(\theta_1), e(\theta_2))|\, P_{z_1}(d\theta_1)P_{z_2}(d\theta_2) \\ &\leqslant \log\Big(\int_0^1\int_0^1 |\tilde F(e(\theta_1), e(\theta_2))|\, P_{z_1}(d\theta_1)P_{z_2}(d\theta_2)\Big) \\ &\leqslant \log\Big(\int_0^1\int_0^1 |\tilde F(e(\theta_1), e(\theta_2))|\, d\theta_1 d\theta_2\, \Big\|\frac{P_{z_1}(d\theta)}{d\theta}\Big\|_\infty \Big\|\frac{P_{z_2}(d\theta)}{d\theta}\Big\|_\infty\Big) \\ &\leqslant \log(B_2) + 2\log\big(\big\|\frac{d\theta}{d\sigma}\big\|_{L^2(\partial\mathcal{R}(q))}\big) + 2\sup_{|w|\leqslant r}\log\Big\|\frac{P_w(d\theta)}{d\theta}\Big\|_\infty \\ &\leqslant \log(B_2) + \log\big(Cq^{-1}\big) + 2\log(2/(1-r)) \end{aligned} \tag{3.9}$$

where $d\sigma$ denotes arc length measure on $\partial\mathcal{R}(q)$, see (2.24). By Lemma 2.1, we can apply Lemma 3.2 to v with $\rho = 1 - \exp(-A/q)$ with some absolute constant A,

$$m = \log B_1,\ M = \log(B_2) + 3Aq^{-1},\ \delta = \exp(-2A/q),\ r = \rho(1-3\delta)^{-1},$$

and $0 < q \leqslant q^* \ll 1$. Thus, for any $H > 0$ there exists a Cartan-2 set $\mathcal{B}$ with parameter H such that for

$$r_1 = 1 - \exp(-A/q) < r(1-2\delta),$$

and any $(z_1, z_2) \in r_1\mathbb{D} \times r_1\mathbb{D}\backslash\mathcal{B}$, we have

$$v(z_1, z_2) \geqslant m - (M - m)(L + 1)^2,$$

where

$$L = 2e^{\frac{6A}{q}} \log(2e^{\frac{2A}{q}}) + e^{\frac{4A}{q}} \log(2e/H) < e^{\frac{8A}{q}} + e^{\frac{4A}{q}} \log(2e/H) - 1.$$

Returning to the original geometry, and analytic function F, we conclude the following via Lemmas 2.1 and 3.1: with $K := (e^{\frac{8A}{q}} + e^{\frac{4A}{q}} \log(2e/H))^2$,

$$B_1^{K+1} \leqslant e^{\frac{3AK}{q}} |F(x_1, x_2)| B_2^K,$$

for all $(x_1, x_2) \in [1/4, 3/4] \times [1/4, 3/4]\backslash\mathcal{E}$, where $\mathcal{E} \subset \mathbb{R}^2$ and $|\mathcal{E}| \leqslant e^{\frac{5A}{q}} H$.

We now pick H so that $e^{\frac{5A}{q}} H = \frac{1}{2}|J|$, and integrate over J, we obtain

$$B_1^{K+1} (|J|/2)^{\frac{1}{2}} \leqslant e^{\frac{3AK}{q}} B_2^K \|F\|_{L^2(J)}$$

or

$$B_1 \leqslant e^{\frac{3A}{q}} (|J|/2)^{-\frac{\kappa}{2}} B_2^{1-\kappa} \|F\|_{L^2(J)}^{\kappa}, \qquad \kappa \leqslant (1 + K)^{-1}. \tag{3.10}$$

We write $\kappa \leqslant (1 + K)^{-1}$ instead of $\kappa = (1 + K)^{-1}$ since we could increase K. One easily checks that $(|J|/2)^{-\frac{\kappa}{2}} \lesssim 1$, and

$$K \leqslant e^{\frac{C_1}{q}} (\log(1/|J|))^2 - 1,$$

with some absolute constant C_1. Taking $C := \max(4A, C_1)$, and

$$K_0 := e^{\frac{C}{q}} (\log(1/|J|))^2.$$

We conclude from (3.10) with the estimate $K \leqslant K_0 - 1$ that

$$B_1 \leqslant e^{\frac{C}{q}} B_2^{1-\kappa} \|F\|_{L^2(J)}^{\kappa}, \qquad \kappa \leqslant K_0^{-1},$$

as claimed. □

In analogy with the one-dimensional case in Proposition 2.7, we can deduce the following L^2 localization result.

Proposition 3.4. *Fix $\lambda \in (0, \frac{1}{2}]$ and for each integers n_1, n_2 let*

$$I_{n_1,n_2} \subset [n_1, n_1 + 1] \times [n_2, n_2 + 1]$$

be some Borel set with $|I_{n_1,n_2}| = \lambda$. Let $f \in L^2(\mathbb{R}^2)$ be band-limited, i.e., $\hat{f}$ is of compact support. Then for each $0 < q \leqslant q^*$

$$\|f\|_{L^2(\mathbb{R}^2)}^2 \leqslant e^{\frac{2C}{q}} \Big(\sum_{(n_1,n_2)\in\mathbb{Z}^2} \|f\|_{L^2(I_{n_1,n_2})}^2 \Big)^{\kappa} \|e^{2\pi q(|\xi_1|+|\xi_2|)} \hat{f}(\xi)\|_{L^2(\mathbb{R}^2)}^{2(1-\kappa)} \tag{3.11}$$

with $0 < \kappa \leqslant e^{-\frac{C}{q}} (-\log\lambda)^{-2}$, and C some absolute constant.

Proof. Let F be the entire function with $F = f$ on $\mathbb{R}^2$. Fix $0 \leqslant t_1, t_2 \leqslant 1$ and for $j = 1, 2$ denote $\mathcal{R}_{n,t_j}(q)$ be the rectangle with vertices $n - 1 - t_j \pm iq$, $n + 2 + t_j \pm iq$. We obtain from Proposition 3.3 that for any $n_1, n_2 \in \mathbb{Z}$:

$$\|f\|_{L^2([n_1,n_1+1]\times[n_2,n_2+1])} \leqslant e^{\frac{5C}{q}} \|F\|_{L^2(\partial\mathcal{R}_{n_1,t_1}(q)\times\partial\mathcal{R}_{n_2,t_2}(q))}^{1-\kappa} \|f\|_{L^2(I_{n_1,n_2})}^{\kappa},$$

with $\kappa \leqslant e^{-\frac{5C}{q}}(\log((3+2t_1)(3+2t_2)/|I_{n_1,n_2}|))^{-2}$, and C being the absolute constant in Proposition 3.3. Squaring, summing, and applying Hölder's inequality, we have

$$\|f\|^2_{L^2(\mathbb{R}^2)} \leqslant e^{\frac{10C}{q}}\Big(\sum_{(n_1,n_2)\in\mathbb{Z}^2}\|F\|^2_{L^2(\partial\mathcal{R}_{n_1,t_1}(q)\times\partial\mathcal{R}_{n_2,t_2}(q))}\Big)^{1-\kappa}$$
$$\Big(\sum_{(n_1,n_2)\in\mathbb{Z}^2}\|f\|^2_{L^2(I_{n_1,n_2})}\Big)^{\kappa}.$$

Taking expectation of the previous line with respect to $0 \leqslant t_1, t_2 \leqslant 1$, we obtain

$$\begin{aligned}&\|f\|^2_{L^2(\mathbb{R}^2)}\\ \leqslant& e^{\frac{10C}{q}}\Big(\sum_{(n_1,n_2)\in\mathbb{Z}^2}\mathbb{E}_{t_1}\mathbb{E}_{t_2}\|F\|^2_{L^2(\partial\mathcal{R}_{n_1,t_1}(q)\times\partial\mathcal{R}_{n_2,t_2}(q))}\Big)^{1-\kappa}\Big(\sum_{(n_1,n_2)\in\mathbb{Z}^2}\|f\|^2_{L^2(I_{n_1,n_2})}\Big)^{\kappa}.\end{aligned} \tag{3.12}$$

By decomposing each $\partial\mathcal{R}_{n,t}(q)$ into its four sides, we decompose

$$\sum_{(n_1,n_2)\in\mathbb{Z}^2}\mathbb{E}_{t_1}\mathbb{E}_{t_2}\|F\|^2_{L^2(\partial\mathcal{R}_{n_1,t_1}(q)\times\partial\mathcal{R}_{n_2,t_2}(q))} \tag{3.13}$$

into the following three parts:

Part 1. Vertical and Horizontal mixed terms. This part contains eight terms, each can be bounded in the same way. Taking the left vertical side of $\mathcal{R}_{n_1,t_1}(q)$ and upper horizontal side of $\mathcal{R}_{n_2,t_2}(q)$ for example, we have

$$\begin{aligned}&\sum_{(n_1,n_2)\in\mathbb{Z}^2}\mathbb{E}_{t_2}\int_{\mathbb{R}}\mathbb{1}_{[n_2-1-t_2,n_2+2+t_2)}\mathbb{E}_{t_1}\int_{-q}^{q}|F(n_1-1-t_1+is,x_2+iq)|^2\,dsdx_2\\ \leqslant&5\sum_{n_1\in\mathbb{Z}}\mathbb{E}_{t_1}\int_{\mathbb{R}}\int_{-q}^{q}|F(n_1-1-t_1+is,x_2+iq)|^2\,dsdx_2\\ =&5\int_{-q}^{q}\int_{\mathbb{R}^2}|F(x_1+is,x_2+iq)|^2\,dx_1dx_2ds\\ \leqslant&5\int_{-q}^{q}\int_{\mathbb{R}^2}e^{4\pi(s\xi_1+q\xi_2)}|\hat{f}(\xi_1,\xi_2)|^2\,d\xi_1d\xi_2ds\\ \leqslant&10q\|e^{2\pi q(|\xi_1|+|\xi_2|)}\hat{f}(\xi)\|^2_{L^2(\mathbb{R}^2)},\end{aligned}$$

in which we used (2.31) in the first step. Hence, part 1 contributes in total at most

$$80q\|e^{2\pi q(|\xi_1|+|\xi_2|)}\hat{f}(\xi)\|^2_{L^2(\mathbb{R}^2)}. \tag{3.14}$$

Part 2. Vertical+Vertical sides. This part contains four terms. Taking the left vertical sides of $\mathcal{R}_{n_1,t_1}(q)$ and $\mathcal{R}_{n_2,t_2}(q)$ for example, we have

$$\begin{aligned}&\sum_{(n_1,n_2)\in\mathbb{Z}^2}\mathbb{E}_{t_1}\mathbb{E}_{t_2}\int_{-q}^{q}\int_{-q}^{q}|F(n_1-1-t_1+is_1,n_2-1-t_2+is_2)|^2\,ds_1ds_2\\ =&\int_{-q}^{q}\int_{-q}^{q}\int_{\mathbb{R}^2}|F(x_1+is_1,x_2+is_2)|^2\,dx_1dx_2ds_1ds_2\\ \leqslant&4q^2\|e^{2\pi q(|\xi_1|+|\xi_2|)}\hat{f}(\xi)\|^2_{L^2(\mathbb{R}^2)}.\end{aligned}$$

Hence, part 2 contributes in total at most

$$16q^2 \| e^{2\pi q(|\xi_1|+|\xi_2|)} \hat{f}(\xi) \|_{L^2(\mathbb{R}^2)}^2. \tag{3.15}$$

Part 3. Horizontal+Horizontal sides. This part also contains four terms. Taking the upper horizontal sides of $\mathcal{R}_{n_1,t_1}(q)$ and $\mathcal{R}_{n_2,t_2}(q)$ for example, we have

$$\begin{aligned}
\sum_{(n_1,n_2)\in\mathbb{Z}^2} \mathbb{E}_{t_1}\mathbb{E}_{t_2} \int_{\mathbb{R}^2} & \mathbb{1}_{[n_1-1-t_1,n_1+2+t_1)} \mathbb{1}_{[n_2-1-t_2,n_2+2+t_2)} |F(x_1+iq, x_2+iq)|^2 dx_1 dx_2 \\
&\leqslant 25 \int_{\mathbb{R}^2} |F(x_1+iq, x_2+iq)|^2 dx_1 dx_2 \\
&\leqslant 25 \| e^{2\pi q(|\xi_1|+|\xi_2|)} \hat{f}(\xi) \|_{L^2(\mathbb{R}^2)}^2,
\end{aligned}$$

in which we used (2.31) in the first step. Hence, the contribution of part (3) is at most

$$100 \| e^{2\pi q(|\xi_1|+|\xi_2|)} \hat{f}(\xi) \|_{L^2(\mathbb{R}^2)}^2. \tag{3.16}$$

Plugging the estimates in (3.14), (3.15) and (3.16) into (3.13), we obtain

$$\begin{aligned}
\sum_{(n_1,n_2)\in\mathbb{Z}^2} \mathbb{E}_{t_1}\mathbb{E}_{t_2} \|F\|_{L^2(\partial\mathcal{R}_{n_1,t_1}(q)\times\partial\mathcal{R}_{n_2,t_2}(q))}^2 &\leqslant (4q+10)^2 \| e^{2\pi q(|\xi_1|+|\xi_2|)} \hat{f}(\xi) \|_{L^2(\mathbb{R}^2)}^2 \\
&\leqslant 144 \| e^{2\pi q(|\xi_1|+|\xi_2|)} \hat{f}(\xi) \|_{L^2(\mathbb{R}^2)}^2,
\end{aligned} \tag{3.17}$$

for $q \leqslant 1/2$. Plugging (3.17) into (3.12) yields

$$\|f\|_{L^2(\mathbb{R}^2)}^2 \leqslant 144 e^{\frac{10C}{q}} \Big(\sum_{(n_1,n_2)\in\mathbb{Z}^2} \|f\|_{L^2(I_{n_1,n_2})}^2 \Big)^{\kappa} \| e^{2\pi q(|\xi_1|+|\xi_2|)} \hat{f}(\xi) \|_{L^2(\mathbb{R}^2)}^{2(1-\kappa)},$$

as claimed. □

In general dimensions one can proceed similarly. First, we inductively define Cartan sets in higher dimensions.

Definition 3.2. We say that $\mathcal{B} \subset \mathbb{C}^2$ is a Cartan-d set with parameter $H > 0$ if for all $(z_1, z_2, \ldots, z_d) \in \mathcal{B}$ one has either

- $z_1 \in \bigcup_j D(\zeta_j, s_j)$ with $\sum_j s_j \leqslant 5H$ or for all other z_1 one has
- $(z_2, \ldots, z_d)$ belongs to a Cartan-$(d-1)$ set with parameter $H > 0$ depending on z_1.

By arguments analogous to those used above for $d = 2$, one can exploit these Cartan sets in higher dimensions to obtain the following result. We leave the details to the reader. Throughout, we let $C(d) \geqslant 1$ be a constant depending only on the dimension d. It is allowed to change its values from line to line.

Proposition 3.5. *Fix $\lambda \in (0, \frac{1}{2}]$ and for each integer vector $n = (n_1, \ldots, n_d) \in \mathbb{Z}^d$, $d \geqslant 2$, let*

$$I_n \subset \prod_{j=1}^{d} [n_j, n_j+1)$$

be some Borel set with $|I_n| = \lambda$. Let $f \in L^2(\mathbb{R}^d)$ be band-limited, i.e., $\hat{f}$ is of compact support. Then for each $0 < q \leqslant q^* = q^*(d) \ll 1$

$$\|f\|^2_{L^2(\mathbb{R}^d)} \leqslant e^{\frac{2C(d)}{q}} \Big(\sum_{n \in \mathbb{Z}^d} \|f\|^2_{L^2(I_n)} \Big)^{\kappa} \|e^{2\pi q|\xi|_1} \hat{f}(\xi)\|^{2(1-\kappa)}_{L^2(\mathbb{R}^d)} \tag{3.18}$$

with $0 < \kappa \leqslant e^{-\frac{C(d)}{q}}(-\log\lambda)^{-d}$, $C(d) \geqslant 1$ some absolute constant depending on d.

As a precursor to the results of the next section, which involve L^2 functions with Fourier support in thin sets, we now establish an uncertainty principle for $L^2(\mathbb{R}^d)$ functions under a quantitative decay assumption on their Fourier transforms.

Corollary 3.6. *Let $\Theta(\xi) = \Theta(|\xi|_1) = (\log(2 + |\xi|_1))^{-\alpha}$, $0 < \alpha < 1$. Let $\mathcal{S} := \bigcup_{n \in \mathbb{Z}^d} I_n$ be as in Proposition 3.5. Then*

$$\|f\|_2 \leqslant C(d, \alpha, A, \lambda) \|f\|_{L^2(\mathcal{S})} \tag{3.19}$$

for all $f \in L^2(\mathbb{R}^d)$ with $\|e^{\Theta(\xi)|\xi|_1} \hat{f}\|_{L^2(\mathbb{R}^d)} \leqslant A\|f\|_{L^2(\mathbb{R}^d)}$.

Proof. With $0 < q$ small to be determined, we fix $R \geqslant 1$ so that $2\pi q = \Theta(R)$. Split $f = f_1 + f_2$, $\hat{f}_1(\xi) = \hat{f}(\xi)\mathbb{1}_{[|\xi|_1 \leqslant R]}$. Then by (3.18), and since $2\pi q \leqslant \Theta(\xi)$ for $|\xi|_1 \leqslant R$,

$$\begin{aligned} \|f_1\|_2^2 &\leqslant e^{\frac{2C(d)}{q}} \|f_1\|^{2\kappa}_{L^2(\mathcal{S})} \|e^{\Theta(\xi)|\xi|_1} \hat{f}_1\|_2^{2(1-\kappa)} \\ &\leqslant e^{\frac{2C(d)}{q}} \|f_1\|^{2\kappa}_{L^2(\mathcal{S})} (A\|f\|_2)^{2(1-\kappa)} \end{aligned}$$

with

$$\kappa = e^{-\frac{C(d)}{q}}(-\log\lambda)^{-d} = e^{-\frac{2\pi C(d)}{\Theta(R)}}(-\log\lambda)^{-d}$$

Moreover, since

$$\begin{aligned} \|f\|_2^2 &= \|f_1\|_2^2 + \|f_2\|_2^2 \\ &\leqslant e^{\frac{2C(d)}{q}} (\|f\|_{L^2(\mathcal{S})} + \|f_2\|_2)^{2\kappa} (A\|f\|_2)^{2(1-\kappa)} + \|f_2\|_2^2 \end{aligned}$$

and

$$\|f_2\|_2 \leqslant e^{-\Theta(R)R} \|e^{\Theta(\xi)|\xi|_1} \hat{f}\|_2 \leqslant A e^{-\Theta(R)R} \|f\|_2 \leqslant \frac{1}{2}\|f\|_2$$

where we chose R large enough depending on $A \geqslant 1$. It follows that

$$\|f\|_2^2 \leqslant 2e^{\frac{2C(d)}{q}} (\|f\|_{L^2(\mathcal{S})} + Ae^{-\Theta(R)R}\|f\|_2)^{2\kappa} (A\|f\|_2)^{2(1-\kappa)}$$

whence

$$\begin{aligned} \|f\|_2 &\leqslant 2^{\frac{1}{2\kappa}} A^{\frac{1-\kappa}{\kappa}} e^{\frac{C(d)}{\kappa q}} (\|f\|_{L^2(\mathcal{S})} + Ae^{-\Theta(R)R}\|f\|_2) \\ &= 2^{\frac{1}{2\kappa}} A^{\frac{1-\kappa}{\kappa}} e^{\frac{C(d)}{\kappa q}} \|f\|_{L^2(\mathcal{S})} + \exp\big(-T(R)\big)\|f\|_2 \end{aligned}$$

with

$$\begin{aligned} T(R) &= \Theta(R)R - \frac{C(d)}{\kappa q} - \kappa^{-1}\log(\sqrt{2}A) \\ &= \Theta(R)R - \Big(\frac{2\pi C(d)}{\Theta(R)} + \log(\sqrt{2}A)\Big) e^{\frac{2\pi C(d)}{\Theta(R)}} (-\log\lambda)^d \end{aligned}$$

In addition to $2A \leqslant e^{\Theta(R)R}$ we require that $T(R) \geqslant 1$. These conditions hold for sufficiently large R. □

The proof of the corollary gives an explicit and effective dependence of the constant $C(d,\alpha,A,\lambda)$ on A,λ, but we have no need for it. Corollary 3.6 follows (perhaps with a different dependence on the constants) from a quantitative version of the Logvinenko-Sereda theorem (see e.g. [Kov, MusSch]). The results in the next section, however, do not.

4. Uncertainty principle with thin Fourier support

We begin with the concept of a damping function.

Definition 4.1. Let Θ be as in Corollary 3.6, with $\alpha\in(0,1)$ fixed. Let $Y\subset\mathbb{R}^d$. We say that Y *admits a damping function* with parameters c_1,c_2,c_3, all falling into the interval $(0,1)$, if there exists a function $\psi\in L^2(\mathbb{R}^d)$ satisfying

- $\mathrm{supp}(\psi)\subset[-c_1,c_1]^d$,
- $\|\widehat{\psi}\|_{L^2([-1,1]^d)}\geqslant c_2$,
- $|\widehat{\psi}(\xi)|\leqslant\langle\xi\rangle^{-d}$ for all $\xi\in\mathbb{R}^d$,
- $|\widehat{\psi}(\xi)|\leqslant\exp\big(-c_3\Theta(|\xi|_1)|\xi|_1\big)$ for all $\xi\in Y$.

Lemma 4.1. *Fix $c_1\in(0,\frac12]$ and for each integer vector $n=(n_1,\dots,n_d)\in\mathbb{Z}^d$, $d\geqslant 2$, let*

$$I_n\subset\prod_{j=1}^d[n_j,n_j+1)$$

be a square with side length $2c_1$. Define $\mathcal{S}:=\bigcup_{n\in\mathbb{Z}^d}I_n$. Suppose $Y\subset\mathbb{R}^d$ is such that $Y+[-2,2]^d$ admits a damping function with parameters c_1, and $c_2,c_3\in(0,1)$. Then every $f\in L^2(\mathbb{R}^d)$ with $\mathrm{supp}(\hat f)\subset Y$ satisfies

$$\begin{aligned}\|\hat f\|^2_{L^2([-1,1]^d)}\leqslant C(d)c_2^{-2}\left\langle R\right\rangle^{2d}e^{\frac{4\pi C(d)}{c_3\Theta(R)}}\big(&\|\mathbb{1}_{\mathcal{S}}f\|^{2\kappa}_{H^{-d}}\|f\|^{2(1-\kappa)}_{H^{-d}}\\ &+\exp(-2c_3\kappa\Theta(R)R)\,\|f\|^2_{H^{-d}}\big)\end{aligned}\tag{4.1}$$

and $\kappa=e^{-\frac{2\pi C(d)}{c_3\Theta(R)}}(-d\log c_1)^{-d}$, provided $R\geqslant(2d/c_3)^2$ and $0<c_3\leqslant c_3^*(d):=2\pi q_*$ where q_* is as in Proposition 3.5.

Proof. Let $\eta\in[-2,2]^d$. Set $f_\eta(x):=e^{2\pi ix\cdot\eta}f(x)$, and $g_\eta:=f_\eta*\psi$ where ψ is the damping function as in Definition 4.1 associated with $Y+[-2,2]^d$. Split

$$\begin{aligned}g_\eta&=g_1+g_2,\\ \mathrm{supp}(\widehat{g_1})&\subset\{\xi\in\mathbb{R}^d\ :\ |\xi|_1\leqslant R\}\\ \mathrm{supp}(\widehat{g_2})&\subset\{\xi\in\mathbb{R}^d\ :\ |\xi|_1>R\}\end{aligned}\tag{4.2}$$

where $2\pi q=c_3\Theta(R)$. Note that our assumption $c_3\leqslant 2\pi q_*$ guarantees that $q\leqslant q_*$ holds for any $R\geqslant 1$. Note also that since $\mathrm{supp}(\psi)\subset[-c_1,c_1]^d$, we have $\mathbb{1}_{\mathcal{S}'}g_\eta=\mathbb{1}_{\mathcal{S}'}(\mathbb{1}_{\mathcal{S}}f_\eta*\psi)$ where $\mathcal{S}':=\bigcup_{n\in\mathbb{Z}^d}I_n'$ with I_n' a square with the same center as I_n, but half the side length. By Proposition 3.5 with $\lambda=c_1^d$ one has

$$\begin{aligned}\|g_\eta\|_2^2&=\|g_1\|_2^2+\|g_2\|_2^2\\ &\leqslant e^{\frac{2C(d)}{q}}\big(\|g_\eta\|_{L^2(\mathcal{S}')}+\|g_2\|_2\big)^{2\kappa}\|e^{2\pi q|\xi|_1}\widehat{g_1}\|_2^{2(1-\kappa)}+\|g_2\|_2^2\end{aligned}\tag{4.3}$$

with

$$0<\kappa\leqslant e^{-\frac{C(d)}{q}}(-d\log c_1)^{-d}=e^{-\frac{2\pi C(d)}{c_3\Theta(R)}}(-d\log c_1)^{-d},$$

$C(d)$ some absolute constant. By construction, $\mathrm{supp}(\widehat{f_\eta}) \subset Y + \eta \subset Y + [-2,2]^d$, hence

$$|\widehat{g_\eta}(\xi)| \leqslant |\widehat{f_\eta}(\xi)| \exp\big(-c_3\Theta(|\xi|_1)|\xi|_1\big) \qquad \forall\, \xi \in \mathbb{R}^d$$

whence

$$\begin{aligned}
\|e^{2\pi q|\xi|_1}\widehat{g_1}\|_2 = \|e^{c_3\Theta(R)|\xi|_1}\widehat{g_1}\|_2 &\leqslant \sup_{|\xi|_1\leqslant R} \langle\xi\rangle^d \,\|f_\eta\|_{H^{-d}} \leqslant \langle R\rangle^d \,\|f_\eta\|_{H^{-d}} \\
\|g_2\|_2 &\leqslant \sup_{|\xi|_1\geqslant R} \exp(-c_3\Theta(|\xi|_1)|\xi|_1)\langle\xi\rangle^d \,\|f_\eta\|_{H^{-d}} \\
&\leqslant \exp(-c_3\Theta(R)R)\langle R\rangle^d \,\|f_\eta\|_{H^{-d}}
\end{aligned}$$

where we used that $|\xi|_2 \leqslant |\xi|_1$, and that $r \mapsto \exp(-c_3\Theta(r)r)\langle r\rangle^d$ is decreasing for large r. To be specific,

$$\begin{aligned}
\exp(-c_3\Theta(r)r)\langle r\rangle^d &= \exp(-h(r)) \\
h(r) &= c_3\,(\log(2+r))^{-\alpha}\, r - \frac{d}{2}\log(1+r^2)
\end{aligned}$$

Differentiating, we obtain

$$\begin{aligned}
h'(r) &= c_3\,(\log(2+r))^{-\alpha}\Big[1 - \frac{\alpha r}{2+r}(\log(2+r))^{-1}\Big] - \frac{dr}{1+r^2} \\
&\geqslant \frac{c_3}{2}\,(\log(2+r))^{-\alpha} - dr^{-1} \geqslant \frac{c_3}{2}\,(\log(2+r))^{-1} - dr^{-1}
\end{aligned}$$

where we used that $\frac{\alpha r}{2+r}(\log(2+r))^{-1} \leqslant \frac{1}{2}$ for all $r \geqslant 0$. One has $u > \log(2+u^2)$ for $u \geqslant 2$, say. Hence, if $r \geqslant (2d/c_3)^2$, then

$$\frac{c_3}{2}\,(\log(2+r))^{-1} - dr^{-1} > 0$$

and thus $h'(r) > 0$. So it suffices to assume that $R \geqslant (2d/c_3)^2$.

Inserting these bounds into (4.3) yields

$$\begin{aligned}
\|g_\eta\|_2^2 \leqslant{}& e^{\frac{2C(d)}{q}}\big(\|\mathbb{1}_{\mathcal{S}} f_\eta\|_{H^{-d}} + \exp(-c_3\Theta(R)R)\langle R\rangle^d\,\|f_\eta\|_{H^{-d}}\big)^{2\kappa}\big(\langle R\rangle^d\,\|f_\eta\|_{H^{-d}}\big)^{2(1-\kappa)} \\
&+ \exp(-2c_3\Theta(R)R)\langle R\rangle^{2d}\,\|f_\eta\|_{H^{-d}}^2
\end{aligned}$$

Since $\sup_{\eta\in[-2,2]^d}\|f_\eta\|_{H^{-d}} \leqslant C(d)\|f\|_{H^{-d}}$, we can simplify this further:

$$\|g_\eta\|_2^2 \leqslant C(d)\langle R\rangle^{2d}\, e^{\frac{4\pi C(d)}{c_3\Theta(R)}}\big(\|\mathbb{1}_{\mathcal{S}} f\|_{H^{-d}}^{2\kappa}\|f\|_{H^{-d}}^{2(1-\kappa)} + \exp(-2c_3\kappa\Theta(R)R)\,\|f\|_{H^{-d}}^2\big). \tag{4.4}$$

Finally,

$$\begin{aligned}
\|\widehat{f}\|_{L^2([-1,1]^d)}^2 &\leqslant c_2^{-2}\int_{[-1,1]^d}|\widehat{f}(\zeta)|^2\,d\zeta \int_{[-1,1]^d}|\widehat{\psi}(\xi)|^2\,d\xi \\
&\leqslant c_2^{-2}\int_{[-1,1]^d}\int_{[-2,2]^d}|\widehat{f}(\xi-\eta)|^2|\widehat{\psi}(\xi)|^2\,d\eta d\xi \\
&\leqslant c_2^{-2}\int_{[-2,2]^d}\int_{\mathbb{R}^d}|\widehat{f}(\xi-\eta)|^2|\widehat{\psi}(\xi)|^2\,d\xi d\eta \\
&= c_2^{-2}\int_{[-2,2]^d}\|g_\eta\|_2^2\,d\eta
\end{aligned}$$

and we are done. $\square$

We now remove the localization in Fourier space on the left-hand side of (4.1) in order to obtain the main result of this section.

Corollary 4.2. *Fix $c_1 \in (0, \frac{1}{2}]$ and for each integer vector $n = (n_1, \ldots, n_d) \in \mathbb{Z}^d$, $d \geqslant 2$, let*

$$I_n \subset \prod_{j=1}^{d} [n_j, n_j + 1)$$

be a square with side length $2c_1$. Define $\mathcal{S} := \bigcup_{n \in \mathbb{Z}^d} I_n$. Suppose $Y \subset [-\alpha_1, \alpha_1]^d \subset \mathbb{R}^d$ with $\alpha_1 \geqslant 1$ is such that $Y + [-2, 2]^d + \eta$ admits a damping function with parameters c_1, and $c_2, c_3 \in (0, 1)$ for each $\eta \in [-\alpha_1 - 1, \alpha_1 + 1]^d$. Assume further that $0 < c_3 < c_3^*(d) \ll 1$, with $c_3^*(d)$ be as in Lemma 4.1. Then every $f \in L^2(\mathbb{R}^d)$ with $\mathrm{supp}(\hat{f}) \subset Y$ satisfies

$$\|f\|_2 \leqslant C_* \|f\|_{L^2(\mathcal{S})} \tag{4.5}$$

with constant C_* depending only on d, c_1, c_2, c_3, α explicitly as in (4.15).

Proof. Let $\ell \in (2\mathbb{Z})^d$ be such that $\ell + [-1, 1]^d \cap [-\alpha_1, \alpha_1]^d \neq \varnothing$ and define $f_\ell(x) := e^{2\pi i x \cdot \ell} f(x)$ so that $\widehat{f_\ell}(\xi) = \widehat{f}(\xi - \ell)$ and $\mathrm{supp}(\widehat{f_\ell}) \subset Y + \ell$. In order to apply Lemma 4.1, we also need to ensure that $Y + [-2, 2]^d + \ell$ admits a damping function. This, however, follows from our assumptions. Hence, for each such ℓ,

$$\begin{aligned} \|\hat{f}\|^2_{L^2([-1,1]^d + \ell)} \leqslant C(d) c_2^{-2} \langle R \rangle^{2d} e^{\frac{4\pi C(d)}{c_3 \Theta(R)}} \big(&\|\mathbb{1}_{\mathcal{S}} f_\ell\|^{2\kappa}_{H^{-d}} \|f_\ell\|^{2(1-\kappa)}_{H^{-d}} \\ &+ \exp(-2c_3 \kappa \Theta(R) R)\, \|f_\ell\|^2_{H^{-d}} \big) \end{aligned} \tag{4.6}$$

and $\kappa = e^{-\frac{2\pi C(d)}{c_3 \Theta(R)}} (-d \log c_1)^{-d}$, provided $R \geqslant (2d/c_3)^2$. Summing over $\ell \in (2\mathbb{Z})^d$, and using Hölder's inequality yields

$$\begin{aligned} \|f\|_2^2 &\leqslant C(d) c_2^{-2} \langle R \rangle^{2d} e^{\frac{4\pi C(d)}{c_3 \Theta(R)}} \big(\|\mathbb{1}_{\mathcal{S}} f\|_2^{2\kappa} \|f\|_2^{2(1-\kappa)} + \exp(-2c_3 \kappa \Theta(R) R)\, \|f\|_2^2 \big) \\ &= C(d) c_2^{-2} \langle R \rangle^{2d} e^{\frac{4\pi C(d)}{c_3 \Theta(R)}} \|\mathbb{1}_{\mathcal{S}} f\|_2^{2\kappa} \|f\|_2^{2(1-\kappa)} \\ &\quad + C(d) c_2^{-2} \langle R \rangle^{2d} e^{\frac{4\pi C(d)}{c_3 \Theta(R)}} e^{-2c_3 \kappa \Theta(R) R} \|f\|_2^2. \end{aligned} \tag{4.7}$$

Suppose further that R satisfies,

$$R \geqslant R_0(d, c_1, c_2, c_3, \alpha) := \max \begin{cases} (i).\ \exp\left[\left(\frac{16\pi C(d)}{c_3} \right)^{\frac{1}{1-\alpha}} \right], \\ (ii).\ \exp\left(4^{\frac{1}{1-\alpha}} \right), \\ (iii).\ \left(\frac{(-d \log c_1)^d}{c_3} \right)^8, \\ (iv).\ \left(4 \log \frac{2C(d)}{c_2^2} \right)^2, \\ (v).\ (8d)^4. \end{cases} \tag{4.8}$$

Note that (i), (ii), (iii) of (4.8) imply

$$\begin{aligned} e^{-\frac{2\pi C(d)}{c_3 \Theta(R)}} (R+2)^{\frac{1}{4}} &\geqslant 1, \\ \Theta(R)(R+2)^{\frac{1}{8}} &\geqslant 1, \text{ and} \\ \frac{c_3}{(-d \log c_1)^d} (R+2)^{\frac{1}{8}} &\geqslant 1, \end{aligned} \tag{4.9}$$

respectively. Hence multiplying the three inequalities of (4.9) yields

$$\begin{aligned} c_3\kappa\Theta(R)(R+2) &\geqslant \sqrt{R+2}, \quad \text{or} \\ \kappa &\geqslant (c_3\Theta(R)\sqrt{R+2})^{-1}, \end{aligned} \tag{4.10}$$

and thus

$$e^{2c_3\kappa\theta(R)R} \geqslant e^{c_3\kappa\theta(R)(R+2)} \geqslant e^{\sqrt{R+2}}. \tag{4.11}$$

One also derives from (iv), (v) and (i) that

$$\begin{aligned} &\frac{1}{4}\sqrt{R+2} \geqslant \log\frac{2C(d)}{c_2^2}, \\ &\frac{1}{2}\sqrt{R+2} \geqslant 2d\log(R+2) \geqslant \log\langle R\rangle^{2d}, \quad \text{and,} \\ &\frac{1}{4}\sqrt{R+2} \geqslant \frac{4\pi C(d)}{c_3\Theta(R)}, \end{aligned} \tag{4.12}$$

respectively. Hence by summing up the three inequalities of (4.12), and exponentiating, we obtain

$$e^{\sqrt{R+2}} \geqslant 2C(d)c_2^{-2}\langle R\rangle^{2d} e^{\frac{4\pi C(d)}{c_3\Theta(R)}}. \tag{4.13}$$

Combining (4.11) with (4.13), we arrive at

$$C(d)c_2^{-2}\langle R\rangle^{2d} e^{\frac{4\pi C(d)}{c_3\Theta(R)}} e^{-2c_3\kappa\Theta(R)R} \leqslant \frac{1}{2}.$$

Thus (4.7) yields

$$\|f\|_2 \leqslant \Big(2C(d)c_2^{-2}\langle R\rangle^{2d} e^{\frac{4\pi C(d)}{c_3\Theta(R)}}\Big)^{\frac{1}{2\kappa}} \|\mathbb{1}_{\mathcal{S}}f\|_2.$$

Combining the estimate of κ in (4.10) with (4.13), we obtain

$$\Big(2C(d)c_2^{-2}\langle R\rangle^{2d} e^{\frac{4\pi C(d)}{c_3\Theta(R)}}\Big)^{\frac{1}{2\kappa}} \leqslant e^{\frac{c_3\Theta(R)(R+2)}{2}}.$$

Now we take R_0 as in (4.8) and define R_1 as follows

$$R_1(d,c_1,c_2,c_3,\alpha) := \max\big((2d/c_3)^2, R_0(d,c_1,c_2,c_3,\alpha)\big). \tag{4.14}$$

Then

$$\|f\|_2 \leqslant C_*(d,c_1,c_2,c_3,\alpha)\|\mathbb{1}_{\mathcal{S}}f\|_2,$$

with

$$C_*(d,c_1,c_2,c_3,\alpha) = e^{\frac{c_3\Theta(R_1)(R_1+2)}{2}}, \tag{4.15}$$

as claimed. □

5. FUP assuming damping functions on Y

In section we prove, by the same iteration as in [BouDya], the fractal uncertainty principle for sets $X \subset [-1,1]^d$ and $Y \subset [-N,N]^d$. On Y we do not impose a geometric condition. Rather, in this section we still restrict ourselves to assuming the existence of damping functions living on Y, as well as on sets derived from Y through translations and dilations, see Definition 4.1. On X we impose a certain tree structure "with gaps", cf. [BouDya, Lemma 2.10].

Definition 5.1. We say that $X \subset [-1,1]^d \subset \mathbb{R}^d$ *is porous at scale* $L \geqslant 3$ *with depth* n, where L is an integer, if the following holds: denote by $\mathcal{C}_n$ the cubes obtained from $[-1,1]^d$ by partitioning it into congruent cubes of side length L^{-n}. Thus, $\#\mathcal{C}_n = 2^d L^{nd}$. The condition on X is that for all $Q \in \mathcal{C}_n$ with $Q \cap X \neq \varnothing$, there exists $Q' \in \mathcal{C}_{n+1}$ so that $Q' \subset Q$ and $Q' \cap X = \varnothing$.

It is shown in [BouDya] that sets $X \subset \mathbb{R}$ obeying the δ-regularity condition on scales N^{-1} to 1 (see Definition 6.1) satisfy this porosity property at depth n for all $n \geqslant 0$ with $L^{n+1} \leqslant N$. We include a d-dimensional analogy in Appendix A, see Lemma A.7. We can now formulate the Fractal Uncertainty Principle, conditionally on the existence of damping functions in Y. As in [BouDya] the argument is based on an induction on scales, where at each step a small gain is achieved by means of Corollary 4.2. Recall that $\alpha \in (0,1)$ is the parameter from the damping function.

Theorem 5.1. *Let $X \subset [-1,1]^d \subset \mathbb{R}^d$ be porous at scale $L \geqslant 3$ with depth n, for all $n \geqslant 0$ with $L^{n+1} \leqslant N$. Suppose $Y \subset [-N,N]^d$ is such that for all $n \geqslant 0$ with $L^{n+1} \leqslant N$ one has that for all*

$$\eta \in [-NL^{-n} - 3, NL^{-n} + 3]^d$$

the set

$$L^{-n}Y + [-4,4]^d + \eta \tag{5.1}$$

admits a damping function with parameters $c_1 = (2L)^{-1} \in (0, \frac{1}{2}]$, and $c_2, c_3 \in (0,1)$. Assume $0 < c_3 < c_3^*(d)$ as in Corollary 4.2. Then there exists $\beta = \beta(L, c_2, c_3, d, \alpha) > 0$ and $\tilde{C} = \tilde{C}(L, c_2, c_3, d, \alpha) > 0$ so that any $f \in L^2(\mathbb{R}^d)$ with $\mathrm{supp}(\widehat{f}\,) \subset Y$ satisfies

$$\|f\|_{L^2(X)} \leqslant \tilde{C} N^{-\beta} \|f\|_{L^2(\mathbb{R}^d)} \tag{5.2}$$

for all $N \geqslant N_0(L, c_2, c_3, d, \alpha)$.

Proof. We pick a nonnegative Schwarz function ϕ in $\mathbb{R}^d$ with $\mathrm{supp}(\widehat{\phi}) \subset [-1,1]^d$ and $\widehat{\phi}(0) = 1$. With $T \in \mathbb{N}$ to be determined, we set $\psi(x) := L^{Td}\phi(L^T x)$ so that $\mathrm{supp}(\widehat{\psi}) \subset [-L^T, L^T]^d$. Let

$$\begin{aligned} \mathcal{S}_n &:= \bigcup_{\substack{Q \in \mathcal{C}_n \\ Q \cap X \neq \varnothing}} Q \\ \mathcal{S}_n^* &:= \mathcal{S}_n + [-L^{-n}/10, L^{-n}/10]^d \end{aligned} \tag{5.3}$$

and define $\Psi_n := \psi_n * \mathbb{1}_{\mathcal{S}_{n+1}^*}$ where $\psi_k(x) := L^{kd}\psi(L^k x)$. There exists a constant C_ϕ depending only on ϕ such that for any $n \geqslant 0$,

$$\Psi_n \geqslant \Big(1 - \frac{C_\phi}{L^{T-1}}\Big)\mathbb{1}_X.$$

Thus, for all $m \geqslant 1$,

$$\prod_{n=0}^{m-1} \Psi_n \geqslant \Big(1 - \frac{C_\phi}{L^{T-1}}\Big)^m \mathbb{1}_X. \tag{5.4}$$

Moreover, if $Q \in \mathcal{C}_{n+1}$ with $n \geqslant 0$ satisfies $Q \cap X = \varnothing$, denote by Q^* the cube with the same center as Q, but half the side length, i.e., of side length $L^{-(n+1)}/2$. Denote the collection of all such cubes Q^* by U_{n+1}. By the definitions of S^*_{n+1} and Q^*, we clearly have

$$S^*_{n+1} \cap \Big(U_{n+1} + [-L^{-(n+1)}/10, L^{-(n+1)}/10]^d\Big) = \varnothing.$$

Then for $x \in U_{n+1}$, and a constant c_ϕ that depends on ϕ only, we have

$$\begin{aligned} \Psi_n(x) &= \int_{\mathbb{R}^d} \phi_{n+T}(x) \mathbb{1}_{S^*_{n+1}}(x-y)\,dy \\ &= \int_{\mathbb{R}^d} \phi(y) \mathbb{1}_{S^*_{n+1}}(x - L^{-(n+T)}y)\,dy \\ &\leqslant \int_{\mathbb{R}^d \setminus [-L^{T-1}/10,\ L^{T-1}/10]^d} \phi(y)\,dy \leqslant \frac{c_\phi}{L^{T-1}}, \end{aligned} \tag{5.5}$$

uniformly in n.

Let $f \in L^2(\mathbb{R}^d)$ with $\operatorname{supp}(\widehat{f}\,) \subset Y$. Then for $m \geqslant 1$,

$$f_m := \prod_{n=0}^{m-1} \Psi_{nT} \cdot f$$

satisfies

$$\begin{aligned} \operatorname{supp}(\widehat{f_m}) &\subset Y + \sum_{n=0}^{m-1} \operatorname{supp}(\widehat{\psi_{nT}}) \\ &\subset Y + \sum_{n=0}^{m-1} [-L^{(n+1)T}, L^{(n+1)T}]^d = Y + \ell_m[-1,1]^d \end{aligned} \tag{5.6}$$

where

$$\ell_m := L^T \frac{L^{mT}-1}{L^T-1}.$$

One has $f_{m+1} = \Psi_{mT} f_m$ for all $m \geqslant 0$ with $f_0 = f$. We claim that there exists $\gamma_0 = \gamma_0(L, d, c_1, c_2, c_3) \in (0,1)$ with

$$\|f_{m+1}\|_{L^2([-1,1]^d)} \leqslant (1-\gamma_0)\|f_m\|_{L^2([-1,1]^d)} \tag{5.7}$$

Define $g_m(x) := f_m(L^{mT}x)$. Then

$$\begin{aligned} \operatorname{supp}(\widehat{g_m}) &\subset L^{-mT}Y + \ell_m L^{-mT}[-1,1]^d \\ &\subset L^{-mT}Y + [-2,2]^d, \end{aligned} \tag{5.8}$$

where we used

$$\ell_m L^{-mT} \leqslant \frac{L^T}{L^T-1} \leqslant 2.$$

In particular, assuming also that $L^{mT} \leqslant N$,

$$\operatorname{supp}(\widehat{g_m}) \subset [-NL^{-mT}, NL^{-mT}]^d + [-2,2]^d = [-NL^{-mT}-2, NL^{-mT}+2]^d,$$

where $NL^{-mT}+2$ will be our parameter α_1 in Corollary 4.2.

Under this rescaling, the cubes in $\mathcal{C}_{mT}$ turn into unit cubes. Assuming further $L^{mT+1} \leqslant N$, the porosity condition at scale L with depth mT ensures that we always have a "missing cube" of side length L^{-1} inside. In view of our definition of Q^*, we only use the concentric cube of half that side length. In view of the conditions on Y in the theorem we can apply Corollary 4.2 to g_m to obtain the following: with all norms being taken locally on $[-1,1]^d$, and with U_{mT+1}, the missing cubes of the next generation as above,

$$\begin{aligned}\|\Psi_{mT} f_m\|_2^2 &\leqslant \|\Psi_{mT}\|_\infty^2 \|f_m\|_{L^2([-1,1]^d\setminus U_{mT+1})}^2 + \|\Psi_{mT}\|_{L^\infty(U_{mT+1})}^2 \|f_m\|_{L^2(U_{mT+1})}^2 \\ &\leqslant \|f_m\|_{L^2([-1,1]^d\setminus U_{mT+1})}^2 + \|\Psi_{mT}\|_{L^\infty(U_{mT+1})}^2 \|f_m\|_{L^2(U_{mT+1})}^2 \\ &= \|f_m\|_{L^2([-1,1]^d)}^2 - (1 - \|\Psi_{mT}\|_{L^\infty(U_{mT+1})}^2) \|f_m\|_{L^2(U_{mT+1})}^2 \\ &\leqslant \Big(1 - C_*^{-2}\big(1 - c_\phi^2/L^{2(T-1)}\big)\Big)\|f_m\|_2^2\end{aligned} \tag{5.9}$$

To obtain this estimate, we used that

$$\|\Psi_{mT}\|_\infty \leqslant 1, \quad \|\Psi_{mT}\|_{L^\infty(U_{mT+1})} \leqslant \frac{c_\phi}{L^{T-1}},$$

and

$$\|f_m\|_{L^2(U_{mT+1})} \geqslant C_*^{-1}\|f_m\|_2^2,$$

with $C_* = C_*(d, L, c_2, c_3, \alpha)$ by Corollary 4.2. Choosing

$$\gamma_0(T) := \frac{1 - c_\phi^2/L^{2(T-1)}}{2C_*^2}, \tag{5.10}$$

and using $(1-x)^{1/2} \leqslant 1 - x/2$ for $0 \leqslant x \leqslant 1$, we have

$$\Big(1 - C_*^{-2}\big(1 - c_\phi^2/L^{2(T-1)}\big)\Big)^{\frac{1}{2}} \leqslant 1 - \gamma_0(T).$$

This establishes the claim (5.7).

Applying (5.7) iteratively and using (5.4), we obtain

$$\begin{aligned}\|f\|_{L^2(X)} &\leqslant \Big(1 - \frac{C_\phi}{L^{T-1}}\Big)^{-(m+1)} \|\prod_{n=0}^m \Psi_n f\|_{L^2(X)} \\ &\leqslant \Big[\Big(1 - \frac{C_\phi}{L^{T-1}}\Big)^{-1}(1 - \gamma_0(T))\Big]^{m+1} \|f\|_2 \\ &\leqslant \Big(1 - \frac{\gamma_0(T)}{2}\Big)^{m+1}\|f\|_2.\end{aligned} \tag{5.11}$$

In the last inequality we used

$$1 - \gamma_0(T) \leqslant 1 - \frac{\gamma_0(T)}{2} - \frac{C_\phi}{L^{T-1}} \leqslant \Big(1 - \frac{\gamma_0(T)}{2}\Big)\Big(1 - \frac{C_\phi}{L^{T-1}}\Big),$$

which requires

$$\begin{aligned}&L^{T-1} - \frac{c_\phi^2}{L^{T-1}} \geqslant 4C_\phi C_*^2, \quad \text{or} \\ &T \geqslant T_0(d, L, c_2, c_3, \alpha) := \left\lceil \frac{\log(2C_\phi C_*^2 + \sqrt{4C_\phi^2 C_*^4 + c_\phi^2})}{\log L} \right\rceil.\end{aligned} \tag{5.12}$$

Finally, for any $T \geqslant T_0$, taking $m \in \mathbb{N}$ be such that $L^{mT+1} \leqslant N < L^{(m+1)T+1}$, (5.11) yields (5.2) with

$$\beta = -\frac{\log(1-\gamma_0(T)/2)}{T\log L}, \tag{5.13}$$

and

$$\tilde{C} = \left(1 - \frac{\gamma_0(T)}{2}\right)^{-1/T}. \tag{5.14}$$

as claimed. In the current theorem, we could simply choose $T = T_0$. The flexibility of choosing T will simplify our computations in our proof of Theorem 1.2. □

6. Geometry of Y and damping functions

6.1. **Regular sets.** We will call a set $I = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d]$ of equal side length a *d-dimensional cube* in $\mathbb{R}^d$, we denote its side length by r_I.

Recall the notion of δ-regularity from [BouDya, Definition 1.1], below is a d-dimensional analogy.

Definition 6.1. Suppose $X \subset \mathbb{R}^d, X \neq \varnothing$ is closed, and $0 < \delta < d$, $C_R \geqslant 1$, $0 \leqslant \alpha_0 \leqslant \alpha_1 \leqslant \infty$. Then X is δ-regular on scales α_0 to α_1, with constant C_R, if there exists a Borel measure μ_X with the following properties:

- μ_X is supported on X
- $\mu_X(I) \leqslant C_R r_I^\delta$ for each d-dimensional cube I of side length $\alpha_0 \leqslant r_I \leqslant \alpha_1$
- $\mu_X(I) \geqslant C_R^{-1} r_I^\delta$ for each d-dimensional cube $I \subset \mathbb{R}^d$, centered at a point in X and of side length $\alpha_0 \leqslant r_I \leqslant \alpha_1$

See [BouDya, Section 2.2] for the geometry of such sets in $\mathbb{R}$. Loosely speaking, they behave like δ-dimensional fractal sets. The properties of δ-regular sets carry over to higher dimensions. We include some properties in Appendix A.

6.2. **Geometry of Y and damping functions.** Bourgain and Dyatlov observed that δ-regular sets on $\mathbb{R}$ admit damping functions as in Definition 4.1 above with $\alpha = (1 + \delta)/2$. They obtained these functions as a consequence of the Beurling-Malliavin theorem [BeuMal]. However, one does not need the full strength of this theorem. To be more precise, in place of the original Beurling-Malliavin condition $\|(\log\omega)'\|_\infty < \infty$, with ω the weight, a much easier proof is possible (via outer functions) if we assume instead that $\|(H\log\omega)'\|_\infty \ll 1$ where H is the Hilbert transform on $\mathbb{R}$, see [KhaMasNaz, Section 1.14, Theorem 1]. By means of this technique, Jin and Zhang [JinZha, Lemma 4.1] proved the following quantitative result on damping functions.

Lemma 6.1. *Let $S \geqslant 1$ be a constant. Let $Y \subset [-SN, SN]$ be δ_1-regular on scales 2 to N, with constant C_R, $0 < \delta_1 < 1$. For any $0 < c_1 < 1$, Y admits a damping function with $\alpha = (1 + \delta_1)/2$ and parameters c_1,*

$$c_2 = \iota\, c_1^6, \quad c_3 = \iota\, c_1 C_R^{-2} \delta_1(1 - \delta_1), \tag{6.1}$$

where $\iota > 0$ is some small constant that depends on S. Instead of the pointwise global decay of $\langle\xi\rangle^{-1}$ in Definition 4.1, we have

$$|\widehat{\psi}(\xi)| \leqslant \exp(-c_3\,\langle\xi\rangle^{1/2}) \quad \forall\, \xi \in \mathbb{R} \tag{6.2}$$

In this paper we need a slightly different version, where we have pointwise lower bound of $|\widehat{\psi}(\xi)|$ on $[-3/4, 3/4]$. The advantage of a pointwise lower bound over a L^2 bound is that it leads to a lower bound of the product of several $\widehat{\psi}$'s. Let us also note that in Lemma 4.1 of [JinZha], $S = 1$. But it is clear from their proof that it works for any $S \geqslant 1$. We will briefly discuss the changes of constants caused by S in Appendix B. We need the extra factor S in our proof of Lemma 6.3.

Lemma 6.2. *Let $S \geqslant 1$ be a constant. Assume that $Y \subset [-SN, SN]$ is a δ_1-regular set with constant C_R on scales 2 to N and $\delta_1 \in (0,1)$. Fix $0 < c_1 < 1$, then there exists a function $\psi \in L^2(\mathbb{R})$ such that*

$$\begin{aligned}
&\operatorname{supp}\psi \subset \left[-\frac{c_1}{10}, \frac{c_1}{10}\right],\\
&|\widehat{\psi}(\xi)| \leqslant \exp(-c_3\langle\xi\rangle^{1/2}), \quad \forall \xi \in \mathbb{R},\\
&|\widehat{\psi}(\xi)| \leqslant \exp(-c_3\Theta(|\xi|)|\xi|), \quad \forall \xi \in Y, \ |\xi| \geqslant 10,
\end{aligned}$$

and

$$|\widehat{\psi}(\xi)| \geqslant c_2, \quad \forall \xi \in [-3/4, 3/4], \tag{6.3}$$

with

$$\alpha = (1+\delta_1)/2, \quad c_2 = \iota\, c_1^{10}, \quad c_3 = \iota\, c_1 C_R^{-2}\delta_1(1-\delta_1),$$

where $\iota > 0$ is some small constant that depends on S.

We include the proof of Lemma 6.2 in Appendix B.

In higher dimensions, we reduce ourselves to this one-dimensional setting by taking finite unions of products. For simplicity, we restrict ourselves to two dimensions, although the exact analogue can be done in any finite dimension.

Definition 6.2. Pick some $\varepsilon_0 \in (0,1)$ and let $Y \subset \mathbb{R}^2$ be of the form

$$\begin{aligned}
&Y \subset \bigcup_{j=1}^{m} Y_j,\\
&Y_j = \{\xi_1\vec{e}_{j,1} + \xi_2\vec{e}_{j,2} \ : \ \xi_i \in Y_{j,i}, \ i = 1, 2\}
\end{aligned} \tag{6.4}$$

Here $\vec{e}_{j,i} \in \mathbb{S}^1$ with $|\vec{e}_{j,1} \cdot \vec{e}_{j,2}| < 1 - \varepsilon_0$ for all $1 \leqslant j \leqslant m$, and $Y_{j,i}$ are δ_1-regular on scales α_0 to α_1 with constant C_R, where $0 < \delta_1 < 1$. In that case Y is called *admissible on scales α_0 to α_1* with parameters $\delta_1, C_R, \varepsilon_0, m$. In general dimensions, we require that $\vec{e}_{j,i}$ are unit vectors with $|\det(\vec{e}_{j,1}, \ldots, \vec{e}_{j,d})| \geqslant \varepsilon_0$, cf. (1.3).

Throughout, we will freeze ε_0 and constants are allowed to depend on it. The admissible sets on scale 2 to N that are contained in $[-N, N]^d$ carry damping functions.

We note that for our proof of Theorem 1.2, we only need $m = 1$. We give a construction with arbitrary $m \geqslant 1$ here, since the construction itself may be of independent interest.

Lemma 6.3. *Let $Y \subset [-N, N]^2$ be admissible on scales 2 to N as in Definition 6.2. Then Y admits a damping function with parameters c_1,*

$$\begin{aligned}
c_2 &= \iota^{2m+4} c_1^{20m+4}\, m^{-20m}\, C_R^{-8}(\delta_1(1-\delta_1))^4\\
c_3 &= \iota\, c_1\, m^{-1} C_R^{-2}\delta_1(1-\delta_1)
\end{aligned}$$

where $\iota > 0$ is a small constant that depends on ε_0.

Remark 6.4. *For general dimension d, we can take*

$$\begin{aligned} c_2 &= \iota^m c_1^{(10m+2)d}\, m^{-10md}\, C_R^{-4d}(\delta_1(1-\delta_1))^{2d} \\ c_3 &= \iota\, c_1\, m^{-1} C_R^{-2}\delta_1(1-\delta_1) \end{aligned}$$

where $\iota > 0$ is a small constant that depends on ε_0 and d.

Proof. Let $\psi_{j,i}$ be the damping function associated with $Y_{j,i} \subset [-SN, SN]$, with $S = S(\varepsilon_0) \geqslant 1$, via Lemma 6.2 with parameters $\tilde{c}_1 := \varepsilon_1 c_1 m^{-1}$ where ε_1 is a small parameter depending on ε_0, and c_2, c_3 as given by Lemma 6.2, but in terms of $\tilde{c}_1$. I.e.,

$$c_2 = \iota\, \varepsilon_1^{10}\, c_1^{10} m^{-10}, \quad c_3 = c_1\, m^{-1} \iota\, \varepsilon_1\, C_R^{-2}\delta_1(1-\delta_1),$$

where ι depends ε_0. We will absorb the constant ε_1 into ι. In the following we will also allow ι to change its value from line to line, as long as it only depends on ε_0.

Denote the coordinates associated with the basis $\vec{e}_{j,1}$, $\vec{e}_{j,2}$ by $(\xi_{j,1}, \xi_{j,2})$. We set, with $\xi \in \mathbb{R}^2$,

$$\widehat{\psi}(\xi) := \prod_{j=1}^{m} \widehat{\psi_j}(\xi), \qquad \widehat{\psi_j}(\xi) := \widehat{\psi_{j,1}}(\xi_{j,1})\widehat{\psi_{j,2}}(\xi_{j,2})$$

Then

$$\begin{aligned} |\widehat{\psi}_j(\xi)| &\leqslant \exp(-c_3 \langle \xi_{j,1} \rangle^{\frac{1}{2}}) \exp(-c_3 \langle \xi_{j,2} \rangle^{\frac{1}{2}}) \\ &\leqslant \exp(-c_3 \langle \xi \rangle^{\frac{1}{2}}), \end{aligned} \tag{6.5}$$

where c_3, more precisely, ι, can change its value in the last line depending on ε_0. Taking products gives

$$|\widehat{\psi}(\xi)| \leqslant \exp(-mc_3 \langle \xi \rangle^{\frac{1}{2}}) = \exp(-c_1 \nu \langle \xi \rangle^{\frac{1}{2}}), \quad \nu = \iota\, C_R^{-2}\delta_1(1-\delta_1) \tag{6.6}$$

In particular, $\psi \in L^2(\mathbb{R}^2)$ as well as $\psi_j \in L^2(\mathbb{R}^2)$. Since ψ_j are also compactly supported functions, $\psi_j \in L^1(\mathbb{R}^2)$. Hence in the sense of L^1 functions,

$$\psi = \ast_{j=1}^{m}\, \psi_j$$

whence

$$\operatorname{supp}(\psi) \subset \sum_{j=1}^{m} \operatorname{supp}(\psi_j) \subset \sum_{j=1}^{m} [-c_1 m^{-1}, c_1 m^{-1}]^2 \subset [-c_1, c_1]^2,$$

where we used that each $\psi_{j,i}$ is a damping function with $\tilde{c}_1 = \varepsilon_1 c_1 m^{-1}$. Next, if $\xi \in Y_j$, then

$$\begin{aligned} |\widehat{\psi}_j(\xi)| &\leqslant \exp\big(-c_3 \Theta(|\xi_{j,1}|)|\xi_{j,1}|\big) \exp\big(-c_3 \Theta(|\xi_{j,2}|)|\xi_{j,2}|\big) \\ &\leqslant \exp\big(-c_3 \Theta(|\xi|_1)|\xi|_1\big) \end{aligned}$$

where again ι is allowed to change in the second line. Since Y is covered by the union of Y_j, we have

$$|\widehat{\psi}(\xi)| \leqslant \exp\big(-c_3 \Theta(|\xi|_1)|\xi|_1\big) \quad \forall\, \xi \in Y. \tag{6.7}$$

Finally, from (6.3), for each $1 \leqslant j \leqslant m$,

$$|\widehat{\psi_j}(\xi)| \geqslant c_2^2, \quad \forall \xi_{j,1}, \xi_{j,2} \in [-3/4, 3/4].$$

Hence,

$$\|\widehat{\psi}\|_{L^2([-1,1]^2)} \geqslant c_2^{2m} |E|^{\frac{1}{2}},$$

where E is the subset of $[-1,1]^2$ where all conditions $\xi_{j,i} \in [-3/4, 3/4]$, $\quad i = 1, 2$, $1 \leqslant j \leqslant m$, are met. Clearly, $|E|^{\frac{1}{2}}$ is some number depending on ε_0. It follows that

$$\|\widehat{\psi}\|_{L^2([-1,1]^2)} \geqslant \iota^{2m}\, c_1^{20m} m^{-20m} \tag{6.8}$$

where ι depends on ε_0.

We required $|\widehat{\psi}(\xi)| \leqslant \langle\xi\rangle^{-2}$ in our definition of damping function, see Definition 4.1. Since for any $0 < \rho < 1$

$$\exp\big(-\rho\langle\xi\rangle^{\frac{1}{2}}\big) \leqslant 5\rho^{-4}\langle\xi\rangle^{-2}$$

It follows from (6.6) that $\widetilde{\psi} := \frac{1}{5}(c_1\nu)^4\psi$ is a damping function in the sense of the definition. Since $\frac{1}{5}(c_1\nu)^4 \leqslant 1$, the decay (6.7) remains intact, as does the support condition. However, (6.8) needs to be modified:

$$\begin{aligned}\|\widehat{\widetilde{\psi}}\|_{L^2([-1,1]^2)} &\geqslant \frac{1}{5}(c_1\nu)^4\iota^{2m}\, c_1^{20m} m^{-20m}\\ &= \frac{1}{5}\iota^{2m+4} c_1^{20m+4} m^{-20m} C_R^{-8}(\delta_1(1-\delta_1))^4.\end{aligned}$$

Absorbing the $\frac{1}{5}$ into ι, the lemma is proved. $\square$

Finally, we need to check that Y remains admissible if it is transformed by the similarities in (5.1).

Lemma 6.5. *Let $Y \subset [-N,N]^d$ with $N \geqslant 10$ be admissible on scales 2 to N with parameters $\delta_1, C_R, \varepsilon_0, m$. Let $L \geqslant 4$ be an integer. Then for all integers $n \geqslant 0$ with $L^{n+1} \leqslant N$ and for all*

$$\eta \in [-NL^{-n} - 3, NL^{-n} + 3]^d$$

the set

$$L^{-n}Y + [-4,4]^d + \eta \subset [-(2NL^{-n} + 7), 2NL^{-n} + 7]^d$$

is admissible at scale $S(2NL^{-n} + 7)$ with parameters $\delta_1, 576S^2C_R, \varepsilon_0, m$, where $S = S(\varepsilon_0, d) \geqslant 1$.

Proof. First,

$$L^{-n}Y + [-4,4]^d + \eta \subset [-2NL^{-n} - 7, 2NL^{-n} + 7]^d$$

for all η as above. Second, by (6.4),

$$L^{-n}Y + [-4,4]^d + \eta \subset \bigcup_{j=1}^{m}\Big(L^{-n}Y_j + [-4,4]^d + \eta\Big),$$

$$L^{-n}Y_j = \Big\{\sum_{k=1}^{d}\xi_k\vec{e}_{j,k} \;:\; \xi_k \in L^{-n}Y_{j,k},\ k = 1, 2, ..., d.\Big\},$$

and

$$L^{-n}Y_j + [-4,4]^d + \eta \subset \Big\{\sum_{k=1}^{d}\xi_k\vec{e}_{j,k} : \xi_k \in L^{-n}Y_{j,k} + [-4S, 4S] + \eta_{j,k},\ k = 1, 2, ..., d.\Big\},$$

where $S = S(\varepsilon_0, d) \geqslant 1$ and $|\eta_{j,k}| \leqslant S(NL^{-n} + 3)$. By Lemmas 2.1, 2.2, 2.3 in [BouDya], see also Lemmas A.2, A.3, A.4 with $d = 1$, the sets

$$L^{-n}Y_{j,k} + [-4S, 4S] + \eta_{j,k} \subset [-S(2NL^{-n} + 7), S(2NL^{-n} + 7)]$$

are δ_1-regular with constant $576S^2C_R$ on scales 2 to $S(2NL^{-n}+7)$. Indeed, for $n \geqslant 1$, Lemma A.2 implies that $L^{-n}Y_{j,k}$ is δ_1-regular on scales $2L^{-n} \leqslant 1/2$ to $L^{-n}N$ with constant C_R. Lemma A.4 implies that

$$L^{-n}Y_{j,k} + [-4S, 4S] = L^{-n}Y_{j,k} + 8S[-1/2, 1/2]$$

is δ_1-regular on scales 1 to $L^{-n}N$ with constant $32SC_R$. Lemma A.3 allows us to increase the upper scale from $L^{-n}N$ to $9SL^{-n}N \geqslant S(2L^{-n}N+7)$, with changing the constant from $32SC_R$ to $576S^2C_R$. Note that shifting a set does not change its δ_1-regularity, hence $L^{-n}Y_{j,k}+[-4S,4S]+\eta_{j,k}$ is δ_1-regular with constant $576S^2C_R$. The proof for $n=0$ is similar.

The lemma now follows from Definition 6.2. □

6.3. **Proof of Theorem 1.2.**

Proof. The proof of Theorem 1.2 is now a corollary to Theorem 5.1 and the considerations in this section, with $m=1$. We will keep track of various constants in order to obtain the effective exponent β.

First, let

$$L := \lceil (2^{\frac{d}{2}}\sqrt{2d+1}C_R)^{\frac{2}{d-\delta}} \rceil \geqslant 4,$$

be as in (A.3). Lemma A.7 implies that for all $n \geqslant 0$ with $L^{n+1} \leqslant N$, X is porous at scale L with depth n. This verifies the porosity condition on X in Theorem 5.1.

Combining Lemma 6.3, more specifically Remark 6.4, with Lemma 6.5, we obtain that for any $n \in \mathbb{N}$ such that $L^{n+1} \leqslant N$, and for all $\eta \in [-L^{-n}N-3, L^{-n}N+3]^d$, the set

$$L^{-n}Y + [-4,4]^d + \eta$$

admits a damping function with parameters c_1,

$$\begin{aligned} c_2 &= \iota c_1^{12d}(576S^2C_R)^{-4d}(\delta_1(1-\delta_1))^{2d}, \\ c_3 &= \iota c_1(576S^2C_R)^{-2}\delta_1(1-\delta_1), \end{aligned}$$

where ι and S are constants depending on ε_0. We absorb the constant S into ι, and allow ι to depend on d as well. Hence we can simply write

$$\begin{aligned} c_2 &= \iota c_1^{12d}C_R^{-4d}(\delta_1(1-\delta_1))^{2d}, \\ c_3 &= \iota c_1 C_R^{-2}\delta_1(1-\delta_1). \end{aligned}$$

Note that this verifies the condition on Y in Theorem 5.1.

Before applying Theorem 5.1, let us first determine the constant C_* in Corollary 4.2 with c_1, c_2, c_3 defined above. Recall that

$$C_* = e^{\frac{c_3\Theta(R_1)(R_1+2)}{2}},$$

with $\alpha = (1+\delta_1)/2$ and

$$R_1 = \max \begin{cases} \exp\Big[\Big(\frac{C_R^2}{\iota c_1\delta_1(1-\delta_1)}\Big)^{\frac{2}{1-\delta_1}}\Big] \\ \exp\Big(4^{\frac{2}{1-\delta_1}}\Big) \\ \Big(\frac{C_R^2(-\log c_1)^d}{\iota c_1\delta_1(1-\delta_1)}\Big)^8 \\ \Big[4\log\Big(\frac{C_R^{8d}}{\iota c_1^{24d}(\delta_1(1-\delta_1))^{4d}}\Big)\Big]^2 \\ (8d)^4 \\ \frac{C_R^4}{\iota c_1^2(\delta_1(1-\delta_1))^2} \end{cases} \tag{6.9}$$

be as in (4.14), in which we absorb all the d-dependent constants into ι.

Now we can apply Theorem 5.1 with

$$c_1 = (2L)^{-1} = \Big(2\big\lceil (2^{\frac{d}{2}}\sqrt{2d+1}C_R)^{\frac{2}{d-\delta}}\big\rceil\Big)^{-1}.$$

We need to trace out the constant β.

Plugging c_1 into (6.9), and making ι smaller if necessary (depending only on d and ε_0), we have

$$R_1 \leqslant \exp\Big[\Big(\frac{(C_R^2/\iota)^{\frac{2d-2\delta+2}{d-\delta}}}{\delta_1(1-\delta_1)}\Big)^{\frac{2}{1-\delta_1}}\Big] =: R_2.$$

This implies

$$C_* = \exp\Big(c_1 C_R^{-2}\delta_1(1-\delta_1)\Theta(R_1)(R_1+2)\Big) \leqslant \exp(R_2).$$

Recall T_0 as in (5.12) and γ_0 as in (5.10). We compute that,

$$\begin{aligned} T_0 = \left\lceil \frac{\log(2C_\phi C_*^2 + \sqrt{4C_\phi^2 C_*^4 + c_\phi^2})}{\log L} \right\rceil &\leqslant \frac{2\log C_* + \log(5C_\phi)}{\log L} \\ &\leqslant \frac{2R_2 + \log(5C_\phi)}{\log L} =: T_1, \end{aligned} \tag{6.10}$$

and

$$\gamma_0(T_1) = \frac{1 - c_\phi^2/L^{2(T_1-1)}}{2C_*^2} \geqslant \frac{1}{4C_*^2} \geqslant \frac{1}{4}\exp(-2R_2). \tag{6.11}$$

In both inequalities above, we used $C_* \leqslant \exp(R_2)$.

Recall β as in (5.13). Use that $-\log(1-x) \geqslant x$ for $x<1$, we have

$$\beta = -\frac{\log(1-\gamma_0(T_1)/2)}{T_1\log L} \geqslant \frac{\gamma_0(T_1)}{2T_1\log L}.$$

Combining this with the estimates of T_1 and $\gamma_0(T_1)$ as in (6.10) and (6.11), we have

$$\beta \geqslant \exp\Big\{-\exp\Big[\Big(\frac{(C_R^2/\iota)^{\frac{2d-2\delta+2}{d-\delta}}}{\delta_1(1-\delta_1)}\Big)^{\frac{2}{1-\delta_1}}\Big]\Big\},$$

with ι being a small constant depending on ε_0 and d. This finishes the proof. $\square$

Corollary 1.3 follows from Theorem 1.2 by the triangle inequality.

Remark 6.6. *If we try to combine the construction of a damping function for m covers as in Lemma 6.3, with Theorem 5.1, we could allow m to grow in N like* $\log\log\log N$. *This is worse than the power law growth obtained via the triangle inequality.*

6.4. **Distortion of Y by diffeomorphisms.**

Let $\mathcal{F}_\hbar$ be the unitary semiclassical Fourier transform on $L^2(\mathbb{R}^d)$ defined by

$$\mathcal{F}_\hbar f(\xi) = \hbar^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi/\hbar} f(x)\, dx = \hbar^{-\frac{d}{2}} \hat{f}(\xi/\hbar).$$

We will use the following proposition which roughly says that the intersection of an admissible set with a cube is still admissible. We only work with admissible sets with $m = 1$ throughout this section.

Proposition 6.7. *Let $Y \subset \mathbb{R}^d$ be an admissible set on scales N^{-1} to 1 with parameters $\delta_1, C_R, \varepsilon_0$. Let $Q \subset \mathbb{R}^d$ be a cube of side length $r_Q \leqslant r_0$. Then*

$$Y \cap Q \subseteq \bigcup_{j=1}^{C(\varepsilon_0,d,r_0)} W_j,$$

where each W_j is contained in a cube of side length $C(\varepsilon_0, d)$, and is admissible on scales N^{-1} to 1 with parameters $\delta_1, (4C_R)^{\frac{2}{1-\delta_1}} C_R, \varepsilon_0$.

Proof. Let $Y = \{\sum_{k=1}^d \xi_k \vec{e}_k,\ \xi_k \in Y_k\}$, where $\vec{e}_k \in \mathbb{S}^1$ and $|\det(\vec{e}_1, ..., \vec{e}_d)| \geqslant \varepsilon_0$. We cover Q by the smallest parallelepiped $\widetilde{Q}$, whose edges are determined by $\vec{e}_1, ..., \vec{e}_d$, that contain Q. We can write $\widetilde{Q} = \{\sum_{k=1}^d \xi_k \vec{e}_k,\ \xi_k \in \widetilde{Q}_k\}$.

By Lemma A.1, there exist disjoint intervals $\mathcal{J}_k$ such that

$$Y_k = \bigcup_{J_{k,\ell}\in\mathcal{J}_k} (Y_k \cap J_{k,\ell});$$

$$(4C_R)^{-\frac{2}{1-\delta_1}} \leqslant |J_{k,\ell}| \leqslant 1 \quad \text{for all } J_{k,\ell} \in \mathcal{J}_k,$$

where $(Y_k \cap J_{k,\ell})$'s are δ_1-regular sets with constant $\widetilde{C}_R = (4C_R)^{\frac{2}{1-\delta_1}} C_R$ on scales N^{-1} to 1. For any $\underline{\ell} \in \mathbb{N}^d$, let $Y_{\underline{\ell}} := \{\sum_{k=1}^d \xi_k \vec{e}_k,\ \xi_k \in Y_k \cap J_{k,\ell_k}\}$. Hence $Y_{\underline{\ell}}$ is admissible on scales N^{-1} to 1 with parameters $\delta_1, \widetilde{C}_R, \varepsilon_0$. Furthermore, $Y_{\underline{\ell}}$ is contained in a cube of side length $C(\varepsilon_0, d)$. Finally note that $\widetilde{Q}_k$ intersects at most with finitely many $J_{k,\ell}$'s, and this number depends only on ε_0, d and r_0. $\square$

In this section we prove Theorem 1.4. We need to show that Theorem 1.2 remains valid if an admissible set Y is distorted by a diffeomorphism $\Phi_N(x)$ from the cube $[-N,N]^d \to [-N,N]^d$, cf. (1.5). The argument is related to Section 4 of [BouDya]. Thus, let $Y = \Phi_N(\widetilde{Y})$ where $\widetilde{Y} \subset [-N,N]^d$ is an admissible set with constants C_R, ε_0 on scales 1 to N. Suppose $f \in L^2(\mathbb{R}^d)$ with $\operatorname{supp}(\widehat{f}) \subset Y$ and set $\widehat{g} := \widehat{f} \circ \Phi_N$ so that $\operatorname{supp}(\widehat{g}) \subset \widetilde{Y}$. Furthermore,

$$\begin{aligned} f(x) &= \int_{[-N,N]^d} e^{2\pi i x\cdot\xi}\, \widehat{f}(\xi)\, d\xi = \int_{[-N,N]^d} e^{2\pi i x\cdot\xi}\, \widehat{g}(\Phi_N^{-1}(\xi))\, d\xi \\ &= \int_{[-N,N]^d} e^{2\pi i x\cdot\Phi_N(\eta)}\, \widehat{g}(\eta)\, |\det(D\Phi_N(\eta))|\, d\eta \end{aligned} \tag{6.12}$$

We claim that for some $\beta > 0$ and $C > 0$ depending on all the same parameters in Theorem 1.2 as well as on D_0,

$$\Big\| \int_{[-N,N]^d} e^{2\pi i x\cdot \Phi_N(\eta)}\, \widehat{h}(\eta)\, d\eta \Big\|_{L^2(X)} \leqslant CN^{-\beta}\|h\|_2 \tag{6.13}$$

for all $h \in L^2$ with $\mathrm{supp}(\widehat{h}) \subset \widetilde{Y}$, in which $\widetilde{Y} \subset [-N,N]^d$ is an admissible set with constants C_R, ε_0 on scales 1 to N. Setting $\widehat{h}(\eta) := \widehat{g}(\eta)\ |\det(D\Phi_N(\eta))|$, we conclude from (6.13) that

$$\|f\|_{L^2(X)} \leqslant CN^{-\beta}\|\widehat{h}\|_2 \leqslant CN^{-\beta}\|\widehat{f}\|_2 = CN^{-\beta}\|f\|_2$$

with possibly a different constant. So it remains to prove the claim (6.13). We will prove it from another statement, namely

$$\Big\| \int_{[-N,N]^d} e^{2\pi i x\cdot \Phi_N(\eta)}\, \mathbb{1}_{\widetilde{Y}}(\eta)\, h(\eta)\, d\eta \Big\|_{L^2(X)} \leqslant CN^{-\beta}\|h\|_2 \tag{6.14}$$

for all $h \in L^2$. Notice that by Plancherel we could remove the Fourier transform from h.

To prove (6.14), divide $[-N,N]^d = \bigcup_k Q_k$ into congruent cubes of side length L_N with $\frac{1}{2}\sqrt{N} \leqslant L_N \leqslant \sqrt{N}$. Let $\{\chi_k\}_k$ be a partition of unity adapted to these cubes. With η_k being the center of Q_k,

$$\begin{aligned}
&\int_{[-N,N]^d} e^{2\pi i x\cdot \Phi_N(\eta)}\, \mathbb{1}_{\widetilde{Y}}(\eta)\, h(\eta)\, d\eta \\
&= \sum_k \int_{\mathbb{R}^d} e^{2\pi i x\cdot \Phi_N(\eta)}\, \chi_k(\eta)\, \mathbb{1}_{\widetilde{Y}}(\eta)\, h(\eta)\, d\eta \\
&= \sum_k \int_{\mathbb{R}^d} e^{2\pi i x\cdot (\Phi_N(\eta_k) + D\Phi_N(\eta_k)(\eta-\eta_k))}\, a_k(x,\eta)\, \mathbb{1}_{\widetilde{Y}}(\eta)\, h(\eta)\, d\eta =: \sum_k (T_k h)(x)
\end{aligned} \tag{6.15}$$

where

$$\begin{aligned}
a_k(x,\eta) &:= e^{2\pi i x\cdot R_k(\eta)}\, \chi_k(\eta) \\
R_k(\eta) &:= \int_0^1 (1-t)\langle D^2\Phi_N(\eta_k + t(\eta-\eta_k))(\eta-\eta_k), \eta-\eta_k\rangle\, dt
\end{aligned} \tag{6.16}$$

the latter being the error in the second order Taylor expansion (we are suppressing the parameter N here). Then

$$\begin{aligned}
\|R_k\|_{L^\infty(\mathrm{supp}\chi_k)} &\leqslant C = C(d, D_0) \\
\|\partial_x^\alpha\, a_k(x,\eta)\|_{L^\infty([-1,1]^d\times \mathrm{supp}\chi_k)} &\leqslant C(d, D_0, \alpha), \quad \mathrm{diam}\,\mathrm{supp}\chi_k \leqslant C\sqrt{N},
\end{aligned} \tag{6.17}$$

for every multi-index α. By Hörmander's variable coefficient Plancherel theorem,

$$\max_k \|T_k\|_{2\to 2} \leqslant C(d, D_0) \tag{6.18}$$

This follows by the usual T^*T argument:

$$\begin{aligned}
\|T_k h\|_2^2 &= \langle T_k^* T_k h, h\rangle \\
(T_k^* T_k h)(\eta') &= \int_{\mathbb{R}^d} K_k(\eta', \eta)\, h(\eta)\, d\eta \\
K_k(\eta', \eta) &= \int_{\mathbb{R}^d} e^{2\pi i x\cdot (\Phi_N(\eta) - \Phi_N(\eta'))}\, \mathbb{1}_{\widetilde{Y}}(\eta)\, \mathbb{1}_{\widetilde{Y}}(\eta')\, \chi_k(\eta)\chi_k(\eta')\, dx
\end{aligned} \tag{6.19}$$

Since $\|\Phi_N(\eta) - \Phi_N(\eta')\| \geqslant D_0^{-1}\|\eta - \eta'\|$ in the sense of Euclidean lengths, repeated integrations by parts yield the decay

$$|K_k(\eta', \eta)| \leqslant C(d, D_0)\langle \eta - \eta' \rangle^{-d-1}$$

whence (6.18) follows by Schur's test. In particular, $\|\mathbb{1}_X T_k\|_{2\to 2} \leqslant C$ with the same constant as in (6.18).

Next, we would like to show that $\mathbb{1}_X T_k$ and $\mathbb{1}_X T_\ell$ do not interact much for all cubes Q_k, Q_ℓ which are not nearest neighbors. In order to integrate by parts in x, cf. (6.19), we need to smooth out $\mathbb{1}_X$ at the correct scale. Define

$$X(N^{-\frac{1}{2}}) := X + [-N^{-\frac{1}{2}}, N^{-\frac{1}{2}}]^d.$$

By [DyaZah, Lemma 3.3] there exists a smooth ψ taking values in $[0,1]$ with $\psi = 1$ on X and with $\operatorname{supp}(\psi) \subset X(N^{-\frac{1}{2}})$, as well as so that

$$\|\partial_x^\alpha \psi\|_\infty \leqslant C(\alpha) N^{\frac{|\alpha|}{2}} \tag{6.20}$$

for all multi-indices. Define $S_k := \psi\, T_k$. On the one hand, S_k still obeys (6.18). On the other hand, for any cubes Q_k, Q_ℓ which are not nearest neighbors one has

$$\|S_k^* S_\ell\|_{2\to 2} \leqslant C(d, D_0, p)\, N^{\frac{p}{2}} \operatorname{dist}(Q_k, Q_\ell)^{-p} \tag{6.21}$$

for every positive integer p. This follows from the fact that the kernel of $S_k^* S_\ell$ equals

$$K_{k,\ell}(\eta', \eta) = \int_{\mathbb{R}^d} e^{2\pi i x\cdot(\Phi_N(\eta) - \Phi_N(\eta'))}\, \mathbb{1}_{\tilde{Y}}(\eta)\, \mathbb{1}_{\tilde{Y}}(\eta')\, \chi_k(\eta)\chi_\ell(\eta')\, \psi(x)^2\, dx$$

Using the differential operator

$$\mathcal{L} = \frac{1}{2\pi i}\frac{\Phi_N(\eta) - \Phi_N(\eta')}{\|\Phi_N(\eta) - \Phi_N(\eta')\|^2} \cdot \nabla_x,$$

which obeys

$$\mathcal{L}\, e^{2\pi i x\cdot(\Phi_N(\eta) - \Phi_N(\eta'))} = e^{2\pi i x\cdot(\Phi_N(\eta) - \Phi_N(\eta'))},$$

repeated integration by parts now yields (6.21). Finally, given any k, only a uniformly bounded number of choices of ℓ will satisfy

$$S_k S_\ell^* = \psi\, T_k T_\ell^*\, \psi \neq 0$$

This is due to the fact that $\chi_k(\eta)\chi_\ell(\eta) = 0$ up to a bounded number of choices of ℓ given k. If we label the cubes by lattice points $\underline{k} \in \mathbb{Z}^d$, then $\eta_{\underline{k}} = L_N \underline{k}$ whence

$$N^{\frac{p}{2}} \operatorname{dist}(Q_{\underline{k}}, Q_{\underline{\ell}})^{-p} \lesssim N^{\frac{p}{2}} (L_N |\underline{k} - \underline{\ell}|)^{-p} \lesssim |\underline{k} - \underline{\ell}|^{-p}$$

which is summable over $\mathbb{Z}^d$ provided $p > d$. On the other hand, we also have

$$\|S_k^* S_\ell\|_{2\to 2} \leqslant \|S_k\|_{2\to 2}\|S_\ell\|_{2\to 2} \leqslant B^2, \quad B := \sup_j \|S_j\|_{2\to 2}$$

Combining these two estimates we infer that for any $0 < \varepsilon < 1$,

$$\|S_{\underline{k}} S_{\underline{\ell}}^*\|_{2\to 2} + \|S_{\underline{k}}^* S_{\underline{\ell}}\|_{2\to 2} \leqslant C(d, D_0, \varepsilon)\, B^{2(1-\varepsilon)} \langle \underline{k} - \underline{\ell} \rangle^{-2(d+1)}$$

for all $\underline{k}, \underline{\ell} \in \mathbb{Z}^d$. Note that $B \leqslant C(d, D_0)$ by Hörmander's bound (6.18). Hence by Cotlar's lemma,

$$\Big\| \int_{[-N,N]^d} e^{2\pi i x\cdot\Phi_N(\eta)}\, \mathbb{1}_{\tilde{Y}}(\eta)\, h(\eta)\, d\eta \Big\|_{L^2(X)} \leqslant C(\varepsilon, d, D_0)\, \max_k \|S_k\|_{2\to 2}^{1-\varepsilon}. \tag{6.22}$$

The claim (6.14) will now follow from (6.22) by applying the fractal uncertainty principle of Theorem 1.2 to each S_k. For this we need to linearize the phase as in (6.15) which in turn makes the localization to scales $\sqrt{N}$ necessary.

To be specific, we reduce (6.14) to the following estimate. Let ψ_0 be compactly supported functions satisfying the bounds

$$\|\partial_x^\alpha \psi_0\|_\infty \leqslant C_s\, N^s \qquad \forall\, |\alpha| = s \geqslant 0 \tag{6.23}$$

where $N \geqslant 1$ is arbitrary and all constant are independent of N. We assume that ψ_0 is supported in a δ-regular set in $[-1,1]^d$ on scales $1/N$ to 1, and with $0<\delta<d$. Let

$$Z = N^{-1} Y_1$$

be a rescaled version of an admissible set Y_1 with constants $C_R, \delta_1, \varepsilon_0$ on scales 1 to N. The point is Y_1 is not assumed to be contained in $[-N,N]^d$, hence Theorem 1.2 does not apply directly. Hence we need to use Proposition 6.7 instead, for which we need to make assumptions on $\operatorname{supp} a$. Suppose that the symbol a is smooth and compactly supported with the bounds

$$\|\partial_x^\alpha\, a(x,\xi)\|_\infty \leqslant C(\alpha) \ \text{ for } \forall\ \alpha, \ \text{ and } \ \operatorname{supp} a(x,\cdot) \subset Q, \tag{6.24}$$

where Q is a cube in $\mathbb{R}^d$ that is independent of x, and is of side length $r_Q \leqslant r_0$. Then for some $\beta>0$ and C as above,

$$\Big\|\psi_0\, A\, \mathbb{1}_Z\, h\Big\|_2 \leqslant C N^{-\beta} \|h\|_2, \tag{6.25}$$

where

$$(Ah)(x) := N^{\frac{d}{2}} \int_{\mathbb{R}^d} e^{2\pi i N x\cdot\xi}\ a(x,\xi) h(\xi)\, d\xi.$$

Indeed,

$$\begin{aligned}
&\Big\| \int_{\mathbb{R}^d} e^{2\pi i x\cdot(\Phi_N(\eta_k)+D\Phi_N(\eta_k)(\eta-\eta_k))}\ \psi(x)\, a_k(x,\eta)\, \mathbb{1}_{\tilde{Y}}(\eta)\, h(\eta)\, d\eta\Big\|_2 \\
\lesssim &\Big\| \int_{\mathbb{R}^d} e^{2\pi i x\cdot\zeta}\ \psi(x)\, a_k(x, D\Phi_N(\eta_k)^{-1}\zeta+\eta_k)\, \mathbb{1}_{\tilde{Y}-\eta_k}(D\Phi_N(\eta_k)^{-1}\zeta) \\
&\qquad\qquad\qquad\qquad h(D\Phi_N(\eta_k)^{-1}\zeta+\eta_k)\, d\zeta\Big\|_2 \\
= &N^{\frac{d}{4}} \Big\| \int_{\mathbb{R}^d} e^{2\pi i N^{\frac12} x\cdot\xi}\ \psi(x)\, \widetilde{a}_k(x, N^{\frac12}\xi)\, N^{\frac{d}{4}} \mathbb{1}_{Y_1}(N^{\frac12}\xi)\, \widetilde{h}(N^{\frac12}\xi)\, d\xi\Big\|_2.
\end{aligned}$$

Here $\widetilde{a}$, $\widetilde{h}$ signify the functions on the second line but with the linear isomorphism $D\Phi_N(\eta_k)^{-1}$ and the shift η_k included, and $Y_1 = D\Phi_N(\eta_k)(\widetilde{Y}-\eta_k)$ is an admissible set on scales 1 to N with constants that depend on D_0. Note that $\mathbb{1}_{Y_1}(N^{\frac12}\xi) = \mathbb{1}_Z(\xi)$, with $Z = N^{-\frac12} Y_1$ which is an admissible set on scales $N^{-\frac12}$ to 1. By (6.20), $\psi_0(x) := \psi(x)$ satisfies the required bound, furthermore ψ_0 is supported on a $X(N^{-\frac12})$ which is a δ regular set on scales $N^{-\frac12}$ to 1, see Lemma A.4. As for the amplitude, and ignoring the distinction between $\widetilde{a}_k$ and a_k,

$$\begin{aligned}
a_k(x, N^{\frac12}\xi) &:= e^{2\pi i x\cdot R_k(N^{\frac12}\xi)}\, \chi_k(N^{\frac12}\xi) \\
R_k(N^{\frac12}\xi) &:= N \int_0^1 (1-t)\langle D^2\Phi_N(\eta_k + t(N^{\frac12}\xi-\eta_k))(\xi-\eta_k'), \xi-\eta_k'\rangle\, dt
\end{aligned}$$

where $\eta_k' = N^{-\frac{1}{2}}\eta_k$. Setting $a(x,\xi) = a_k(x, N^{\frac{1}{2}}\xi)$, we conclude from (6.17) that a satisfies (6.24) with constant $r_0 = C$, which is an absolute constant. Finally,

$$\|N^{\frac{d}{4}}\widetilde{h}(N^{\frac{1}{2}}\xi)\|_2 \simeq \|h\|_2.$$

Thus, we can apply (6.25) with N replaced by $N^{\frac{1}{2}}$ to obtain a gain of $N^{-\beta/2}$, and we are done.

It remains to prove (6.25). Note that this is equivalent to proving

$$\left\|\psi_0\, A\, \mathbb{1}_{Z\cap Q}\, h\right\|_2 \leqslant CN^{-\beta}\|h\|_2. \tag{6.26}$$

By Proposition (6.7), we can cover $Z \cap Q$ by $C(\varepsilon_0, d, r_0)$ many admissible sets W_j's with constants $\delta_1, \widetilde{C}_R := (4C_R)^{\frac{2}{1-\delta_1}} C_R$, $\tilde{\varepsilon}_0 = \tilde{\varepsilon}_0(\varepsilon_0, D_0)$. Hence, via triangle inequality, it suffices to prove (6.26) with $Z \cap Q$ replaced by W_j.

If $a = 1$ on the $\mathrm{supp}(\psi_0) \times W_j$, then this follows immediately from Theorem 1.2 by a rescaling. Indeed, one has by that theorem

$$\begin{aligned}
&\left\|N^{\frac{d}{2}} \int_{\mathbb{R}^d} e^{2\pi i N x\cdot\xi}\ \psi_0(x)\, \mathbb{1}_{W_j}(\xi)\, h(\xi)\, d\xi\right\|_2 \\
=&\left\| \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi}\ \psi_0(x)\, \mathbb{1}_{W_j}(\xi/N)\, N^{-\frac{d}{2}} h(\xi/N)\, d\xi\right\|_2 \\
\lesssim& N^{-\beta}\|N^{-\frac{d}{2}}h(\xi/N)\|_2 = N^{-\beta}\|h\|_2.
\end{aligned}$$

Let us now consider general a satisfying (6.24). Let $\rho \in (0,1)$ with its value determined later.

Let us note that by the usual A^*A argument, we have Hörmander's bound,

$$\|A\|_{2\to 2} \leqslant C. \tag{6.27}$$

Next we decompose $\psi_0\, A\, \mathbb{1}_{W_j}$ into the following

$$\begin{aligned}
&\psi_0\, A\, \mathbb{1}_{W_j} = \psi_0\, \mathcal{F}_\hbar^{-1}\, A_1 + A_2\, \mathcal{F}_\hbar\, A\, \mathbb{1}_{W_j}, \\
&A_1 := \mathbb{1}_{\mathbb{R}^d\setminus W_j(N^{-\rho})}\, \mathcal{F}_\hbar\, A\, \mathbb{1}_{W_j}, \quad A_2 := \psi_0\, \mathcal{F}_\hbar^{-1}\, \mathbb{1}_{W_j(N^{-\rho})},
\end{aligned}$$

where $\hbar = N^{-1}$. Clearly, by (6.27), we have

$$\|\psi_0\, A\, \mathbb{1}_{W_j}\|_{2\to 2} \lesssim \|A_1\|_{2\to 2} + \|A_2\|_{2\to 2}. \tag{6.28}$$

Thus it suffices to bound $\|A_1\|_{2\to 2}$ and $\|A_2\|_{2\to 2}$.

We compute the integral kernel of A_1:

$$K_{A_1}(\xi,\eta) = \mathbb{1}_{\mathbb{R}^d\setminus W_j(N^{-\rho})}(\xi)\ \mathbb{1}_{W_j}(\eta)\ N^d \int_{\mathbb{R}^d} e^{2\pi i N x\cdot(\eta-\xi)} a(x,\eta)\, dx.$$

Note that the Euclidean distance $\|\eta - \xi\| \geqslant N^{-\rho}$ on the support of K_{A_1}. Hence by repeated integration by parts in x, we obtain that

$$|K_{A_1}(\xi,\eta)| \leqslant C_{d,\rho} N^{d-\lceil\frac{d+10}{1-\rho}\rceil}\langle \eta-\xi\rangle^{-\lceil\frac{d+10}{1-\rho}\rceil} \leqslant C_{d,\rho}N^{-10}.$$

By Schur's test, we arrive at

$$\|A_1\|_{2\to 2} \leqslant CN^{-10}. \tag{6.29}$$

In view of A_2. Note that

$$W_j(N^{-\rho}) \subset \bigcup_{\substack{\|k\|_\infty \leqslant N^{1-\rho} \\ k\in\mathbb{Z}^d}} (W_j(N^{-1}) + k).$$

Note that

$$W_j(N^{-1}) \subset \widehat{W}_j := \{\sum_{\ell=1}^{d} \xi_\ell \vec{e}_\ell, \quad \xi_\ell \in N^{-1} \cdot W_{j,\ell}(2)\},$$

which is an admissible set on scales $2N^{-1}$ to 1. Thus by Theorem 1.2 and triangle inequality, we have for $f \in L^2(\mathbb{R}^d)$,

$$\begin{aligned}\|A_2 f\| \leqslant &\sum_{\|k\| \leqslant N^{1-\rho}} \left\|\psi_0 \ \mathcal{F}_\hbar^{-1} \ \mathbb{1}_{\widehat{W}_j+k} \, f\right\|_2\\ \lesssim &\sum_{\|k\| \leqslant N^{1-\rho}} \left\|\mathbb{1}_{\operatorname{supp}\psi_0} \ \mathcal{F}_\hbar^{-1} \ \mathbb{1}_{\widehat{W}_j+k} \, f\right\|_2\\ \leqslant &CN^{-\beta+d(1-\rho)}\|f\|_2,\end{aligned}$$

where $\hbar = N^{-1}$. Hence for $\rho = 1 - \beta/2d$,

$$\|A_2\|_{2\to 2} \leqslant CN^{-\frac{\beta}{2}}. \tag{6.30}$$

Combining (6.28), (6.29) with (6.30), we obtain (6.25). This concludes the proof of Theorem 1.4.

6.5. **Fourier integral operator.** In this section, we prove a fractal uncertainty principle for Fourier integral operators on $\mathbb{R}^d$. The proof follows that of the one dimensional case in [BouDya, Section 4], thus we shall be very brief.

Let

$$(B(\hbar)f)(x) := \hbar^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-2\pi i \Phi(x,y)/\hbar} b(x,y) f(y)\, dy, \tag{6.31}$$

where for some open set $U \subset \mathbb{R}^{2d}$,

$$\begin{aligned}&\Phi \in C^\infty(U;\mathbb{R}), \qquad b \in C_0^\infty(U), \qquad \det\left(\frac{\partial^2\Phi}{\partial x_j \partial y_k}\right) \neq 0 \ \text{ on } U\\ &\text{and} \quad \left(\sup_U \left\|\left(\frac{\partial^2\Phi}{\partial x_j \partial y_k}\right)\right\|\right) \cdot \left(\sup_U \left\|\left(\frac{\partial^2\Phi}{\partial x_j \partial y_k}\right)^{-1}\right\|\right) \leqslant C_\Phi,\end{aligned} \tag{6.32}$$

for some constant $C_\Phi \geqslant 1$, in which $\|\cdot\|$ is the matrix norm.

Proposition 6.8. Let $X, Y \subset [-1,1]^d$. Assume that X is a δ-regular set on scales 0 to 1 with constant C_R, and Y is an admissible set on scales 0 to 1 with parameters $\delta_1, C_R, \varepsilon_0$. Assume (6.32) holds. Then there exists $\beta > 0$, $\rho \in (0,1)$ depending only on $\delta, \delta_1, C_R, \varepsilon_0, d, C_\Phi$, and $C > 0$ depending only on $\delta, \delta_1, C_R, \varepsilon_0, d, \Phi, b$ such that for $0 < \hbar < h_0(\Phi) < 1$,

$$\|\mathbb{1}_{X(\hbar^{\rho/2})} B(\hbar) \mathbb{1}_{Y(\hbar^\rho)}\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \leqslant C\hbar^\beta.$$

Proof. As was pointed out in [BouDya], it is enough to prove Proposition 6.8 under the assumption that

$$1 < \left|\det\left(\frac{\partial^2\Phi}{\partial x_j \partial y_k}\right)\right| < 2 \ \text{ on } U. \tag{6.33}$$

Let $\tilde{h} := \hbar^{1/2}$. Divide $[-2,2]^d = \bigcup_k Q_k$ into congruent cubes of side length L with $\tilde{h}/2 \leqslant L < \tilde{h}$. Let $\{\chi_k\}_k$ be a partition of unity adapted to these cubes. With y_k

being the center of Q_k, we have

$$\begin{aligned}
&\hbar^{-\frac{d}{2}}\int_{\mathbb{R}^d} e^{-2\pi i\Phi(x,y)/\hbar}b(x,y)\mathbb{1}_{Y(\hbar^\rho)}(y)f(y)\,dy\\
&=\sum_k \hbar^{-\frac{d}{2}}\int_{\mathbb{R}^d} e^{-2\pi i\Phi(x,y)/\hbar}b(x,y)\chi_k(y)\mathbb{1}_{Y(\hbar^\rho)}(y)f(y)\,dy\\
&=\sum_k e^{-2\pi i\Phi(x,y_k)/\hbar}\hbar^{-\frac{d}{2}}\int_{\mathbb{R}^d} e^{-2\pi i\nabla_y\Phi(x,y_k)\cdot(y-y_k)/\hbar}\tilde{b}_k(x,y)\mathbb{1}_{Y(\hbar^\rho)}(y)f(y)\,dy\\
&=:\sum_k (T_kf)(x),
\end{aligned}$$

where

$$\begin{aligned}
&\tilde{b}_k(x,y)=e^{-2\pi i\Psi_k(x,y)/\hbar}\chi_k(y)b(x,y),\\
&\Psi_k(x,y)=\int_0^1(1-t)\langle(y-y_k),\mathrm{H}\Phi(x,y_k+t(y-y_k))(y-y_k)\rangle\,dt,
\end{aligned}\tag{6.34}$$

in which $\mathrm{H}\Phi(x,\cdot)$ is the Hessian of $\Phi(x,\cdot)$ in the y-variable.

We will prove

$$\|\mathbb{1}_{X(\tilde{\hbar}^\rho)}T_k\|_{L^2\to L^2}\leqslant C\tilde{\hbar}^\beta,\tag{6.35}$$

and the estimate for $\sum_k \mathbb{1}_{X(\tilde{\hbar}^\rho)}T_k$ follows from almost orthogonality and Cotlar's lemma, see the proof of Proposition 4.3 in [BouDya].

Let

$$\phi(x):=\nabla_y\Phi(x,y_k).$$

By (6.33), the Jacobian matrix $\mathrm{J}\phi$ satisfies $1<|\det(\mathrm{J}\phi(x))|<2$, hence ϕ admits an inverse function.

We have, by a change variable $x\to\phi^{-1}(x)$,

$$\begin{aligned}
&\|\mathbb{1}_{X(\tilde{\hbar}^\rho)}(x)(T_kf)(x)\|_{L^2}\\
=&\|\mathbb{1}_{\phi(X(\tilde{\hbar}^\rho))}(x)|\det(\mathrm{J}\phi^{-1}(x))|^{\frac12}\hbar^{-\frac{d}{2}}\\
&\qquad\int_{\mathbb{R}^d}e^{-2\pi i x\cdot y/\hbar}\tilde{b}_k(\phi^{-1}(x),y+y_k)\mathbb{1}_{Y(\hbar^\rho)-y_k}(y)f(y+y_k)\,dy\|_{L^2}\\
=&\|\mathbb{1}_{\phi(X(\tilde{\hbar}^\rho))}(x)|\det(\mathrm{J}\phi^{-1}(x))|^{\frac12}\\
&\qquad\int_{\mathbb{R}^d}e^{-2\pi i x\cdot y/\tilde{\hbar}}\tilde{b}_k(\phi^{-1}(x),\tilde{\hbar}y+y_k)\mathbb{1}_{Y(\hbar^\rho)-y_k}(\tilde{\hbar}y)f(\tilde{\hbar}y+y_k)\,dy\|_{L^2}\\
\leqslant&\|\mathbb{1}_{\phi(X(\tilde{\hbar}^\rho))}\,A(\hbar)\,\mathbb{1}_{\tilde{\hbar}^{-1}(Y(\tilde{\hbar}^{2\rho})-y_k)}\|_{L^2\to L^2}\cdot\|\tilde{\hbar}^{\frac{d}{2}}f(\tilde{\hbar}y+y_k)\|_{L^2}\\
=&\|\mathbb{1}_{\phi(X(\tilde{\hbar}^\rho))}\,A(\hbar)\,\mathbb{1}_{Y(\tilde{\hbar}^{2\rho-1})-\tilde{\hbar}^{-1}y_k}\|_{L^2\to L^2}\cdot\|f\|_{L^2},
\end{aligned}$$

where

$$\begin{aligned}
(A(\hbar)f)(x)=&\tilde{\hbar}^{-\frac{d}{2}}\int_{\mathbb{R}^d}e^{-2\pi i x\cdot y/\tilde{\hbar}}\hat{b}_k(x,y)f(y)\,dy\\
\tilde{b}(x,y)=&|\det(\mathrm{J}\phi^{-1}(x))|^{\frac12}\,\tilde{b}_k(\phi^{-1}(x),\tilde{\hbar}y+y_k).
\end{aligned}\tag{6.36}$$

Now it suffices to bound

$$\|\mathbb{1}_{\phi(X(\tilde{\hbar}^\rho))}\,A(\hbar)\,\mathbb{1}_{Y(\tilde{\hbar}^{2\rho-1})-\tilde{\hbar}^{-1}y_k}\|_{L^2\to L^2}.\tag{6.37}$$

Let $\tilde{X} := \phi(X)$. By our assumption 6.32,

$$(\sup \|\mathrm{J}\phi\|) \cdot (\sup \|(\mathrm{J}\phi)^{-1}\|) \leqslant C_\Phi.$$

Note (6.33) implies $C_1 := \sup \|\mathrm{J}\phi\| \geqslant 1$ and hence $C_2 := \sup \|(\mathrm{J}\phi)^{-1}\| \leqslant C_\Phi$. By Lemma A.5, $\tilde{X}$ is δ-regular with constant $C_R(d\,C_\Phi)^{\delta/2}$ on scales 0 to $d^{-1/2}C_2^{-1}$.

If $d^{-1/2}C_2^{-1} < 1$, Lemma A.3 implies $\tilde{X}$ is δ-regular with constant

$$2(d^{1/2}C_2)^d C_R(d\,C_\Phi)^{\delta/2} \leqslant 2d^{\frac{d+\delta}{2}} C_R C_\Phi^{d+\frac{\delta}{2}} =: \widetilde{C}_R$$

on scales 0 to 1. If $d^{-1/2}C_2^{-1} \geqslant 1$, let $\widetilde{C}_R := C_R(d\,C_\Phi)^{\delta/2}$. Hence $\tilde{X}$ is always δ-regular with constant $\widetilde{C}_R$ on scales 0 to 1.

It is also easy to see that $\phi(X(\tilde{h}^\rho)) \subseteq \tilde{X}(C(\Phi)\tilde{h}^\rho)$, where $C(\Phi)$ is a constant depending on Φ. For $0 < \hbar < h_0(\Phi)$, we have $C(\Phi)\tilde{h}^\rho < \tilde{h}^{2\rho-1}$, hence

$$\|\mathbb{1}_{\phi(X(\tilde{h}^\rho))} A(\hbar) \mathbb{1}_{(Y(\tilde{h}^{2\rho-1})-\tilde{h}^{-1}y_k)}\|_{L^2\to L^2} \leqslant \|\mathbb{1}_{\tilde{X}(\tilde{h}^{2\rho-1})} A(\hbar) \mathbb{1}_{(Y(\tilde{h}^{2\rho-1})-\tilde{h}^{-1}y_k)}\|_{L^2\to L^2}$$

Next note that

$$\begin{aligned} \tilde{X}(\tilde{h}^{2\rho-1}) &\subseteq \bigcup_{\substack{j\in\mathbb{Z} \\ \|j\|\leqslant \tilde{h}^{2\rho-2}}} (\tilde{X}(\tilde{h}) + \tilde{h}j) =: \bigcup_{\substack{j\in\mathbb{Z} \\ \|j\|\leqslant \tilde{h}^{2\rho-2}}} \tilde{X}_j \\ Y(\tilde{h}^{2\rho-1}) - \tilde{h}^{-1}y &\subseteq \bigcup_{\substack{k\in\mathbb{Z} \\ \|p\|\leqslant \tilde{h}^{2\rho-2}}} (Y(\tilde{h}) - \tilde{h}^{-1}y + \tilde{h}p) =: \bigcup_{\substack{k\in\mathbb{Z} \\ \|p\|\leqslant \tilde{h}^{2\rho-2}}} Y_p. \end{aligned} \tag{6.38}$$

Hence, it is eventually reduced to estimating each $\|\mathbb{1}_{\tilde{X}_j} A(\hbar) \mathbb{1}_{Y_p}\|_{L^2\to L^2}$.

It is easy to check that $\hat{b}_k(x,y)$ satisfy (6.24), hence by (6.25), we have

$$\|\mathbb{1}_{\tilde{X}_j} A(\hbar) \mathbb{1}_{Y_p}\|_{L^2\to L^2} \leqslant C\tilde{h}^\beta,$$

for some $\beta > 0$. Choosing $2d(\rho-1) < \beta/2$, we conclude that

$$\|\mathbb{1}_{\tilde{X}(\tilde{h}^{2\rho-1})} A(\hbar) \mathbb{1}_{(Y(\tilde{h}^{2\rho-1})-\tilde{h}^{-1}y_k)}\|_{L^2\to L^2} \leqslant C\tilde{h}^{\frac{\beta}{2}}$$

by triangle inequality. This proves the claimed result. □

Appendix A. Regular sets

We show that certain operations preserve the class of δ-regular sets if we allow to increase the regularity constant and shrink the scales.

The first lemma is from [BouDya]. It shows a δ-regular set in $\mathbb{R}^1$, $0 < \delta < 1$, can be split into smaller δ-regular sets.

Lemma A.1. *Let $X \subset \mathbb{R}^1$ be a δ-regular set with constant C_R on scales α_0 to α_1, and assume that $0 < \delta < 1$ and $(4C_R)^{\frac{2}{1-\delta}}\alpha_0 \leqslant \rho \leqslant \alpha_1$. Then there exists a collection of disjoint intervals $\mathcal{J}$ such that*

$$X = \bigcup_{J\in\mathcal{J}} (X \cap J);\ (4C_R)^{-\frac{2}{1-\delta}}\rho \leqslant |J| \leqslant \rho \text{ for all } J \in \mathcal{J},$$

and each $X \cap J$ is δ-regular with constant $\tilde{C}_R := (4C_R)^{\frac{2}{1-\delta}} C_R$ on scales α_0 to ρ.

The rest of this section concerns δ-regular sets in $\mathbb{R}^d$. We show that certain operations preserve the class of δ-regular sets if we allow to increase the regularity constant and shrink the scales.

Lemma A.2. *Let X be a δ-regular set with $\delta \in (0,d)$ and constant C_R, on scales α_0 to α_1. Fix $\lambda > 0$ and $y \in \mathbb{R}^d$. Then the set $\tilde{X} := y + \lambda X$ is a δ-regular set with constant C_R on scales $\lambda\alpha_0$ to $\lambda\alpha_1$.*

Proof. Taking the measure

$$\mu_{\tilde{X}}(A) := \lambda^\delta \mu_X(\lambda^{-1}(A-y)),$$

it is easy to verify. □

Lemma A.3. *Let X be a δ-regular set with constant C_R on scales α_0 to α_1. Fix $T > 1$. Then X is δ-regular with constant $\tilde{C}_R := 2T^d C_R$ on scales α_0 to $T\alpha_1$.*

Proof. Let I be a cube such that $\alpha_0 \leqslant r_I \leqslant T\alpha_1$. For $\alpha_0 \leqslant r_I \leqslant \alpha_1$, the upper bound is immediate. For $\alpha_1 < r_I \leqslant T\alpha_1$, I can be covered by $\lceil T\rceil^d \leqslant 2T^d$ cubes of side length α_1 each, therefore

$$\mu_X(I) \leqslant 2T^d C_R \alpha_1^\delta \leqslant \tilde{C}_R r_I^\delta.$$

In view of the lower bound estimate, we assume I is centered at a point in X. As before, we may assume $\alpha_1 < r_I \leqslant T\alpha_1$. Let $I' \subset I$ be the cube with the same center and $r_{I'} = \alpha_1$. Then

$$\mu_X(I) \geqslant \mu_X(I') \geqslant C_R^{-1}\alpha_1^\delta \geqslant \tilde{C}_R^{-1} r_I^\delta,$$

as claimed. □

Lemma A.4. *Let X be a δ-regular set with constant C_R on scales α_0 to α_1. Fix $T \geqslant 1$.*

(1) Suppose $\alpha_1 \geqslant 2\alpha_0$, the neighborhood $X + [-T\alpha_0, T\alpha_0]^d$ is δ-regular with constant $\tilde{C}_R := 4^d T^d C_R$ on scales $2\alpha_0$ to α_1.

(2) Suppose that $\alpha_1 \geqslant T\alpha_0$, then $X + [-T\alpha_0, T\alpha_0]^d$ is δ-regular with constant $C_R' = 4^d C_R$ on scales $T\alpha_0$ to α_1.

Proof. Let $\tilde{X} := X + [-T\alpha_0, T\alpha_0]^d$ and define $\mu_{\tilde{X}}$ supported on $\tilde{X}$ by convolution

$$\mu_{\tilde{X}}(A) := \frac{1}{(T\alpha_0)^d}\int_{[-T\alpha_0,T\alpha_0]^d} \mu_X(A+y)\,dy.$$

Let I be a cube such that $M\alpha_0 \leqslant r_I \leqslant \alpha_1$ with $M \geqslant 1$. Then

$$\mu_{\tilde{X}}(I) \leqslant 2^d C_R r_I^\delta,$$

which proves the upper bound estimates for both cases.

Now assume that I is centered at a point $x_1 \in \tilde{X}$. Take $x_0 \in X$ such that $x_0 \in x_1 + [-T\alpha_0, T\alpha_0]^d$, and I' be the cube centered at x_0 with side length $r_{I'} = r_I/2$. Then

$$\mu_X(I') \geqslant C_R^{-1}(r_I/2)^\delta \geqslant 2^{-d} C_R^{-1} r_I^\delta.$$

Let $J = x_0 - x_1 + [-\alpha_0/2, \alpha_0/2]^d$, then $J \cap [-T\alpha_0, T\alpha_0]^d$ contains a cube with side length at least $\alpha_0/2$. Clearly, $I' \subset I + y$ for any $y \in J$. Hence

$$\mu_{\tilde{X}}(I) \geqslant \frac{1}{(2T)^d}\mu_X(I') \geqslant \tilde{C}_R^{-1} r_I^\delta,$$

which proves the lower bound estimate for (1).

Let $J = x_0 - x_1 + [-T\alpha_0/2, T\alpha_0/2]^d$, then $J \cap [-T\alpha_0, T\alpha_0]^d$ contains a cube with side length at least $T\alpha_0/2$. Clearly, $I' \subset I + y$ for any $y \in J$. Hence

$$\mu_{\tilde{X}}(I) \geqslant \frac{1}{2^d}\mu_X(I') \geqslant (C'_R)^{-1} r_I^\delta,$$

which proves the lower bound estimate for (2). □

Lemma A.5. *Assume $F : \mathbb{R}^d \to \mathbb{R}^d$ is C^1 diffeomorphism. Let $C_1 := \sup_{x\in\mathbb{R}^d} \|\mathrm{J}F(x)\|$ and $C_2 := \sup_{x\in\mathbb{R}^d} \|\mathrm{J}F^{-1}(x)\|$, where $\mathrm{J}F$ is the Jacobian matrix and $\|\cdot\|$ is the matrix norm. Assume that for some constant $C_F \geqslant 1$, we have*

$$C_1 C_2 \leqslant C_F. \tag{A.1}$$

Let X be a δ-regular set with constant C_R on scales α_0 to $\alpha_1 \geqslant C_F^2\alpha_0$. Then $F(X)$ is a δ-regular set with constant $\widetilde{C}_R := C_R(d\,C_F)^{\delta/2}$ on scales $d^{1/2}C_1\alpha_0$ to $d^{-1/2}C_2^{-1}\alpha_1$.

Proof. Let $\widetilde{X} := F(X)$ and define measure $\mu_{\widetilde{X}}$ supported on $\widetilde{X}$ as

$$\mu_{\widetilde{X}}(A) := C_F^{-\frac{\delta}{2}} C_1^\delta\, \mu_X(F^{-1}(A)).$$

Let $\widetilde{I}$ be a cube with side length $r_{\tilde{I}}$ with

$$d^{\frac{1}{2}} C_1 \alpha_0 \leqslant r_{\tilde{I}} \leqslant d^{-\frac{1}{2}} C_2^{-1}\alpha_1. \tag{A.2}$$

Clearly, $F^{-1}\tilde{I}$ is contained in a cube of side length r, where $r \leqslant \sqrt{d}\,C_2 r_{\tilde{I}}$. Indeed, let y be the center of $\tilde{I}$. Then for any $x \in \tilde{I}$, we have

$$\|F^{-1}(x) - F^{-1}(y)\| \leqslant C_2\|x-y\| \leqslant \frac{\sqrt{d}}{2} C_2 r_{\tilde{I}}.$$

Let I be the cube centered at $F^{-1}y$ of side length $\sqrt{d}\,C_2 r_{\tilde{I}} \leqslant \alpha_1$. Then

$$\mu_{\widetilde{X}}(\tilde{I}) \leqslant \mu_X(I) \leqslant C_F^{-\frac{\delta}{2}} C_1^\delta C_R(\sqrt{d}\,C_2 r_{\tilde{I}})^\delta = C_R(d\,C_F)^{\frac{\delta}{2}} r_{\tilde{I}}^\delta.$$

If, in addition, $y \in \widetilde{X}$. Let $y = F(z)$, where $z \in X$. Then the cube Q centered at z of side length $r = d^{-\frac{1}{2}} C_1^{-1} r_{\tilde{I}} \geqslant \alpha_0$ is contained in $F^{-1}(\tilde{I})$. Indeed, for any $x \in Q$, we have

$$\|F(x) - F(z)\| \leqslant \frac{\sqrt{d}}{2} C_1 r = \frac{r_{\tilde{I}}}{2}.$$

Hence

$$\mu_{\widetilde{X}}(\tilde{I}) = C_F^{-\frac{\delta}{2}} C_1^\delta \mu_X(F^{-1}(\tilde{I})) \geqslant C_F^{-\frac{\delta}{2}} C_1^\delta C_R^{-1}(d^{-\frac{1}{2}} C_1^{-1} r_{\tilde{I}})^\delta = C_R^{-1}(d\,C_F)^{-\frac{\delta}{2}} r_{\tilde{I}}^\delta.$$

This proves the claim. □

Lemma A.6. *Let X be a δ-regular set with constant C_R on scales α_0 to α_1, and $0 < \delta < d$. Fix an integer*

$$L \geqslant (2^{d/2}\sqrt{2d+1}\,C_R)^{\frac{2}{d-\delta}}. \tag{A.3}$$

Assume that I is a cube with $\alpha_0 \leqslant r_I/L \leqslant r_I \leqslant \alpha_1$ and $I_1, \ldots, I_{L^d}$ is the partition of I into cubes of side length r_I/L. Then there exists ℓ such that $X \cap I_\ell = \varnothing$.

Proof. Using Lemma A.2, it suffices to consider $I = [0, L]^d$, $\alpha_0 \leqslant 1 \leqslant L \leqslant \alpha_1$. We argue by contradiction. Assume that each I_ℓ intersects X. Then $I'_\ell := I_\ell + [-1/2, 1/2]^d$ contains a unit cube centered at a point in X and thus

$$\mu_X(I'_\ell) \geqslant C_R^{-1}, \quad \forall 1 \leqslant \ell \leqslant L^d.$$

On the other hand,

$$\bigcup_{\ell=1}^{L^d} I'_\ell = \left[-\frac{1}{2}, L + \frac{1}{2}\right]^d,$$

and each point in $[-1/2, L + 1/2]^d$ can be covered by at most $2d + 1$ of the cubes I'_ℓ. Therefore

$$C_R^{-1} L^d \leqslant \sum_{\ell=1}^{L^d} \mu_X(I'_\ell) \leqslant (2d+1)\mu_X\Big(\left[-\frac{1}{2}, L + \frac{1}{2}\right]^d\Big) \leqslant (2d+1) C_R (L+1)^\delta,$$

which contradicts (A.3). □

Recall our definition of $\mathcal{C}_n$ and porosity in Definition 5.1.

Lemma A.7. *Let $X \subset [-1, 1]^d$ be a δ-regular set with constant C_R on scales α_0 to α_1. Let L satisfy* (A.3)*, and take $n \in \mathbb{Z}$ such that $\alpha_0 \leqslant L^{-n-1} \leqslant L^{-n} \leqslant \alpha_1$. Then X is porous at scale L with depth n.*

Lemma A.8. *Let X be a δ-regular set with constant C_R on scales α_0 to α_1. Let $C >\geqslant 1$ be a constant. Let I be a cube of side length r_I satisfying $\alpha_0 \leqslant r_I \leqslant C\alpha_1$. Let $\rho > 0$ satisfy $\alpha_0 \leqslant \rho \leqslant \min(r_I, \alpha_1)$. Then there exists a non-overlapping* [1] *collection $\mathcal{J}$ of $N_{\mathcal{J}}$ cubes of side length ρ each such that*

$$X \cap I \subset \bigcup_{J \in \mathcal{J}} J, \quad N_{\mathcal{J}} \leqslant \left(6\left\lceil\frac{3+C}{2}\right\rceil\right)^d C_R^2 \left(\frac{r_I}{\rho}\right)^\delta.$$

We will only use this lemma in dimension 1. Note that in [BouDya], this is formulated with $C = 1$. We use this form with a constant C in the proof of Lemma 6.2.

Proof. Let $\mathcal{J}$ consist of all cubes of the form $\times_{k=1}^d \rho[j_k, j_k + 1]$, $(j_1, j_2, ..., j_d) \in \mathbb{Z}^d$, which intersect $X \cap I$. Then $X \cap I \subset \bigcup_{J \in \mathcal{J}} J$. Next, we will prove the upper bound on $N_{\mathcal{J}}$.

For each $J \in \mathcal{J}$, let $J' \supset J$ be the cube with the same center and is of side length 2ρ. Since J intersects X, J' contains a cube of side length ρ centered at a point in X. Therefore

$$\mu_X(J') \geqslant C_R^{-1}\rho^\delta.$$

It is also clear that $\bigcup_{J \in \mathcal{J}} J' \subset I(\frac{3}{2}\rho)$, and each point lies in at most 3^d of the cubes J'.

If $r_I \leqslant \alpha_1$, $I(\frac{3}{2}\rho)$ can be covered by 4^d cubes of side length r_I. If $\alpha_1 < r_I \leqslant C\alpha_1$, $I(\frac{3}{2}\rho)$ can be covered by $2^d\lceil\frac{3+C}{2}\rceil^d$ cubes of side length α_1. Therefore, we always have

$$N_{\mathcal{J}} \cdot C_R^{-1}\rho^\delta \leqslant \sum_{J \in \mathcal{J}} \mu_X(J') \leqslant 3^d \mu_X\left(\bigcup_{J \in \mathcal{J}} J'\right) \leqslant \left(6\left\lceil\frac{3+C}{2}\right\rceil\right)^d C_R r_I^\delta,$$

[1] A collection of cubes is non-overlapping if the intersection of each two different cubes has empty interior.

this proves the upper bound on $N_{\mathcal{J}}$. $\square$

Appendix B. Proof of Lemma 6.2

We follow the proofs of Theorem 3.2 and Lemma 4.1 in [JinZha]. Let us start with introducing some notations.

B.1. **Hilbert transform.** Let $\mathcal{H}_0$ be the standard Hilbert transform defined as convolution with p.v.$\frac{1}{\pi x}$: For $f \in C_0^\infty(\mathbb{R})$, (or more generally, $f \in L^1(\mathbb{R}, \langle x \rangle^{-1}\, dx)$)

$$\mathcal{H}_0(f)(x) := \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \int_{|x-t| \geqslant \varepsilon} \frac{f(t)}{x-t}\, dt.$$

Let $\mathcal{H}$ be the modified Hilbert transform with the integral kernel that decays like $|x|^{-2}$ as $|x| \to \infty$:

$$\mathcal{H}(f)(x) := \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \int_{|x-t| \geqslant \varepsilon} f(t) \Big(\frac{1}{x-t} + \frac{t}{t^2+1} \Big)\, dt, \quad f \in L^1(\mathbb{R}, \langle x \rangle^{-2}\, dx).$$

The advantage of $\mathcal{H}$ is that it applies to a larger space that contains $L^\infty(\mathbb{R})$ as well as functions the grow like $|x|^{1-\epsilon}$ as $|x| \to \infty$.

If $f \in L^1(\mathbb{R}, \langle x \rangle^{-1}\, dx)$, then $\mathcal{H}(f)$ differs from $\mathcal{H}_0(f)$ by a constant. Moreover, we have the inversion formula for all $f \in L^1(\mathbb{R}, \langle x \rangle^{-2}\, dx)$ with $\mathcal{H}(f) \in L^1(\mathbb{R}, \langle x \rangle^{-2}\, dx)$:

$$\mathcal{H}(\mathcal{H}(f)) = -f + c(f), \tag{B.1}$$

where $c(f)$ is a real constant depending on f.

We will use the following example later in the proof.

Example B.1. [JinZha, Example 2.3] Let $f(x) = \log(x^2+1)$, then we can compute

$$\mathcal{H}(f)'(x) = \mathcal{H}_0(f')(x) = -\frac{2}{x^2+1}. \tag{B.2}$$

B.2. **Hardy space and outer functions.** We recall the definition of Hardy space on the real line

$$H^2 = H^2(\mathbb{R}) = \{ f \in L^2(\mathbb{R}) : \operatorname{supp} \hat{f} \subset [0, \infty) \}.$$

If $f \in L^2(\mathbb{R})$, then $f + i\mathcal{H}_0(f) \in H^2(\mathbb{R})$.

The space of modulus of functions in H^2 can be characterized by the logarithmic integral: for $\omega \in L^2$, $\omega \geqslant 0$, we define

$$\mathcal{L}(\omega) := \int_{\mathbb{R}} \frac{\log \omega(x)}{1+x^2}\, dx.$$

Theorem B.2. [HavJör, Sec.1.5] *If $f \in H^2$, and $\mathcal{L}(|f|) = -\infty$, then $f \equiv 0$. On the other hand, if $\omega \in L^2$, and $\mathcal{L}(\omega) > -\infty$, then there exists a function $f \in H^2$ with $|f| = \omega$, unique up to a multiplication by a complex constant with unit modulus.*

If $\mathcal{L}(\omega) > -\infty$. Let $\Omega = -\log \omega$, then $\Omega \in L^1(\mathbb{R}, \langle x \rangle^{-2}\, dx)$. Therefore we can define $\tilde{\Omega} = \mathcal{H}(\Omega)$ and take

$$f = a e^{-(\Omega + i\tilde{\Omega})}, \quad |a| = 1. \tag{B.3}$$

We call functions of the form (B.3) for general $\Omega \in L^1(\mathbb{R}, \langle x \rangle^{-2}\, dx)$ *outer functions.* The class of outer functions is closed under multiplications. Moreover if two outer functions have the same modulus, then they differ by a complex constant with unit modulus.

The following lemma gives a sufficient condition of a function to be the modulus of the Fourier transform of a function supported in $[0, \sigma]$.

Lemma B.3. [KhaMasNaz, Theorem 1] *Assume that* $\omega = e^{-\Omega} \in L^2$ *and* $\mathcal{L}(\omega) > -\infty$. *In addition, we assume that* $\omega^2 e^{2\pi i \sigma x}$ *is an outer function. Then there exsits* $\psi \in L^2$ *with* $\operatorname{supp}\psi \subset [0, \sigma]$ *and* $|\widehat{\psi}| = \omega$.

B.3. **An effective multiplier theorem.** We prove an effective multiplier theorem. This proof is essentially in [JinZha, Section 3], the only change we make lies in the definition of $k(x)$ below. Our modified definition makes sure that $k(x)$ is a constant function in a neighborhood of 0, which leads to a pointwise lower bound of $\widehat{\psi}(x)$ on the whole interval $[-3/4, 3/4]$.

Theorem B.4. *Assume that* $0 < \omega \leqslant 1$ *satisfies* $\mathcal{L}(\omega) > -\infty$, *and*

$$\|\mathcal{H}(\Omega)'\|_{L^\infty} \leqslant \frac{\pi}{2}\sigma,$$

where $0 < \sigma < 1/10$, $\Omega = -\log \omega$. *Then there exists* $\psi \in L^2(\mathbb{R})$ *with*

$$\operatorname{supp}\psi \subset [0, \sigma], \quad |\widehat{\psi}| \leqslant \omega,$$

and

$$|\widehat{\psi}| \geqslant \frac{\sigma^{10}}{4 \times 10^{11}}\omega, \quad \text{on } [-3/4, 3/4].$$

Proof. We first set

$$\omega_0(x) = \frac{\omega(x)}{(x^2 + T^2)^5}, \quad \Omega_0(x) = -\log(\omega_0(x)),$$

with constant T that will be specified later. We then have

$$\Omega_0 = \Omega + 5\log(x^2 + T^2).$$

We compute

$$\mathcal{H}(\log(x^2 + T^2))(0) = \lim_{\varepsilon \to 0^+} \int_{|t| \geqslant \varepsilon} \log(t^2 + T^2)\Big(\frac{1}{-t} - \frac{t}{t^2 + 1}\Big)\, dt,$$

in which the Integrand is an odd function. Hence the integration is zero. Therefore we have

$$\mathcal{H}(\Omega_0)(0) = \mathcal{H}(\Omega)(0) + 5\mathcal{H}(\log(x^2 + T^2)) = \mathcal{H}(\Omega)(0). \tag{B.4}$$

By (B.2), we compute

$$\mathcal{H}(\log(x^2 + T^2))' = T^{-1}\mathcal{H}(\log(x^2 + 1))'(\cdot/T) = -\frac{2T}{x^2 + T^2}.$$

Thus if we choose $T = \frac{20}{\pi\sigma} \geqslant \frac{200}{\pi} \geqslant 60$, we have

$$\|\mathcal{H}(\Omega_0)'\|_{L^\infty} \leqslant \|\mathcal{H}(\Omega)'\|_{L^\infty} + 5\|\mathcal{H}(\log(x^2 + T^2))'\|_{L^\infty} \leqslant \pi\sigma. \tag{B.5}$$

Let us define

$$s_0(x) = \pi\sigma x + \mathcal{H}(\Omega_0)(x).$$

Hence by (B.4),

$$s_0(0) = \mathcal{H}(\Omega)(0),$$

depending only on ω.

Let $s(x)$ be defined as follows

$$s(x) = s_0(x) - \pi k(x) - \frac{\pi}{2},$$

in which

$$k(x) = \begin{cases} \lfloor \frac{s_0(x)}{\pi} \rfloor, & \text{if } \frac{s_0(0)}{\pi} \in \left[\frac{1}{4}, \frac{3}{4}\right] \mod 1, \\ \\ \lfloor \frac{s_0(x)}{\pi} - \frac{1}{2} \rfloor, & \text{if } \frac{s_0(0)}{\pi} \in \left[0, \frac{1}{4}\right) \bigcup \left(\frac{3}{4}, 1\right) \mod 1, \end{cases} \tag{B.6}$$

Note that our definition of $k(x)$ is different from that in [JinZha]. We modify the definition in order to make sure $k(x)$ is a constant near $x = 0$. This will be explained and used later in the proof.

By (B.5), $s_0(x)$ is a non-decreasing function and so is k. Note also that by our definition of $s(x)$, we have

$$\|s\|_{L^\infty} \leqslant \pi. \tag{B.7}$$

Let $m = e^{-M}$ where $M = \mathcal{H}(s)$. Next, we will estimate $M(x) = \mathcal{H}(s)(x)$. We split the integral into three parts $M(x) = J_1(x) + J_2(x) + J_3(x)$, where

$$\begin{aligned} J_1(x) &= \frac{1}{\pi} \int_{|x-t|<1/2} \frac{s(t) - s(x)}{x - t} \, dt; \\ J_2(x) &= \frac{1}{\pi} \int_{|x-t|<1/2} s(t) \frac{t}{t^2 + 1} \, dt; \\ J_3(x) &= \frac{1}{\pi} \int_{|x-t|\geqslant 1/2} s(t) \Big(\frac{1}{x - t} + \frac{t}{t^2 + 1} \Big) \, dt. \end{aligned}$$

We estimate J_2 and J_3 in the same way as in [JinZha]. By (B.7), we have

$$|J_2(x)| \leqslant \frac{1}{\pi} \cdot \|s\|_{L^\infty} \cdot \frac{1}{2} \leqslant \frac{1}{2}. \tag{B.8}$$

Also, we have

$$|J_3(x)| \leqslant \frac{1}{\pi} \cdot \|s\|_{L^\infty} \int_{|x-t|\geqslant 1/2} \left| \frac{1}{x - t} + \frac{t}{t^2 + 1} \right| dt \leqslant 6 \log(|x| + 2). \tag{B.9}$$

Finally, we need to bound $|J_1|$. By (B.5), $s_0(x) = \pi\sigma x + \mathcal{H}(\Omega_0)(x)$ is non-decreasing with $\|s_0'\|_{L^\infty} \leqslant 2\pi\sigma$. Since we assume $0 < \sigma < 1/10$, we have

$$\|\pi^{-1} s_0'\|_{L^\infty} < \frac{1}{5}.$$

This leads to the following

- if $\pi^{-1} s_0(0) \in [1/4, 3/4] \mod 1$,

$$\frac{s_0(x)}{\pi} \in (0, 1) \mod 1, \quad \forall x \in \left[-\frac{5}{4}, \frac{5}{4}\right].$$

- if $\pi^{-1} s_0(0) \in [0, 1/4) \bigcup (3/4, 1) \mod 1$,

$$\frac{s_0(x)}{\pi} - \frac{1}{2} \in (0, 1) \mod 1, \quad \forall x \in \left[-\frac{5}{4}, \frac{5}{4}\right].$$

Recall our definition of $k(x)$ in (B.6), we know in each case $k(x)$ is a constant function on the interval $[-5/4, 5/4]$.

Thus for $x \in [-3/4, 3/4]$, we have

$$|J_1(x)| \leqslant \frac{1}{\pi} \int_{|x-t|<1/2} \left| \frac{s_0(t) - s_0(x)}{x-t} \right| dt \leqslant \frac{1}{\pi} \|s_0'\|_{L^\infty} \leqslant 2\sigma. \tag{B.10}$$

For all x, we only have a lower bound of J_1. Since k is non-decreasing, we have

$$J_1(x) \geqslant \frac{1}{\pi} \int_{|x-t|<1/2} \frac{s_0(t) - s_0(x)}{x-t} dt \geqslant -2\sigma. \tag{B.11}$$

Now combining (B.8), (B.9) with (B.10), we have the following estimate of M on $[-3/4, 3/4]$:

$$|M(x)| \leqslant 2\sigma + \frac{1}{2} + 6 \log \frac{11}{4} < 7. \tag{B.12}$$

Using (B.11) instead of (B.10), we obtain that for all x,

$$M(x) \geqslant -2\sigma - \frac{1}{2} - 6 \log (|x| + 2) > -1 - 6 \log(|x| + 2). \tag{B.13}$$

Next we will apply Lemma B.3 to $\tilde{\omega} = \frac{1}{3} m \omega_0$. We check that $\tilde{\omega}$ satisfies all the assumptions. First, by (B.13), we have

$$0 \leqslant \tilde{\omega} \leqslant \frac{e}{3} (|x| + 2)^6 \omega_0 \leqslant \frac{\omega}{x^2 + T^2}.$$

Hence $0 \leqslant \tilde{\omega} \leqslant \omega$ and $\tilde{\omega} \in L^2$. Moreover

$$\mathcal{L}(\tilde{\omega}) = \mathcal{L}(m/3) + \mathcal{L}(\omega_0) > -\infty.$$

By the construction $M = \mathcal{H}(s)$ and the inversion formula (B.1), we have

$$\mathcal{H}(-2M - 2\Omega_0) = 2s - 2\mathcal{H}(\Omega_0) - 2c(M) = 2\pi\sigma x - 2\pi k(x) - \pi - 2c(M),$$

where $k(x) \in \mathbb{Z}$ and $c(M)$ is a real constant. Therefore for some constant a with $|a| = 1$, we have

$$\tilde{\omega}^2 e^{2\pi i \sigma x} = \frac{1}{9} e^{-2M - 2\Omega_0 + 2\pi i \sigma x} = \frac{a}{9} e^{-2M - 2\Omega_0 + i\mathcal{H}(-2M - 2\Omega_0)},$$

which shows $\tilde{\omega}^2 e^{2\pi i \sigma x}$ is an outer function.

By Lemma B.3, there exists $\psi \in L^2$ with $\mathrm{supp}(\psi) \subset [0, \sigma]$ and $|\widehat{\psi}| \leqslant \tilde{\omega} \leqslant \omega$. Furthermore, on $[-3/4, 3/4]$, by (B.12), and since $T = \frac{20}{\pi\sigma}$, we have

$$|\widehat{\psi}(x)| = \tilde{\omega}(x) \geqslant \frac{1}{3} (1 + T^2)^{-5} e^{-7} \omega(x) \geqslant \frac{\sigma^{10}}{4 \times 10^{11}} \omega(x),$$

as claimed. □

B.4. **Multiplier adapted to the regular sets.** Now we are in the place to finish the proof of Lemma 6.2.

Proof. The proof is the essentially same as Lemma 4.1 of [JinZha]. We briefly go through the various constants below.

We define $n_1 \in \mathbb{N}$ by $2^{n_1} < S\alpha_1 \leqslant 2^{n_1+1}$. For $1 \leqslant n \leqslant n_1$, let $A_n := [-2^{n+1}, -2^n] \bigcup [2^n, 2^{n+1}]$, then by Lemma A.8, we have a collection $\mathcal{J}_n$ of N_n

intervals of size $\rho_n := n^{-\frac{1+\delta}{2}} 2^n$ such that each element is of the form $[j, j+1]$, $j \in \mathbb{Z}$, intersects A_n and

$$Y \cap A_n \subset \bigcup_{J \in \mathcal{J}_n} J.$$

Moreover, the number N_n satisfies

$$N_n \leqslant 6 \left\lceil \frac{3+S}{2} \right\rceil C_R^2 \left(\frac{2^n}{\rho_n} \right)^{\delta} = 6 \left\lceil \frac{3+S}{2} \right\rceil C_R^2 n^{\delta(1+\delta)/2}. \tag{B.14}$$

Following the proof of [JinZha], we a weight function ω such that

$$\begin{aligned}
&\omega(\xi) = \exp(-\langle \xi \rangle^{1/2}) \geqslant 0.3, \quad \forall \xi \in [-1, 1], \\
&\omega(\xi) \leqslant \exp(-\langle \xi \rangle^{1/2}), \quad \forall \xi \in \mathbb{R}, \\
&\omega(\xi) \leqslant \exp(-\Theta(|\xi|)|\xi|), \quad \forall \xi \in Y, \ |\xi| \geqslant 10, \\
&\|\mathcal{H}(\omega)'\|_{L^\infty} \leqslant \frac{\iota^{-1} C_R^2}{\delta_1(1-\delta_1)},
\end{aligned}$$

where $0 < \iota < 1$ is a constant depending only on S. The dependence comes from the upper bound of N_n in (B.14).

Applying Theorem B.4 to ω^{c_3} with

$$\sigma = \frac{c_1}{5}, \quad c_3 = \frac{\pi}{10} \iota\, c_1 C_R^{-2} \delta_1 (1-\delta_1) < 1.$$

We obtain ψ with

$$\begin{aligned}
&\operatorname{supp} \psi \subset \left[0, \frac{c_1}{5} \right], \\
&|\widehat{\psi}(\xi)| \geqslant \frac{c_1^{10}}{4 \times 10^{18}} \omega(\xi)^{c_3} \geqslant \frac{3}{4 \times 10^{19}} c_1^{10}, \quad \forall \xi \in [-3/4, 3/4], \\
&|\widehat{\psi}(\xi)| \leqslant \exp(-c_3 \langle \xi \rangle^{1/2}), \quad \forall \xi \in \mathbb{R}, \\
&|\widehat{\psi}(\xi)| \leqslant \exp(-c_3 \Theta(|\xi|)|\xi|), \quad \forall \xi \in Y, \ |\xi| \geqslant 10.
\end{aligned}$$

Finally, shifting ψ by $c_1/10$ yields the desired function. □

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