

ROLE OF LOCAL GEOMETRY IN ROBUSTNESS OF POWER GRID NETWORKS

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ABSTRACT

We introduce a novel approach to study robustness of a power grid network employing the tools of topological data analysis (TDA). This approach not only enables one to incorporate intrinsic network properties such as electrical conductance but more importantly also offers a systematic and comprehensive framework to study the role of topology in its functionality and robustness. This is achieved by viewing the network as a weighted graph, equipping it with a nested simplicial complex structure and extracting topological summaries in the form of the Betti numbers and persistent diagrams. These summaries are then used to characterize network vulnerability under critical conditions such as targeted attacks.

Index Terms— Power grids, complex networks, topological data analysis, Betti numbers, power system vulnerability

1. INTRODUCTION

A power grid can be intrinsically modeled as a complex network, where nodes are either transformers, substations or generators, and edges represent physical cables connecting two nodes [1, 2, 3]. Methods of complex network analysis have provided new insights into the fundamental and intrinsic characteristics of power system efficiency, vulnerability and resilience. In particular, numerous recent results indicate that both the topological and functional structure of power grid networks can dramatically impact power system reliability and the effectiveness of associated risk mitigation strategies [4, 5, 6]. The electrical engineering (EE) concepts, e.g. impedance, maximum power, etc., can also be incorporated into the CN analysis through a hybrid approach [2]. Though the common vulnerability metrics of a power grid networks are primarily lower-order connectivity features, e.g., degree distribution, average path length (APL), clustering coefficient (CC), etc., a number of recent studies suggest that the power grid robustness is also associated with higher-order network features, e.g., network *motifs* [7, 8, 9].

In the present work, we study power grid functionality, more specifically its robustness under targeted attacks, by introducing concepts of topological data analysis (TDA), which systematically allow to account for both fundamental CN and EE properties of the networks. TDA has recently been recognized as an indispensable tool for analysis of many complex systems, from nanotechnologies to genetics to neuroscience to music [10, 11, 12]. The tools of TDA, most notably persistent homology, are shown to unveil some critical characteristics behind functionality of many complex systems and interactions of their components at multi-scale levels, which are otherwise largely inaccessible with conventional analytical approaches. However, despite its success in many domains, TDA is a novel and yet uncharted research area in application to analysis of power grids. In this paper, we aim to bridge this gap and open up novel research directions and fresh insights on how the arsenal of emerging methods of TDA can be used for inference on power flow networks.

2. BACKGROUND ON GRAPHS AND TOPOLOGICAL DATA ANALYSIS

We start by providing a brief mathematical overview of the key concepts.

Preliminaries on Power Grids as Graphs We consider a graph $G = (V, E)$ as a model for a power grid network, with node set V and set of edges $E \subset V \times V$. Here $e_{uv} \in E$ represents an edge, e.g., a transmission line between nodes u and v , and nodes represent, e.g., generators, transformers, and load buses. We assume that G is *undirected* i.e., for all $e_{uv} \in E$, $e_{uv} \equiv e_{vu}$. Since the topological structure of G does not reflect the functional information about the power grid [2, 5], we can also consider an *(edge)-weighted* graph, or a pair (G, ω) . Here $\omega : V \times V \mapsto \mathbb{R}_{\geq 0}$ is an *(edge) weight* function such that each edge $e_{uv} \in E$ has a weight ω_{uv} . Also, for technical purposes, we assume $e_{uu} \in E$ and $\omega_{uu} = 0$ for all $u \in V$.

Persistent Homologies and Their Measures While re-current multi-node subgraph pattern i.e., network motifs has a high potential in providing an invaluable insight into local higher-order properties of power grid networks and their

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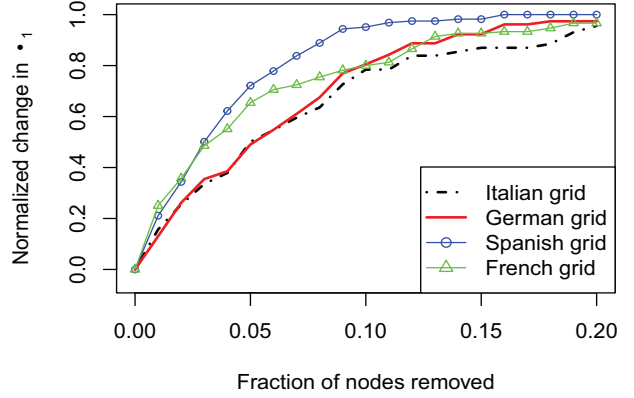
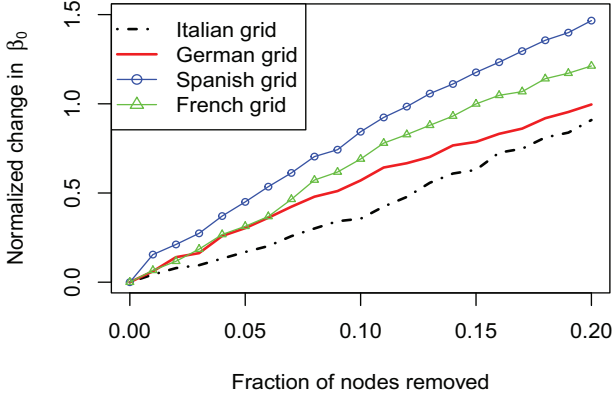


Fig. 1: Dynamics of the Betti numbers under degree based attacks.

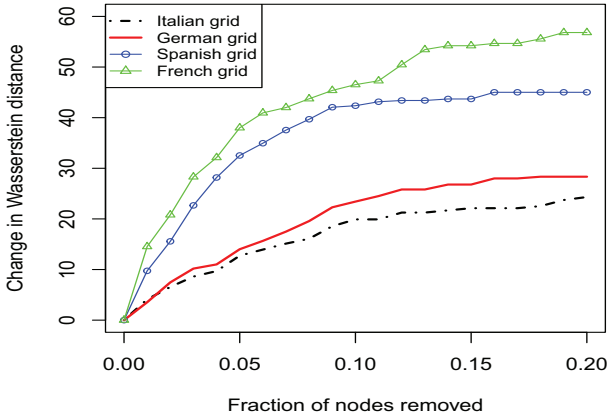


Fig. 2: Dynamics of the Wasserstein distances among the persistent diagrams, based on one-dimensional holes, under degree based attacks.

role in power grid vulnerability and resilience [7, 8, 9], motif analysis is largely restricted to unweighted graphs and hence, does not allow the systematic incorporation of such important real-world characteristics of power grids as, for instance, weights due to time evolving operating conditions and different node characteristics, e.g., generators, substations, transformers, etc. As an alternative, we propose to bring the tools of TDA, in particular, persistent homology to the inference on weighted power grid networks [10, 11, 12, 13, 14].

Consider an (edge)-weighted graph (G, ω) as a representation of a power grid network. If we select a certain threshold (or scale) $\nu_j > 0$ and keep only edges with weights $\omega_{uv} \leq \nu_j$, we obtain a graph G_j with an associated adjacency ma-

trix $A_{uv} = \mathbb{1}_{\omega_{uv} \leq \nu_j}$. Now, changing the threshold values $\nu_1 < \nu_2 < \dots < \nu_n$ results in a hierarchical nested sequence of graphs $G_1 \subseteq G_2 \subseteq \dots \subseteq G_n$ that is called as a *network filtration*. As a result, we can associate an abstract simplicial complex with each $G_j, j = 1, \dots, n$, which, in turn, allows to approximate the underlying geometry of (G, ω) with a combinatorial structure. For instance, the Vietoris-Rips (VR) complex is one of the most popular choices in TDA due to its simplicity and computational advantages [14, 15]. In particular, if we view weights ω_{uv} as some “distance” measure between nodes u and v , then the Vietoris-Rips complex at threshold ν_j is defined as $VR_j = \{\sigma \subset V | \omega_{uv} \leq \nu_j \text{ for all } u, v \in \sigma\}$. That is, VR_j contains all the k -node subsets of G_j , $k = 1, \dots, K$, which are pairwise connected by an edge as simplices of dimension $k - 1$ (i.e., a node is a 0-simplex, an edge is a 1-simplex, a triangle is a 2-simplex, etc).

Now, armed with the associated VR filtration, $VR_1 \subseteq VR_2 \subseteq \dots \subseteq VR_n$, we can quantitatively track the change in topological features such as the numbers of connected components, one-dimensional holes and higher-order structures, that appear and disappear with an increasing threshold ν_j . In turn, the analysis of evolution and lifespan of such topological features provides a multi-scale quantitative insight into network geometry and its role in network organization, functionality, and interactions among network components under varying operating conditions. Systematic evaluation of patterns and dynamics of multi-scale network geometry can be approached via an algebraic tool, based on the adaptation of a homology theory to applied data analysis and known as *persistent homology*. That is, the idea is to detect features which are long-lived, or persistent over varying thresholds ν_j . Such persistent features are likelier to impact power grid network functionality. The most widely used topological summaries of persistent features are the *Betti numbers*, *persistent bar-*

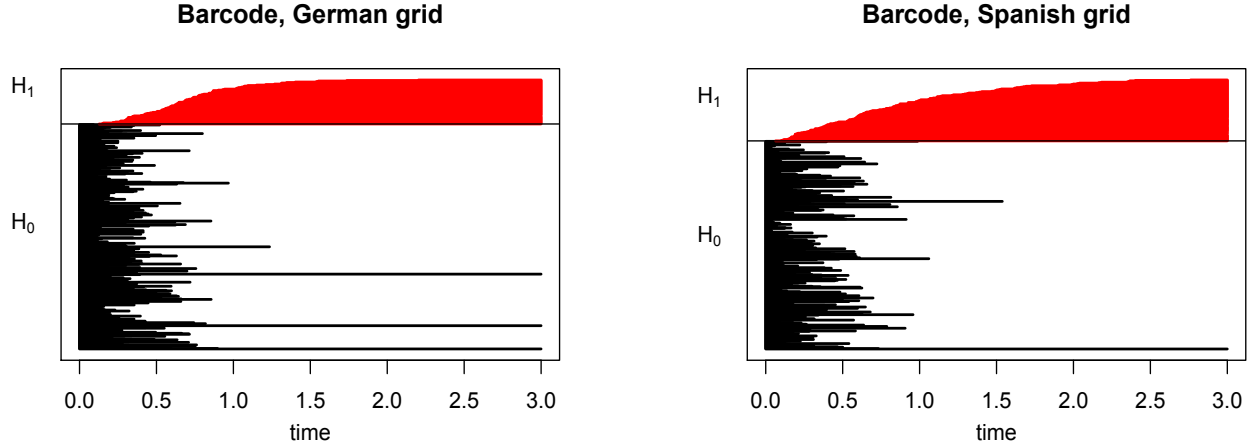


Fig. 3: Barcode for the number of connected components (H_0) and one-dimensional holes (H_1), where X-axis (Time) represents the threshold ν .

codes, and persistent diagrams. Though the definition of Betti numbers is rather abstract, they have a very tangible meaning. Namely, for a given simplicial complex, Betti-0 (β_0) represents the number of its connected components; Betti-1 (β_1) is the number of one-dimensional holes, Betti-2 (β_2) is the number of two-dimensional holes etc. Barcodes can be viewed as a persistence analog of the Betti numbers and depict the birth and death times of each topological feature as a bar (see Fig. 3). As a result, barcodes provide an insight on the significance of various topological features and allow to distinguish persistent features from topological noise [16]. Another visual representation of topological properties is a persistent diagram where each topological feature is denoted by a point and the (x, y) -coordinates of the point corresponding to the birth and death times, respectively. Hence, features with a longer lifespan, i.e. a stronger persistence, are those points that are far from the main diagonal.

3. PERSISTENT HOMOLOGY BASED NETWORK SIMILARITY MEASURES

To assess the similarity of two power grid networks (G_1, ω_1) and (G_2, ω_2) , we can compare their persistence diagrams. Indeed, persistence diagrams are often viewed as topological proxies for the input data [17, 10]. Hence, similarity between the persistent diagrams can be viewed as proxy similarity measures between the two power grids (G_1, ω_1) and (G_2, ω_2) . Given two diagrams D_1 and D_2 of (G_1, ω_1) and (G_2, ω_2) , respectively, we can compare their similarity using their bottleneck distance

$$W_\infty(D_1, D_2) = \inf_{\gamma} \sup_{x \in D_1} \|x - \gamma(x)\|_\infty,$$

where γ ranges over all bijections between D_1 and D_2 , and $\|z\|_\infty = \max_i |z_i|$. The intuitive idea behind the bottleneck distance is to superimpose two persistent diagrams and to quantify the shift required to force the points at the two diagrams to match [12]. Alternatively, we can use more conventional matching metrics such as Wasserstein distance

$$W_r(D_1, D_2) = \inf_{\gamma} \left(\sum_{x \in D_1} \|x - \gamma(x)\|_\infty^r \right)^{1/r}.$$

Both the bottleneck and Wasserstein distances are feasible choices as power grid matching metrics due to their robustness with respect to minor perturbation of the input data and associated topological noise [17].

Vulnerability of Power Grids Many factors, both known and hidden, impact power grid robustness. Moreover, there exist several notions of grid robustness and therefore one can observe different vulnerability properties from the same grid under various types of intentional attacks and random failures. [8, 18] empirically show that stability of unweighted power grid networks is related to local network properties and, particularly, network motifs. Most recently, [9] provide a motif-based analysis of local power grid vulnerability under random and intentional attacks. [9] have found that the dynamics of distributions of 4-node motifs under various attacks differ with respect to the global tail-based grid classification of power grid fragility proposed in [18]. Moreover, robust and fragile power systems are shown to exhibit different degrees of local sensitivity and degradation with respect to the type of attack and the type of motif. However, these studies are limited to unweighted power grid networks and do not allow to incorporate critical information on EE properties of power systems.

We investigate how the summaries of topological features

such as the Betti numbers β_p , $p \in \mathbb{Z}^+$, and persistent diagrams evolve under intentional targeted attacks. Also, we evaluate sensitivity of these topological summaries to the attack. Note that many of the introduced TDA concepts are abstract and may not have straightforward explanation on the relations between, e.g., the number of p -dimensional holes and the power system robustness. Our main postulate here is that a system can be called more resilient if it tends to preserve longer its original properties under the attack, and our primary focus is on analysis of geometric properties of the system. In turn, the Betti numbers β_p , $p \in \mathbb{Z}^+$, and persistent diagrams provide systematic characterization of the system geometry. As a result, multiple power grid networks can be classified in terms of their resilience to the attack at a multi-scale level.

4. A CASE STUDY

We present a study on robustness/vulnerability of electricity transmission networks of four European countries, e.g., Germany, Italy, France and Spain, where nodes indicate power stations/sub-stations. The data are obtained from the Union for the Coordination of the Transmission of Electricity (UCTE). Table 1 presents the number of nodes, edges of the networks.

Table 1: *Network descriptions*

Power grid	# of nodes	# of edges
Germany	445	567
Italy	273	375
France	677	913
Spain	461	664

The edge weight function of the grids is defined as the reciprocal of electrical conductance [19], i.e. a ratio of the normalized geographic distance between nodes u and v to the number of direct transmission lines between them. (Conventionally, we set up the weight ω_{uv} to infinity if $e_{uv} \notin E$.) Furthermore, since the observed power grids do not typically contain inter-connected groups of four nodes, we limit the dimension of VR complexes by three.

Having equipped the observed power grid networks with the corresponding VR complex filtration, we now calculate the Betti numbers β_0 and β_1 (one for each threshold value) for the original power grid networks and the power grid networks under degree based attacks, that is, when a fraction of nodes with the highest degrees are removed. The changes in the Betti numbers under attacks are calculated relative to the initial Betti numbers prior to the attack, i.e.,

$$\|\vec{\beta}_k^0 - \vec{\beta}_k^p\|_2 / \|\vec{\beta}_k^0\|_2, \quad k \in \{0, 1\},$$

where $\|\cdot\|_2$ denotes the Euclidean distance, p is the fraction of removed nodes, $\vec{\beta}_k^0 = \{\beta_{k1}^0, \dots, \beta_{kn}^0\}$ and $\vec{\beta}_k^p = \{\beta_{k1}^p, \dots, \beta_{kn}^p\}$

are the sequences of Betti numbers of a power grid network before and after the attack respectively. As Fig. 1 indicates, the power grid networks of Spain and France exhibit the steepest (relative) change in the Betti-0 and Betti-1 numbers. This implies that the targeted attacks change the topology underlying these two networks more adversely than that of Italy and Germany. Hence, the power grid networks of Italy and Germany appear to be more robust in terms of their topological structure. Furthermore, Fig. 2 shows the dynamics of the Wasserstein distances $W_2(D_0, D_p)$ between the persistent diagrams D_0 and D_p for each power grid network, where D_0 and D_p are the persistent diagrams of a power grid network prior to and after the attack, respectively. Fig. 1 and 2 confirm the earlier observation that the Spanish and French power grids are likely to be more vulnerable than the German and Italian power systems.

Fig. 3 compares the barcode plots of the German and Spanish power grid networks. Here, the barcodes of H_0 (shown in black) and H_1 (shown in red) correspond to the number of connected components and one-dimensional holes, respectively, as a function of a changing threshold $\nu \in [0, 3]$. Notice that the number of persistent one-dimensional holes in the Spanish power grid network is greater than the corresponding number in the German power grid network. That is, the Spanish power grid network exhibits a heavier left tail in the barcode plot of H_1 for lower thresholds.

In this case study we limited ourselves to a degree-based attack. The proposed framework, however, is general enough to handle other types of targeted attacks, such as the one where edges with the highest weight are removed.

5. CONCLUSION AND DISCUSSION

We have introduced a TDA-based framework to study power grid network functionality, and in particular its behavior under targeted attacks. To our knowledge, this is the first application of TDA to the analysis of power grids. In our pilot study, we have investigated robustness properties of power grid networks in four European countries. Our results have demonstrated that the extracted topological summaries such as the Betti numbers and persistent diagrams can be used to characterize network vulnerability under targeted attacks.

Given that Figs. 1 and 2 suggest that the Spanish power system tends to be more fragile under attacks, we hypothesize that vulnerability properties of grids tend to be linked (at least) to the persistence of one-dimensional holes. Remarkably, one-dimensional holes are also intrinsically connected to a number of cycles and related network motifs in the unweighted networks [20]. In turn, such cyclical network motifs are known to impact stability of unweighted power grid network [8, 9, 18]. However, a deeper understanding of the role of persistent homologies, their functional relationship with the power grid organization and measures of associated uncertainties requires a more systematic analysis.

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