



On convergence criteria for the coupled flow of Li–Yuan–Zhang

Teng Fei¹ · Bin Guo¹ · Duong H. Phong¹

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Abstract

A one-parameter family of coupled flows depending on a parameter $\kappa > 0$ is introduced which reduces when $\kappa = 1$ to the coupled flow of a metric ω with a $(1, 1)$ -form α due recently to Y. Li, Y. Yuan, and Y. Zhang. It is shown in particular that, for $\kappa \neq 1$, estimates for derivatives of all orders would follow from C^0 estimates for ω and α . Together with the monotonicity of suitably adapted energy functionals, this can be applied to establish the convergence of the flow in some situations, including on Riemann surfaces. Very little is known as yet about the monotonicity and convergence of flows in presence of couplings, and conditions such as $\kappa \neq 1$ seem new and may be useful in the future.

1 Introduction

The main goal of this paper is to study the following flow of a Kähler metric ω_t coupled to a closed $(1, 1)$ -form α_t , on a given compact complex manifold X ,

$$\partial_t \omega_t = -Ric(\omega_t) + \lambda \omega_t + \alpha_t, \quad (1.1)$$

$$\partial_t \alpha_t = -\kappa \bar{\partial} \bar{\partial}_{\omega_t}^{\dagger} \alpha_t. \quad (1.2)$$

Here $Ric(\omega) = -i\partial\bar{\partial} \log \omega^n$ is the Ricci form of ω , n is the complex dimension of X , and κ a positive constant. When $\kappa = 1$ and $\lambda = -1$, this is the flow introduced recently by Li, Yuan, and Zhang [12]. However, the stationary points are independent of κ , and as we shall see in the present paper, it may be advantageous to consider a different value of κ . We shall refer to the flow (1.1, 1.2) as the κ -LYZ flow. The original motivation for the κ -LYZ flow is

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✉ Duong H. Phong
phong@math.columbia.edu

Teng Fei
tfei@math.columbia.edu

Bin Guo
bguo@math.columbia.edu

¹ Department of Mathematics, Columbia University, New York, NY 10027, USA

that its stationary points are given by a Kähler metric of constant scalar curvature together with a harmonic $(1, 1)$ -form. As such, it provides a natural approach to the well-known open problem of finding Kähler metrics of constant scalar curvature [12].

Our interest in the κ -LYZ flow comes from a different source. A leading contender for a unified theory of the forces of nature at their most fundamental level is M theory and its limiting string and supergravity theories [1, 10, 23]. These theories all incorporate the gravitational field together with other fields, and the resulting equations, whether in the original dimension or upon compactification, are all Einstein's equation modified by interactions with other fields or higher string modes. The corresponding parabolic flows are then all essentially Ricci flows, modified by couplings to other fields. Some explicit examples are the renormalization group flow for the bosonic string considered in [11], the Anomaly flow arising from compactifications of the heterotic string [7, 20, 21] and the flows found in [6] arising from compactifications of eleven-dimensional supergravity. However, while there has been considerable progress in recent years in the understanding of the Ricci flow, there are still very few tools available for the study of non-linear coupled systems in general, and in particular of the long-time behavior of the Ricci flow when it is coupled to other fields. Even for the simplest coupled Ricci flow, namely the Ricci flow coupled to a scalar field, only a criterion for the development of singularities [14] and a Perelman-type pseudo-locality Theorem [8] are known. A generalization of the first of these results to the Ricci flow coupled to the harmonic map flow is in [16]. The κ -LYZ flow is arguably the simplest example of a coupled Ricci flow in Kähler geometry, since the additional field α is a closed $(1, 1)$ -form, and the closedness of both the metric form ω and of the additional field α are preserved along the flow. It can also be viewed as an Abelian model for the Anomaly flow [20, 21], in which the role of α is played by a Hermitian metric on a holomorphic vector bundle, and higher powers of the curvature appear in the couplings.

The study of a partial differential equation usually begins with the identification of a minimum number of estimates from which the existence and/or C^∞ regularity of the equation would follow. Fundamental examples are the $C^{2,\alpha}$ estimates for linear uniformly elliptic second order equations, the C^0 estimate for the complex Monge–Ampère equation on a compact Kähler manifold, and the uniform estimate for the metrics in the method of continuity for the problem of constant scalar curvature Kähler metrics. In all these cases estimates for the derivatives of any order would follow, implying in turn both existence and regularity properties for the equation. Even such basic results are not yet available for either general non-linear elliptic or parabolic systems, or even more specifically for the Ricci flow coupled with other fields.

The main goal of the present paper is to initiate such a study for the κ -LYZ flow. One of our main results is that, for $\kappa \neq 1$, if the metrics ω_t are all equivalent and the form α_t are uniformly bounded, then the derivatives of both ω_t and α_t of all orders are uniformly bounded (Theorem 1). Thus the uniform boundedness of ω_t and α_t plays the role in the κ -LYZ system of the $C^{2,\alpha}$ estimate for linear elliptic equations, the C^0 estimate for the Monge–Ampère equation, and the uniform estimates for metrics in the continuity method for the problem of constant scalar curvature. The condition $\kappa \neq 1$ seems to be new and it is essential to our proof. It can be viewed as a restriction on how the Ricci flow can be coupled to the flow of the form α_t , and such restrictions may be important in future investigations of other coupled systems. The uniform bounds on ω_t and α_t in Theorem 1 are strong conditions, but we can show that they hold for suitable data and suitable values of κ in the case of Riemann surfaces (Theorem 2). A key tool in this case is an apparently new Liouville-entropy type energy for the coupled system which is monotone along the κ -LYZ flow. We can then establish the convergence of the κ -LYZ flow in this case. While this result is special, convergence

results are rather rare in general for coupled systems, and this is to our knowledge the only result on the convergence of a coupled Ricci flow available so far. When the manifold X has higher dimension, we introduce instead a modified Mabuchi energy which incorporates the additional field, and which can be shown to be monotone along the κ -LYZ flow. We can then show that, under the same hypotheses as in Theorem 1 and the additional assumption that the modified Mabuchi energy is bounded from below, the κ -LYZ flow converges smoothly to a smooth stationary point (Theorem 3).

It may be instructive to relate our results to some recent ones in the literature. In the original paper of Li et al.[12], Shi-type estimates were established for all derivatives of the curvature, assuming that the curvature is bounded. But Shi-type estimates hold only for finite-time intervals, while we are here interested in bounds that are uniform for all time. Another closely related problem is the problem of Kähler metrics of constant scalar curvature, which has already been mentioned several times. It is an elliptic equation of 4th-order in the potential, which can also be expressed as a coupled elliptic system in the potential and the volume, viewed as two separate unknowns.¹ It is then well-known that the condition of uniform boundedness for the metrics in the method of continuity would imply estimates of all orders and the solvability of the equation. The extension of this result to the parabolic case does not seem available in the literature as yet. Additional difficulties result from the need to estimate the partial derivatives in time, and the resulting complicated mixing of the two unknowns. Theorem 1 in the present paper can be interpreted as such a parabolic result for the κ -LYZ equation. Very recently, Chen and Cheng [2–4] made a breakthrough on the constant scalar curvature problem by showing in particular how the uniform boundedness of the metrics in the elliptic approach follows from the properness of the Mabuchi functional. It is however not known at this moment how to adapt their methods to the parabolic setting. For example, the classical parabolic approach to the constant scalar curvature problem, namely by the Calabi flow, still remains an open problem at this time.

2 Estimates for higher order derivatives of ω and α

We begin with some simplifying remarks on the normalization of the κ -flow (1.1, 1.2). Clearly, if $(\omega_\infty, \alpha_\infty)$ is a stationary point of the flow, then the Kähler classes $[\omega_\infty], [\alpha_\infty]$ must satisfy

$$-2\pi c_1(X) + \lambda[\omega_\infty] + [\alpha_\infty] = 0. \quad (2.1)$$

It is natural then to consider initial data ω_0, α_0 in cohomology classes satisfying the same condition. Since the flow preserves the cohomology class of α_0 , it means that we should assume the following condition on the initial data

$$-2\pi c_1(X) + \lambda[\omega_0] + [\alpha_0] = 0. \quad (2.2)$$

In general, it is not difficult to work out explicitly the dependence of $[\omega_t]$ on time, and to see that (2.2) is also a necessary condition for the convergence of $[\omega_t]$ when $\lambda \geq 0$. Thus it is simplest to assume henceforth the condition (2.2) on the initial data (ω_0, α_0) . As a consequence $[\omega_t]$ is constant in time. Moreover, the κ -LYZ flow is invariant under the scaling $\omega \rightarrow M\omega, t \rightarrow Mt, \alpha \rightarrow \alpha, \lambda \rightarrow \lambda/M$, therefore we may always assume that

$$V = \int_X \frac{\omega_t^n}{n!} = 1. \quad (2.3)$$

¹ A similar decomposition was used by Trudinger and Wang [24] for the affine Plateau problem.

In this section, we shall prove the following theorem:

Theorem 1 *Assume that the κ -LYZ flow exists on a time interval $[0, T)$ and that the following basic assumption holds: there exists a positive constant $K > 1$ so that*

$$K^{-1}\omega_0 \leq \omega_t \leq K\omega_0, \quad -K\omega_t \leq \alpha_t \leq K\omega_t \quad (2.4)$$

for all $t \in [0, T)$. Let the endomorphisms h_t be defined by $h^p_q = \omega_0^{p\bar{m}} \omega_{\bar{m}q}$. If $0 < \kappa \neq 1$, then for any integer $k \geq 0$, there exists a constant C_k depending only on κ, K, k, n, λ and the initial data ω_0 and α_0 so that

$$\sup_{X \times [0, T)} (|\nabla^k h| + |\nabla^k \alpha|) \leq C_k. \quad (2.5)$$

These estimates are similar in spirit to the ones for Kähler–Einstein metrics, in the sense that all estimates in that case follow from the C^0 estimate. The difference resides in the fact that the estimates in the Kähler–Einstein case are estimates for the potential, while here they are for the metrics and $(1, 1)$ -form α . The fact that we are dealing here with a system also creates many new difficulties.

As in [12], we shall make use of the reformulation of the flow in terms of potentials. Thus set

$$\omega_t = \omega_0 + i\partial\bar{\partial}\varphi_t, \quad \alpha_t = \alpha_0 - i\partial\bar{\partial}F_t, \quad (2.6)$$

for smooth functions φ_t and F_t determined up to additive terms which are constants in space, but which may depend on time. We can fix these additive terms by requiring that the κ -LYZ be equivalent to the following coupled flow of the system (φ_t, F_t) ,

$$\frac{\partial \varphi}{\partial t} = \log \left(\frac{\omega_t^n}{\omega_0^n} \right) - H_0 + \lambda \varphi - F, \quad (2.7)$$

$$\frac{\partial F}{\partial t} = \kappa \Delta_{\omega_t} F - \kappa \operatorname{tr}_{\omega_t} \alpha_0 + \kappa b. \quad (2.8)$$

Here $H_0 \in C^\infty$ is the Ricci potential defined by

$$\operatorname{Ric}(\omega_0) - \lambda \omega_0 - \alpha_0 = i\partial\bar{\partial}H_0 \quad \int_X e^{H_0} \omega_0^n = \int_X \omega_0^n = n!V, \quad (2.9)$$

and b is a time-independent constant chosen so that

$$\int_X \dot{F} \omega_t^n = 0, \quad \text{hence } b = n \int_X \alpha_0 \wedge \omega_0^{n-1}. \quad (2.10)$$

In this form, the κ -LYZ flow with $\lambda = 0$ and α the Ricci form of some metric is a parabolic version of the elliptic system of equations for the potentials of ω and α considered by Chen and Cheng [2–4].

By the assumption (2.4), we know that $\partial\bar{\partial}\varphi_t$ and $\partial\bar{\partial}F_t$ are both uniformly bounded with respect to the fixed Kähler metric ω_0 . Our goal is to derive higher order derivative estimates of ω_t and α_t , beginning with the third order derivative estimate for φ_t . For the Monge–Ampère equation, C^3 estimates for the potential are obtained using the maximum principle and the Calabi identity (see e.g. [25]). Here the Calabi identity and the maximum principle do not suffice, because the coupling between ω_t and α_t results in terms that involve the derivatives of α_t and hence are not a priori bounded. We overcome these difficulties by combining the method of [18], which considers instead the evolution of the connection defined by ω_t , with a parabolic Moser iteration argument. For this parabolic Moser argument, it is crucial to have

a uniform lower bound for the scalar curvature of ω_t , and this is where the condition $\kappa \neq 1$ is needed.

To begin with, we establish a lower bound for the scalar curvature R .

Lemma 1 *For any $0 < \kappa \neq 1$, there exists a constant $C = C(\omega_0, \kappa, K, \lambda) > 0$ such that*

$$R \geq -C,$$

where $R = R(\omega_t)$ is the scalar curvature of ω_t .

Proof From the definition of the κ -LYZ flow, it is easy to derive the flows for R and $\text{tr}_\omega \alpha$,

$$\begin{aligned} \frac{\partial R}{\partial t} &= \frac{\partial}{\partial t} \left(-g^{i\bar{j}} \partial_i \partial_{\bar{j}} \log \det g \right) \\ &= -\Delta (-R + n\lambda + \text{tr}_\omega \alpha) + R_{\bar{j}i} \left(R_{i\bar{j}} - \lambda g_{i\bar{j}} - \alpha_{i\bar{j}} \right) \\ &= \Delta R - \Delta \text{tr}_\omega \alpha + |\text{Ric}|^2 - \lambda R - R_{\bar{j}i} \alpha_{i\bar{j}} \end{aligned} \quad (2.11)$$

while

$$\frac{\partial}{\partial t} \text{tr}_\omega \alpha = \kappa \Delta \text{tr}_\omega \alpha + \left(R_{i\bar{j}} - \lambda g_{i\bar{j}} - \alpha_{i\bar{j}} \right) \alpha_{\bar{j}i}.$$

For $\kappa \neq 1$, we can combine these two equations to get

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta \right) \left(R + \frac{\text{tr}_\omega \alpha}{\kappa - 1} \right) &= |\text{Ric}|^2 - \lambda R - \lambda \text{tr}_\omega \alpha - |\alpha|^2 - \frac{\kappa - 2}{\kappa - 1} R_{\bar{j}i} \alpha_{i\bar{j}} \\ &\geq \frac{R^2}{2n} - \lambda R - C \\ &\geq \frac{1}{4n} \left(R + \frac{\text{tr}_\omega \alpha}{\kappa - 1} \right)^2 - C, \end{aligned}$$

where in the first inequality we use the fact that $n|\text{Ric}|^2 \geq R^2$. We can apply now the maximum principle to the function $R + \text{tr}_\omega \alpha / (\kappa - 1)$ and obtain a uniform lower bound for $R + \text{tr}_\omega \alpha / (\kappa - 1)$. Since by assumption $\text{tr}_\omega \alpha$ is bounded, the lower bound for R follows.

We are now ready to prove the Calabi C^3 -estimates of φ , under the assumption (2.4).

Lemma 2 *There exists a constant $C = C(\omega_0, \kappa, K, \lambda) > 0$ such that*

$$\sup_{X \times [0, T]} |\nabla \partial \bar{\partial} \varphi|_{\omega_0} \leq C.$$

Proof For notation convenience we will denote $\hat{g}$ the metric associated to the fixed Kähler form ω_0 . We define

$$S_{i\bar{j}}^k := \Gamma_{i\bar{j}}^k - \hat{\Gamma}_{i\bar{j}}^k$$

where $\Gamma_{i\bar{j}}^k = g^{k\bar{l}} \frac{\partial g_{i\bar{l}}}{\partial z^j}$ is the Christoffel symbol of $\omega = \omega_t$ and $\hat{\Gamma}_{i\bar{j}}^k$ is that of $\hat{g}$. Because of the equivalence of ω_t and ω_0 by the assumption (2.4), to prove the lemma it suffices to show that $|S|_\omega^2 = S_{i\bar{j}}^k \overline{S_{p\bar{q}}^k} g^{i\bar{p}} g^{j\bar{q}} g_{r\bar{k}}$ is uniformly bounded.

We calculate under the normal coordinates of $\omega = \omega_t$ at some fixed point,

$$\begin{aligned}
\frac{\partial}{\partial t}|S|^2 &= 2 \left(R_{\bar{i}p} - \lambda g_{\bar{i}p} - \alpha_{\bar{i}p} \right) S_{ij}^k \overline{S_{pj}^k} + (-R_{\bar{r}k} + \lambda g_{\bar{r}k} + \alpha_{\bar{r}k}) S_{ij}^k \overline{S_{ij}^r} + 2Re \left(\frac{\partial S_{ij}^k}{\partial t} \overline{S_{ij}^k} \right) \\
&= 2 \left(R_{\bar{i}p} - \lambda g_{\bar{i}p} - \alpha_{\bar{i}p} \right) S_{ij}^k \overline{S_{pj}^k} + (-R_{\bar{r}k} + \lambda g_{\bar{r}k} + \alpha_{\bar{r}k}) S_{ij}^k \overline{S_{ij}^r} \\
&\quad + 2Re \left(\nabla_j (-R_{\bar{k}i} + \alpha_{\bar{k}i}) \overline{S_{ij}^k} \right).
\end{aligned}$$

And if we denote $S_{ij,p}^k = \nabla_p S_{ij}^k$ and $S_{ij,p\bar{p}}^k = \bar{\nabla}_{\bar{p}} \nabla_p S_{ij}^k$, then

$$\begin{aligned}
\Delta|S|^2 &= \left(S_{ij}^k \overline{S_{ij}^k} \right)_{p\bar{p}} \\
&= S_{ij,p\bar{p}}^k \overline{S_{ij}^k} + S_{ij,p}^k \overline{S_{ij,p}^k} + S_{ij,\bar{p}}^k \overline{S_{ij,\bar{p}}^k} + S_{ij}^k \overline{S_{ij,p\bar{p}}^k} \\
&= \overline{S_{ij}^k} \left(S_{ij,p\bar{p}}^k + S_{mj}^k R_{\bar{m}i} + S_{im}^k R_{\bar{m}j} - S_{ij}^m R_{\bar{k}m} \right) + S_{ij}^k \overline{S_{ij,p\bar{p}}^k} + |\nabla S|^2 + |\bar{\nabla} S|^2.
\end{aligned}$$

Note that

$$\begin{aligned}
S_{ij,p\bar{p}}^k &= \nabla_p \bar{\nabla}_{\bar{p}} S_{ij}^k = \nabla_p \left(\bar{\partial}_{\bar{p}} \left(\Gamma_{ij}^k - \hat{\Gamma}_{ij}^k \right) \right) \\
&= \nabla_p \left(-R_{i\bar{p}j}^k + \hat{R}_{i\bar{p}j}^k \right) \\
&= -\nabla_j R_{i\bar{p}}^k + \left(\nabla_p - \hat{\nabla}_p \right) \hat{R}_{i\bar{p}j}^k + \hat{\nabla}_j \hat{R}_{i\bar{p}}^k
\end{aligned}$$

and the middle term on the RHS can be written as the form $S * \hat{R}m$.

Therefore we have

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta \right) |S|^2 &= -2\alpha_{\bar{i}p} S_{ij}^k \overline{S_{pj}^k} - \lambda |S|^2 + \alpha_{\bar{r}k} S_{ij}^k \overline{S_{ij}^r} + 2Re \left(\nabla_j \alpha_{\bar{k}i} \overline{S_{ij}^k} \right) \\
&\quad - 2Re \left(\left(\nabla_p - \hat{\nabla}_p \right) \hat{R}_{i\bar{p}j}^k \overline{S_{ij}^k} \right) - 2Re \left(\hat{\nabla}_j \hat{R}_{i\bar{p}}^k \overline{S_{ij}^k} \right) - |\nabla S|^2 - |\bar{\nabla} S|^2 \\
&\leq C|S|^2 + C|S| + 2Re \left(\nabla_j \alpha_{\bar{k}i} \overline{S_{ij}^k} \right) - |\nabla S|^2 - |\bar{\nabla} S|^2, \tag{2.12}
\end{aligned}$$

for some $C = C(\omega_0, K, \lambda) > 0$. Since $\nabla_j \alpha_{\bar{k}i}$ is not apriorily bounded, we cannot apply maximum principle as usual to estimate $|S|^2$. Instead we will bound $|S|^2$ by Moser iteration. We denote $\hat{u} = \max(|S|^2, 1) \geq 1$ and for any times $t_0 < t_1 \leq T' < T$ define a Lipchitz function $\xi(t)$ such that $\xi(t) = 0$ for $t \leq t_0$, $\xi(t) = 1$ for $t \geq t_1$ and $\xi(t) = \frac{t-t_0}{t_1-t_0}$ for $t \in [t_0, t_1]$. Multiplying both side of (2.12) by $\xi(t)\hat{u}^p$ for $p \geq 1$, we get for any $s \in [t_1, T]$

$$\begin{aligned}
&\int_X \frac{\xi \hat{u}^{p+1}}{p+1} \omega_t^n \Big|_{t=s} + \int_{t_1}^s \int_X \frac{4p}{(p+1)^2} |\nabla \hat{u}^{\frac{p+1}{2}}|^2 \omega_t^n dt \\
&\leq \int_{t_0}^T \int_X \left(C \hat{u}^{p+1} + 2\xi \hat{u}^p Re \left(\nabla_j \alpha_{\bar{k}i} \overline{S_{ij}^k} \right) - \xi \hat{u}^p (|\nabla S|^2 + |\bar{\nabla} S|^2) \right) \omega_t^n dt \\
&\quad + \int_{t_0}^T \xi'(t) \int_X \hat{u}^{p+1} \omega_t^n dt + \int_{t_0}^T \int_X \frac{\hat{u}^{p+1}}{p+1} (-R + \lambda + \text{tr}_\omega \alpha) \omega_t^n dt.
\end{aligned}$$

Because of the lower bound of R in Lemma 1, the last integral on the RHS is bounded above by

$$C \int_{t_0}^T \int_X \frac{1}{p+1} \hat{u}^{p+1} \omega_t^n dt.$$

To deal with the term involving $\nabla_j \alpha_{\bar{k}i} \overline{S_{ij}^k}$ we make use of integration by parts and Cauchy–Schwarz inequality as follows

$$\begin{aligned} \int_{t_0}^T \int_X 2\xi \hat{u}^p \operatorname{Re} \left(\nabla_j \alpha_{\bar{k}i} \overline{S_{ij}^k} \right) \omega_t^n dt &= \int_{t_0}^T \int_X -2\xi \hat{u}^p \operatorname{Re} \left(\alpha_{\bar{k}i} \overline{S_{ij,\bar{j}}^k} \right) - 2\xi p \hat{u}^{p-1} \operatorname{Re} \left(\alpha_{\bar{k}i} \nabla_j \hat{u} \overline{S_{ij}^k} \right) \\ &\leq \int_{t_0}^T \int_X C\xi \hat{u}^p |\bar{\nabla} S| + Cp\xi \hat{u}^p (|\nabla S| + |\bar{\nabla} S|) \\ &\leq \int_{t_0}^T \int_X Cp^2 \xi \hat{u}^p + \frac{1}{2} \xi \hat{u}^p (|\nabla S|^2 + |\bar{\nabla} S|^2), \end{aligned}$$

where in the first inequality we apply Kato's inequality $|\nabla|S|| \leq |\nabla S| + |\bar{\nabla} S|$. Combining the estimates and varying $s \in [t_1, T']$, we conclude that

$$\begin{aligned} \sup_{t \in [t_1, T']} \int_X \hat{u}^{p+1} \omega_t^n + \int_{t_1}^{T'} \int_X |\nabla \hat{u}^{\frac{p+1}{2}}|^2 \omega_t^n dt \\ \leq C \left((p+1)^3 + \frac{1}{t_1 - t_0} \right) \int_{t_0}^{T'} \int_X \hat{u}^{p+1} \omega_t^n dt =: A. \end{aligned}$$

We can now use the standard parabolic Moser iteration to conclude a sub-mean-value inequality (noting that the Sobolev inequality holds for ω_t by the assumption on the equivalence of the metrics ω_t and ω_0).

By Hölder inequality and the Sobolev inequality, we have

$$\begin{aligned} \int_{t_1}^{T'} \int_X \hat{u}^{(p+1)(1+\frac{1}{n})} &\leq \int_{t_1}^{T'} \left(\int_X \hat{u}^{p+1} \right)^{1/n} \left(\int_X \left(\hat{u}^{\frac{p+1}{2}} \right)^{\frac{2n}{n-1}} \right)^{(n-1)/n} dt \\ &\leq A^{1/n} \int_{t_1}^{T'} C_0 \left(\int_X |\nabla \hat{u}^{\frac{p+1}{2}}|^2 + \hat{u}^{p+1} \right) \\ &\leq 2C_0 A^{(n+1)/n}. \end{aligned} \quad (2.13)$$

Therefore

$$\left(\int_{t_1}^{T'} \int_X \hat{u}^{\frac{n+1}{n}(p+1)} \right)^{\frac{n}{(n+1)(p+1)}} \leq C_3^{1/(p+1)} \left((p+1)^3 + \frac{1}{t_1 - t_0} \right)^{1/(p+1)} \left(\int_{t_0}^{T'} \int_X \hat{u}^{p+1} \right)^{1/(p+1)}. \quad (2.14)$$

If we denote $G(p, t) = \left(\int_t^{T'} \int_X \hat{u}^p \right)^{1/p}$, the inequality (2.14) yields that

$$G \left(\frac{n+1}{n}(p+1), t_1 \right) \leq C_3^{1/(p+1)} \left((p+1)^3 + \frac{1}{t_1 - t_0} \right)^{1/(p+1)} G(p+1, t_0). \quad (2.15)$$

For simplicity we denote $\eta = (n+1)/n > 1$, and define a sequence of numbers $p_k + 1 = (p_{k-1} + 1)\eta = \dots = (p_0 + 1)\eta^k$. For any $0 < s_1 < s_2 < T'$, we define a sequence of times $t_k = s_1 + (1 - \eta^{-k})(s_2 - s_1)$, $t_0 = s_1$ and $t_\infty = s_2$. Applying (2.15) for the pairs $(p_k + 1, t_k)$, we get

$$G(p_{k+1} + 1, t_{k+1}) \leq C_3^{\frac{1}{(p_0+1)\eta^k}} \left((p_0 + 1)^3 + \frac{1}{s_2 - s_1} \right)^{\frac{1}{(p_0+1)\eta^k}} \eta^{\frac{3k}{(1+p_0)\eta^k}} G(p_k + 1, t_k),$$

iterating this inequality we get

$$\begin{aligned} G(\infty, s_2) &\leq C_3^{\sum_k \frac{1}{(1+p_0)\eta^k}} \left((p_0 + 1)^3 + \frac{1}{s_2 - s_1} \right)^{\sum_k \frac{1}{(p_0+1)\eta^k}} \eta^{\sum_k \frac{3k}{(1+p_0)\eta^k}} G(p_0 + 1, s_1) \\ &\leq C_4 C_3^{\frac{n+1}{1+p_0}} \left((p_0 + 1)^3 + \frac{1}{s_2 - s_1} \right)^{\frac{n+1}{(p_0+1)}} G(p_0 + 1, s_1). \end{aligned}$$

Let $p_0 = 1$, then the estimate above implies that

$$\sup_{X \times [s_2, T']} \hat{u} \leq C_5 \left(1 + \frac{1}{s_2 - s_1} \right)^{(n+1)/2} \left(\int_{s_1}^{T'} \int_X \hat{u}^2 \right)^{1/2}. \quad (2.16)$$

Since we do not know the L^4 -norm bound of $|S|$, we will use another iteration argument to replace the L^2 -norm of $\hat{u}$ on the RHS to the L^1 -norm. We fix a $p \in (0, 2)$, and denote

$$h(s) = \sup_{X \times [s, T']} \hat{u},$$

then (2.16) implies that

$$\begin{aligned} h(s_2) &\leq C_5 \left(1 + \frac{1}{s_2 - s_1} \right)^{(n+1)/2} h(s_1)^{\frac{2-p}{2}} \left(\int_{s_1}^{T'} \int_X \hat{u}^p \right)^{1/2} \\ &\leq \frac{1}{2} h(s_1) + C_5^{p/2} \left(1 + \frac{1}{s_2 - s_1} \right)^{(n+1)/p} \left(\int_{s_1}^{T'} \int_X \hat{u}^p \right)^{1/p}. \end{aligned} \quad (2.17)$$

Now for any $0 \leq t_1 < t_2 \leq T'$, we define a sequence of times $r_k = t_2 - (1 - \delta^k)(t_2 - t_1)$, for some $\delta \in (0, 1)$ to be determined later. Clearly $r_0 = t_2$ and $r_\infty = t_1$, and $r_k - r_{k+1} = (t_2 - t_1)(1 - \delta)\delta^k$. Iterating (2.17), we get

$$\begin{aligned} h(t_2) = h(r_0) &\leq \frac{1}{2} h(r_1) + C_5^{1/p} \left(1 + \frac{1}{r_0 - r_1} \right)^{(n+1)/p} \left(\int_{t_1}^{T'} \int_X \hat{u}^p \right)^{1/p} \\ &\leq \frac{1}{2^k} h(r_k) + C_5^{1/p} \left(\int_{t_1}^{T'} \int_X \hat{u}^p \right)^{1/p} \sum_{i=0}^{k-1} 2^{-i} \left(1 + \frac{1}{r_i - r_{i+1}} \right)^{(n+1)/p} \\ &\leq \frac{1}{2^k} h(r_k) + C_6^{1/p} \left(\int_{t_1}^{T'} \int_X \hat{u}^p \right)^{1/p} \frac{\sum_{i=0}^{k-1} 2^{-i} \delta^{-(n+1)i/p}}{(t_2 - t_1)^{(n+1)/p} (1 - \delta)^{(n+1)/p}}. \end{aligned}$$

If we choose $\delta \in (0, 1)$ such that $2\delta^{(n+1)/p} > 1$, then the summation on the RHS of the above converges. Noting that h is apriorily bounded (may not be uniform), and letting $k \rightarrow \infty$, we arrive at the desired estimate

$$h(t_2) \leq C_7^{1/p} \frac{1}{(t_2 - t_1)^{(n+1)/p}} \left(\int_{t_1}^{T'} \int_X \hat{u}^p \right)^{1/p}.$$

Setting $p = 1$, we get

$$\sup_{X \times [t_2, T']} \hat{u} \leq \frac{C}{(t_2 - t_1)^{n+1}} \int_{t_1}^{t_2} \int_X \hat{u} \omega_i^n dt \leq \frac{C}{(t_2 - t_1)^{n+1}} \int_{t_1}^{T'} \int_X (|S|^2 + 1) \omega_i^n dt,$$

for any times $0 \leq t_1 < t_2 < T' < T$. It suffices to bound the L^2 -norm of $|S|$ from above. By the Chern–Lu inequality [25], we have

$$\left(\frac{\partial}{\partial t} - \Delta\right) \text{tr}_{\omega_0} \omega \leq \lambda \text{tr}_{\omega_0} \omega + \text{tr}_{\omega_0} \alpha + B \text{tr}_{\omega} \omega_0 \text{tr}_{\omega_0} \omega - \delta_K |S|^2 \leq -\delta_K |S|^2 + C_K$$

where $\delta_K > 0$ is a constant depending only on K and the dimension n , and $-B$ is the lower bound of the bisectional curvature of ω_0 . Integrating the equation above we obtain

$$\begin{aligned} \frac{d}{dt} \int_X (\text{tr}_{\omega_0} \omega) \omega^n &\leq -\delta_K \int_X |S|^2 \omega^n + \int_X \text{tr}_{\omega_0} \omega (-R + \lambda + \text{tr}_{\omega} \alpha) \omega^n + C_K \\ &\leq -\delta_K \int_X |S|^2 \omega^n + C_K \end{aligned}$$

where in the last inequality we use the lower bound of R as in Lemma 1. Integrating over $t \in [t_1, T']$ we get the desired L^2 -bound of $|S|$ on $X \times [t_1, T']$. The proof of the upper bound $|S|^2$ is complete.

Now that we have the third order estimates for the potentials, we can come to the proof of the estimate for $\partial_t \omega$, which is a key estimate distinguishing the parabolic case from the elliptic case. Even so, we do not have as yet an estimate for the Riemann curvature tensor, which we circumvent below by a local Moser iteration argument:

Lemma 3 *There is a constant $C = C(\omega_0, \kappa, K, \lambda) > 0$ such that*

$$\sup_X \left| \frac{\partial \omega_t}{\partial t} \right| \leq C, \quad \forall t \in [0, T).$$

Proof Taking $\frac{\partial}{\partial t}$ on both sides of the Eq. (2.7) and adding the resulting equation to $\frac{1}{\kappa-1} \times$ (2.8), we get

$$\left(\frac{\partial}{\partial t} - \Delta\right) \left(\dot{\varphi} + \frac{\kappa}{\kappa-1} F\right) = \lambda \dot{\varphi} - \frac{\kappa}{\kappa-1} \text{tr}_{\omega_t} \alpha_0 + \frac{\kappa}{\kappa-1} b.$$

We denote $\Phi = \dot{\varphi} + \frac{\kappa}{\kappa-1} F$, then we calculate (the norm of $\partial \bar{\partial} \Phi$ is under the metric $\omega = \omega_t$)

$$\begin{aligned} &\left(\frac{\partial}{\partial t} - \Delta\right) |\partial \bar{\partial} \Phi|^2 \\ &= 2\Phi_{i\bar{j}} \left(\lambda \Phi - \frac{\lambda \kappa}{\kappa-1} F - \frac{\kappa}{\kappa-1} \text{tr}_{\omega_t} \alpha_0 \right)_{\bar{k}l} g^{j\bar{k}} g^{l\bar{i}} - 2\lambda |\partial \bar{\partial} \Phi|^2 \\ &\quad - 2\alpha_{i\bar{j}} \Phi_{\bar{k}l} \Phi_{\bar{p}q} g^{q\bar{k}} g^{j\bar{p}} g^{l\bar{i}} + \Phi^{j\bar{i}} \Phi^{l\bar{k}} R_{i\bar{j}k\bar{l}} - 2|\nabla \partial \bar{\partial} \Phi|^2. \end{aligned} \quad (2.18)$$

Since the Riemannian curvature $Rm(\omega_t)$ on the RHS of (2.18) is not a priori bounded, we cannot apply global (in space X) Moser iteration as in the proof of Lemma 2 to bound $|\partial \bar{\partial} \Phi|^2$. Instead we will use the local expression of Rm , the Calabi C^3 -estimate in Lemma 2 and a local Moser iteration argument. To begin with, we can choose a cover of X by Euclidean balls $\{B_a, z_a^i\}_a$, where z_a^i are the complex coordinates. Without loss of generality we may assume B_a are Euclidean balls with radius 3 and the balls $\{\frac{1}{3}B_a\}$ also cover X , here $\frac{1}{3}B_a$ denotes the concentric ball with radius $1/3$ of that of B_a . Recall that the curvature Rm is given by (in $B = B_a$)

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{i\bar{j}}}{\partial z^l \partial \bar{z}^k} + g^{p\bar{q}} \frac{\partial g_{q\bar{j}}}{\partial z^l} \frac{\partial g_{i\bar{p}}}{\partial \bar{z}^k},$$

and the second term is uniformly bounded in B by Lemma 2. We will use integration by parts to deal with the first term in $R_{i\bar{j}k\bar{l}}$. We first observe that

$$\nabla_p \Phi_{i\bar{j}} = \frac{\partial^3 \Phi}{\partial z^p \partial \bar{z}^i \partial \bar{z}^j} - \Gamma_{p\bar{j}}^k \Phi_{i\bar{k}},$$

$\Gamma_{p\bar{j}}^k$ is bounded by Lemma 2 so

$$|\nabla \partial \bar{\partial} \Phi|^2 \geq \frac{1}{2} |D \partial \bar{\partial} \Phi|^2 - C |\partial \bar{\partial} \Phi|^2,$$

where we use D to denote the ordinary derivatives in (B, z^i) .

We fix times $t_1 < t_2 < T' < T$ and radii $r_2 < r_1 \leq 3$, and define cut-off functions $\xi(t)$ as in the proof of Lemma 2 and $\rho(z) = \rho(|z|)$ such that $\rho = 1$ on $B(0, r_2)$ and $\rho = 0$ outside $B(0, r_1)$, and $|D\rho| \leq \frac{2}{r_1 - r_2}$. We denote $w = \max(|\partial \bar{\partial} \Phi|^2, 1)$. Multiply both sides of the Eq. (2.18) by $\xi \rho^2 w^p$ (for $p \geq 1$) and do integration by parts, then we get

$$\begin{aligned} & \frac{d}{dt} \left(\int_B \frac{\xi \rho^2 w^{p+1}}{p+1} \omega_t^n \right) + \int_B \frac{p\xi}{(p+1)^2} \left| \nabla \left(\rho w^{\frac{p+1}{2}} \right) \right|^2 - 2 \int_B \xi w^{p+1} |\nabla \rho|^2 \omega_t^n \\ & \leq \int_B \frac{\xi \rho^2}{p+1} w^{p+1} (-R + \lambda n + \text{tr}_{\omega_t} \alpha_t) \omega_t^n + \int_B \frac{\rho^2 w^{p+1} \xi'(t)}{p+1} + \int_B C w^{p+1} \\ & \quad - \frac{1}{2} |D \partial \bar{\partial} \Phi|^2 - \frac{2\kappa\xi}{\kappa-1} \rho^2 w^p \Phi^{j\bar{i}} (\lambda F + \text{tr}_{\omega_t} \alpha_0)_{i\bar{j}} - \xi \rho^2 w^p \Phi^{j\bar{i}} \Phi^{k\bar{l}} \frac{\partial}{\partial z^k} \left(\frac{\partial g_{i\bar{j}}}{\partial \bar{z}^{\bar{l}}} \right), \end{aligned} \quad (2.19)$$

we deal with the last two terms in the integral by integration by parts as follows: the second last term is equal to

$$\begin{aligned} & \int_B \frac{2\kappa\xi}{\kappa-1} \rho^2 (2\rho^{-1} w^p \rho_j \Phi^{j\bar{i}} + p w^{p-1} w_j \Phi^{j\bar{i}} + w^p \partial_j \Phi^{j\bar{i}} + w^p \Phi^{j\bar{i}} \partial_j \log \det g) (\lambda F + \text{tr}_{\omega_t} \alpha_0)_{i\bar{j}} \\ & \leq \int_B C p^2 \xi \rho^2 w^{p+1} + C \xi w^{p+1} |\nabla \rho|^2 + \frac{1}{10} \xi \rho^2 w^p |D \partial \bar{\partial} \Phi|^2, \end{aligned}$$

where in the inequality above we use the known estimates that $|\nabla F| \leq C$, $|\nabla \text{tr}_{\omega_t} \alpha_0| \leq C$ and $|\nabla \phi| \leq C$ which follows from the Eq. (2.7) and Calabi C^3 -estimates. By integration by parts the last integral in (2.19) is equal to

$$\begin{aligned} & \int_B \xi \rho^2 \left(\rho^{-1} w^p \rho_k \Phi^{j\bar{i}} \Phi^{k\bar{l}} + p w^{p-1} w_k \Phi^{j\bar{i}} \Phi^{k\bar{l}} + w^p \partial_k \left(\Phi^{j\bar{i}} \Phi^{k\bar{l}} \right) + w^p \Phi^{j\bar{i}} \Phi^{k\bar{l}} \partial_k \log \det g \right) \frac{\partial g_{i\bar{j}}}{\partial \bar{z}^{\bar{l}}} \\ & \leq \int_B C p^2 \xi \rho^2 w^{p+1} + C \xi w^{p+1} |\nabla \rho|^2 + \frac{1}{10} \xi \rho^2 w^p |D \partial \bar{\partial} \Phi|^2. \end{aligned}$$

Substituting the above inequalities to (2.19) and arguing as in the proof of Lemma 2, we get

$$\begin{aligned} & \sup_{[t_2, T']} \int_{B_{r_2}} w^{p+1} \omega_t^n + \int_{t_2}^{T'} \int_{B_{r_2}} \left| \nabla \left(\rho w^{\frac{p+1}{2}} \right) \right|^2 \omega_t^n dt \\ & \leq C \left((p+1)^3 + \frac{1}{(r_1 - r_2)^2} + \frac{1}{-t_1 + t_2} \right) \int_{t_1}^{T'} \int_{B_{r_1}} w^{p+1} \omega_t^n dt. \end{aligned} \quad (2.20)$$

By Holder inequality and Sobolev inequality as in (2.13), we have

$$H((p+1)\eta, t_2, r_2) \leq C^{1/(p+1)} \left((p+1)^3 + \frac{1}{(r_1 - r_2)^2} + \frac{1}{-t_1 + t_2} \right)^{1/(p+1)} H(p+1, t_1, r_1)$$

where $H(p, t', r) = \left(\int_{t'}^{T'} \int_{B_r} w^p \omega_t^n dt \right)^{1/p}$. Taking $p_k + 1 = \eta^k(p_0 + 1)$, $r_k = 1 + \eta^{-k/2}$ and $t_k = T' - 1 - \eta^{-k}$, we can do the iteration argument and the trick as in the proof of Lemma 2 to conclude that

$$\sup_{[T'-1, T'] \times B_1} w \leq C \int_{T'-2}^{T'} \int_{B_2} w \omega_t^n dt.$$

As the last step, we have the following equation from Bochner formula

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta \right) |\nabla \Phi|^2 &= -|\partial \bar{\partial} \Phi|^2 - |\nabla \nabla \Phi|^2 - \lambda |\nabla \Phi|^2 - \alpha (\nabla \Phi, \bar{\nabla} \Phi) \\ &\quad - 2 \operatorname{Re} \left\langle \nabla \Phi, \bar{\nabla} \left(-\dot{\varphi} - \frac{\kappa}{\kappa - 1} \operatorname{tr}_{\omega} \alpha_0 \right) \right\rangle \leq -|\partial \bar{\partial} \Phi|^2 + C, \end{aligned}$$

integrating this equation over $[T' - 2, T'] \times X$, and by the lower bound of $-R$ we can obtain the $L^2([T' - 2, T'] \times X)$ -norm bound of $|\partial \bar{\partial} \Phi|$, hence that of w . This completes the proof of the bound of $|\partial \bar{\partial} \Phi|$ since the balls B_1 cover X by the original choice. Since $\partial \bar{\partial} F$ is uniformly bounded by assumption, this implies that $\partial \bar{\partial} \dot{\varphi}$ is bounded, so is $\frac{\partial \omega_t}{\partial t}$. The proof of Lemma 3 is complete.

We apply the linear theory of parabolic equations, which requires the coefficients $a^{ij}(x, t)$ in (2.21) below to be continuous with uniform modulus of continuity or uniform Hölder regularity in *both space and time directions*. Lemma 2 gives the uniform Lipschitz continuity of $g^{i\bar{j}}$ in space directions and Lemma 3 provides the Lipschitz regularity of $g^{i\bar{j}}$ in the time direction.

To continue with the higher order estimates, we use the standard linear theory for parabolic equations. For convenience we state the following $W_p^{2,1}$ -estimate (see Theorem 7.22 in [13]). For a smooth bounded domain $\Omega \subset \mathbf{R}^m$ and $T > 0$ we denote the parabolic cylinder

$$\Omega_T := \Omega \times [0, T] \subset \mathbf{R}^{m+1},$$

and the $W_p^{2,1}(\Omega_T)$ -norm of a function u in Ω_T is defined as

$$\|u\|_{W_p^{2,1}(\Omega_T)} = \sum_{j=0}^2 \|D^j u\|_{L^p(\Omega_T)} + \|\partial_t u\|_{L^p(\Omega_T)}.$$

Lemma 4 Let $u \in W_{p,loc}^{2,1}(\Omega_T) \cap L^p(\Omega_T)$ for some $p \geq 2$ satisfy the parabolic equation

$$\frac{\partial u}{\partial t}(x, t) = a^{ij}(x, t) u_{ij}(x, t) + f(x, t), \quad \text{in } \Omega_T \quad (2.21)$$

where a^{ij} is strictly elliptic in the sense that $\lambda |\xi|^2 \leq a^{ij} \xi_i \xi_j \leq \Lambda |\xi|^2$ for some $\lambda, \Lambda > 0$ and any $\xi \in \mathbf{R}^m$ and a^{ij} is continuous in Ω_T . Suppose $f \in L^p(\Omega_T)$, then for any domain $\Omega' \subset \subset \Omega$ and $\varepsilon_0 > 0$ there exists a constant $C = C(\lambda, \Lambda, d(\Omega', \partial\Omega), \omega_a, \varepsilon_0, m, p) > 0$ such that

$$\|D^2 u\|_{L^p(\Omega' \times [\varepsilon_0, T])} + \left\| \frac{\partial u}{\partial t} \right\|_{L^p(\Omega' \times [\varepsilon_0, T])} \leq C (\|f\|_{L^p(\Omega_T)} + \|u\|_{L^p(\Omega_T)}),$$

where ω_a denotes the modulus of continuity of a^{ij} .

In particular, if $f \in L^\infty(\Omega_T)$ and $u \in L^\infty(\Omega_T)$, by Sobolev embedding, we have $u \in C^{1+\beta, \frac{1+\beta}{2}}(\Omega_T \setminus \partial_P \Omega_T)$ for any $\beta \in (0, 1)$. Here $\partial_P \Omega_T$ denotes the parabolic boundary of Ω_T . It follows that we have the uniform estimate

$$\|u\|_{C^{1+\beta, \frac{1+\beta}{2}}(\Omega' \times [\varepsilon_0, T])} \leq C(\|f\|_{L^\infty(\Omega_T)} + \|u\|_{L^\infty(\Omega_T)}),$$

where the constant C depends in addition on β .

We make the following elementary observation from elliptic theory.

Lemma 5 *Under the assumptions (2.4), there is a constant $C = C(K, \omega_0) > 0$ such that*

$$\sup_{X \times [0, T]} (|\nabla F| + |\nabla \varphi|) \leq C.$$

Proof By the assumption (2.4), we have

$$|\Delta_{\omega_0}(\varphi - \bar{\varphi})| \leq C, \quad |\Delta_{\omega_0}(F - \bar{F})| \leq C,$$

where $\bar{\varphi}$ and $\bar{F}$ denote the average of φ and F with respect to ω_0^n , respectively. It is elementary to see the C^0 bound of $\varphi - \bar{\varphi}$ and $F - \bar{F}$. The desired gradient bound follows standard linear theory for elliptic equations. In fact we can also get the C^β -bound of $\nabla \varphi$ and ∇F for any $\beta \in (0, 1)$, though this is not needed for later calculations.

We choose (and fix) finite coordinate charts of X , $\{U_a\}$, $\{U'_a\}$ and $\{U''_a\}$ such that $U''_a \subset \subset U'_a \subset \subset U_a$ and $X = \cup_a U''_a$, and $\{z^i_a\}$ the complex coordinate functions on U_a . In the following we will work on any chosen U_a and omit the subscript a in U_a , U'_a , U''_a and z^i_a . We will fix times $t_0 \geq 0$, $t_1 = t_0 + 1$ and $t_2 = t_0 + 2$.

Lemma 6 *For any $\beta \in (0, 1)$, there exists a constant $C = C(\omega_0, \kappa, K, \beta) > 0$ such that*

$$\|\partial \bar{\partial} \varphi\|_{C^{1+\beta, \frac{1+\beta}{2}}(X \times [t_1, t_2])} \leq C$$

or equivalently

$$\|\omega_t\|_{C^{1+\beta, \frac{1+\beta}{2}}(X \times [t_1, t_2])} \leq C.$$

Proof We work on a coordinate chart $U = U_a$ as before. Taking $\frac{\partial^2}{\partial z^i \partial \bar{z}^j}$ on both sides of (2.7) we get on $U \times [t_0, t_2]$

$$\begin{aligned} \frac{\partial \varphi_{\bar{j}i}}{\partial t} &= \Delta_{\omega_t} \varphi_{\bar{j}i} - g^{p\bar{m}} \partial_{\bar{j}} g_{\bar{m}k} g^{k\bar{q}} \varphi_{i\bar{q}p} + g^{p\bar{q}} \partial_{\bar{j}i}^2 \hat{g}_{\bar{q}p} - g^{p\bar{m}} \partial_{\bar{j}} g_{\bar{m}k} g^{k\bar{q}} \partial_i \hat{g}_{\bar{q}p} + \hat{R}_{\bar{j}i} \\ &\quad - h_{0, \bar{j}i} - \varphi_{\bar{j}i} - F_{\bar{j}i} =: \Delta_{\omega_t} \varphi_{\bar{j}i} + f_1 \end{aligned} \quad (2.22)$$

where $f_1 \in L^\infty(U \times [t_0, t_2])$ by assumptions and Lemma 2. Applying the linear theory in Lemma 4 and Sobolev embedding theorem, we conclude that for any $\beta \in (0, 1)$ there is a uniform $C = C(\kappa, K, \beta, \omega_0) > 0$ such that ($\hat{t}_1 = t_1 - 1/2$ and $p = \frac{2n+2}{1-\beta}$)

$$\begin{aligned} \|\varphi_{\bar{j}i}\|_{C^{1+\beta, \frac{1+\beta}{2}}(U' \times [t_1, t_2])} &\leq \|D^2 \varphi_{\bar{j}i}\|_{L^p(U' \times [\hat{t}_1, t_2])} + \|\partial_t \varphi_{\bar{j}i}\|_{L^p(U' \times [\hat{t}_1, t_2])} \\ &\leq C \left(\|f_1\|_{L^\infty(U \times [t_0, t_2])} + \|\varphi_{\bar{j}i}\|_{L^\infty(U \times [t_0, t_2])} \right) \leq C. \end{aligned} \quad (2.23)$$

Since $\{U'_a\}$ covers X , the lemma follows.

Lemma 7 For any $\beta \in (0, 1)$, there exists a constant $C = C(\omega_0, \kappa, K, \beta) > 0$ such that

$$\|\nabla F\|_{C^{1+\beta, \frac{1+\beta}{2}}(X \times [t_1, t_2])} \leq C, \quad \text{and} \quad \|\alpha\|_{C^{\beta, \frac{\beta}{2}}(X \times [t_1, t_2])} \leq C.$$

Proof We work on a coordinate chart $U = U_a$. Taking $\frac{\partial}{\partial \bar{z}^i}$ on both sides of the Eq. (2.8) we have the following equation holds on $U \times [t_0, t_2]$

$$\frac{\partial F_i}{\partial t} = \kappa \Delta_{\omega_t} F_i - \kappa g^{p\bar{m}} \partial_i g_{\bar{m}k} g^{k\bar{q}} F_{\bar{q}p} - \kappa \partial_i (\text{tr}_{\omega_t} \alpha_0) =: \kappa \Delta_{\omega_t} F_i + f. \quad (2.24)$$

By the Calabi C^3 -estimates in Lemma 2 we see that $\|f\|_{L^\infty(U \times [t_0, t_2])} \leq C$ for a uniform constant $C > 0$. Since the coefficients of $\kappa \Delta_{\omega_t}$ is Lipschitz continuous with a uniform Lipschitz constant by Lemmas 2 and 3, we can now apply the linear theory in Lemma 4 to conclude that for any $p \geq 2$

$$\begin{aligned} & \|D^2 F_i\|_{L^p(U' \times [t_1 - 1/2, t_2])} + \|\partial_t F_i\|_{L^p(U' \times [t_1 - 1/2, t_2])} \\ & \leq C(p)(\|f\|_{L^\infty(U \times [t_0, t_2])} + \|F_i\|_{L^\infty(U \times [t_0, t_2])}) \leq C, \end{aligned} \quad (2.25)$$

thanks to Lemma 5 on the gradient bound of F . By Sobolev embedding then we get $\|F_i\|_{C^{1+\beta, \frac{1+\beta}{2}}(U' \times [t_1, t_2])} \leq C$. Taking $\frac{\partial}{\partial \bar{z}^j}$ on both sides of (2.24) we get the equation for $F_{\bar{j}i}$

$$\frac{\partial F_{\bar{j}i}}{\partial t} = \kappa \Delta_{\omega_t} F_{\bar{j}i} + \kappa \partial_{\bar{j}} g^{m\bar{k}} F_{i\bar{k}m} + f_{\bar{j}} =: \kappa \Delta_{\omega_t} F_{\bar{j}i} + f_2. \quad (2.26)$$

We observe that the terms in f_2 in (2.26) involve either $\partial_{\bar{k}l}^2 g_{\bar{p}q}$ or $DF_{\bar{k}l}$, besides the bounded factors. By (2.23) and (2.25) we know $\partial_{\bar{k}l}^2 g_{\bar{p}q}$ and $DF_{\bar{k}l}$ are both bounded in $L^p(U' \times [t_1 - 1/2, t_2])$ for any $p \geq 2$, so is f_2 by Holder inequality. Applying linear theory in Lemma 4 and Sobolev embedding theorem again, we conclude that $(\hat{t}_1 = t_1 - 1/2$ and $p = 2 \cdot \frac{2n+2}{1-\beta})$

$$\begin{aligned} \|F_{\bar{j}i}\|_{C^{1+\beta, \frac{1+\beta}{2}}(U'' \times [t_1, t_2])} & \leq C(\|D^2 F_{\bar{j}i}\|_{L^p(U' \times [t_1, t_2])} + \|\partial_t F_{\bar{j}i}\|_{L^p(U' \times [t_1, t_2])}) \\ & \leq C\left(\|F_{\bar{j}i}\|_{L^p(U \times [\hat{t}_1, t_2])} + \|f_2\|_{L^p(U \times [\hat{t}_1, t_2])}\right) \leq C. \end{aligned}$$

Since $\{U''_a\}$ covers X , the lemma follows from the fact that $\alpha_t = \alpha_0 + i\partial\bar{\partial}F$.

Proof of Theorem 1 We can now apply the standard bootstrap argument to finish the proof of Theorem 1. Taking $\frac{\partial}{\partial \bar{z}^k}$ on both sides of (2.22), the lower order terms in the resulted equation are bounded uniformly in L^p for any $p \geq 2$ by Lemmas 6 and 7, and this gives rise to $W_{p,loc}^{2,1}$ estimates of $\varphi_{\bar{j}i\bar{k}}$. Feed these estimates into the equation obtained by taking $\frac{\partial}{\partial \bar{z}^k}$ on both sides of (2.26). By similar argument this yields $W_{p,loc}^{2,1}$ estimates of $F_{\bar{j}i\bar{k}}$. We can repeat this process any finite number of times to conclude that $D^k \varphi_{\bar{j}i}$ and $D^k F_{\bar{j}i}$ are both in $W_{p,loc}^{2,1}$ for any $p \geq 2$ and $k \in \mathbf{Z}_+$. The higher order estimates of $\varphi_{\bar{j}i}$ and $F_{\bar{j}i}$ then follows from Sobolev embedding theorem, so do the higher order derivative estimates of ω_t and α_t . Theorem 1 is proved.

We remark that perhaps the most important consequence of the assumption (2.4) is that the Sobolev constant is uniformly bounded along the flow. This is known to be true, but highly non-trivial, along the Kähler–Ricci flow [17,26]. It would clearly be desirable to determine whether, or under what conditions, the Sobolev constant will be uniform along the κ -LYZ flow.

3 The case of Riemann surfaces

Both the Ricci flow and the Calabi flow have been completely solved on Riemann surfaces [5, 9, 22]. Thus the case of Riemann surfaces provides an excellent laboratory where to examine the complications arising from couplings to an additional field. It is easy to see that stationary points for the κ -LYZ flow always exist. Even so, the convergence of the κ -LYZ flow is not evident. What we can prove is the following:

Theorem 2 *Consider the κ -LYZ flow on a compact Riemann surface X with negative Euler characteristic. Write $\alpha_t = \tau_t \omega_t$ for a smooth function τ_t . Assume that at initial time, we have*

$$R(\omega_0) < 0 \text{ and } \tau_0 > 0 \quad (3.1)$$

everywhere on X . Then there exists a positive constant κ_0 such that for any $\kappa > \kappa_0$, $\kappa \neq 0$, the κ -LYZ flow converges in C^∞ to a pair $(\omega_\infty, \alpha_\infty)$ with

$$R(\omega_\infty) = \lambda + \tau_\infty, \quad \tau_\infty = \int_X \alpha_0 \quad (3.2)$$

both constants. Actually, κ_0 can be taken to be any constant with $R_0 + \frac{1}{\kappa_0 - 1} \tau_0 < 0$.

We observe that the conditions on the initial data of the theorem imply that $\lambda < 0$. Assuming this, it is easy to construct many initial data satisfying the conditions. Given a negatively curved compact Riemann surface (X, ω_0) , there exists a sufficiently negative constant λ such that the cohomology class $-\lambda[\omega_0] + 2\pi c_1(X)$ lies in the Kähler cone. We can then choose a positive closed $(1, 1)$ -form α_0 in this class.

Proof of Theorem 2 The κ -LYZ flow had been written earlier in terms of potentials (2.7) and (2.8)). Here, we shall exploit the setting of Riemann surfaces in order to write the flow in terms of the conformal factor ϕ and the trace τ defined by

$$\omega = e^\phi \omega_0, \quad \tau \equiv \text{tr}_\omega \alpha = g^{z\bar{z}} \alpha_{\bar{z}z}. \quad (3.3)$$

Clearly the flow (1.1) can be rewritten in terms of ϕ as

$$\dot{\phi} = \frac{\partial \phi}{\partial t} = -R + \lambda + \tau, \quad (3.4)$$

while the flow (1.2) for α can be rewritten in terms of τ as

$$\partial_t \tau = \kappa \Delta \tau - \tau(-R + \lambda + \tau). \quad (3.5)$$

We can rewrite the Eq. (2.11) for the evolution equation of the scalar curvature R as follows

$$\partial_t R = \Delta R - \Delta \tau - R(-R + \lambda + \tau). \quad (3.6)$$

We have seen the advantage of taking the coefficient $\kappa \neq 1$ in (1.2) in obtaining the heat inequality (2.12). For Riemann surfaces, this inequality is strengthened into a heat equation for $R + \frac{1}{\kappa - 1} \tau$,

$$\partial_t \left(R + \frac{1}{\kappa - 1} \tau \right) = \Delta \left(R + \frac{1}{\kappa - 1} \tau \right) - \left(R + \frac{1}{\kappa - 1} \tau \right) (-R + \lambda + \tau). \quad (3.7)$$

3.1 Estimates for τ and R

From the Eqs. (3.5) and (3.7), it follows by the maximum principle that

$$\max_X \left(R(\omega_t) + \frac{1}{\kappa - 1} \tau_t \right) < 0, \quad \text{and} \quad \min_X \tau_t > 0,$$

therefore $\max_X R(\omega_t) < 0$. From (3.5) we have at the maximum point of τ_t

$$\begin{aligned} \frac{d\tau_{\max}}{dt} &\leq \tau_{\max} R - \tau_{\max} \lambda - \tau_{\max}^2 \\ &\leq -\frac{\kappa}{\kappa - 1} \tau_{\max}^2 - \lambda \tau_{\max} = -\frac{\kappa}{\kappa - 1} \tau_{\max} \left(\tau_{\max} + \frac{\kappa - 1}{\kappa} \lambda \right), \end{aligned} \quad (3.8)$$

- if $\tau_{\max}|_{t=0} < (\kappa - 1)|\lambda|/\kappa$, by (3.8), it follows that $\tau_{\max} + (\kappa - 1)\lambda/\kappa < 0$ for all $t > 0$.
- if $\tau_{\max}|_{t=0} \geq (\kappa - 1)|\lambda|/\kappa$, by solving the ODE

$$f' = -\frac{\kappa}{\kappa - 1} f^2 - \lambda f, \quad f(0) = \tau_{\max}|_{t=0} =: a \geq (\kappa - 1)|\lambda|/\kappa,$$

and comparing $\tau_{\max}$ with this f we get

$$\tau_{\max}(t) \leq f(t) = (\kappa - 1) \frac{|\lambda|}{\kappa} \frac{\frac{a}{a + (\kappa - 1)\lambda/\kappa}}{\frac{a}{a + (\kappa - 1)\lambda/\kappa} - e^{\lambda t}} \leq (\kappa - 1) \frac{|\lambda|}{\kappa} \frac{\frac{a}{a + (\kappa - 1)\lambda/\kappa}}{\frac{a}{a + (\kappa - 1)\lambda/\kappa} - 1}.$$

Therefore we get in both cases $\tau_{\max} \leq C(a, \lambda)$.

Similarly we deal with $R + \frac{1}{\kappa - 1} \tau$. Applying maximum principle to $R + \frac{1}{\kappa - 1} \tau$ in (3.7) we get

$$\frac{d}{dt} \left(R + \frac{1}{\kappa - 1} \tau \right)_{\min} \geq - \left(R + \frac{1}{\kappa - 1} \tau \right)_{\min} \left(- \left(R + \frac{1}{\kappa - 1} \tau \right)_{\min} + \lambda + \frac{\kappa}{\kappa - 1} \tau \right).$$

For notation simplicity we denote $h = -(R + \frac{1}{\kappa - 1} \tau)_{\min} > 0$, then

$$\frac{dh}{dt} \leq -h^2 - h \left(\lambda + \frac{\kappa}{\kappa - 1} \tau \right) \leq -h^2 + |\lambda|h = -h(h - |\lambda|),$$

by similar argument as above, if $h|_{t=0} < |\lambda|$, then $h \leq |\lambda|$ for all $t > 0$; otherwise we have

$$h(t) \leq |\lambda| \frac{\frac{b'}{b' - |\lambda|}}{\frac{b'}{b' - |\lambda|} - 1} = C(\lambda, b'), \quad \text{if } h_{t=0} := b' \geq |\lambda|.$$

So we find that $-C(\lambda, b') \leq R + \frac{1}{\kappa - 1} \tau < 0$ for all $t > 0$. On the other hand, from $0 < \tau \leq C(\lambda, a)$, we derive that the scalar curvature R satisfies $-C(\lambda, a, b') \leq R < 0$. Let us summarize what we have proven so far:

Lemma 8 *Under the conditions in Theorem 2 on the initial values, there is a uniform constant C depending only on the initial values, λ and $\kappa > \kappa_0$ such that*

$$0 < \tau \leq C, \quad -C \leq R < 0.$$

3.2 A Liouville-entropy functional

Motivated by the Liouville energy studied in conformal geometry (see for example [22]), we define a functional $E(\phi, \tau)$ associated to the LYZ flow as follows, assuming that τ is a positive function:

$$E(\phi, \tau) = \int_X \left(\frac{1}{2} |\nabla \phi|_{\omega_0}^2 + R_0 \phi - \lambda e^\phi - \tau e^\phi \right) \omega_0 + \int_X (\tau \log \tau) e^\phi \omega_0, \quad (3.9)$$

where $R_0 = R(\omega_0)$ is the scalar curvature of ω_0 . Clearly E is the sum of an “energy part” and an “entropy part”. We have

Lemma 9 *We consider the κ -LYZ flow (1.1, 1.2). Then*

- (a) *The functional $E(\phi, \tau)$ is monotone non-increasing along the flow.*
- (b) *Under the assumptions of Theorem 2 on the initial data (ω_0, α_0) and the choice of $\kappa > \kappa_0$, there is a constant $C > 0$ such that*

$$E(\phi, \tau) \geq \int_X \frac{1}{4} |\nabla \phi|_{\omega_0}^2 \omega_0 + 2\pi |\chi| \cdot |\bar{\phi}| - C, \quad \forall t \geq 0. \quad (3.10)$$

where χ is the Euler characteristic of X , and $\bar{\phi}$ is the average of the function ϕ with respect to the initial metric ω_0 .

- (c) *For any $p \geq 1$, there exists a constant C_p so that, along the κ -LYZ flow, we have*

$$\int_X e^{p|\phi|} \omega_0 \leq C_p. \quad (3.11)$$

Proof To prove (a), we calculate the variation of E along the flow: (we denote $\dot{\phi} = \frac{\partial \phi}{\partial t}$)

$$\begin{aligned} \frac{d}{dt} E(\phi, \tau) &= \int_X (-\Delta_{\omega_0} \phi + R_0 - \lambda e^\phi - \tau e^\phi) \dot{\phi} \omega_0 - \int_X \dot{\tau} e^\phi \omega_0 \\ &\quad + \int_X ((\dot{\tau} \log \tau) e^\phi \omega_0 + \dot{\tau} e^\phi \omega_0 + (\tau \log \tau) \dot{\phi} e^\phi \omega_0) \\ &= \int_X -(\dot{\phi})^2 e^\phi \omega_0 + \int_X (\kappa e^{-\phi} \Delta_{\omega_0} \tau (\log \tau) e^\phi \omega_0 - \tau \dot{\phi} (\log \tau) e^\phi \omega_0 + (\tau \log \tau) \dot{\phi} e^\phi \omega_0) \\ &= - \int_X (\dot{\phi})^2 e^\phi \omega_0 - \kappa \int_X \frac{|\nabla \tau|_{\omega_0}^2}{\tau} \omega_0 \\ &= - \int_X (\dot{\phi})^2 \omega_t - \kappa \int_X \frac{|\nabla \tau|_{\omega_t}^2}{\tau} \omega_t \leq 0. \end{aligned}$$

This proves (a).

To prove (b), we note that the volume of ω_t has been normalized to be 1, and thus

$$\int_X e^\phi \omega_0 = 1. \quad (3.12)$$

Since $\tau \log \tau$ is bounded from below, it follows that the contribution of the entropy term is bounded from below. Furthermore, we have seen that under the conditions of the lemma, λ is negative and τ is bounded. Thus we can write

$$\begin{aligned} E(\phi, \tau) &\geq \int_X \frac{1}{2} |\nabla_0 \phi|^2 \omega_0 + R_0 \phi \omega_0 - C \\ &= \int_X \frac{1}{2} |\nabla_0 \phi|^2 \omega_0 + R_0 (\phi - \bar{\phi}) \omega_0 + 2\pi \chi \bar{\phi} - C \geq \int_X \frac{1}{4} |\nabla_0 \phi|^2 \omega_0 + 2\pi \chi \bar{\phi} - C \end{aligned}$$

where we changed constants and applied the Poincaré inequality to absorb the integral of $R_0(\phi - \bar{\phi})$ into the first term on the right hand side. Applying Jensen's inequality to the normalization Eq. (3.12) shows that $\bar{\phi} \leq 0$. Since $\chi < 0$, we obtain (b).

Finally, (c) follows from (b) and the Trudinger inequality in two dimensions, since (b) implies that both $\|\nabla\phi\|_{\omega_0}$ and $\bar{\phi}$ are bounded. The proof of the lemma is complete.

3.3 Uniform bounds for ϕ

With the uniform bound for R and the L^p estimates for e^ϕ in (3.11), we can apply the arguments of Struwe [22] to obtain uniform bounds for ϕ ,

$$\|\phi\|_{C^0} \leq C. \quad (3.13)$$

The details are as follows. First, we state the concentration/compactness theorem due to Struwe [22, Theorem 3.2] as a lemma.

Lemma 10 *Let $\omega_j = e^{\phi_j}\omega_0$ be a sequence of metrics on a compact Riemann surface X , with $[\omega_j] = 1$ and Calabi energy $\|R_j\|_{L^2(\omega_j)}^2$ uniformly bounded. Then*

- (a) *Either there exists a constant C so that $\|\phi_j\|_{H_{(2)}(\omega_0)} \leq C$ for all j ;*
- (b) *or there exists a finite set Z of points $z_1, \dots, z_k$ such that, for any $z_k \in Z$ and any neighborhood V around z_k , we have*

$$\liminf_{j \rightarrow \infty} \int_V |R_j| \omega_j \geq 2\pi. \quad (3.14)$$

The cardinality of Z can be bounded in terms of the uniform upper bound on the Calabi energy.

Returning to the proof of the uniform bound for ϕ , it suffices now to show that the L^p bound for $e^{|\phi|}$ and the uniform bound for R rule out the existence of points z_k satisfying the condition (3.14). The above lemma of Struwe would imply then that $\|u_j\|_{H_{(2)}(\omega_j)} \leq C$, which implies, in two real dimensions, that there exists a positive constant $\alpha \in (0, 1)$ with $\|\phi\|_{C^\alpha} \leq C$.

For each z in X and each $r > 0$, let $B(z, r)$ be the ball centered at z and of radius r with respect to the fixed metric ω_0 . Then we write

$$\int_{B(z,r)} |R_j| \omega_j \leq \left(\int_{B(z,r)} |R_j|^2 \omega_j \right)^{\frac{1}{2}} \left(\int_{B(z,r)} \omega_j \right)^{\frac{1}{2}} \leq A \left(\int_{B(z,r)} \omega_j \right)^{\frac{1}{2}} \quad (3.15)$$

where A is the uniform upper bound for the Calabi energy. Next,

$$\begin{aligned} \int_{B(z,r)} \omega_j &= \int_{B(z,r)} e^{\phi_j} \omega_0 \leq \left(\int_{B(z,r)} e^{2\phi_j} \omega_0 \right)^{\frac{1}{2}} \left(\int_{B(z,r)} \omega_0 \right)^{\frac{1}{2}} \\ &\leq \left(\int_{B(z,r)} e^{2\phi_j} \omega_0 \right)^{\frac{1}{2}}. \end{aligned} \quad (3.16)$$

But the uniform bound for the integral of $e^{2|\phi_j|}$ obtained in (3.11) shows that the right hand side tends to 0 as $r \rightarrow 0$. This rules out the concentration of the curvature, and the proof of the C^α bounds for ϕ_j is complete.

3.4 Convergence of $\|\nabla\tau\|_{L^2}^2$ and $\|\dot{\phi}\|_{L^2}^2$ to 0

The bound from below for the functional $E(\phi, \tau)$ implies the existence of a sequence t_j , $j \leq t_j \leq j+1$ with

$$\int_X (\dot{\phi})^2 + \int_X |\nabla\tau|^2 \rightarrow 0 \quad (3.17)$$

as $j \rightarrow \infty$. (Here the integrals are with respect to the metric ω_{t_j} , which we do not write explicitly for notational simplicity.) Indeed, for any time T , it follows from the formula for the time derivative of $E(\phi, \tau, t)$ that

$$E(T) - E(0) = - \int_0^T \int_X (\dot{\phi})^2 \omega_t + \kappa \int_X \frac{|\nabla\tau|_{\omega_t}^2}{\tau} \omega_t dt \quad (3.18)$$

and hence, applying the lower bound for $E(T)$ and letting $T \rightarrow \infty$,

$$\int_0^\infty \left(\int_X (\dot{\phi})^2 \omega_t + \kappa \int_X \frac{|\nabla\tau|_{\omega_t}^2}{\tau} \omega_t \right) dt < \infty. \quad (3.19)$$

Since $1/\tau$ is uniformly bounded from below, the convergence of the integral in t implies the desired statement.

Next, we improve this sequential convergence to a full convergence as $t \rightarrow \infty$:

Lemma 11 Define the function $Q(t)$ by

$$Q(t) = \int_X (-R + \lambda + \tau)^2 \omega + \int_X |\nabla\tau|^2 \omega. \quad (3.20)$$

Then

- (a) There exists a sequence t_j , $j \leq t_j \leq j+1$, with $Q(t_j) \rightarrow 0$ as $j \rightarrow \infty$.
 (b) There exist positive constants A, ϵ so that

$$\frac{dQ}{dt} \leq -\epsilon \int_X ((\Delta\tau)^2 + |\nabla R|^2) \omega + A Q(t). \quad (3.21)$$

- (c) We have $Q(t) \rightarrow 0$ as $t \rightarrow \infty$.

Proof The statement (a) is a reformulation of the properties of the sequence t_j that we just obtained, so we concentrate on the proof of (b). We evaluate

$$\begin{aligned} \partial_t \int_X |\nabla\tau|^2 \omega &= \partial_t \int_X \partial_z \tau \partial_{\bar{z}} \tau = 2 \int_X \partial_z \dot{\tau} \partial_{\bar{z}} \tau \\ &= 2 \int_X \partial_z (\kappa \Delta\tau - \tau(-R + \lambda + \tau)) \partial_{\bar{z}} \tau \\ &= -2\kappa \int_X (\Delta\tau)^2 \omega + 2 \int_X \tau(-R + \lambda + \tau) \partial_z \partial_{\bar{z}} \tau. \end{aligned} \quad (3.22)$$

Since τ is uniformly bounded, we can estimate the second term on the right hand side by

$$\begin{aligned} \left| \int_X \tau(-R + \lambda + \tau) \partial_z \partial_{\bar{z}} \tau \right| &\leq M \left| \int_X (-R + \lambda + \tau) \Delta\tau \right| \omega \\ &\leq \epsilon \int_X |\Delta\tau|^2 \omega + C_\epsilon \int_X (-R + \lambda + \tau)^2 \omega \end{aligned} \quad (3.23)$$

and hence, for ϵ sufficiently small,

$$\partial_t \int_X |\nabla \tau|^2 \omega \leq -\kappa \int_X (\Delta \tau)^2 \omega + C Q(t). \quad (3.24)$$

Next,

$$\partial_t \left(\int_X (-R + \lambda + \tau)^2 \omega \right) = \int_X 2(-\partial_t R + \partial_t \tau)(-R + \lambda + \tau) \omega + \int_X (-R + \lambda + \tau)^2 \partial_t \omega$$

We estimate first the second term on the right hand side

$$\int_X (-R + \lambda + \tau)^2 \partial_t \omega = \int_X (-R + \lambda + \tau)^3 \omega \leq C Q \quad (3.25)$$

since $|R|$ and τ are bounded. Next, the first term on the right hand side can be written as

$$\begin{aligned} & 2 \int_X (\partial_t R - \partial_t \tau)(R - \lambda - \tau) \omega \\ &= 2 \int_X (\Delta R - (1 + \kappa) \Delta \tau - (R - \tau)(-R + \lambda + \tau))(R - \lambda - \tau) \omega \\ &= 2 \int_X (-|\nabla R|^2 + \nabla R \cdot \nabla \tau) \omega - 2(1 + \kappa) \int_X \Delta \tau (R - \lambda - \tau) \omega \\ &\quad + 2 \int_X (R - \tau)(R - \lambda - \tau)^2 \omega. \end{aligned} \quad (3.26)$$

We can then write

$$2 \int_X (-|\nabla R|^2 + \nabla R \cdot \nabla \tau) \omega \leq - \int_X |\nabla R|^2 + \int_X |\nabla \tau|^2 \leq - \int_X |\nabla R|^2 + Q(t). \quad (3.27)$$

and

$$\begin{aligned} \left| 2(1 + \kappa) \int_X \Delta \tau (R - \lambda - \tau) \omega \right| &\leq \epsilon \int_X (\Delta \tau)^2 \omega + C_\epsilon \int_X (R - \lambda - \tau)^2 \omega \\ &\leq \epsilon \int_X (\Delta \tau)^2 \omega + C_\epsilon Q(t). \end{aligned} \quad (3.28)$$

Finally, since both R and τ are uniformly bounded, we have

$$\left| 2 \int_X (R - \tau)(R - \lambda - \tau)^2 \omega \right| \leq C \int_X \int_X (R - \lambda - \tau)^2 \omega \leq C Q(t). \quad (3.29)$$

The statement (b) is proved.

The statement (c) is an easy consequence of (b). Dropping the negative terms in the estimate for dQ/dt and integrating from t_j to any t with $t_j \leq t \leq j + 1$, we obtain

$$Q(t) \leq Q(t_j) e^{A(t-t_j)} \leq e^A Q(t_j) \quad (3.30)$$

It follows that $Q(t) \rightarrow 0$ as $t \rightarrow \infty$, and (c) is proved.

3.5 Convergence of the flow

By Lemma 8 and Sect. 3.3, ϕ and τ are uniformly bounded. Thus we can apply Theorem 1 to get uniform C^k bounds for ϕ and τ for any k . It follows that for any sequence of times $t_j \rightarrow \infty$, there is a convergent subsequence in C^∞ converging to $(\omega_\infty, \tau_\infty)$. To get the full convergence instead of just subsequential convergence, it suffices to show that the limit is unique.

This can be seen as follows. By the previous section, $\|\nabla \tau_j\|_{L^2(\omega_0)} \rightarrow 0$, and hence τ_∞ is a constant. It follows that ω_∞ is a metric of constant scalar curvature

$$\text{Ric}(\omega_\infty) = (\lambda + \tau_\infty)\omega_\infty \quad (3.31)$$

with volume 1. For $\chi < 0$, this specifies the metric ω_∞ uniquely, and the convergence of the flow is established.

3.6 A brief discussion of the case $\kappa = 1$

In this subsection we briefly discuss the κ -LYZ flow on compact Riemann surfaces with $\kappa = 1$. Furthermore we assume that the initial data τ_0 is positive as in Theorem 2. Again by the maximum principle we know that τ remains positive along the flow, however we no longer have a quantity like in (3.7) to apply the maximum principle. The new idea here is to introduce an enhanced functional

$$\hat{E}(\phi, \tau) = E(\phi, \tau) + \int_X \frac{R^2}{\tau} \omega.$$

When $\kappa = 1$, it is easy to see that

$$\frac{d}{dt} \hat{E}(\phi, \tau) = - \int_X (\dot{\phi})^2 \omega - \int_X \frac{|\nabla \tau|^2}{2\tau} \omega - \int_X \frac{2}{\tau} \left| \nabla R - \left(\frac{1}{2} + \frac{R}{\tau} \right) \nabla \tau \right|^2 \omega \leq 0,$$

hence $\hat{E}(\phi, \tau)$ is monotone non-increasing along the flow. Given the initial data, by using a finer version of the Moser–Trudinger inequality, one can show that $\hat{E}(\phi, \tau)$ is uniformly bounded from below and the integral

$$\int_X \frac{R^2}{\tau} \omega$$

is uniformly bounded. Therefore if we assume that τ is uniformly bounded from above as in the second part of (2.4), it follows that the Calabi energy $\int_X R^2 \omega$ is uniformly bounded and we can apply Lemma 10 to prove that ϕ is uniformly bounded. So we conclude that in the case of Riemann surfaces with $\kappa = 1$ and $\tau_0 > 0$, the second assumption in (2.4) implies the first, i.e., the evolving metric ω_t is uniformly equivalent to ω_0 .

4 A Mabuchi-type energy for the κ -LYZ coupled flow

In this section, we introduce a Mabuchi-type energy functional $M_{\omega_0}(\varphi, F)$ associated to the coupled system of scalar equations (2.7) and (2.8). If in addition to the assumptions in Theorem 1, we assume that this functional M_{ω_0} is bounded from below along the flow, then we can establish the smooth convergence of (2.7) and (2.8), and hence of the metrics and closed forms in (1.1) and (1.2). More precisely,

Theorem 3 Under the same assumptions as Theorem 1 with $T = +\infty$ and $\lambda < 0$, if $M_{\omega_0}(\varphi_t, F_t) \geq -M_0$ for some $M_0 > 0$, then the solution (φ_t, F_t) of the system (2.7, 2.8) converges smoothly to a limit $(\varphi_\infty, F_\infty)$ satisfying the equations:

$$\log \frac{\omega_{\varphi_\infty}^n}{\omega_0^n} = H_0 - \lambda \varphi_\infty + F_\infty, \quad \Delta_{\omega_{\varphi_\infty}} F_\infty = \text{tr}_{\omega_{\varphi_\infty}} \alpha_0 - b.$$

Here $\omega_{\varphi_\infty} = \omega_0 + i\partial\bar{\partial}\varphi_\infty$ and $\alpha_\infty = \alpha_0 - i\partial\bar{\partial}F_\infty$ are stationary solutions of the Eqs. (1.1) and (1.2), and b is the constant defined in (2.10).

Moreover, if X does not admit any nontrivial holomorphic vector field, then the convergence is exponential.

4.1 A Mabuchi-type energy functional

Motivated by the Mabuchi K -energy [15], we introduce the following functional for the coupled system (2.7, 2.8)²

$$\mu_{\omega_0}(\varphi) = -\frac{n}{V} \int_0^1 \int_X \varphi'_s (\text{Ric}(\omega_{\varphi_s}) - \lambda \omega_{\varphi_s} - \alpha_0) \wedge \omega_{\varphi_s}^{n-1} ds, \quad (4.1)$$

where φ_s is a path in the space of Kähler potentials of ω_0 connecting 0 and φ , and $\omega_{\varphi_s} = \omega_0 + i\partial\bar{\partial}\varphi_s$. It can be checked that the integral above is independent of the choice of the path φ_s . Furthermore, if φ varies in a family of Kähler potentials, then we have

$$\frac{d}{dt} \mu_{\omega_0}(\varphi_t) = -\frac{n}{V} \int_X \dot{\varphi} (\text{Ric}(\omega_\varphi) - \lambda \omega_{\varphi_t} - \alpha_0) \wedge \omega_{\varphi_t}^{n-1}.$$

Let φ and F be the potentials of ω_t and α_t defined in (2.6). We define a functional associated with the flow Eqs. (2.7) and (2.8) by

$$M_{\omega_0}(\varphi, F) = \mu_{\omega_0}(\varphi) - \frac{1}{V} \int_X F \omega_\varphi^n. \quad (4.2)$$

Lemma 12 The functional M_{ω_0} defined in (4.2) is monotone and non-increasing along the flow (2.7) and (2.8).

Proof This follows from direct calculations:

$$\begin{aligned} \frac{d}{dt} M_{\omega_0}(\varphi, F) &= -\frac{n}{V} \int_X \dot{\varphi} (\text{Ric}(\omega_t) - \lambda \omega_t - \alpha_0) \wedge \omega_t^{n-1} - \frac{1}{V} \int_X (\dot{F} \omega_t^n + n F \partial \bar{\partial} \dot{\varphi} \wedge \omega_t^{n-1}) \\ &= -\frac{n}{V} \int_X \dot{\varphi} (\text{Ric}(\omega_t) - \lambda \omega_t - \alpha_0) \wedge \omega_t^{n-1} - \frac{n}{V} \int_X \dot{\varphi} i \partial \bar{\partial} F \wedge \omega_t^{n-1} \\ &= \frac{n}{V} \int_X \dot{\varphi} i \partial \bar{\partial} \dot{\varphi} \wedge \omega_t^{n-1} \\ &= -\frac{1}{V} \int_X |\nabla \dot{\varphi}|_{\omega_t}^2 \omega_t^n \leq 0. \end{aligned}$$

Next we recall the well-known I and J functionals of Kähler geometry,

$$I(\varphi) = \frac{1}{V} \int_X \varphi (\omega_0^n - \omega_\varphi^n) = \frac{1}{V} \sum_{k=0}^{n-1} \int_X i \partial \varphi \wedge \bar{\partial} \varphi \wedge \omega_0^{n-1-k} \wedge \omega_\varphi^k,$$

² In keeping with the traditional notation for the Mabuchi functional itself, we have denoted the volume $\int_X \omega_0^n$ of ω by V instead of normalizing it to 1 as in the previous sections. This will also allow normalizing λ to -1 in subsequent sections.

and

$$J(\varphi) = \frac{1}{V} \int_0^1 \int_X \varphi'_s (\omega_0^n - \omega_{\varphi_s}^n) ds = \frac{1}{V} \sum_{k=0}^{n-1} \int_X \frac{n-k}{n+1} i \partial \varphi \wedge \bar{\partial} \varphi \wedge \omega_0^{n-1-k} \wedge \omega_{\varphi}^k,$$

where φ_s is chosen as in the definition of $\mu_{\omega_0}(\varphi)$ in (4.1). It is well-known that

$$I(\varphi) - J(\varphi) \geq \frac{1}{n+1} I(\varphi) \geq 0.$$

Recall that H_0 satisfies $\text{Ric}(\omega_0) - \lambda \omega_0 - \alpha_0 = i \partial \bar{\partial} H_0$ and is normalized by $\int e^{H_0} \omega_0^n = V$. For any given path φ_s connecting φ and 0, we define a family of smooth functions:

$$H_s = -\log \left(\frac{\omega_{\varphi_s}^n}{\omega_0^n} \right) - \lambda \varphi_s + H_0 + C(s)$$

where $C(s)$ is a normalizing function (constant in X for each $s \in [0, 1]$) such that $\int e^{H_s} \omega_{\varphi_s}^n = V$. We can check that H_s is a potential satisfying

$$\text{Ric}(\omega_{\varphi_s}) - \lambda \omega_{\varphi_s} - \alpha_0 = i \partial \bar{\partial} H_s.$$

By the following calculation,

$$\begin{aligned} \mu_{\omega_0}(\varphi) &= -\frac{n}{V} \int_0^1 \int_X \varphi'_s i \partial \bar{\partial} H_s \wedge \omega_{\varphi_s}^{n-1} ds \\ &= -\frac{n}{V} \int_0^1 \int_X H_s i \partial \bar{\partial} \varphi'_s \wedge \omega_{\varphi_s}^{n-1} ds \\ &= -\frac{1}{V} \int_0^1 \int_X H_s \frac{\partial}{\partial s} \omega_{\varphi_s}^n ds = -\frac{1}{V} \int_0^1 \left(\frac{d}{ds} \int_X H_s \omega_{\varphi_s}^n - \int_X \frac{\partial H_s}{\partial s} \omega_{\varphi_s}^n \right) ds \\ &= -\frac{1}{V} \int_X H_1 \omega_{\varphi}^n + \frac{1}{V} \int_X H_0 \omega_0^n + \frac{1}{V} \int_0^1 \int_X (-\Delta_{\omega_{\varphi_s}} \varphi'_s - \lambda \varphi'_s + C'(s)) \omega_{\varphi_s}^n ds \\ &= \frac{1}{V} \int_X \log \left(\frac{\omega_{\varphi}^n}{\omega_0^n} \right) \omega_{\varphi}^n + \frac{\lambda}{V} \int_X \varphi \omega_{\varphi}^n - \frac{1}{V} \int_X H_0 \omega_{\varphi}^n + \frac{1}{V} \int_X H_0 \omega_0^n - \frac{\lambda}{V} \int_0^1 \int_X \varphi'_s \omega_{\varphi_s}^n ds \\ &= \frac{1}{V} \int_X \log \left(\frac{\omega_{\varphi}^n}{\omega_0^n} \right) \omega_{\varphi}^n - \lambda (I(\varphi) - J(\varphi)) - \frac{1}{V} \int_X H_0 \omega_{\varphi}^n + \frac{1}{V} \int_X H_0 \omega_0^n, \end{aligned} \tag{4.3}$$

we can see that $\mu_{\omega_0}(\varphi)$ is equal to the sum of an “entropy part”, an “energy part” and some bounded terms.

4.2 Convergence of the flow when $M(\varphi, F) > -\infty$ and $\lambda < 0$

We can give now the proof of Theorem 3. Recall that we are making the basic assumption (2.4) and also that $M(\varphi, F) \geq -M_0$ is bounded from below along the flow and $\lambda < 0$. For notational simplicity we set $\lambda = -1$.

Lemma 13 *There exists a constant $C = C(\omega_0, \kappa, K) > 0$ such that*

$$-C \leq \varphi_t + F_t \leq C.$$

Proof We calculate

$$\left(\frac{\partial}{\partial t} - \kappa \Delta \right) (F + \varphi) = -\kappa \operatorname{tr}_{\omega_t} \alpha_0 + \kappa b + \log \frac{\omega_t^n}{\omega_0^n} - (\varphi + F) - \kappa n + \kappa \operatorname{tr}_{\omega_t} \omega_0,$$

the desired bound for $F + \varphi$ then follows from the maximum principle and the assumption (2.4). The lemma is proved.

Since $\dot{\varphi} + \varphi + F$ is uniformly bounded by the Eq. (2.7) and the assumption (2.4), we conclude from the above lemma that $\|\dot{\varphi}\|_{L^\infty} \leq C(\omega_0, \kappa, K)$.

Lemma 14 *There is a constant $C = C(\omega_0, \kappa, K, M_0)$ such that*

$$\sup_X (|F_t| + |\varphi_t|) \leq C, \quad \forall t \in [0, T).$$

Proof From the expression of μ_{ω_0} in (4.3) for $\lambda = -1$ we get $\mu_{\omega_0}(\varphi) \geq -C(H_0, \omega_0)$. Then the monotonicity of $M_{\omega_0}(\varphi, F)$ yields a uniform lower bound of $\int_X F_t \omega_t^n \geq -C(H_0, \omega_0, \varphi_0, F_0)$, where φ_0 and F_0 denote the initial values of φ and F , respectively. On the other hand, since $I - J$ is given by linear combinations of integrals of the form $\int_X i \partial \varphi \wedge \bar{\partial} \varphi \wedge \omega_0^j \wedge \omega_\varphi^{n-1-j}$, it is uniformly bounded by Lemma 5. The entropy part

$$\int_X \log \left(\frac{\omega_\varphi^n}{\omega_0^n} \right) \omega_\varphi^n \quad (4.4)$$

is also bounded above by assumption. Thus the assumption that $M(\varphi, F) \geq -M_0$ implies a uniform upper bound of $\int_X F_t \omega_t^n \leq C$. Combined with the gradient bound of F in Lemma 5, this gives a C^0 -bound for F_t , and by Lemma 13, we also get $\sup_X |\varphi_t| \leq C$. The lemma is proved.

Combining Lemma 14 and Theorem 1, we obtain a uniform estimate for all derivatives of φ_t and F_t :

$$\sup_X |\partial_t^k \nabla^l F_t| + |\partial_t^k \nabla^l \varphi_t| \leq C(k, l, \kappa, K, M_0), \quad \forall t \in [0, T).$$

Recall that we are considering the case of infinite maximum existence time, $T = \infty$. From the proof of Lemma 12, we can deduce that

$$\int_0^\infty \int_X |\nabla \dot{\varphi}|^2 \omega_t^n dt < \infty,$$

and this implies that

$$\lim_{i \rightarrow \infty} \int_i^{i+2} \int_X |\nabla \dot{\varphi}|^2 \omega_t^n dt = 0.$$

From the equation

$$\left(\frac{\partial}{\partial t} - \Delta \right) |\nabla \dot{\varphi}|^2 = -|\nabla \nabla \dot{\varphi}|^2 - |\nabla \bar{\nabla} \dot{\varphi}|^2 - 2 \operatorname{Re} \langle (\nabla \dot{\varphi}, \bar{\nabla}(\dot{\varphi} + \dot{F})) \rangle + |\nabla \dot{\varphi}|^2 - \alpha(\nabla \dot{\varphi}, \bar{\nabla} \dot{\varphi}),$$

we can use either Moser iteration or arguments similar to the ones in Sect. 3.4 to deduce that

$$\sup_{X \times [i+1, i+2]} |\nabla \dot{\varphi}|^2 \leq C \int_i^{i+2} \int_X |\nabla \dot{\varphi}|^2 \omega_t^n dt \rightarrow 0, \quad \text{as } i \rightarrow \infty.$$

Therefore the smooth limit of $\dot{\varphi}_t$ as $t \rightarrow \infty$ must be constant. If X does not admit any nontrivial holomorphic vector fields, we can use the argument of [19] to get the exponential convergence of $\int_X |\nabla \dot{\varphi}_t|^2 \omega_t^n \leq C e^{-\delta t}$ for some $\delta > 0$ depending on the lower bound of positive eigenvalue of $-\Delta_{\omega_t}$ on vector fields in $T^{1,0}X$.

By the L^∞ -bounds of φ_t and F_t , we conclude that the smooth limits of $\dot{\varphi}_t$ and $\dot{F}_t$ must be zero. The limiting functions φ_∞ and F_∞ satisfy the equations

$$\log \frac{\omega_{\varphi_\infty}^n}{\omega_0^n} = H_0 + \varphi_\infty + F_\infty, \quad \Delta_{\omega_{\varphi_\infty}} F_\infty - \text{tr}_{\omega_{\varphi_\infty}} \alpha_0 + b = 0,$$

i.e. the $(1, 1)$ -forms $\omega_{\varphi_\infty} = \omega_0 + i\partial\bar{\partial}\varphi_\infty$ and $\alpha_\infty = \alpha_0 - i\partial\bar{\partial}F_\infty$ define stationary solutions of the flow (1.1) and (1.2). The proof of Theorem 3 is complete.

References

1. Becker, K., Becker, M., Schwarz, J.H.: String Theory and M Theory. Cambridge University Press, Cambridge (2007)
2. Chen, X.-X., Cheng, J.: On the constant scalar curvature Kähler metrics, apriori estimates (2017). [arXiv:1712.06697](https://arxiv.org/abs/1712.06697)
3. Chen, X.-X., Cheng, J.: On the constant scalar curvature Kähler metrics, existence results (2018). [arXiv:1801.00656](https://arxiv.org/abs/1801.00656)
4. Chen, X.-X., Cheng, J.: On the constant scalar curvature Kähler metrics, general automorphism group (2018). [arXiv:1801.05907](https://arxiv.org/abs/1801.05907)
5. Chow, B.: The Ricci flow on the 2-sphere. J. Differ. Geom. **33**(2), 325–334 (1991)
6. Fei, T., Guo, B., Phong, D.H.: Parabolic dimensional reductions of 11D supergravity (2018). [arXiv:1806.00583](https://arxiv.org/abs/1806.00583)
7. Fei, T., Huang, Z., Picard, S.: The anomaly flow over Riemann surfaces (2017). [arXiv:1711.08186](https://arxiv.org/abs/1711.08186)
8. Guo, B., Huang, Z., Phong, D.H.: Pseudo-locality for a coupled Ricci flow. Commun. Anal. Geom. **26**(3), 585–626 (2018)
9. Hamilton, R.S.: The Ricci flow on surfaces. In: Mathematics and General Relativity (Santa Cruz, CA, 1986), Contemp. Math., vol. 71, pp. 237–262. Amer. Math. Soc. (1988)
10. Horava, P., Witten, E.: Heterotic and type I string dynamics from eleven dimensions. Nucl. Phys. B **460**, 506–524 (1996)
11. Li, Y.: Generalized Ricci flow I: higher derivative estimates for compact manifolds. Anal. PDE **5**(4), 747–775 (2012)
12. Li, Y., Yuan, Y., Zhang, Y.: A new geometric flow over Kähler manifolds. Commun. Anal. Geom. (2018)
13. Lieberman, G.M.: Second Order Parabolic Differential Equations, p. xii+439. World Scientific, Singapore (1996)
14. List, B.: Evolution of an extended Ricci flow system. Commun. Anal. Geom. **16**(5), 1007–1048 (2008)
15. Mabuchi, T.: Some symplectic geometry on compact Kähler manifolds I. Osaka J. Math. **24**, 227–252 (1987)
16. Müller, R.: Ricci flow coupled with harmonic map flow. Ann. Sci. Éc. Norm. Supér. (4) **45**(1), 101–142 (2012)
17. Perelman, G.: The entropy formula for the Ricci flow and its geometric applications (2002). [arXiv:math/0211159](https://arxiv.org/abs/math/0211159)
18. Phong, D.H., Sesum, N., Sturm, J.: Multiplier ideal sheaves and the Kähler–Ricci flow. Commun. Anal. Geom. **15**(3), 613–632 (2007)
19. Phong, D.H., Song, J., Sturm, J., Weinkove, B.: The Kähler–Ricci flow and the $\bar{\partial}$ -operator on vector fields. J. Differ. Geom. **81**(3), 631–647 (2009)
20. Phong, D.H., Picard, S., Zhang, X.W.: New curvature flows in complex geometry. Surv. Differ. Geom. **22**(1), 331–364 (2017)
21. Phong, D.H., Picard, S., Zhang, X.W.: Geometric flows and Strominger systems. Math. Z. **288**(1–2), 101–113 (2018)
22. Struwe, M.: Curvature flows on surfaces. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **1**(2), 247–274 (2002)
23. Townsend, P.: The eleven-dimensional supermembrane revisited. Phys. Lett. B **350**, 184–188 (1995)
24. Trudinger, N.S., Wang, X.J.: The affine Plateau problem. J. Am. Math. Soc. **18**(2), 253–289 (2005)

25. Yau, S.-T.: On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation. I. *Commun. Pure Appl. Math.* **31**(3), 339–411 (1978)
26. Zhang, Q.S.: A uniform Sobolev inequality under Ricci flow. *Int. Math. Res. Not.* (2007). <https://doi.org/10.1093/imrn/rnm056>

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