Distributed Connectivity Maintenance in Multi-Agent Systems with Field of View Interactions

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***Abstract*— In this paper, we consider the problem of coordinating a multi-agent team to maintain a connected topology while equipped with limited field of view sensors. Applying the potential-based control framework and assuming agent interaction is encoded by a triangular geometry, we derive a distributed control law based on two non-linear functions, the point-to-line distance and the multivariate Gaussian, in order to achieve the topology control objective. Furthermore, we demonstrate a condition on digraphs for which the proposed control strategy achieves the system objective. Finally, numerical simulations are provided to corroborate the theoretical findings.**

I. Introduction

In the last two decades, distributed coordination problems for multi-agent systems have been widely investigated. While much attention has been devoted to *symmetric* coordination problems [1]–[7], *asymmetric* (or directed) ones have not been investigated as deeply. This is mostly due to the fact that in directed settings most of the properties that hold easily in the undirected context are difficult to verify and maintain. Nevertheless, recent works like [8]–[11] have investigated the directed coordination problem with success. In [8] the authors derive a class of digraphs for which a team of locally cooperating agents with linear time-invariant dynamics can distributively solve optimization problems that generally require global information in order to be solved. In [9] a consensus tracking problem with bounded input constraints is addressed to achieve current-sharing in a parallel charging system with directed communication. In [10], we derived a distributed potential-based control framework for topology control in which the anti-gradient potential control term assumes the usual quasi-linear, anti-symmetric form. In this work, we are interested in relaxing this quasi-linear, anti-symmetric assumption allowing us to consider nonlinear anti-gradient control terms capable of representing control schemes for topology control with triangular fields of view.

One of the directed coordination problems that has received recent attention [12]–[15] is control with limited *fields of view* (FOVs), for example aerial vehicles equipped with cameras capable of sensing other agents in a visually restricted area. In [12], a consensus and containment problem for a network of single-integrator agents with limited angular fields of view is considered. Under the connectivity assumption of the interaction digraph the authors manage to prove the convergence of the proposed control strategy considering first semi-circular FOVs and then heterogeneous angular FOVs. The authors in [13] consider an extremum-seeking problem for a mobile robot equipped with limited FOV sensing. A control scheme is proposed such that the robot orients its FOV to maximize an objective function subject to nonholonomic constraints. In this regard, our proposed control framework will be able to achieve a global objective, such as topology control, while encoding the best interactions among the agents. Moreover, in [14] a formation control problem for a team of agents equipped with limited field of view sensors is addressed. The authors, after designing a distance estimator scheme, manage to prove the convergence of the formation control protocol under the presence of time delays and discontinuous image issues.

In this paper, we propose a distributed potential-based coordination framework for agents equipped with limited field of view sensors. In particular, considering as a case study triangular fields of view rigidly attached to the agents, we derive a potential field to control agent motion in order to: i) keep the neighbors of each agent inside its field of view and ii) try to obtain the best FOV interactions using a quality map given by the sensors. Differently from the classical potential-based control, the proposed potential field does not require the usual *quasi-linear and anti-symmetric* assumptions allowing us to use generic non-linear functions such as the point-to-line distance and a multivariate Gaussian. Simulation results involving a team of six agents are then provided to prove the effectiveness of the proposed control strategy.

II. Preliminaries

A. Agent and Network Modeling

Let us consider a multi-agent system composed of n agents and assume that each agent i has the a first-order dynamics $\dot{s}_i(t) = u_i(t)$ with $s_i(t) = [p_i(t)^T, \theta_i(t)]^T \in \mathbb{R}^2 \times (-\pi, \pi]$ the state of the agent i composed of the position $p_i(t) = [x_i(t), y_i(t)]^T \in \mathbb{R}^2$ and the orientation $\theta_i \in (-\pi, \pi]$, while $u_i(t) \in \mathbb{R}^3$ denotes the control input. Stacking agent states and inputs yields the overall system

$$\dot{\mathbf{s}}(t) = \mathbf{u}(t) \tag{1}$$

with $\mathbf{s}(t) = [s_1(t)^T, \ldots, s_n(t)^T]^T \in \mathbb{R}^{2n} \times (-\pi, \pi]^n$ and $\mathbf{u}(t) = [u_1(t)^T, \ldots, u_n(t)^T]^T \in \mathbb{R}^{2n} \times (-\pi, \pi]^n$ the stacked vector of states and control inputs, respectively. In the sequel, time-dependence will be omitted for the sake of clarity.

Let us assume that each agent i possesses a limited field of view that is encoded by a triangle geometry $\mathcal{T}_i$ rigidly fixed

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to the agent. This kind of sensing yields asymmetric agent interactions that we will describe through a directed graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ with node set $\mathcal{V} = \{q_1, \ldots, q_n\}$ and edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. In particular, we will say that an edge $e_{ij} \in \mathcal{E}$ connects agent i and agent j if $p_j \in \mathcal{T}_i$. In addition, when referencing single edges we will use the convention e_k meaning that we are referencing the k-th directed edge out of $|\mathcal{E}|$ total edges[1]. Moreover, we will denote by $\mathcal{N}_i^+ = \{j \in \mathcal{V} : (i,j) \in \mathcal{E}\}$ the set of *out-neighbors* of agent i and $\mathcal{N}_i^- = \{j \in \mathcal{V} : (j,i) \in \mathcal{E}\}$ the set of *in-neighbors*. Note that since the graph is directed $(i,j) \in \mathcal{E}$ does not imply $(j,i) \in \mathcal{E}$.

A useful representation for a directed graph $\mathcal{G}$ is the *incidence matrix* $\mathcal{B}(\mathcal{G}) \in \mathbb{R}^{n \times |\mathcal{E}|}$, that is a matrix with rows indexed by agents and columns indexed by edges, such that $\mathcal{B}_{ij} = 1$ if the edge e_j leaves vertex v_i, -1 if it enters vertex v_i, and 0 otherwise. The *outgoing incidence matrix* $\mathcal{B}_+$ contains only the outgoing parts of the incidence matrix $\mathcal{B}$, with incoming parts set to zero. We will also make use of the *directed edge Laplacian* $\mathcal{L}_\mathcal{E}^d \in \mathbb{R}^{|\mathcal{E}| \times |\mathcal{E}|}$ given by $\mathcal{L}_\mathcal{E}^d = \mathcal{B}^T \mathcal{B}_+$. For properties of the edge Laplacian see for example [7], [16]. Let us also recall the definition of a positive semi-definite matrix that is, given a symmetric $m \times m$ matrix A it holds that $x^T A x \geq 0 \ \forall x \in \mathbb{R}^m$ in which the strict inequality implies the positive definiteness of A. This definition can be extended to a non-symmetric matrix B by considering the positive (semi)-definiteness of its symmetric part $B^s = (B + B^T)/2$ since $x^T B^s x \geq 0$ implies $x^T B x \geq 0$.

III. Directed Coordination Framework

A. Field of View Modeling

In this section we derive the mathematical modeling of the agents' fields of view that we assume here to take on a triangle geometry. The triangle $\mathcal{T}_i$ encoding the sensing of the agent i is described by three points $v_i^k \in \mathcal{T}_i \subset \mathbb{R}^2$ with $k \in \{1,2,3\}$ and by an angle offset η_i with respect to the agent orientation θ_i. This modeling allows us to represent drones or ground robots equipped with limited field of view sensors that are not aligned with the agent's body-fixed reference frame. In this work, as a case study we assume that the triangle $\mathcal{T}_i$ is an isosceles triangle with sides of length $l_i \in \mathbb{R}_+$ and angle $2\,\alpha_i \in \mathbb{R}_+$. This leads to the following formal description

$$\begin{aligned} v_i^1 &= p_i \\ v_i^2 &= p_i + l_i\, R(\eta_i)\, t_i^2 \\ v_i^3 &= p_i + l_i\, R(\eta_i)\, t_i^3 \end{aligned} \tag{2}$$

where $R(\eta_i)$ is a rotation matrix with respect to the angle η_i and t_i^k with $k \in \{2,3\}$ are translation vectors defined as

$$t_i^2 = \begin{pmatrix} \cos(\theta_i - \alpha_i) \\ \sin(\theta_i - \alpha_i) \end{pmatrix} \qquad t_i^3 = \begin{pmatrix} \cos(\theta_i + \alpha_i) \\ \sin(\theta_i + \alpha_i) \end{pmatrix} \tag{3}$$

A graphical representation of the triangular field of view can be found in Figure 1 in which the following angles have been used $\theta_i = 60°, \eta_i = 75°$, and $\alpha_i = 35°$.

[1]In order to reference the k-th edge, the edge set $\mathcal{E}$ needs to be sorted. A simple sort can be obtained enumerating the edges $(1,j)$ of the first agent as $e_1, e_2, \ldots$, then the edges $(2,h)$ of the second agent and so on.

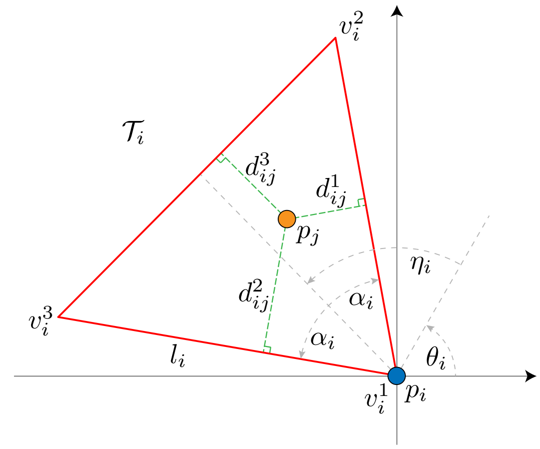


Fig. 1. Modeling of the triangular field of view $\mathcal{T}_i$ with side l_i of the agent s_i in which the angles $\theta_i = 60°$, $\eta_i = 75°$, and $\alpha_i = 35°$ have been used. In dashed green lines are reported the distances d_{ij}^1, d_{ij}^2, and d_{ij}^3 introduced in eq. (6).

B. Potential Fields for Topology Maintenance

As a case study let us consider the maintenance of the interactions (topology) among the agents in the network, i.e., we want to preserve the initial graph $\mathcal{G}$. In other words, we want to derive a distributed potential-based control law such that each agent i maintains its neighbor j inside the triangle $\mathcal{T}_i$. For this purpose, we will make use of the perpendicular distance from a point, the agent j, to a line, the side of the triangle $\mathcal{T}_i$. Specifically, the distance D from a point $P_0 = (x_0, y_0)$ to a line $\overline{P_1\,P_2}$ that passes through two points $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$ is defined as:

$$D(P_0, \overline{P_1\,P_2}) = \frac{(y_2 - y_1)x_0 - (x_2 - x_1)y_0 + x_2 y_1 - y_2 x_1}{\sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}} \tag{4}$$

A natural potential field that keeps the agent j inside the field of view of the agent i is then the following

$$\Phi_{ij}\left(s_i, s_j\right) = \sum_{k=1}^{3} \left(\frac{1}{d_{ij}^k}\right)^2 \tag{5}$$

where the terms d_{ij}^k with $k \in \{1,2,3\}$ are the distance between the agent j and the three sides of field of view of the agent i respectively (Figure 1), that is

$$\begin{aligned} d_{ij}^1 &= D\left(s_j, \overline{v_i^1, v_i^2}\right) \\ d_{ij}^2 &= D\left(s_j, \overline{v_i^1, v_i^3}\right) \\ d_{ij}^3 &= D\left(s_j, \overline{v_i^2, v_i^3}\right) \end{aligned} \tag{6}$$

We now demonstrate a technical result of the potential field Φ_{ij} in (5) that will prove necessary to derive the main result in Theorem 1.

Lemma 1. *The potential field Φ_{ij} in* (5) *satisfies the local potential property $\nabla_{p_i} \Phi_{ij} = -\nabla_{p_j} \Phi_{ij}$.*

Proof. In order to prove this lemma we will show how the gradient of the potential term Φ_{ij} with respect to the positions of the agents i and j are equal but with opposite signs. However, since the derivation is equal for all the terms $\left(d_{ij}^k\right)^{-2}$ with $k \in \{1,2,3\}$, we will only derive the gradient of the term $\left(d_{ij}^1\right)^{-2}$ since the same reasoning applies to the others.

Denoting the points v_i^1, v_i^2 as $v_i^1 = (x_1, y_1)$, $v_i^2 = (x_2, y_2)$, the square of the distance d_{ij}^1 can be derived as

$$(d_{ij}^1)^2 = \frac{\Big((y_2 - y_1)x_j - (x_2 - x_1)y_j + x_2 y_1 - y_2 x_1\Big)^2}{(y_2 - y_1)^2 + (x_2 - x_1)^2} \tag{7}$$

then, using the definition of v_i^1 and v_i^2 given in (2) we obtain

$$\begin{aligned}(d_{ij}^1)^2 &= l_i^{-2}\left(l_i\big((x_i - x_j)\sin\gamma + (y_i - y_j)\cos\gamma\big)\right)^2 \\ &= \left[(p_i - p_j)^T \begin{pmatrix}\sin\gamma \\ \cos\gamma\end{pmatrix}\right]^2\end{aligned} \tag{8}$$

where $\gamma \in \mathbb{R}$ is the angle defined as $\gamma = \alpha_i - 2\eta_i - \theta_i$. Now, computing the gradient with respect to the agent i we obtain

$$\nabla_{p_i}\left(\frac{1}{d_{ij}^1}\right)^2 = -2\begin{pmatrix}\sin\gamma \\ \cos\gamma\end{pmatrix}\left[(p_i - p_j)^T\begin{pmatrix}\sin\gamma \\ \cos\gamma\end{pmatrix}\right]^{-3} \tag{9}$$

while with respect to the agent j we get

$$\nabla_{p_j}\left(\frac{1}{d_{ij}^1}\right)^2 = 2\begin{pmatrix}\sin\gamma \\ \cos\gamma\end{pmatrix}\left[(p_i - p_j)^T\begin{pmatrix}\sin\gamma \\ \cos\gamma\end{pmatrix}\right]^{-3} \tag{10}$$

Then, assuming a similar derivation for the terms d_{ij}^2 and d_{ij}^3 we have that

$$\sum_{k=1}^{3}\nabla_{p_i}\left(\frac{1}{d_{ij}^k}\right)^2 = -\sum_{k=1}^{3}\nabla_{p_j}\left(\frac{1}{d_{ij}^k}\right)^2 \tag{11}$$

which implies that $\nabla_{p_i}\Phi_{ij} = -\nabla_{p_j}\Phi_{ij}$ thus completing the proof. □

Remark 1. It should be noted that the gradient control term $\nabla_{p_i}\Phi_{ij}$ in eq. (9) can be expressed in a generic form as

$$\nabla_{p_i}\Phi_{ij}(p_i, p_j) = f(p_i, p_j) \tag{12}$$

in which $f \in C^0$ is a continuous function. Furthermore, the gradient control term (9) is not always differentiable since the denominator $\sin(\gamma)(x_i - x_j) + \cos(\gamma)(y_i - y_j)$ can be equal to zero in four cases: i) when the distance between the agent i and j goes to zero, ii) when the agents are aligned on the x-axis and the angle $\gamma = 0$, iii) when the agents are aligned on the y-axis and the angle $\gamma = \frac{\pi}{2}$, or iv) when the angle $\gamma = \arctan\left(-\frac{y_i - y_j}{x_i - x_y}\right)$. However, in this work, the aforementioned behaviours are not allowed since the potential field is built in order to keep the neighbor j inside the FOV of the agent i. Future work will investigate this aspect in order to remove such limitation for more general topology control.

Until now we have considered the maintenance of the initial agent interactions, focusing on keeping each neighbor $j \in \mathcal{N}_i^+$ inside the limited field of view of the agent i. We want now to take a further step by modeling the possibility for each agent i to control its field of view in order to have its neighbors in a specific point of the triangle $\mathcal{T}_i$ in which the quality of the interactions is the best. In order to do so, let us assume that each agent is provided with a *quality map* $\mathcal{M}$ of its FOV that gives a quality metric for each interior point and let us also assume that each agent is capable of differentiating this map[2]. This allows us to model the map $\mathcal{M}$ as a potential field that we can use together with the point-to-line potential discussed previously.

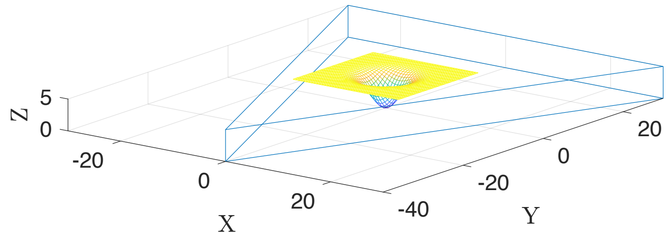


Fig. 2. Example of the potential field Ψ_{ij} centered in $[0,0]^T$ with amplitude $A = 5$ and variance coefficients $a = b = 0.1$.

To this end, let us denote with $t_i^\star \in \mathbb{R}^2$ the desired position that the neighbors of the agent i should reach inside the FOV of the agent i, defined as

$$t_i^\star = p_i + l_i^\star R(\eta_i)\begin{bmatrix}\cos\theta_i \\ \sin\theta_i\end{bmatrix} \tag{13}$$

Now, in order to model the map $\mathcal{M}$ we use a two-dimensional Gaussian function $\Psi_{ij}(s_i, s_j)$ with mean equal to $t_i^\star = (x_0, y_0)$ and variance σ_x^2, σ_y^2, that is

$$\Psi_{ij}(s_i, s_j) = A\left(1 - e^{-\tau}\right) \tag{14}$$

in which A is the amplitude and τ is defined as

$$\tau = \frac{(x_j - x_0)^2}{2\sigma_x^2} + \frac{(y_j - y_0)^2}{2\sigma_y^2} = a(x_j - x_0)^2 + b(y_j - y_0)^2 \tag{15}$$

with $a = (2\sigma_x^2)^{-1}$, $b = (2\sigma_y^2)^{-1}$ the sensor related variances (see Figure 2).

Similar to the FOV maintenance controller, we can move along the anti-gradient of our potential field $\Psi_{ij}(s_i, s_j)$ to orient the FOV of the agents in order to minimize the distance between the neighbors $j \in \mathcal{N}_i$ and the desired position $t_i^\star$. We now demonstrate again a technical result of the potential field Ψ_{ij} in (14) that will prove necessary to derive the main result in Theorem 1.

Lemma 2. *The potential field Ψ_{ij} in* (14) *satisfies the local potential property $\nabla_{p_i}\Psi_{ij} = -\nabla_{p_j}\Psi_{ij}$.*

Proof. In order to prove this result, similar to the proof of the previous lemma, we will show how the gradient of the potential

[2]This map can be either an intrinsic property of the sensors or extrinsic influence. Extrinsic influences may occur for example in vision applications where illumination, occlusion, etc., may induce wide variations in the quality of vision-based agent interactions.

term Ψ_{ij} with respect to the positions of the agents i and j are equal but with opposite signs. Recalling the definition of the desired position $t_i^\star$ in (13), the potential term Ψ_{ij} is

$$\Psi_{ij}(s_i, s_j) = A\left(1 - e^{-\tau}\right) \tag{16}$$

where τ defined in (15) can be expressed as

$$\tau = a\big(l_i^\star \cos\gamma + x_i - x_j\big)^2 + b\big(l_i^\star \sin\gamma + y_i - y_j\big)^2 \tag{17}$$

in which $\gamma \in \mathbb{R}$ is the angle equal to $\gamma = \theta_i + 2\eta_i$. Now, computing the gradient with respect to the agent i we get the following

$$\nabla_{p_i}\Psi_{ij} = 2\,A \begin{pmatrix} a\,e^{-\tau}(x_i - x_j + l_i^\star \cos\gamma) \\ b\,e^{-\tau}(y_i - y_j + l_i^\star \sin\gamma) \end{pmatrix} \tag{18}$$

while the gradient with respect to p_j is

$$\nabla_{p_j}\Psi_{ij} = -2\,A \begin{pmatrix} a\,e^{-\tau}(x_i - x_j + l_i^\star \cos\gamma) \\ b\,e^{-\tau}(y_i - y_j + l_i^\star \sin\gamma) \end{pmatrix} \tag{19}$$

which shows clearly that $\nabla_{p_i}\Psi_{ij} = -\nabla_{p_j}\Psi_{ij}$ thus completing the proof. □

Remark 2. Similar to the term $\nabla_{p_i}\Phi_{ij}$ in eq. (12), the gradient control term $\nabla_{p_i}\Psi_{ij}$ in (18) can be expressed in a generic form as

$$\nabla_{p_i}\Psi_{ij}(p_i, p_j) = g(p_i, p_j) \tag{20}$$

in which $g \in C^0$ is a continuous function.

We are now ready to introduce the control law that each agent i needs to run in order to keep its neighbors in the limited sensing zone $\mathcal{T}_i$ near the desired point $t_i^\star$ as the following

$$\dot{p}_i = -\sum_{j\in\mathcal{N}_i^+} \nabla_{p_i}\Big(\Phi_{ij}(s_i, s_j) + \Psi_{ij}(s_i, s_j)\Big) \tag{21a}$$

$$\dot{\theta}_i = -\sum_{j\in\mathcal{N}_i^+} \nabla_{\theta_i}\Big(\Phi_{ij}(s_i, s_j) + \Psi_{ij}(s_i, s_j)\Big) \tag{21b}$$

where $\Phi_{ij}(s_i, s_j)$ and $\Psi_{ij}(s_i, s_j)$ are the potential fields introduced above in (5) and (14). Note that, as it will be shown later, the potential fields Φ_{ij}, Ψ_{ij} do not depend on the orientation θ_j of the neighbors $j \in \mathcal{N}_i^+$ but only on their position p_j. Finally, stacking the position and the orientation of the agent i we obtain the following form for the control input u_i:

$$\dot{s}_i = \underbrace{-\sum_{j\in\mathcal{N}_i^+} \nabla_{s_i}\Big(\Phi_{ij}(s_i, s_j) + \Psi_{ij}(s_i, s_j)\Big)}_{u_i} \tag{22}$$

C. Theoretical Analysis

In this section we provide the main result of our work, that is the stability analysis of the proposed topology control framework. To this end let us introduce the matrix $\overline{\mathcal{L}}$ defined as

$$\overline{\mathcal{L}} = \begin{bmatrix} \big(\mathcal{B}^T\mathcal{B}_+\big) \otimes I_2 & O_{2|\mathcal{E}|\times|\mathcal{E}|} \\ O_{|\mathcal{E}|\times 2|\mathcal{E}|} & \big(\mathcal{B}_+^T\mathcal{B}_+\big) \end{bmatrix} \tag{23}$$

where $\mathcal{B}$ is the incidence matrix of the graph $\mathcal{G}$, $\mathcal{B}_+$ is the outgoing portion of $\mathcal{B}$, I_2 is the 2×2 identity matrix and $O_{r\times r}$ is a $r \times r$ zeros matrix. We are now ready to state our main result.

Theorem 1. *Consider the multi-agent system* (1) *running control laws* (22)*. Then, if the matrix $\overline{\mathcal{L}}$ defined as* (23) *is positive semi-definite, the system is stable in the sense that if the energy $V(\mathbf{s}(t))$ is finite at time $t = t_0$ then it remains finite for all $t > t_0$.*

Proof. In order to study the stability of our proposed control framework, let us introduce the following Lyapunov function for directed graphs $V : \mathbb{R}^{2n} \times (-\pi, \pi]^n \to \mathbb{R}_+$ defined as

$$V(\mathbf{s}(t)) = \sum_{i=1}^{n}\sum_{j\in\mathcal{N}_i^+} \underbrace{\Big(\Phi_{ij}\left(s_i, s_j\right) + \Psi_{ij}\left(s_i, s_j\right)\Big)}_{V_{ij}(s_i, s_j)} \tag{24}$$

with time derivative $\dot{V}(\mathbf{s}) = \left(\nabla_{\mathbf{s}}V\right)^T \dot{\mathbf{s}}$. Now, by following a similar reasoning as in [10] we want to derive an *edge-based* form of $\nabla_{\mathbf{s}}V$ and $\dot{\mathbf{s}}$ that will reveal the topology of the graph. First, let us consider the general form for the gradient of the Lyapunov function $\nabla_{\mathbf{s}}V$, that is

$$\begin{aligned} \nabla_{\mathbf{s}}V &= \sum_{i=1}^{n}\sum_{j\in\mathcal{N}_i^+} \Big[0, \ldots, \nabla_{s_i}V_{ij}^T, 0, \ldots, 0, \nabla_{s_j}V_{ij}^T, \ldots, 0\Big]^T \\ &= \Big[\sum_{j\in\mathcal{N}_1^+} \nabla_{s_1}V_{1j}^T + \sum_{j\in\mathcal{N}_1^-} \nabla_{s_1}V_{j1}^T \quad , \;\ldots\;, \\ &\qquad + \sum_{j\in\mathcal{N}_n^+} \nabla_{s_n}V_{nj}^T + \sum_{j\in\mathcal{N}_n^-} \nabla_{s_n}V_{jn}^T\Big]^T \end{aligned} \tag{25}$$

Let us now analyze the gradient contributions related to the agent h, that is

$$\begin{aligned} \nabla_{s_h}V &= \sum_{j\in\mathcal{N}_h^+} \nabla_{s_h}V_{hj} + \sum_{j\in\mathcal{N}_h^-} \nabla_{s_h}V_{jh} \\ &= \sum_{j\in\mathcal{N}_h^+} \begin{bmatrix} \nabla_{x_h}V_{hj} \\ \nabla_{y_h}V_{hj} \\ \nabla_{\theta_h}V_{hj} \end{bmatrix} + \sum_{j\in\mathcal{N}_h^-} \begin{bmatrix} \nabla_{x_h}V_{jh} \\ \nabla_{y_h}V_{jh} \\ \nabla_{\theta_h}V_{jh} \end{bmatrix} \\ &= \begin{bmatrix} \sum_{j\in\mathcal{N}_h^+} \nabla_{x_h}(\Phi_{hj} + \Psi_{hj}) \\ \sum_{j\in\mathcal{N}_h^+} \nabla_{y_h}(\Phi_{hj} + \Psi_{hj}) \\ \sum_{j\in\mathcal{N}_h^+} \nabla_{\theta_h}(\Phi_{hj} + \Psi_{hj}) \end{bmatrix} - \begin{bmatrix} \sum_{j\in\mathcal{N}_h^-} \nabla_{x_j}(\Phi_{jh} + \Psi_{jh}) \\ \sum_{j\in\mathcal{N}_h^-} \nabla_{y_j}(\Phi_{jh} + \Psi_{jh}) \\ 0 \end{bmatrix} \end{aligned} \tag{26}$$

where we have exploited the local properties proven in Lemma 1 and Lemma 2 and the fact that $\sum_{j\in\mathcal{N}_h^-} \nabla_{\theta_h}(\Phi_{jh} + \Psi_{jh})$ is equal to zero since the orientation θ_h of the agent h is not involved in the potential terms Φ_{jh} and Ψ_{jh}.

Since the contributions in eqs. (25) and (26) are related to the edge endpoints we can restate $\nabla_{\mathbf{s}}V$ using the stacked vector of potential field gradients $\xi \in \mathbb{R}^{3|\mathcal{E}|} = \left[\xi_{xy}^T, \, \xi_\theta^T\right]^T$ where

$\xi_{xy} \in \mathbb{R}^{2|\mathcal{E}|}$ and $\xi_\theta \in \mathbb{R}^{|\mathcal{E}|}$ are defined as

$$
\begin{aligned}
\xi_{xy} &= \Big[\nabla_{x_{e_1(1)}} V_{e_1}^T, \nabla_{y_{e_1(1)}} V_{e_1}^T, \dots, \nabla_{x_{e_{|\mathcal{E}|}(1)}} V_{e_{|\mathcal{E}|}}^T, \nabla_{y_{e_{|\mathcal{E}|}(1)}} V_{e_{|\mathcal{E}|}}^T\Big]^T \\
\xi_\theta &= \Big[\nabla_{\theta_{e_1(1)}} V_{e_1}^T, \dots, \nabla_{\theta_{e_{|\mathcal{E}|}(1)}} V_{e_{|\mathcal{E}|}}^T\Big]^T
\end{aligned}
\tag{27}
$$

where $e_k(1)$ denotes the starting vertex q_i of the k-th edge (i, j), and thus $\nabla_{w_{e_k(1)}} V_{e_k} \in \mathbb{R}$ denotes the gradient with respect to the state variable $w_i \in \{x_i, y_i, \theta_i\}$ of potential function V_{ij}. This allows us to write $\nabla_{\mathbf{s}} V$ in a block diagonal matrix form as

$$
\nabla_{\mathbf{s}} V = \begin{bmatrix} \mathcal{B} \otimes I_2 & O_{2n\times|\mathcal{E}|} \\ O_{n\times 2|\mathcal{E}|} & \mathcal{B}_+ \end{bmatrix} \xi \tag{28}
$$

At this point we notice that $\dot{\mathbf{s}}$ can be rewritten as

$$
\dot{\mathbf{s}} = -\Big[\sum_{j\in\mathcal{N}_1^+} \nabla_{s_1} V_{1j}^T, \dots, \sum_{j\in\mathcal{N}_n^+} \nabla_{s_n} V_{nj}^T\Big]^T \tag{29}
$$

which leads to the following diagonal block matrix

$$
\dot{\mathbf{s}} = -\begin{bmatrix} \mathcal{B}_+ \otimes I_2 & O_{2n\times|\mathcal{E}|} \\ O_{n\times 2|\mathcal{E}|} & \mathcal{B}_+ \end{bmatrix} \xi \tag{30}
$$

since (29) has contributions only from the starting vertex of each edge. At this point, by combining eqs. (28) and (30) we obtain

$$
\begin{aligned}
\dot{V}(\mathbf{s}) &= -\left(\begin{bmatrix} \mathcal{B} \otimes I_2 & O_{2n\times|\mathcal{E}|} \\ O_{n\times 2|\mathcal{E}|} & \mathcal{B}_+ \end{bmatrix} \xi\right)^T \left(\begin{bmatrix} \mathcal{B}_+ \otimes I_2 & O_{2n\times|\mathcal{E}|} \\ O_{n\times 2|\mathcal{E}|} & \mathcal{B}_+ \end{bmatrix} \xi\right) \\
&= -\xi^T \begin{bmatrix} \left(\mathcal{B}^T\mathcal{B}_+\right) \otimes I_2 & O_{2|\mathcal{E}|\times|\mathcal{E}|} \\ O_{|\mathcal{E}|\times 2|\mathcal{E}|} & \left(\mathcal{B}_+^T\mathcal{B}_+\right) \end{bmatrix} \xi = -\xi^T\, \overline{\mathcal{L}}\, \xi
\end{aligned}
\tag{31}
$$

Since the Lyapunov time derivative $\dot{V}(\mathbf{s})$ is in a quadratic form in which the matrix $\overline{\mathcal{L}}$ is positive semi-definite then it is negative semi-definite, proving that if the energy of the system is finite at time $t = t_0$ then it remains finite for all $t > t_0$. $\square$

The Lyapunov time derivative shown in (31) is in a typical quadratic form and its characteristics depend on the block diagonal matrix $\overline{\mathcal{L}}$. Since the first diagonal entry of $\overline{\mathcal{L}}$ is a Kronecker product between an identity matrix and an asymmetric and in general indefinite matrix, i.e. the directed edge Laplacian $\mathcal{L}_{\mathcal{E}}^d$, studying the generic stability properties of the system is difficult (i.e., for which class of topologies is this system stable)[3].

Corollary 1 (Topology Maintenance)**.** *Consider the multi-agent system* (1) *running control laws* (22) *with an initial topology $\mathcal{G}$. Assume that the initial energy configuration is finite. Then, if the matrix $\overline{\mathcal{L}}$ defined as* (23) *is positive semi-definite, the initial topology $\mathcal{G}$ is preserved for all $t > t_0$.*

Proof. The proof is a directed consequence of the construction of the potential terms V_{ij} along with the results of Theorem 1. In particular, since the potentials V_{ij} are built in a way that would make the energy of the system infinite when an edge e is about to be removed from the edge set $\mathcal{E}$ and, as proven in Theorem 1 if the system starts from a finite-energy configuration then the energy must remain finite, then the proposed control framework succeeds in achieving the global objective that in our case is the maintenance of the initial topology $\mathcal{G}$ for all time $t > t_0$. $\square$

[3]Future work will focus on a characterization of topological conditions ensuring that $\overline{\mathcal{L}}$ is positive semidefinite. The characteristics of the matrix $\overline{\mathcal{L}}$ can be evaluated in a distributed fashion using the *Lanczos Biorthogonalization algorithm* as done in [10].

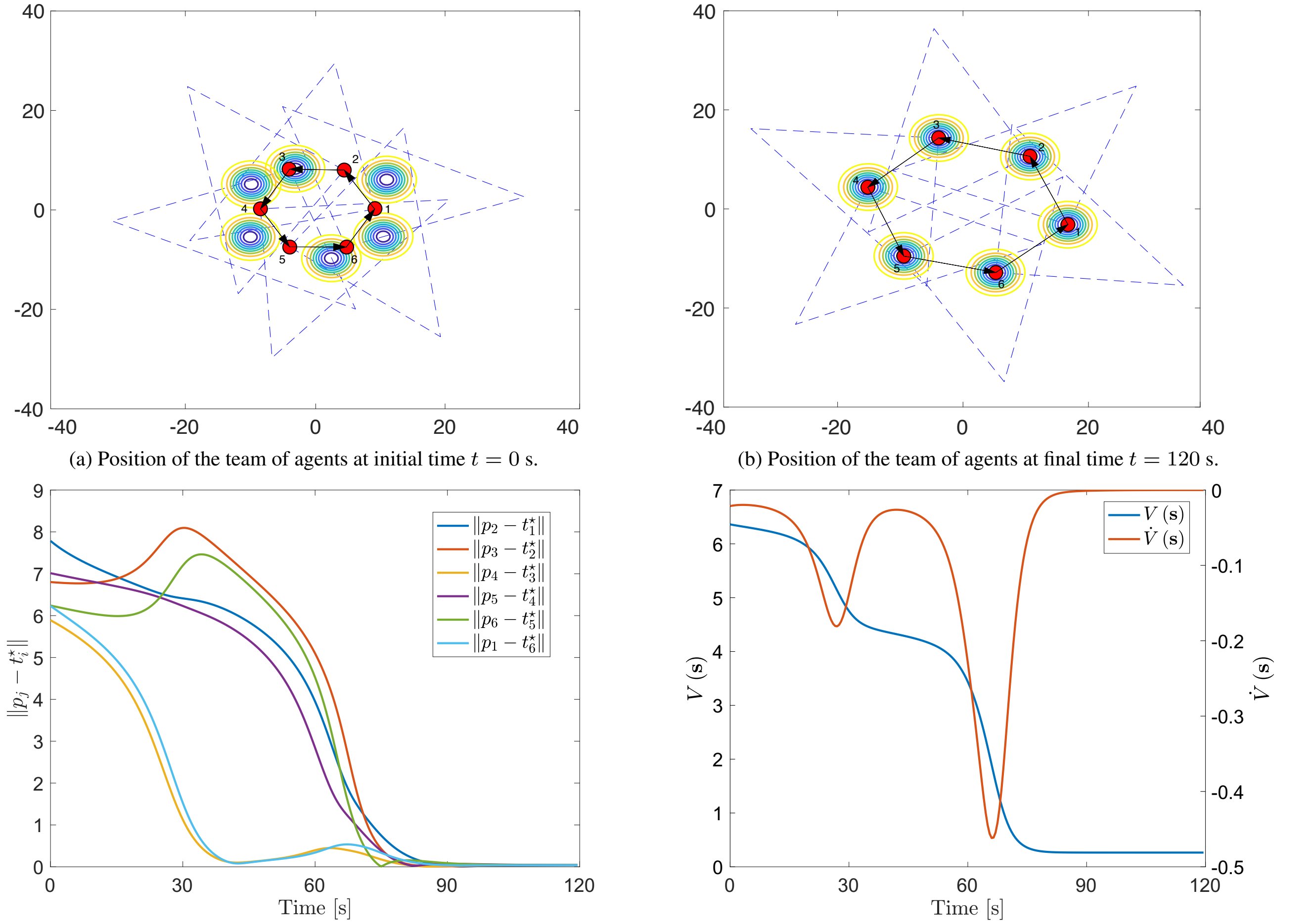
IV. Simulation Results

For the numerical validation we consider a team composed of 6 locally interacting agents that are performing dispersion while maintaining the initial topology $\mathcal{G}$ and encoding the best possible FOV interactions. Each of the agents' field of view is described by a triangle $\mathcal{T}_i$ with angle $\alpha_i = 45°$ and side $l_i = 30$. For the sake of simplicity here we assume that there is not an offset between the orientation of the agent and the orientation of the sensor, i.e., the angle η_i is equal to zero. The quality map $\mathcal{M}$ encoding the best desired configurations for the neighbors of each agent i is selected according to eqs. (13) to (15) in which $l_i^\star = 15$, $A = 1$, and $a = b = 0.1$.

Fig. 3a and 3b show the initial and final positions of the team of agents along with each field of view (dashed blue lines) and the surface levels (circles) of the potential field Ψ_{ij} with $i, j \in \{1, \dots, 6\}$. Note how at the start of the dispersion action there is no agent inside the center of the circles while at the end of it each neighbor is at the desired position $t_i^\star$. In this regard, Fig. 3c depicts the evolution over time of the error $\|p_j - t_i^\star\|$ between the position of the neighbors p_j and the desired position $t_i^\star$. Notice also that the interactions depicted with black arrows are only the ones selected among the possible interactions encoded by the limited fields of view for control purpose, i.e., chosen in order to make the matrix $\overline{\mathcal{L}}$ positive semi-definite. Finally, in Fig. 3d is shown the decrease over time of the Lyapunov function $V(\mathbf{s})$ (orange) and the negative semi-definiteness of its time derivative $\dot{V}(\mathbf{s})$ (light blue). Note that the left axis is referencing the Lyapunov function V while the right one is referencing the time derivative $\dot{V}$. Around time $t = 40$ s it can be seen how a sort of local minimum is overcome: at that time, as shown in Fig. 3c, the errors between the desired positions $t_i^\star$ of the agents $3, 6$ and the position of its neighbors $4, 1$ (yellow and cyan lines, respectively) have reached the zero while the errors of the other agents have not. Then, as the local minimum is overcome, they start going quickly towards the zero reaching it at time $t = 90$ s. This also causes a small rearrangement in the formation of the team as shown by the temporary increase of other error variables. After time $t = 90$ s, as shown in Fig. 3d, the time derivative $\dot{V}$ stabilizes its value to zero and the energy of the system stabilizes itself to a finite value as expected from Theorem 1.

V. Conclusion

In this paper, we proposed a distributed multi-agent coordination framework for directed interactions encoded by arbitrary fields of view. Considering as case of study an arbitrary field of view represented by a geometrical triangle, we derived a



(a) Position of the team of agents at initial time $t = 0$ s.

(b) Position of the team of agents at final time $t = 120$ s.

(c) Evolution of the error between the agents neighbors and their desired position $t_i^\star$ over time.

(d) Evolution of the Lyapunov function $V(\mathbf{s})$ and its time derivative $\dot{V}(\mathbf{s})$ introduced in eqs. (24) and (31) over time.

Fig. 3. Simulation results involving a team of 6 agents.

control law capable of maintaining the interactions among the agents using two potential fields described by generic non linear functions. Numerical simulations with a team of six agents demonstrated the effectiveness of the proposed potential-based control framework.

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