Networking Theories to Design Dynamic Covariation Techtivities for College Algebra Students

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Covariational reasoning is a challenging form of reasoning for undergraduate students to develop and employ. Yet, students' lack of opportunities to use covariational reasoning may account, in part, for some of their difficulties. Building from the work of mathematics education researchers (e.g., Kaput, Thompson, Moore), we developed a suite of Techtivities–free, accessible, digital media activities linking dynamic animations and graphs. Using a Cannon Man Techtivity to illustrate, we provide four key design components and three theoretically based design principles underlying the Techtivities. To inform design both within and across the Techtivities, we network theories of different grain sizes: Thompson's theory of quantitative reasoning and Marton's variation theory. Developing Techtivities for students in the gatekeeping course, College Algebra, we intend to expand students' opportunities to employ covariational reasoning. We discuss implications stemming from students' opportunities to use free, accessible digital media activites, such as Techtivities, to promote their covariational reasoning.

Keywords: Covariational reasoning, Task Design, Quantitative Reasoning, Variation Theory

Covariational reasoning is a critical form of mathematical reasoning imperative for students' understanding of key concepts of introductory college level mathematics such as functions, rates, and graphs (Thompson & Carlson, 2017). At its core, covariational reasoning entails a twofold conception: conceiving of attributes as capable of varying and possible to measure, then conceiving of a relationship between those attributes (Carlson, Jacobs, Coe, Larson, & Hsu, 2002; Thompson & Carlson, 2017). By investigating situations involving multiple changing attributes, students can have opportunities to employ covariational reasoning (e.g., Moore, Stevens, Paoletti, & Hobson, 2016; Johnson, McClintock, & Hornbein, 2017; Thompson & Carlson, 2017). For example, students might investigate a Cannon Man situation, in which a person is ejected into the air, then falls down to the ground with the help of a parachute. In this situation, students employing covariational reasoning could conceive of two possible attributes as capable of varying and possible to measure: Cannon Man's height from the ground and Cannon Man's total distance traveled while in the air. Students could then conceive of a relationship between Cannon Man's height from the ground and total distance traveled.

Building from the work of mathematics education researchers (e.g., Kaput, Thompson, Moore), we developed a suite of Techtivities—free, accessible, digital media activities linking dynamic animations and graphs. We designed the Techtivities for students in College Algebra, an introductory course that can serve as a gatekeeper for many students (e.g., Gordon, 2008; Herriot & Dunbar, 2009). Using a Cannon Man Techtivity to illustrate, we provide four key design components and three theoretically based design principles underlying the Techtivities. Networking theories of different grain sizes—Thompson's theory of quantitative reasoning (1993, 1994, 2002, 2011) and Marton's variation theory (2015)—we designed both within and across the Techtivities. By designing the Techtivities in Desmos (www.desmos.com), we increase accessibility, and thereby expand students' opportunities to employ covariational reasoning. We conclude with implications for students' use of Techtivities to promote their covariational reasoning and for the networking of theories to design digital media activities.

Background

Despite the importance of covariational reasoning, researchers have documented challenges that undergraduate university students enrolled in calculus and trigonometry courses face when encountering situations calling for covariational reasoning (e.g., Carlson et al., 2002; Oehrtman, Carlson, & Thompson, 2008; Moore, 2014; Moore & Carlson, 2012; Moore, Paoletti, & Musgrave, 2013). Broadly, undergraduate students have difficulty using covariational reasoning to make sense of situations involving variation in change that occurs in a single direction, such as a temperature increasing at a decreasing rate (e.g., Carlson et al., 2002; Oehrtman et al., 2008). In addition, students' impoverished conceptions of the "things" that are changing may decrease their likelihood for covariational reasoning (Moore, 2014; Moore & Carlson, 2012). Furthermore, students' lack of covariational reasoning can impact their ability to view graphs as representing relationships between quantities (Moore & Thompson, 2015; Moore, Stevens, Paoletti, & Hobson, 2016).

Through their programs of research, Thompson and Carlson, together with colleagues, have developed and implemented innovative learning materials to provide opportunities for university students in Calculus and Precalculus to employ covariational reasoning (e.g., Carlson et al., 2002; Carlson, Oehrtman, & Engelke, 2010; Carlson, Oehrtman, & Moore, 2010; Thompson & Ashbrook, 2016a; Thompson & Carlson, 2017). In a PreCalculus course designed to foster university students' covariational reasoning, students encountered instructional tasks designed to provide students opportunities to conceive of change in attributes prior to determining numerical amounts of change (Thompson & Carlson, 2017). In their online Conceptual Calculus textbook, Thompson & Ashbrook (2016a) included a task situation involving a droplet of water landing into a bowl of water and creating circular ripples that increase in size (Thompson & Ashbrook, 2016b). We view this situation as having potential to serve as background for a task requiring students to conceive of and represent change in the area and radius of the ripples. Overall, the research programs of Thompson and Carlson have resulted in opportunities for university students to use innovative learning materials designed to promote covariational reasoning. Yet, we argue that there is room for the development of more accessible and multimodal learning materials, so as to provide digital media that broadens access and learning opportunities to an even wider range of students.

An Approach to Technology Development and Use for Greater Access and Participation

By developing a suite of Techtivities in Desmos, we increase accessibility and opportunities for participation in multiple ways: across operating system platforms (Apple OS, Microsoft Windows), across various browsers (i.e., Google Chrome, Mozilla Firefox, Microsoft Edge), via mobile devices (Desmos is compatible with iOS and Android), and as an app extension via Google's Chrome browser (Desmos has 2.8 million app installations within Chrome). Furthermore, Desmos has low barriers to entry and initial use, which afford more expansive opportunities for student participation. Specifically, learner use of Desmos begins in just a few clicks via a web browser or mobile platform; supports learning in over two dozen languages; complies with WCAG 2.0 accessibility standards for learners who may be blind or visually impaired, with screen reader capability on both web-based and mobile platforms; includes authenticated sign in with Google credentials; and incorporates a robust set of web tutorials on Youtube (over a quarter million views). We have intentionally partnered with Desmos because the development and use of each Techtivity will maintain these technical features for greater access and participation, and also align all Techtivities with the broader "ecosystem" of Desmos users, social media networks, technical supports, and complementary resources.

Networking Theories to Design Techtivities

Rasmussen and Wawro (2017) called for researchers investigating research problems in undergraduate mathematics education to network theories, thereby providing new lenses and tools to study the complexities of learning and teaching mathematics. By networking theories of different grain sizes to design the Techtivities, we respond to the call put forth by Rasmussen and Wawro (2017). Watson (2016) articulated three different grain sizes of theories: grand theories (e.g., Piaget's constructivist theory), intermediate theories (e.g., Marton's variation theory), and domain specific/local theories (e.g., Thompson's theory of quantitative reasoning). Following Johnson and colleagues (Johnson, McClintock, Hornbein, Gardner, and Grieser, 2017; Johnson & McClintock, in press), we networked Thompson's theory of quantitative reasoning and Marton's variation theory to design both within and across the Techtivities.

Thompson's Theory of Quantitative Reasoning

In explicating a theory of quantitative reasoning (e.g., Thompson 1993; 1994; 2002; 2011), Thompson employed a constructivist perspective. Thompson's theory of quantitative reasoning focuses on students' mental operations, which individuals can enact in thought as well as action (e.g., Piaget, 1970, 1985). Drawing on Thompson's theory of quantitative reasoning, by quantity we mean how students conceive of the possibility of measuring some attribute. For example, a student might conceive of using a fixed distance between her thumb and forefinger to measure Cannon Man's height from the ground. Thompson's theory of quantitative reasoning undergirds our perspective on covariational reasoning.

Following Thompson and Carlson (2017), we argue that covariational reasoning entails at least four different kinds of mental operations: students' conceptions of attributes as being possible to measure (quantitative reasoning), students' conceptions of attributes as being capable of varying, students' conceptions of a relationship between attributes capable of varying and possible to measure, and students' images of change. Thompson, Hatfield, Yoon, Joshua, and Byerly (2017) built on Saldanha & Thompson's (1998) term, *multiplicative object*, to specify a conception of a relationship between attributes capable of varying and possible to measure. A student conceiving of a relationship between attributes as a multiplicative object can conceptualize a new attribute, which coordinates the constituent attributes (Saldanha & Thompson, 1998; Thompson et al., 2017). For example, a student could conceive of a new attribute, coordinating Cannon Man's height from the ground and total distance traveled at every value of height and distance. By images of change, we mean more than a mental picture, we mean students' mental operations (see also Thompson, 1996). Castillo-Garsow, Johnson, & Moore (2013) posited two contrasting images of change: chunky and smooth. A smooth image of change refers to a conception of change as occurring in progress. A chunky image of change refers to a conception of change as having occurred in particular increments. For example, a student might conceive of Cannon Man's height as changing continually (smooth image of change) or as having changed to reach a certain amount (chunky image of change). Students' use of smooth images of change correlates to more advanced levels of covariational reasoning (Thompson & Carlson, 2017). Researchers have argued for the utility of students' smooth images of change (e.g., Castillo-Garsow et al., 2013), reporting case studies to demonstrate that utility for both undergraduate and high school students (e.g., Johnson, 2012; Moore, 2014).

Marton's Variation Theory

We used Marton's (2015) variation theory to guide design across the Techtivities. Broadly, Marton (2015) argued that instructional designers should develop task sequences that provide students opportunities to discern critical aspects (Marton, 2015). When interacting with the Techtivities, we view covariation to be a critical aspect for students to discern. Furthermore, covariation is a critical aspect comprised of interrelated aspects. For critical aspects comprised of interrelated aspects, Marton (2015) recommended that task sequences first include variation and invariance in each interrelated aspect, then variation in both aspects. To discern covariation, students need to conceive of two constituent attributes as capable of varying and possible to measure, as well as a relationship between those attributes. Consequently, in designing the Techtivities, we first included variation and invariance in each constituent attribute, then variation in both attributes.

Networking Theories to Move Beyond Existing Theoretical Perspectives

Networking theories can take different forms. We network Thompson's theory of quantitative reasoning and Marton's variation theory to design both within and across the Techtivities. To design within each Techtivity, we drew on Thompson's theory of quantitative reasoning to inform our selection of different attributes to use and to inform our design to promote students' use of smooth images of change. To design across the Techtivities, we drew on Marton's variation theory to include variation and invariance in the type and representation of constituent attributes, then variation in both attributes.

For the purposes of designing the Techtivities, we view Thompson's theory of quantitative reasoning and Marton's variation theory to complement, rather than to compete, with each other. From a constructivist perspective, we do not assume that covariation is something that is "out there" for students to notice (see also Johnson, McClintock, Hornbein, et al., 2017). From a variation theory perspective, Marton (2015) asserted that researchers should not assume that students already attend to the critical aspect prior to encountering a task sequence. We concur with Marton (2015), as we do not assume that students already attend to covariation prior to encountering the task sequence. Furthermore, in the design of the Techtivities, the critical aspect for students to discern—covariation—is a conception (see also Johnson, McClintock, Hornbein, et al., 2017). By discernment, we mean students' engagement in mental operations entailed in covariational reasoning. In the next section, we articulate four key design components, encompassing design decisions both within and across the Techtivities.

Four Key Design Components of Each Techtivity

Building from the work of mathematics education researchers (e.g., Kaput & Roschelle, 1999; Moore et al., 2013; Moore et al., 2016; Saldanha & Thompson, 1998; Thompson, 2002; Thompson, Byerly, & Hatfield, 2013) we provide four key design components of each Techtivity. In explicating these components, we expand on Johnson's previous task design research (2013, 2015). Furthermore, we find our design components to be complementary to the task sequence reported by Moore et al. (2016). In their task sequence, Moore et al. (2016) began first by providing students with a video or animation depicting changing attributes; second, they prompted students to sketch a graph showing a relationship, and third, they prompted students to sketch a second graph, containing either the same or similar attributes. Furthermore, Moore et al. (2016) recommended that tasks not include numerical amounts, concurrent with Johnson's (2013, 2015) recommendations. In our design components, we adapt and expand on the task sequence reported by Moore et al. (2016). We include opportunities for students to vary individual attributes, and we constrain the attributes in the second graph, such that those attributes are the same as the attributes in the first graph.

Dynamic Animations of Situations Involving Changing Attributes

Johnson, McClintock, and Hornbein (2017) articulated a need for task designers to take into account the types of attributes included in tasks. In the suite of Techtivities, we intended to select attributes that we thought students may more readily conceive of as measurable. Furthermore, alongside the animation, we identify attributes which will serve as the focus of the Techtivity (Figure 1). We use an animation in part to provide students opportunities to conceive of attributes in the process of changing, or put another way, to use smooth images of change.

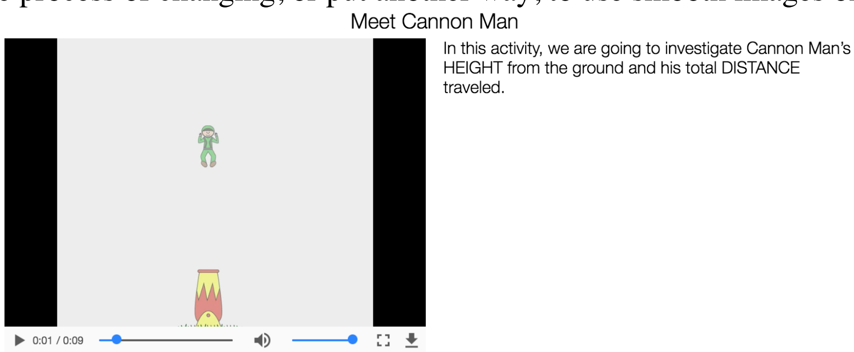

Figure 1. Cannon man animation

Cartesian Graphs Containing Dynamic Segments on the Axes

It is useful for students to use their fingers as tools to represent variation in individual attributes (Thompson, 2002). Through the dynamic segments on each axis (Figure 2, left), we provide students opportunities to use digital media to represent variation in individual attributes. We include freely stretching segments and avoid using numerical amounts to foster students' use of smooth images of change.

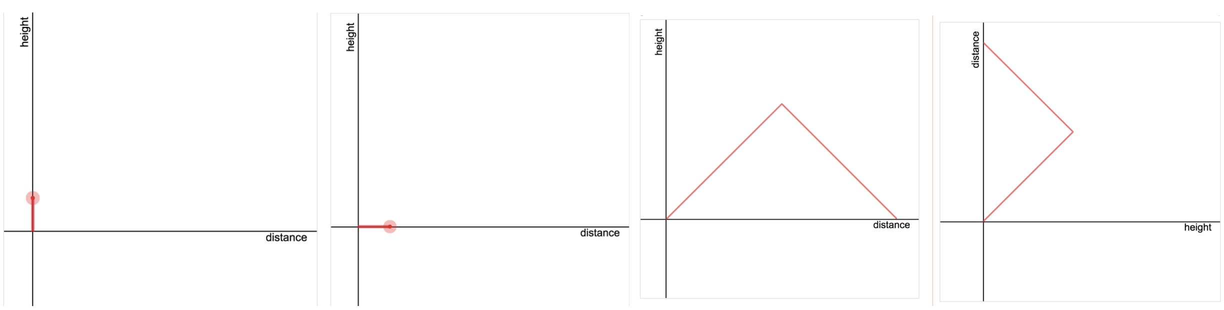

Figure 2. Dynamic segments (left). Graphs varying representation of the same attributes (right).

Opportunities to Sketch a Cartesian Graph after Varying Individual Attributes

Johnson (2015) showed that students' opportunities to conceive of variation in individual attributes impacted their conceptions of covariation. In each Techtivity, after varying individual attributes, students have the opportunity to sketch a Cartesian graph. When working to sketch the graph, students may replay the animation. To sketch a graph, students may select between two digital tools: a free-form pencil or a line segment.

Variation in Representation of Attributes

Students may find it challenging to conceive of graphs as representing relationships between attributes (Moore & Thompson, 2015; Moore et al., 2016). We incorporated Cartesian graphs that represented the same attributes in different ways (Figure 2, right). In so doing, we intended to provide students opportunities to conceive of graphs as representing relationships, rather than forming a particular type of shape (See also Moore & Thompson, 2015).

A Blueprint for a Techtivity

Each Techtivity consists of a series of screens, which students move through in a particular order. Table 1 provides a blueprint for a Techtivity. First, students watch an animation of a situation involving changing attributes (Table 1, Item 1). Second, students move dynamic segments to represent change in each attribute. After moving segments, students view the dynamic segments changing together, appearing in conjunction with the animation (Table 1, Items 2-4). Third, students sketch a Cartesian graph representing how both attributes are changing together. After sketching a graph, students view a computer generated graph, appearing in conjunction with an animation (Table 1, Items 5-6). Fourth, students answer a reflection question (Table 1, Item 7). Fifth, students repeat the process for a new Cartesian graph representing the same situation, with attributes on different axes (Table 1, Item 8).

Table 1. A Blueprint for a Techtivity

A Blueprint for a Techtivity
1. View animation of a situation involving changing attributes. Identify the changing attributes on which to focus in this situation. 2. Move a dynamic segment to show how one attribute is changing. 3. Move a second dynamic segment to show how the other attribute is changing. 4. View both dynamic segments changing together, appearing in conjunction with an animation. (In 2-4, dynamic segments are located on horizontal or vertical axes on a Cartesian Plane.) 5. Sketch a Cartesian graph representing how both attributes are changing together. 6. View a computer-generated Cartesian graph, appearing in conjunction with an animation. 7. Reflect on an aspect of the Cartesian graph. For example, is the graph what you expected? Is there anything about the graph that surprises you? Why might it make sense for a graph to look that way? Is it possible for two different looking graphs to represent the same situation? 8. Repeat 2-7 for a new Cartesian graph representing the same situation, with attributes on different axes.

Design Principles Emerging from the Development of the Techtivities

Increase Accessibility to Expand Students' Opportunities to Employ Covariational Reasoning

Kaput (1994) argued that technology could provide students opportunities to investigate areas of mathematics once reserved only for students at more advanced levels. Covariational reasoning is a critical form of reasoning that cannot be reserved only for students at the upper levels of undergraduate mathematics. At CU Denver, the student population is becoming increasingly diverse. In 2016, 57% of new freshman, and overall 43% of undergraduate students identified as students of color (Williams, 2016). Across Spring and Fall 2016, 70% of students enrolled in College Algebra at CU Denver self-identified as students of color. By designing our Techtivities in Desmos, we increase access for undergraduate students, as well as their instructors. We designed the Techtivities so that students could work in ways that are self-paced, or with direction from their instructors. Furthermore, students and educators have free online access to the suite of Techtivities, to use as a just-in-time curricular resource or as an embedded component of a course, allowing for entire cohorts of students to have opportunities to employ covariational reasoning.

Leverage Domain Specific Theories in Mathematics Education to Design Task Components

By drawing on Thompson's theory of quantitative reasoning, we augment the design of the Techtivities by infusing what we have learned from researchers focusing on students' conceptions. Specific to our focus on covariational reasoning, we leveraged Thompson's theory of quantitative reasoning in three ways. First, provide opportunities for students to conceive of attributes as capable of varying and possible to measure (Table 1, Items 1-3). Second, provide opportunities for students to discern a relationship between attributes, or put another way, to discern covariation (Table 1, Items 4-7). Third, by representing attributes on different axes, provide opportunities for students to conceive of a graph as representing a relationship between attributes capable of varying and possible to measure (Table 1, Item 8).

Network Theories of Different Grain Sizes to Design Both Within and Across Tasks

Networking theories of different grain sizes, we were able to design both within and across the Techtivities (see also Johnson, McClintock, Hornbein, et al., 2017; Johnson & McClintock, in press). Thompson's theory of quantitative reasoning informed our selection of attributes within each Techtivity and across the suite of techtivities. In the Cannon Man Techtivity, total distance traveled is monotonically increasing. In another Techtivity, we include attributes such that neither is monotonically increasing or decreasing (see also Moore et al., 2016). Marton's variation theory informed the sequencing of design across sections of individual Techtivities as well as across the suite of Techtivities. Individual Techtivities include variation in each attribute, then variation in both attributes. The suite of Techtivities provide different backgrounds.

Discussion

Despite the existence of some innovative learning materials for Calculus and Precalculus students (e.g., Carlson, Oehrtman, & Moore, 2010; Thompson & Ashbrook, 2016a), we argue that a broader range of university students need access to such materials. We contend there is an *opportunity* gap for university students to develop and employ covariational reasoning. We view this opportunity gap to be particularly problematic for students enrolled in College Algebra. Furthermore, increasing numbers of College algebra students identify as students of color, and university College Algebra courses have had low success rates (e.g., Gordon, 2008; Herriot & Dunbar, 2009). By developing a suite of Techtivities designed to promote College Algebra students' covariational reasoning, we intend to address this opportunity gap.

Broadly, a dual commitment has motivated our design decisions when developing the Techtivities. We intend to increase students' access to opportunities to employ covariational reasoning, and expand learning opportunities through the development of free, accessible, digital media activities that link dynamic animations with graphs. By attending simultaneously to disciplinary and technical barriers, while foregrounding the expansion of learning opportunities for nondominant students at CU Denver, we make explicit our "researcher positionality" (Aguirre et al., 2017), acknowledging that mathematics education research is both a political and equity-oriented endeavor.

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