

Deep learning for waveform estimation and imaging in passive radar

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Abstract: The authors consider a bistatic configuration with a stationary transmitter transmitting unknown waveforms of opportunity and a single moving receiver and present a deep learning (DL) framework for passive synthetic aperture radar (SAR) imaging. They approach DL from an optimisation based perspective and formulate image reconstruction as a machine learning task. By unfolding the iterations of a proximal gradient descent algorithm, they construct a deep recurrent neural network (RNN) that is parameterised by the transmitted waveforms. They cascade the RNN structure with a decoder stage to form a recurrent auto-encoder architecture. They then use backpropagation to learn transmitted waveforms by training the network in an unsupervised manner using SAR measurements. The highly non-convex problem of backpropagation is guided to a feasible solution over the parameter space by initialising the network with the known components of the SAR forward model. Moreover, prior information regarding the waveform structure is incorporated during initialisation and backpropagation. They demonstrate the effectiveness of the DL-based approach through numerical simulations that show focused, high contrast imagery using a single receiver antenna at realistic signal-to-noise-ratio levels.

1 Introduction

1.1 Motivations

Deep learning (DL) has propelled significant developments in a wide range of applications in science and engineering [1]. These include advancements in medical imaging [2, 3], computer vision [4, 5], and artificial intelligence [6], with an impressive performance in object recognition, natural language processing, and many other applications [1, 7].

Currently, most prominent applications of DL involve establishing complex decision boundaries in high-dimensional parameter spaces using large amounts of training data. We instead consider DL as a joint estimation framework for problems that contain unknown parameters in the measurement model. Passive radar imaging falls into such a class of problems, in which transmitter locations or transmitted waveforms may not be known a priori. Passive radar has been an area of intense research due to the proliferation of transmitters of opportunity and several advantages it offers. These include efficient use of electromagnetic spectrum, increased stealth, and reduced cost among others [8–18]. Existing passive radar methods require two or more antennas, either spatially separated or colocated with sufficient directivity and gain. Specifically, these methods rely on correlating pairs of measurements acquired by two different antennas. These methods can be classified into two major categories: passive coherent localisation (PCL) [19–23] and time difference-of-arrival (TDOA)/frequency difference-of-arrival (FDOA) backprojection [8, 17, 24–32].

The PCL approach attempts to recover a copy of the transmitted waveform by filtering the received signal acquired by an antenna directed towards a transmitter of opportunity. This is followed by matched filtering of the received signal acquired by another antenna directed to a scene of interest [33–36]. This approach relies on an accurate estimation of transmitted waveforms. Recently, several algorithmic advances have been reported in waveform estimation using the structure of digital video broadcasting-terrestrial (DVB-T) signals as illuminators of opportunity [37–41]. In addition to two antennas at each receiver location and prior knowledge of the signal structure, PCL also requires direct line-of-sight to a transmitter of opportunity and high signal-to-noise ratio (SNR) for the received signal from the transmitter.

In the TDOA/FDOA backprojection approach, received signals acquired by two or more sufficiently far-apart receivers are correlated and back-projected based on time or frequency difference of arrival to form an image of a scene [8, 24–30]. As compared to PCL, TDOA/FDOA backprojection does not require direct-line-of-sight to a transmitter, high SNR or the knowledge of transmitter location. However, the method is limited to imaging widely separated point scatterers. To overcome this limitation, an alternative method based on low-rank matrix recovery (LRMR) has been developed [18]. Despite its effectiveness in reconstructing scenes with extended targets, the LRMR-based approach has significant computational and memory requirements, which preclude its applicability to realistically sized images.

Recently, the DL framework has been investigated for signal processing problems, specifically with an emphasis on sparse coding and compressed sensing. In [42], the iterative soft thresholding and coordinate descent algorithms were implemented via a recurrent neural network (RNN), in which each layer of the network corresponded to an iteration. The model was trained in a supervised manner using the desired solutions for sparse codes of the corresponding inputs with the goal of accelerating convergence. This fundamental observation was exploited to implement an approximate message passing algorithm [43], to learn problem-specific gradient descent parameters [44], and to estimate parameterised priors [45]. In [46], we extended the idea of emulating iterative maps using RNNs to image reconstruction problems in a Bayesian framework. We considered the passive imaging problem in which the transmitter location is unknown, and used DL to refine the phase component of the synthetic aperture radar (SAR) forward model. Training was done in an unsupervised manner using SAR signals received directly via complex backpropagation [47]. The method produces focused imagery without increasing the computational complexity of the proximal gradient descent algorithm which it is based on.

1.2 Our approach and its advantages

Following the principles introduced in [46, 48], we develop a DL-based approach for passive SAR image reconstruction when the transmitted waveforms of opportunity are unknown. The key advantage of our approach as compared to other methods is that it

only requires a single receiver antenna, thereby providing reduced cost and increased versatility. Previously, we had presented preliminary results of joint waveform estimation and imaging for passive SAR in [49, 50]. In this study, we present the deep network architecture and unsupervised training scheme to learn transmitted waveforms as a parameter of the SAR imaging problem, while reconstructing focused imagery. Specifically, we extend our preliminary studies by developing the theory of our approach and demonstrating its effectiveness via further numerical simulations.

Unlike traditional PCL, our approach does not require an antenna directed to a transmitter of opportunity. We assume a stationary transmitter transmitting unknown waveforms and a single moving receiver collecting backscattered signal from a scene of interest. We then take an optimisation perspective to DL and interpret image reconstruction as a machine learning task. We derive a proximal gradient descent update to solve for scene reflectivity and formulate a recurrent-auto encoder that is parameterised by unknown waveform coefficients. As a result of our architecture, estimation of the transmitted waveforms is formulated as a parameter learning task via backpropagation. We use complex backpropagation to derive the parameter update equations, making our method applicable to both real and complex waveforms. Our method is based on unsupervised learning, in that; the model is trained solely on received back-scattered signals without using any SAR images as labels. As a result, we avoid upper bounding the quality of reconstructed images by SAR images reconstructed using conventional methods. The highly non-convex problem of backpropagation is guided to a feasible solution over the parameter space by initialising the network with the known components of the SAR forward model.

In our problem formulation, we assume that the structural form of transmitted waveforms is known a priori, and represent them as a linear combination of known basis functions. A wide range of communication and broadcasting waveforms falls into such a class of waveforms including DVB-T and WiMAX signals. We particularly formulate our method for transmitted signals with a flat spectrum, which is applicable to illuminators of opportunity generated from various spread spectrum methods. These include frequency phase shift keying (PSK) modulated, code division multiplexed and orthogonal frequency division multiplexed (OFDM) signals. Recently, in [51], a single receiver PCL methodology was proposed to estimate DVB-T signals for passive imaging, in which the signal structure was taken advantage of in processing the received signal from the surveillance channel. However, in our formulation, such prior knowledge on the waveform is used merely as a functional constraint during backpropagation, and lack thereof is not a limiting factor for the proposed framework. Hence our approach provides flexibility to incorporate any prior information of the waveform structure to improve waveform estimation and imaging.

Finally, in addition to the benefits of deploying a single receiver, our method provides means of estimating the waveform with the task of reconstructing enhanced imagery. This is achieved by formulating waveform estimation as minimisation of the mismatch between the received SAR signal and the synthesised SAR signal from the reconstructed scene. We show that our DL approach produces high contrast imagery when trained under realistic SNR levels. Our results indicate that the performance is strongly correlated to the accuracy of the estimated flat spectrum waveform, which demonstrates the joint estimation capability of the DL approach.

The rest of the paper is organised as follows. In Section 2, we introduce background material on DL. In Section 3, we present the received signal and waveform models. In Section 4, we present the network architecture for passive SAR image reconstruction. In Section 5, we discuss the parameterisation and training of the network. We provide numerical simulations and discussion of the results in Section 6. Section 7 concludes the paper.

We use lower case bolditalic fonts to denote vector quantities in finite-dimensional spaces and upper case bolditalic fonts to denote matrices.

2 DL background

The most fundamental architecture in DL is the artificial neural network, characterised by a cascade of affine mappings followed by point-wise non-linear operations, referred to as layers. Each layer produces a representation $\mathbf{h} \in \mathbb{C}^M$ of its input $\mathbf{d} \in \mathbb{C}^N$, defined as

$$\mathbf{h} = \sigma(\mathbf{A}\mathbf{d} + \mathbf{b}), \quad (1)$$

where $\mathbf{A} \in \mathbb{C}^{M \times N}$ is the weight matrix, $\mathbf{b} \in \mathbb{C}^M$ is the vector of corresponding biases, and $\sigma(\cdot)$ is an element-wise non-linear function, referred to as the activation function of the network. Letting Ω denote the input space, such that $\mathbf{d} \in \Omega$, the layer transforms Ω by (1) to create a feature space containing the representation $\mathbf{h}$.

At each layer, a new representation of the previous layer output is generated, resulting in a hierarchical representation of the input. The output at the end of the k th layer can be written as

$$\mathbf{h}_k = \sigma(\mathbf{A}_k \mathbf{h}_{k-1} + \mathbf{b}_k). \quad (2)$$

Letting $\phi: \Omega_L \rightarrow \Gamma$ be a mapping from the feature space produced by the L th layer, Ω_L , to the output space Γ , and redefining $\mathbf{d} = [\mathbf{d}, 1]^T$ and $\mathbf{W}_k = [\mathbf{A}_k, \mathbf{b}_k]$, $k = 0, \dots, L$, the network output $\mathbf{g}^* \in \Gamma$ becomes

$$\mathbf{g}^* = \phi(\mathbf{W}_L \sigma(\mathbf{W}_{L-1} \dots \sigma(\mathbf{W}_1 \sigma(\mathbf{W}_0 \mathbf{d}))). \quad (3)$$

Equation (3) analytically defines the network operator, $\mathcal{L}(\theta): \Omega \rightarrow \Gamma$, which is the mapping between the input and output spaces, where

$$\mathcal{L}(\theta)[\mathbf{d}] = \mathbf{g}^*, \quad \theta = \{\mathbf{W}_k\}_{k=1}^L. \quad (4)$$

In summary, the weights of the network provide a parameterisation of the operation that the network performs, whereas the non-linear unit introduces the capacity to approximate complex mappings between input and output spaces. The nested non-linear transformations are generally explained in terms of the universal approximation theorem or probabilistic inference [52–54]. The mapping performed by the network operator is referred to as forward propagation.

Learning procedure in the network is the estimation of θ with respect to a figure of merit given a set of training data $\{\mathbf{d}_1, \mathbf{d}_2, \dots, \mathbf{d}_T\}$ and corresponding ground truth data set $\mathbf{G} = \{\mathbf{g}_1, \mathbf{g}_2, \dots, \mathbf{g}_T\}$. This is achieved by optimising a cost function with respect to network parameters θ , which is typically defined as

$$\mathcal{J}_G[\theta] = \frac{1}{2T} \sum_{n=1}^T \|\mathcal{L}(\theta)[\mathbf{d}_n] - \mathbf{g}_n\|_2^2. \quad (5)$$

The analytic method of computing the derivatives through the network with respect to trainable parameters, θ , is referred to as the back-propagation algorithm. Network parameters θ are then updated via gradient descent such that

$$\theta^{l+1} = \theta^l - \eta_l \nabla_{\theta} \mathcal{J}_G[\theta^l], \quad (6)$$

where η_l is the step size of the l th parameter update.

For large training sets, the gradient term $\nabla_{\theta} \mathcal{J}_G[\theta^l]$ is estimated as an average of the gradient values computed over a small subset of the training data. This methodology, referred to as stochastic gradient descent (SGD), performs several gradient updates each time the full data is used. This update cycle is referred to as an ‘epoch.’

The optimisation over $\mathcal{J}_G[\theta]$ is typically a high-dimensional and non-convex problem. The error surface often consists of many saddle points and local minima [1, 55]. As a result, a critical aspect of backpropagation is the initialisation of the network parameters θ , which is typically chosen to guide the network to a desirable

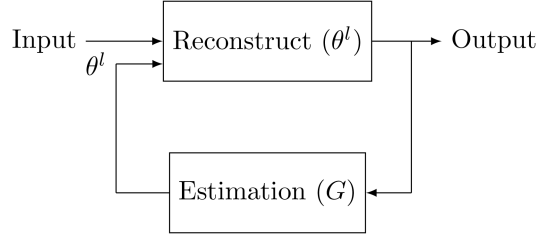


Fig. 1 Process flow of estimation via DL. The forward propagation is modelled as the reconstruction method, whereas estimation is performed at backpropagation which acts as a feedback and updates network parameters

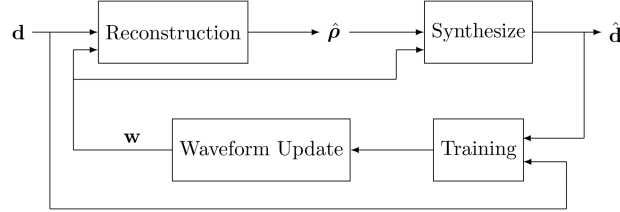


Fig. 2 Process flow of the proposed recurrent auto-encoder. The optimiser reconstructs images, which are used to synthesise the input received signal. The mismatch is back-propagated to update the shared parameters of the two modules

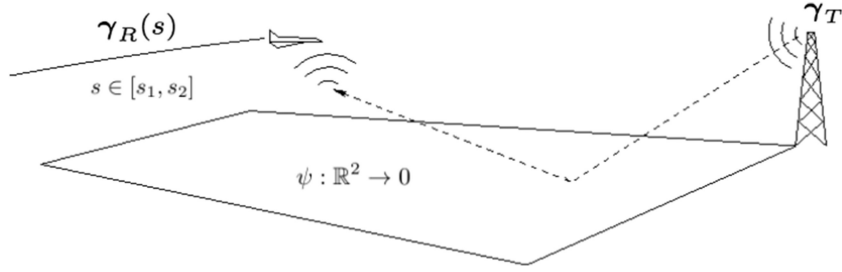


Fig. 3 Depiction of the passive SAR configuration with flat topography $\psi(x)$. A stationary transmitter of opportunity located at γ_T , $\gamma_R: [s_1, s_2] \rightarrow \mathbb{R}^3$ denotes the receiver trajectory

locally optimal solution. We propose a DL framework for problems in which we have unknowns or uncertainties in the measurement model. In general, we can model a measurement mapping as

$$\mathbf{d} = \mathbf{F}(\theta)\boldsymbol{\rho}, \quad (7)$$

where $\mathbf{d} \in \mathbb{C}^M$ are the measurements, $\boldsymbol{\rho} \in \mathbb{C}^N$ is the unknown quantity to be recovered, and $\mathbf{F}(\theta)$ is the measurement map that depends on the parameters θ . Following the DL framework for image reconstruction, we introduced in [46], recovering the unknown $\boldsymbol{\rho}$ can be interpreted as learning a representation of the measurements $\mathbf{d}$ in the image space. In this sense, the DL framework captures the image reconstruction task at the forward propagation step. However, since θ is unknown and arbitrarily initialised, the reconstructed image is initially inaccurate.

The advantages of the DL framework come at the backpropagation step, which allows the unknown parameters θ to be learned and to produce an accurate measurement map, thereby improving the accuracy of image reconstruction [46]. A high-level illustration of the effect of the back-propagation step for imaging is provided in Fig. 1, in which the reconstruction step is parameterised by θ^l and estimation is parameterised by training data G per (5). Further details and discussion of DL framework for image reconstruction are provided in Sections 4 and 5, while the effect of back-propagation in our specific application is illustrated in Fig. 2.

3 Passive SAR imaging

3.1 Imaging geometry

Let $\mathbf{x} = [\mathbf{x}, \psi(\mathbf{x})] \in \mathbb{R}^3$ denote the position of a scatterer, where $\mathbf{x} = [x_1, x_2] \in \mathbb{R}^2$ and $\psi: \mathbb{R}^2 \rightarrow \mathbb{R}$ is a smooth function describing the ground topography.

We assume a stationary transmitter located at position $\gamma_T \in \mathbb{R}^3$ and a moving receiver traversing a trajectory $\gamma_R(s)$, where $s \in [s_1, s_2]$ denotes the slow-time variable. Fig. 3 illustrates the passive SAR configuration under consideration. We assume that both the transmitter and receiver locations are known, but the transmitted waveforms are unknown.

3.2 Illuminators of opportunity

This study considers communication and broadcasting signals as the illuminators of opportunity. These signals are designed based on spread spectrum methodology and characterised by orthogonal division of the time or frequency domain to transmit different communication symbols. One such class of signals are OFDM waveforms. OFDM signals involve different phase or amplitude-modulated symbols that are transmitted using a set of orthogonal waveforms spanning the channel bandwidth [56, 57].

OFDM signals have been widely studied as illuminators of opportunity due to the prevalence of DVB-T and WiMAX standards used throughout the world [9, 11, 13, 14, 58–62]. Notably, the spectrum of an OFDM signal is relatively flat and noise-like over the channel [13, 61]. This characteristic can be observed by expressing the OFDM signal as a sum of random phase modulation symbols by applying the central limit theorem.

We assume that the transmitted waveforms may vary during the receiver's aperture time and model them as slow-time dependent as follows:

$$W(\omega, s) = \sum_{k=1}^K c_k \phi_k(\omega, s), \quad (8)$$

where ϕ_k , $k = 1, \dots, K$, are basis functions and c_k , $k = 1, \dots, K$, are (possibly complex) corresponding coefficients. Equation (8)

provides a representation by which most common waveforms of opportunity can be modelled.

One such structure is that of binary PSK or quadrature PSK (QPSK) modulated signals, which are building blocks of OFDM signals. Specifically, for QPSK, the basis coefficients are sampled uniformly on the unit circle, and the corresponding waveform has a flat spectrum. In our framework, we use this flat-spectrum structure as a statistical prior by the means of a constraint set in waveform estimation. The same process can be used to model more complex waveforms to find constraint sets for estimation.

3.3 Passive SAR forward model

Under the Born approximation and a flat topography assumption, we model the received signal as [63]

$$d(\omega, s) \simeq W(\omega, s) \int e^{-i(\omega/c_0)R(s, \mathbf{x})} a(\mathbf{x}, s) \rho(\mathbf{x}) d\mathbf{x}, \quad (9)$$

where

$$R(s, \mathbf{x}) = |\gamma_T - \mathbf{x}| + |\gamma_R(s) - \mathbf{x}| \quad (10)$$

is the bistatic range, $\omega \in [w_1, w_2]$ is the fast-time frequency, c_0 is the speed of light in free space, $W(\omega, s)$ is the waveform transmitted at $s \in [s_1, s_2]$, and $a(\mathbf{x}, s)$ is the azimuth beam pattern. We let

$$\tilde{\mathcal{F}}(\omega, s)[\rho] := \int e^{-i(\omega/c_0)R(s, \mathbf{x})} a(\mathbf{x}, s) \rho(\mathbf{x}) d\mathbf{x} \quad (11)$$

and write (9) as

$$d(\omega, s) \simeq W(\omega, s) \tilde{\mathcal{F}}(\omega, s)[\rho]. \quad (12)$$

Without loss of generality, we assume $a(\mathbf{x}, s)$ is constant and set it to 1. Placing the origin at the centre of the scene and invoking the far-field and small scene approximations for a sufficiently long aperture, $\tilde{\mathcal{F}}$ can be further approximated to yield the following measurement model:

$$d(\omega, s) \simeq W(\omega, s) e^{-i(\omega/c_0)(|\gamma_R(s)| + |\gamma_T|)} \int e^{-i(\omega/c_0)\mathbf{k} \cdot \mathbf{x}} \rho(\mathbf{x}) d\mathbf{x}, \quad (13)$$

where $\mathbf{k} = \widehat{\gamma_R(s)} + \widehat{\gamma_T}$ ($\mathbf{x}$ denotes the unit vector in the direction of $\mathbf{x}$) and $\tilde{\mathcal{F}}$ is now approximated by the Fourier transform up to a phase factor.

We discretise the scene into an N -pixel grid of $X = \{\mathbf{x}_i\}_{i=1}^N$, and stack $\rho(\mathbf{x}_i)$, $i = 1, \dots, N$ into a vector $\rho \in \mathbb{C}^N$. We discretise the fast-time frequency ω and slow-time s variables into $\{(\omega, s)_m\}_{m=1}^M$ pairs and stack SAR measurements into a vector $\mathbf{d} \in \mathbb{C}^M$. With the same sampling scheme, we stack the sampled waveform elements $W(\omega, s)|_{(\omega, s)_m}$ into a vector $\mathbf{w} \in \mathbb{C}^M$. Thus, we obtain

$$\mathbf{d} \simeq \text{diag}(\mathbf{w}) \tilde{\mathbf{F}} \rho, \quad (14)$$

where $\tilde{\mathbf{F}}$ is the matrix corresponding to the finite-dimensional representation of $\tilde{\mathcal{F}}$ in (11).

3.4 Image formation

Without complete knowledge of the forward model matrix $\mathbf{F} = \text{diag}(\mathbf{w}) \tilde{\mathbf{F}}$, we are no longer able to form bistatic SAR images using a two-layer filtered-backprojection type operation such as the one described in [63]. Similarly, optimisation-based reconstruction approaches are not applicable due to unknowns in the forward model. Since the dependence of the waveform coefficients in the forward model is multiplicative in the frequency domain, the problem of image reconstruction can be viewed as a blind deconvolution problem. A popular approach to solving such problems is to use an alternating minimisation scheme, which

requires the solution of two minimisation problems at each iteration. In this work, we instead propose a data driven approach based on DL. The main advantage of the DL-based method is that it is task-driven. Instead of casting the unknown waveform as a joint parameter of the objective function in optimisation, we cast it as a parameter of the optimiser. This results in conducting waveform estimation to specifically produce accurate imagery.

If the waveforms were known, estimation of the scene reflectivity could be formulated within a Bayesian framework as the following optimisation problem:

$$\rho^* = \underset{\rho}{\text{argmin}} \frac{1}{2} \|\mathbf{d} - \text{diag}(\mathbf{w}) \tilde{\mathbf{F}} \rho\|_2^2 + \lambda \Phi(\rho), \quad (15)$$

where the ℓ_2 -norm term represents the log-likelihood function under an additive white Gaussian noise assumption, $\Phi(\rho)$ is the regulariser capturing the prior information on ρ , and $\lambda > 0$ is the regularisation parameter.

By deploying a convex Φ , the optimisation can be implemented as a forward-backward splitting algorithm [64]. The forward-backward splitting algorithm takes the form of a gradient descent step over the smooth ℓ_2 -norm term, followed by a projection onto the feasible set defined by $\lambda \Phi$ as follows:

$$\rho^{k+1} = \mathcal{P}_{\alpha \lambda \Phi}(\rho^k - \alpha \nabla f[\rho^k]), \quad (16)$$

where

$$f(\rho) := \frac{1}{2} \|\mathbf{d} - \text{diag}(\mathbf{w}) \tilde{\mathbf{F}} \rho\|_2^2, \quad (17)$$

$\alpha > 0$ is the step size and $\mathcal{P}_{\alpha \lambda \Phi}(\cdot)$ is the proximity operator of $\alpha \lambda \Phi(\rho)$ defined as

$$\mathcal{P}_{\alpha \lambda \Phi}(\rho) = \underset{\mathbf{y}}{\text{argmin}} \frac{1}{2} \|\mathbf{y} - \rho\|_2^2 + \alpha \lambda \Phi(\mathbf{y}). \quad (18)$$

When Φ is the indicator function of a closed convex set, the proximity operator is simply an orthogonal projection onto that set. Using (17) and inserting $\nabla f[\rho^k]$ into (16), we have

$$\rho^{k+1} = \mathcal{P}_{\alpha \lambda \Phi}((\mathbf{I} - \alpha \tilde{\mathbf{F}}^H \text{diag}(|\mathbf{w}|^2) \tilde{\mathbf{F}}) \rho^k + \alpha \tilde{\mathbf{F}}^H \text{diag}(\mathbf{w})^H \mathbf{d}), \quad (19)$$

where $|\mathbf{w}|^2$ denotes element-wise magnitude square of $\mathbf{w}$. Clearly, iterative reconstruction in (19) cannot be used without the knowledge of $\mathbf{w}$. Nevertheless, (19) serves as the foundation for the development of our DL-based approach discussed in Sections 4 and 5.

4 Network architecture

Our goal is to implement a deep network that can recover scene reflectivity accurately when the waveform coefficients are unknown. To achieve this, we deploy the methodology proposed in [46] for SAR imaging to simultaneously estimate the waveform and reconstruct the image. Specifically, we use a recurrent auto-encoder architecture [65]. A recurrent auto-encoder consists of a two-stage network: a RNN that emulates the iterative imaging algorithm defined by (19), and a decoder stage that maps the image estimate back to the measurement space. The main advantage of this architecture is that it allows for unsupervised training using the SAR received signal.

4.1 RNN encoder

RNNs fundamentally differ from other architectures since the parameters are shared over layers. As a result, in an RNN, each layer performs the same transformation between feature spaces. The RNN encoder is designed by unfolding the update equation in (19) for a fixed number of iterations, say L . This emulates the imaging algorithm described in Section 3. The corresponding weight matrix and the bias vector are defined as

$$\mathbf{Q} = \mathbf{I} - \alpha \tilde{\mathbf{F}}^H \text{diag}(|\mathbf{w}|^2) \tilde{\mathbf{F}}, \quad \mathbf{b} = \alpha \tilde{\mathbf{F}}^H \text{diag}(\mathbf{w})^H \mathbf{d}, \quad (20)$$

respectively. With this approach, the iterations become the feature transformations performed by the layers in (1) and we get the following update equation:

$$\rho^{k+1} = \mathcal{P}_{\alpha\lambda\Phi}(\mathbf{Q}\rho^k + \alpha\tilde{\mathbf{F}}^H \mathbf{d}), \quad (21)$$

where $\text{diag}(\mathbf{w})\tilde{\mathbf{F}} = \mathbf{F}$. Thus, the image estimates ρ^k , $k = 1, \dots, L$, become representations produced at the k th layer of the network and the proximity operator $\mathcal{P}_{\alpha\lambda\Phi}(\cdot)$ becomes the network activation function. The only condition required on the regulariser to construct a neural network is that it must have a closed-form proximity operator that acts element-wise on its argument (Fig. 4).

While we motivate the network architecture from an optimisation perspective, we modify the forward propagation in a way that deviates from this analogy. Notably, optimisation by (21) is conducted over the complex domain to estimate the scene reflectivity. Furthermore, the proximal gradient method that our optimiser is based on requires a large number of iterations to converge. However, such a large number of layers is not practical from the point of view of training. Hence, we set L much smaller than the number of iterations needed for convergence. Additionally, we require the output of each layer to form visual representations of the received SAR signal. This is achieved by removing the phase of the representations in the network at forward propagation.

In the scope of this study, we consider the recovery of sparse scenes and use the ℓ_1 -norm constraint for Φ as the sparsity inducing prior. The use of ℓ_1 -norm constraint is not a limitation of our framework, as the same iterative form in (21) is obtained with any Φ that has a well-defined, element-wise proximity operator. Furthermore, if the underlying scene is not sparse, a sparsifying transform such as the wavelet transform can be deployed, and optimisation can be re-formulated as recovery of sparse coefficients. Therefore, we define the network non-linearity as a phaseless soft thresholding operator as follows:

$$\mathcal{P}_{\tau\ell_1}(\rho) = \max(|\rho| - \tau, 0), \quad \rho \in \mathbb{C}^N, \quad (22)$$

where $|\rho|$ denotes taking the absolute value of the entries of ρ , τ is the threshold determined by scaling the ℓ_1 -norm constraint. With this modification, every representation in the RNN becomes a visual image, and feature mappings in (21) can be interpreted from an image processing perspective, as discussed in Section 5.

4.2 Decoder

In implementing the decoder, we consider that scene reflectivities may be upper bounded given the operating frequencies of the receiver and typical scene refractive indices. Without loss of generality, we assume that the scene reflectivities vary between 0 and 1 and that the scene consists of at least one strong reflector. Under these assumptions, we normalise the final RNN output ρ^L before projection onto the measurement space as follows:

$$\rho^* = \frac{\rho^L}{\|\rho^L\|_\infty}. \quad (23)$$

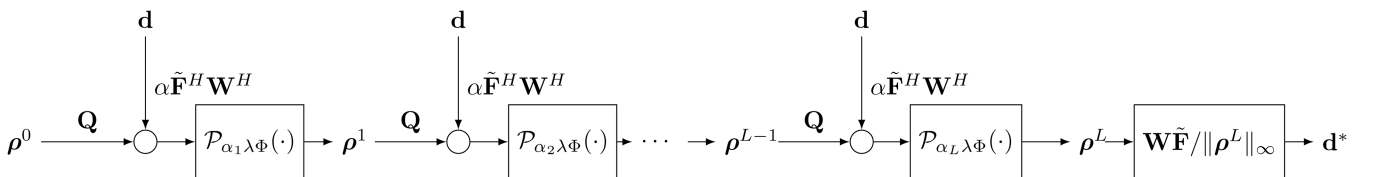


Fig. 4 Proposed recurrent auto-encoder architecture with $\mathbf{W} = \text{diag}(\mathbf{w})$. The initial estimate ρ^0 can be set according to preference (set as a vector of zeros in this work). Following the estimate generated at layer L , a linear decoder maps the estimate back to the data space. The figure depicts the case in which different learning rates (α_i 's) are used at each iteration of the forward solver

The normalisation of the final output enhances the effect of learning in light of the expected range of reflectivity values in the reconstructed image.

Following the normalised image formation step, the decoder maps the estimate ρ^* back to the received signal space and synthesises an estimate of the SAR measurement as follows:

$$\mathbf{d}^* = \text{diag}(\mathbf{w})\tilde{\mathbf{F}}\rho^*. \quad (24)$$

By the insertion of the linear decoding layer (24), the network operator $\mathcal{L}[\theta]$ acts as an approximation to the identity map on the received signal space. Representing ρ^* as a function of network parameters θ and the input $\mathbf{d}$, we write

$$\mathcal{L}[\theta](\mathbf{d}) = \mathbf{d}^* := \text{diag}(\mathbf{w})\tilde{\mathbf{F}}\rho^*(\theta, \mathbf{d}). \quad (25)$$

The final mapping back to received signal space allows the model to be trained in an unsupervised manner. This formulation offers a significant advantage. A supervised training scheme would require ground truth images coupled with SAR measurements. However, in the context of image reconstruction, a large number of SAR images acquired using the same imaging geometry may not be available. Moreover, training the RNN using SAR images would upper bound the performance by the reconstruction quality of conventional imaging algorithms that formed the SAR images in the first place. Our approach avoids these drawbacks by unsupervised training.

5 Network training

5.1 Forward propagation for SAR imaging

Consider the ideal expressions for the network weight matrix and bias terms in (20) with the true underlying waveform. Essentially the map $\mathbf{Q}$ is composed of an all pass filter $\mathbf{I}$ of scale one, and $\mathbf{F}^H \mathbf{F}$, which is a spatially varying filter of low pass characteristic. Having $\alpha M \ll 1$, $\mathbf{Q}$ acts as an image domain filter that gradually suppresses all frequencies by a $1 - \alpha M$ factor, except for the high-frequency bands determined by the cut-offs of the rows of $\mathbf{F}^H \mathbf{F}$. By the definition of $\mathbf{b}$ in (20), the result of the linear filtering operation is biased by the backprojection estimate. Since $\tilde{\mathbf{F}}$ is sampled from a Fourier integral operator [63], we know that backprojection preserves the target placement and edges in the image. Hence, the biasing step practically enhances the foreground of the output of the filtering operation. By repetitive application of $\mathbf{Q}$ and biasing, mid- to low-frequency components are gradually suppressed, whereas the edges get further enhanced due to their high-frequency content. Thereby, pixel-wise absolute-value thresholding at each layer effectively performs background suppression rather than suppressing the foreground, and the composite map of the layers becomes a non-linear enhancement filter.

However, as emphasised throughout the paper, the forward map $\mathbf{F}$ is not fully known and at initialisation, the network cannot perform these operations accurately due to an arbitrary initial $\mathbf{w}$. Let $\mathbf{w} = \mathbf{w}^0$ be a randomly picked waveform coefficient vector to initialise the network parameters $\mathbf{Q}$ and $\mathbf{b}$. Setting the initial image estimate fed to the network as $\rho^0 = 0$, the representation at the first layer becomes

$$\rho^1 = \max\left|\alpha \tilde{\mathbf{F}}^H \text{diag}(\mathbf{w}^0)^H \mathbf{d}\right| - \alpha\lambda, 0). \quad (26)$$

We define $\tilde{\rho}$ as the intermediate estimate resulting from the backprojection with its j th pixel value given as

$$\tilde{\rho}_j := [\tilde{\mathbf{F}}^H \text{diag}(\mathbf{w}^0)^H \mathbf{d}]_j = \sum_{k=1}^M (\tilde{\mathbf{w}}^0)_k (\tilde{\mathbf{F}})_{kj}^* \mathbf{d}_k, \quad j = 1, \dots, N. \quad (27)$$

Plugging in (14) into (27), and breaking down the contributions of pixels with indices $i = j$ and $i \neq j$ into the estimate of the reflectivity at the j th pixel, we obtain

$$\tilde{\rho}_j = \rho_j \left(\sum_{k=1}^M (\tilde{\mathbf{w}}^0)_k (\mathbf{w}_t)_k \right) + \sum_{k=1}^M \sum_{i \neq j} (\tilde{\mathbf{F}})_{kj} (\tilde{\mathbf{F}})_{ki}^* (\tilde{\mathbf{w}}^0)_k (\mathbf{w}_t)_i \rho_i, \quad (28)$$

where $\mathbf{w}_t$ is the vector of true waveform coefficients, and ρ_j , $j = 1, \dots, N$ are the elements of the true scene ρ .

Consider a passive imaging scenario such that the true waveform coefficients are sampled from a QPSK signal, and that we set entries of $\mathbf{w}^0$ with randomly picked symbols from the unit circle in $\mathbb{C}$. Since such a randomly initialised $\mathbf{w}^0$ is highly unlikely to be correlated to the true QPSK waveform, the contribution of the underlying scene pixel ρ_j to its estimate $\tilde{\rho}_j$ diminishes. Furthermore, this contribution is diminished to a level comparable to that of other points in the underlying scene ρ_i , where $i \neq j$. As all scene reflectivities are scaled by low correlated complex exponentials in (28), targets get suppressed and a noisy image is obtained by backprojection. This phenomenon is observed in Fig. 5 from the reconstruction performance with a randomly initialised waveform. Therefore, obtaining a highly correlated waveform coefficient vector $\mathbf{w}$ is the key component of the reconstruction process by (21).

5.2 Backpropagation for waveform estimation

Instead of parameterising the feature maps of the RNN by the weight matrix $\mathbf{Q}$ and the bias vector $\mathbf{b}$, we limit parameterisation to only the waveform vector $\mathbf{w}$. Thereby, backpropagation is cast as a solver for the waveform estimation problem. Furthermore, splitting the waveform as a parameter preserves the known components of $\mathbf{Q}$ and $\mathbf{b}$, which are initialised using $\tilde{\mathbf{F}}$.

In addition to the unknown waveform in the passive SAR forward model, the optimisation hyperparameters can also be learned. For our model, we include the threshold parameter of the network non-linearity, initialised as $\tau = \alpha\lambda$ into learning, and define network parameter as $\theta = \{\mathbf{w}, \tau\}$.

Given training data $D = \{\mathbf{d}_1, \mathbf{d}_2, \dots, \mathbf{d}_T\}$, we search a minimiser over the parameter space $\{\mathbf{w}, \tau\}$ for the following cost function:

$$\mathcal{J}_D(\mathbf{w}, \tau) = \frac{1}{T} \sum_{n=1}^T \ell(\text{diag}(\mathbf{w}) \tilde{\mathbf{F}} \rho_n^*, \mathbf{d}_n), \quad (29)$$

where ℓ is a properly chosen loss function. For ℓ , we pick the ℓ_2 -norm of the mismatch between the data synthesised by the network and the input data. Furthermore, we incorporate constraints into the cost function to enforce prior information on $\mathbf{w}, \tau$. Most significantly, we focus our attention on constraining the underlying waveform by a priori functional form. For our problem, we consider that the unknown waveform coefficients are sampled from a flat spectrum signal, and restrict $\mathbf{w}$ to have unit modulo entries. For τ , we invoke a non-negativity constraint and formulate the backpropagation problem as

$$\text{BP} := \underset{\mathbf{w}, \tau}{\text{minimise}} \frac{1}{T} \sum_{n=1}^T \|\text{diag}(\mathbf{w}) \tilde{\mathbf{F}} \rho_n^* - \mathbf{d}_n\|_2^2 + i_{C_w}(\mathbf{w}) + i_{C_\tau}(\tau), \quad (30)$$

where $i_C(\cdot)$ denotes the indicator function of subscript set C , C_w and C_τ denote constraint sets on parameters $\mathbf{w}$ and τ , respectively.

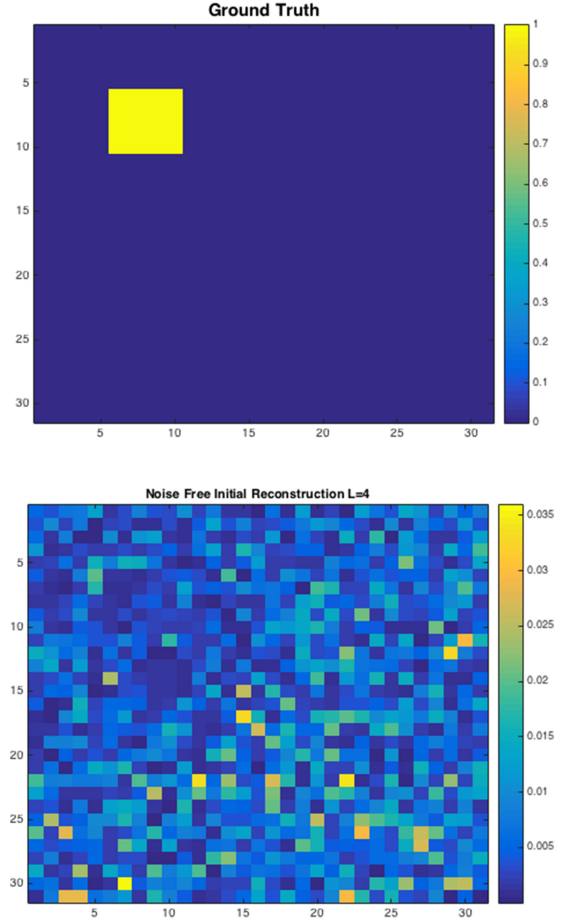


Fig. 5 Ground truth image used in experiments (left) versus the image reconstructed by the RNN encoder with the randomly initialised waveform coefficients (right). The target is completely lost and suppressed in background noise due to the mismatch introduced by the wrong waveform in reconstruction

The minimisation is then performed by projected SGD. Taking gradient steps over the smooth $\mathcal{J}_D$ term, and projecting onto constraint sets C_w and C_τ , we obtain the updates as

$$\mathbf{w}^{l+1} = \mathcal{P}_{C_w} \left(\mathbf{w}^l - \frac{\eta_w}{|B|} \sum_{n \in I_B} \nabla_{\mathbf{w}} \ell(\text{diag}(\mathbf{w}) \tilde{\mathbf{F}} \rho_n^*, \mathbf{d}_n) \Big|_{\theta = \theta^l} \right), \quad (31)$$

$$\tau^{l+1} = \mathcal{P}_{C_\tau} \left(\tau^l - \frac{\eta_\tau}{|B|} \sum_{n \in I_B} \frac{\partial \ell(\text{diag}(\mathbf{w}) \tilde{\mathbf{F}} \rho_n^*, \mathbf{d}_n)}{\partial \tau} \Big|_{\theta = \theta^l} \right), \quad (32)$$

where $\theta^l = \{\mathbf{w}^l, \tau^l\}$ denotes the parameter values at the l th iteration in backpropagation, η_w and η_τ are the learning rates (or step-size), $B \subseteq D$ is the randomly selected subset of the training data, $|B|$ is the cardinality of the subset B , I_B is an index set corresponding to B . $\mathcal{P}_C$ denotes the projection operator corresponding to the constraint set C such that for the specified constraints for $\mathbf{w}$ and τ , we have

$$(\mathcal{P}_{C_w}(\tilde{\mathbf{w}}))_i = \frac{\tilde{w}_i}{|\tilde{\mathbf{w}}|}, \quad \mathcal{P}_{C_\tau}(\tilde{\tau}) = \max(\tilde{\tau}, 0). \quad (33)$$

As mentioned earlier, due to the nested non-linear structure of the recurrent auto-encoder estimator, (29) is a highly non-convex optimisation problem. In addition, first-order methods are prone to converging to local minima, which adds further difficulty in estimating $\mathbf{w}$. However, our parameterisation enforces the problem structure of SAR imaging, and places the network in a neighbourhood over the loss surface only up to a diagonal multiplier of the true forward model, along with any prior knowledge of the functional form of $\mathbf{w}$. Therefore, by the updates

in (6), a stationary point in the neighbourhood of a strong initial point is searched. Hence, backpropagation is used as a tool for refining the SAR forward model and improving image reconstruction implemented by forward propagation.

5.3 Network derivatives

An important consideration is the computation of the backpropagation equation for waveform coefficients. Since the objective being minimised is a real-valued function of a complex variable $\mathbf{w}$, we have to perform complex backpropagation on the parameters defined as

$$\nabla_{\mathbf{w}} \ell = \left(\frac{\partial \ell}{\partial \mathbf{w}} \right), \quad (34)$$

where $\bar{(\cdot)}$ denotes complex conjugation and the partial derivative in (34) is the Wirtinger derivative [66].

Notably, $\mathbf{w}$ parameterises both $\mathbf{Q}$ and $\mathbf{F} = \text{diag}(\mathbf{w})\tilde{\mathbf{F}}$. Writing the partial derivative of the loss function ℓ with respect to $\mathbf{w}$, from the chain rule we have

$$\frac{\partial \ell(\mathbf{d}^*, \mathbf{d})}{\partial \mathbf{w}} = \frac{\partial \mathbf{Q}}{\partial \mathbf{w}} \frac{\partial \ell(\mathbf{d}^*, \mathbf{d})}{\partial \mathbf{Q}} + \frac{\partial \mathbf{F}}{\partial \mathbf{w}} \frac{\partial \ell(\mathbf{d}^*, \mathbf{d})}{\partial \mathbf{F}}. \quad (35)$$

Hence, we compute the derivative with respect to $\mathbf{w}$ by first computing derivatives with respect to $\{\mathbf{Q}, \mathbf{F}\}$ as

$$\begin{aligned} \frac{\partial \ell(\mathbf{d}^*, \mathbf{d})}{\partial \mathbf{Q}} &= \frac{\partial \rho^*}{\partial \mathbf{Q}} \frac{\partial \ell(\mathbf{d}^*, \mathbf{d})}{\partial \rho^*}, \\ \frac{\partial \ell(\mathbf{d}^*, \mathbf{d})}{\partial \mathbf{F}} &= \frac{\partial \rho^*}{\partial \mathbf{F}} \frac{\partial \ell(\mathbf{d}^*, \mathbf{d})}{\partial \rho^*} + \left(\frac{\partial \mathbf{F}}{\partial \mathbf{F}} \rho^* \right) (\bar{\mathbf{d}}^* - \bar{\mathbf{d}}). \end{aligned} \quad (36)$$

The derivative with respect to τ is simply the real-valued partial derivative of the form

$$\frac{\partial \ell(\mathbf{d}^*, \mathbf{d})}{\partial \tau} = \frac{\partial \rho^*}{\partial \tau} \frac{\partial \ell(\mathbf{d}^*, \mathbf{d})}{\partial \rho^*}. \quad (37)$$

The second component of the derivative with respect to $\mathbf{F}$ in (36) originates from having the linear decoder stage that projects the image estimate ρ^* on the received signal space by $\mathbf{F}$. In (36), the partial derivative of $\mathbf{F}$ with respect to itself yields an identity tensor of size $M \times N \times M \times N$, which multiplies the image estimate $\rho^* \in \mathbb{R}^N$ to yield an $M \times N \times M$ tensor. We provide a detailed derivation of network derivatives in Section 10.1.

To back-propagate through the recurrent encoder component, we use the backpropagation through time algorithm (BPTT). Since the parameters $\{\mathbf{Q}, \mathbf{F}, \tau\}$ are shared among layers, and only the error resulting from the final layer is considered in the optimisation, BPTT computes the derivatives in the RNN encoder as

$$\frac{\partial \rho^*}{\partial \theta} = \left(\sum_{i=1}^L \frac{\partial \rho^k}{\partial \theta} \frac{\partial \rho^L}{\partial \rho^k} \right) \frac{\partial \rho^*}{\partial \rho^L}, \quad (38)$$

where ρ^k , $k = 1, \dots, L$ is the network representation generated at the k th layer.

Despite the general form we present in this section, a distinct effect of the flat spectrum constraint we place on $\mathbf{w}$ for signals such as QPSK or OFDM can be observed at backpropagation. As explicitly shown in (20), $\mathbf{Q}$ only has dependence on $\mathbf{w}$ through its elementwise modulus by the $\text{diag}(|\mathbf{w}|^2)$ term. By setting the initial $\mathbf{w}^0$ as a unit modulo entry signal, and projecting $\mathbf{w}$ iterates onto the unit sphere in $\mathbb{C}^M$ by (31), $\text{diag}(|\mathbf{w}|^2)$ is merely the identity matrix and $\mathbf{Q}$ is effectively fixed through the training procedure. Therefore its dependence on the network parameterisation is dropped. Removing the contribution of $\mathbf{Q}$ to the partial derivative in (35) yields the following final update form for the parameter $\mathbf{w}$

$$\begin{aligned} \frac{\partial \ell(\mathbf{d}^*, \mathbf{d})}{\partial \mathbf{w}} &= \frac{\partial \mathbf{F}}{\partial \mathbf{w}} \left[\left(\sum_{i=1}^L \frac{\partial \rho^k}{\partial \mathbf{F}} \frac{\partial \rho^L}{\partial \rho^k} \right) \frac{\partial \rho^*}{\partial \rho^L} \frac{\partial \ell(\mathbf{d}^*, \mathbf{d})}{\partial \rho^*} \right. \\ &\quad \left. + \left(\frac{\partial \mathbf{F}}{\partial \mathbf{F}} \rho^* \right) (\bar{\mathbf{d}}^* - \bar{\mathbf{d}}) \right]. \end{aligned} \quad (39)$$

6 Numerical simulations

6.1 Scene and imaging parameters

We assume isotropic transmit and receive antennas, and simulate a transmitted signal with bandwidth and centre frequency of 8 and 760 MHz, respectively. The simulated waveform is modulated using QPSK, in which the symbols are generated from an i.i.d. uniform distribution. This corresponds to ~ 20 m range resolution for monostatic SAR. Thus, we simulate a 620×620 m² scene and discretise it into 31×31 pixels as shown in Fig. 5.

The receiver antenna traverses a circular trajectory, defined as $\gamma_R(s) = [7\cos(s), 7\sin(s), 6.5]$ km. The transmitter is fixed and located at $\gamma_T = [11.2, 11.2, 0.2]$ km. The aperture is sampled uniformly into 128 uniform samples, and the bandwidth is sampled uniformly into 64 samples.

6.2 Training and testing sets

We generate training samples consisting of randomly generated sparse scenes with a single point or extended target that varies in rectangular shape and location. The length and width of each rectangular target are sampled from a uniform distribution in the range $[1, 6]$. The targets are placed randomly within the range of $[3, 28] \times [3, 28]$ pixels. We then generate received SAR signals for each scene using the full forward model in (14) described in Section 3, and add a realisation of white Gaussian noise vector on each SAR measurement we have generated. We consider six levels of SNRs, -20 , -15 , -10 , -5 , 0 , and 10 dB, and create a training set of noisy received SAR signals of randomly generated scenes at each SNR level, to form six statistically independent sets. The proposed model is trained using the six training sets separately to evaluate the robustness of the model to different noise levels.

In testing, we use measurements collected from a single scene of interest. The test scene is displayed in Fig. 5 and the backscattered field is generated by the forward model in (14). Twenty different realisations of white Gaussian noise at -20 , -15 , -10 , -5 , 0 , and 10 dB SNR are used to form six sets, each consisting of 20 samples of measurements. Each test set is fed into the model trained with the corresponding SNR level of received SAR signals. We evaluate the reconstruction performance as the average over 20 results for statistical accuracy.

For data collection, we envision a two-stage protocol to form training and test sets as proposed in [46]. In the first stage, an airborne receiver collects test data from a scene of interest. In the second stage, arbitrary reflectors are placed in the scene to form either extended or point targets and training data is collected under the same imaging geometry as before. In the data collection procedure, we make the assumption that changes on the transmitted signal are negligible in slow-time. Although our formulation does not require slow-time stationarity, a collection of the training set under a slow time varying waveform is a complication that has to be alleviated and is the main focus of our future work.

6.3 Network design and initialisation

We implement the proposed network with an RNN-encoder of $L = 4$ layers, and the phaseless soft-thresholding activation function introduced in Section 4. The model is trained for ten epochs for each experiment. We limit the number of training samples based on the results of our previous study [46] and set it to 10. We perform batch gradient descent, which corresponds to a single parameter update per epoch. We set the learning rate as $\eta_w = 10^{-4}$ for waveform coefficients, and as $\eta_\tau = 10^{-6}$ for the threshold parameter.

The network weight matrix and bias terms are initialised with the known components of the forward model $\mathbf{F}$ in (25). We set the

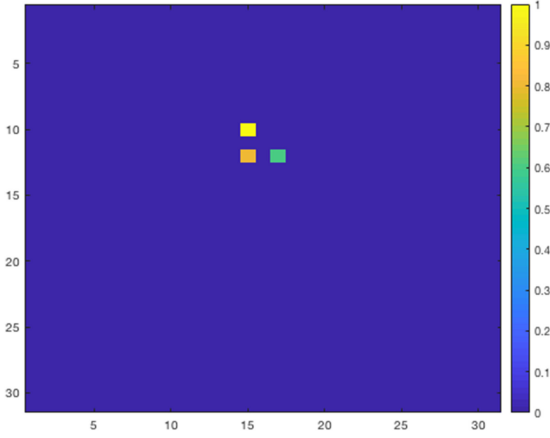


Fig. 6 Phantom used to evaluate resolution. The point targets are located in range bins 15 and 17, cross range bins 10 and 12

initial regularisation parameter as $\lambda = 10$, and set $\alpha = 1 \times 10^{-5}$, upper bounded by the reciprocal of the largest eigenvalue of $\tilde{\mathbf{F}}^H \tilde{\mathbf{F}}$. Accordingly, to the discussion in Section 6.2, we assume that transmitted waveforms are static with respect to the slow-time variable. As demonstrated in Section 5, we assume a priori knowledge is available, and constrain the parameter $\mathbf{w}$ using the QPSK properties, and project the entries of $\mathbf{w}$ at each update onto the unit circle in $\mathbb{C}$, as in (31). We initialise $\mathbf{w}$ by the real-valued flat spectrum signal of all ones instead of random initialisation to standardise our evaluation of different experiments.

6.4 Performance evaluation

We evaluate the reconstruction performance on the image and waveform coefficients using the following figures of merit:

$$L_d(\rho^{*l}) = \frac{\|\text{diag}(\mathbf{w}^l) \tilde{\mathbf{F}}(\rho^{*l}) - \mathbf{d}\|_2^2}{\|\mathbf{d}\|_2^2}, \quad L_\rho(\rho^*) = \frac{\|\rho^* - \rho\|_2^2}{\|\rho\|_2^2}, \quad (40)$$

measures the normalised data domain mismatch and the image domain error of the reconstructed image with respect to the ground truth SAR measurement and testing image, respectively, whereas

$$C_\rho(\rho^*) = \frac{|E[\rho_f^*] - E[\rho_b^*]|^2}{\text{var}[\rho_b^*]}, \quad L_w(\mathbf{w}^l) = \frac{\|\mathbf{w}_t - \mathbf{w}^l\|_2^2}{\|\mathbf{w}_t\|_2^2}, \quad (41)$$

measure the contrast in the reconstructed image and the normalised waveform mismatch with respect to the ground truth QPSK coefficients, respectively. $(\rho^*)^l$ denotes the normalised image generated by the RNN encoder, with the parameters obtained at epoch l , and ρ is the ground truth image. $\mathbf{w}_t$ is the ground truth waveform coefficient vector and $\mathbf{w}^l$ is the learned waveform coefficient vector at epoch l . ρ_f^* and ρ_b^* are the foreground and background components of the reconstructed image, respectively. $E[\cdot]$ stands for statistical expectation, $\text{var}[\cdot]$ stands for statistical variance and $\mathbf{d}$ is the input SAR data. Curves corresponding to each SNR value demonstrate the performance of the model trained on measurements corresponding to that noise level and evaluated on test samples collected at the same SNR.

To evaluate the resolution improvement achieved by learning, we examine the bias terms of the DL-based model. This corresponds to the evaluation of backprojection reconstruction following matched filtering with the learned waveform coefficients. We compare the peak and average background of the reconstructed image with learned waveform coefficients to the one obtained with the true waveform, and to that of reconstruction by backprojection without matched filtering. Essentially, the last case is equivalent to the initial waveform set to all 1's in its bandwidth. Evaluation is performed on the phantom displayed in Fig. 6, with respect to how the two point targets are resolved in horizontal and

vertical directions, as well as how the weak point target in (12) and (17) is resolved from the background noise in our evaluations.

6.5 Results

Our simulations show that the DL-based approach achieves accurate reconstruction performance under SNR scenarios above -20 dB for all metrics under consideration. To display the performance visually, we present the reconstructed images by the model under SNR levels of -15 , -10 , -5 , and 0 dB in Fig. 7. The images displayed in the figure demonstrate the impact of waveform and threshold learning by the DL-based method as described in Section 5.1. By fixing the image-domain filters due to the constraints on $\mathbf{w}$, learning the waveform coefficients become equivalent to refining the backprojection image. Hence, waveform learning directly impacts the placement and strength of target pixels, whereas threshold learning controls the amount of background suppression in the image. It is observed in Fig. 7 that with the exception of the -20 dB case, clear background suppression and geometric fidelity of the extended target are achieved by our method despite initialising the model with a waveform that has poor correlation to the true one. Moreover, image contrast and image mismatch metrics, as well as the decay in waveform error shown in Fig. 8 validate the main arguments of our approach, as the waveform coefficients are learned to the extent of high correlation with the underlying QPSK signal such that the model produces enhanced imagery.

As expected, the performance of the method degrades gracefully as the noise level increases both in image reconstruction and waveform estimation. For the -20 SNR case, the gradual improvement in the waveform is insufficient to impact reconstruction performance, as indicated by negligible changes in image domain metrics as shown in Fig. 8. However, the drastic impact of waveform estimation can be observed in the -10 dB case. Despite no indicative improvement on the data mismatch function over epochs similar to the -20 dB case, the algorithm learns a much more correlated waveform coefficient, which improves the reconstruction performance significantly.

We demonstrate the resolution performance of the DL-based method for the -10 dB case as discussed in Section 6.4. The images reconstructed by backprojection after matched filtering with true, and learned waveform coefficients are provided in Figs. 9 and 10, respectively. From Figs. 9 and 10, we see that the linear reconstruction using the learned waveform produces a nearly identical image as the one produced using the true waveform. This can also be observed in the cross-sections at horizontal and vertical directions that contain target pixels, which are provided in Figs. 11–13 in log scale for reconstruction obtained by match-filtering with true, initial, and learned waveform coefficients, respectively. The peak and average background values at each cross-section of the reconstructed image using learned waveform coefficients are highly consistent with the ones obtained with true waveform coefficients. Notably, the accuracy in image reconstruction is obtained despite a final normalised waveform mismatch of 0.5 as shown in Fig. 8, which indicates the robustness of the method to errors in estimation.

Overall, it can be presumed from our experiments that learning a sufficiently correlated waveform produces improved imagery. The model offers considerable robustness to measurement noise even with a limited number of training samples, which increases the applicability of our method in real-world scenarios. However, the limited performance in the -20 dB scenario can be traced to the limited number of samples used in training. The poor improvement of waveform error suggests the gradient estimates are highly contaminated by noise, which can be avoided by averaging over more samples. Handling such high noise scenarios may require accurate initialisation, or more structural constraints on the functional form of the waveform, as well as increasing the number of training samples.

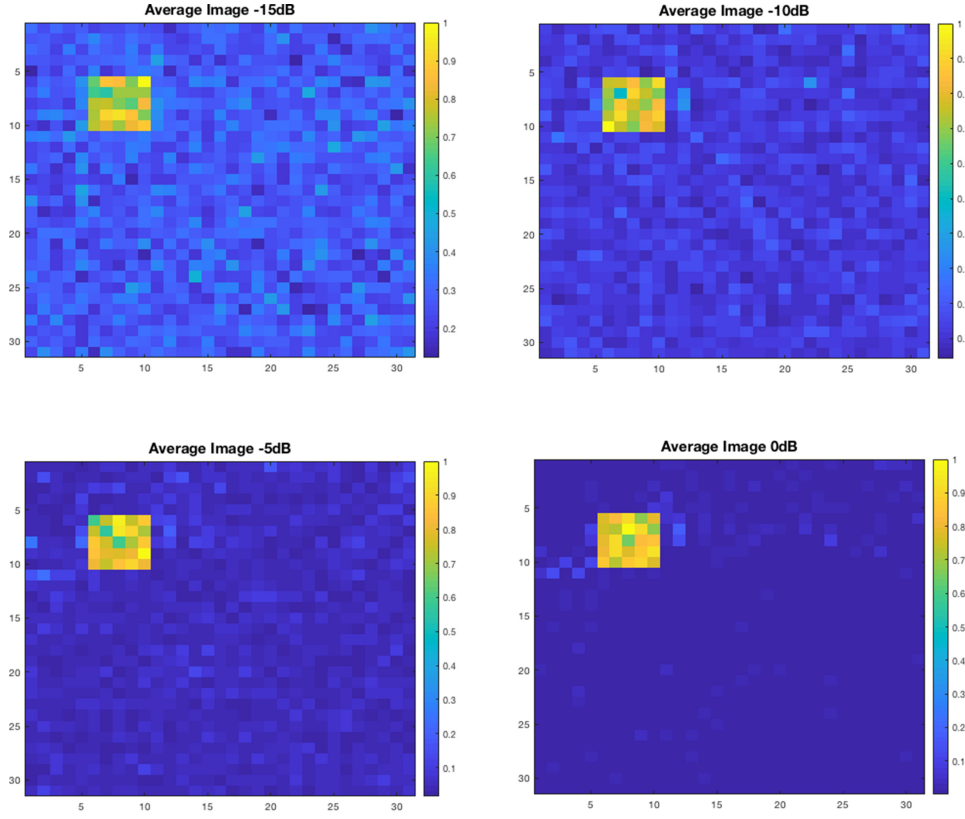


Fig. 7 Reconstructed average images by the proposed model under -15 , -10 , -5 and 0 dB SNR levels at data collection for training and test sets. Each image is formed by averaging the 20 test samples under different realisations at the same SNR. The model has learned suitable parameters such that imaging performance is drastically improved over the initialisation image for every noise level under consideration

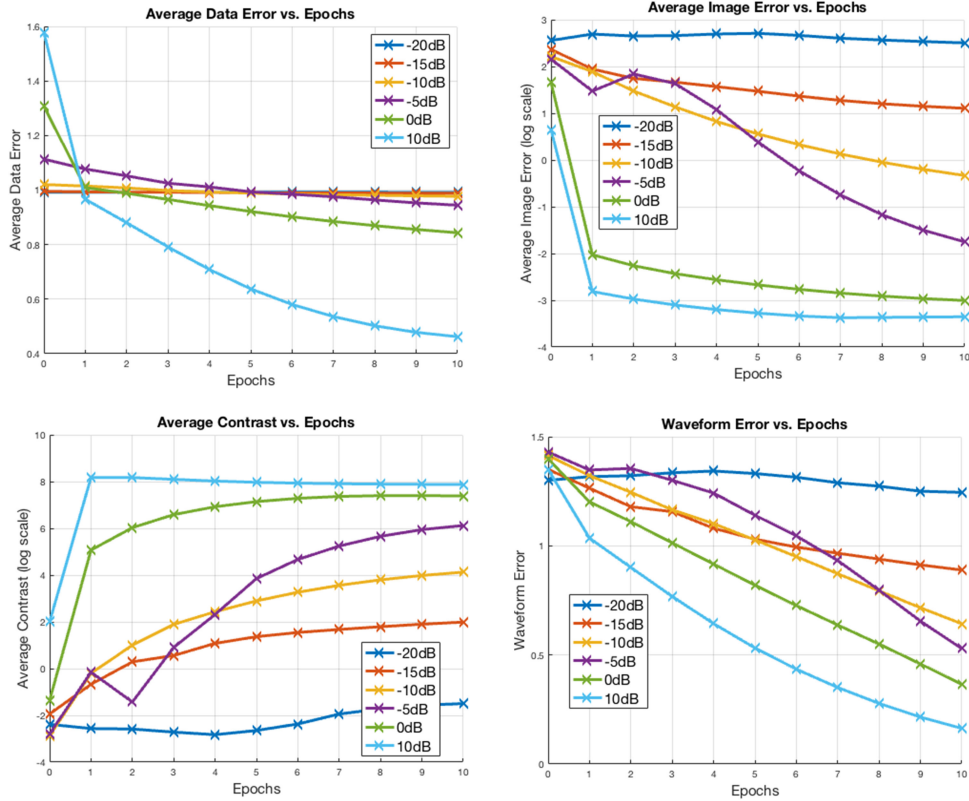


Fig. 8 Average values of the proposed data mismatch, image mismatch (40), image contrast and waveform error metrics (41) over 20 test samples at -20 , -15 , -10 , -5 , 0 , and 10 dB SNR cases. Curves are obtained from reconstruction results with the parameter estimates generated by the network at each epoch, trained with SAR measurements under corresponding SNRs

7 Conclusion

This study presented a novel DL-based approach for simultaneous estimation of the scene reflectivity and the transmitted waveform.

Our method requires a single receiver, providing reduced cost and improved simplicity over existing methods such as PCL and TDOA/FDOA backprojection. We consider a passive imaging scenario in which the transmitter location is known, but the

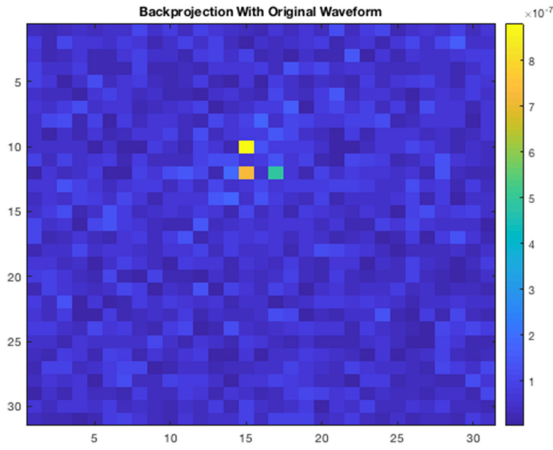


Fig. 9 Linearly reconstructed phantom image by first match filtering with the true underlying waveform coefficients, followed by backprojection, at -10 dB SNR

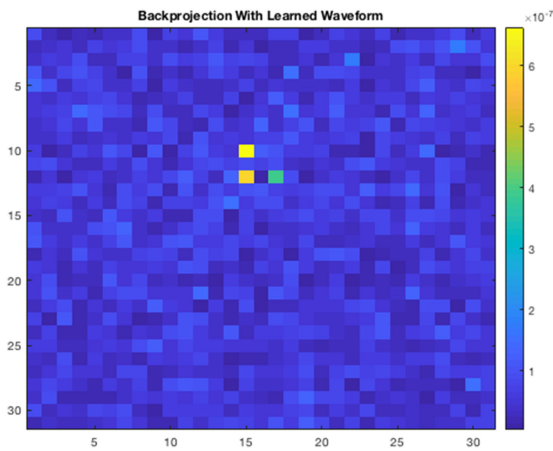


Fig. 10 Linearly reconstructed phantom image by first match filtering with the learned waveform coefficients, followed by backprojection, at -10 dB SNR

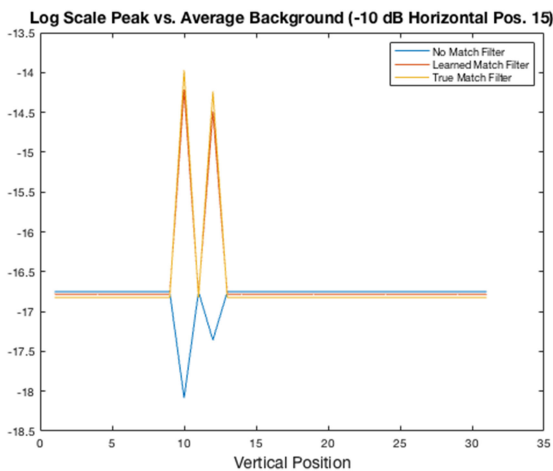


Fig. 11 Cross section of linearly reconstructed images at -10 dB SNR by matched filtering with true, initial (no matched filter), and learned waveform coefficients at horizontal position 15. The background pixels at the cross section are averaged to depict the noise level with respect to peak values. All values displayed in logarithmic scale

transmitted waveform is unknown. We approach image reconstruction in a Bayesian framework and set up an optimisation problem to estimate the scene reflectivity. We formulate a proximal gradient descent algorithm to solve for the scene reflectivity, which we unfold for a fixed number of iterations to formulate an RNN parameterised by waveform coefficients. Hence, for a given waveform coefficient vector, the RNN becomes a solver for the

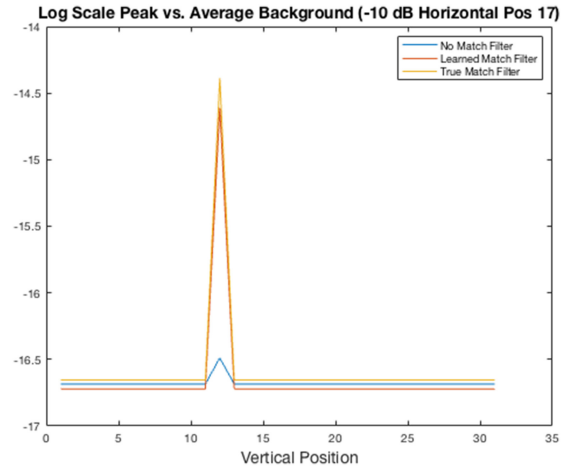


Fig. 12 Cross section of linearly reconstructed images at -10 dB SNR by matched filtering with true, initial (no matched filter), and learned waveform coefficients at horizontal position 17. The background pixels at the cross section are averaged to depict the noise level with respect to peak values. All values displayed in logarithmic scale

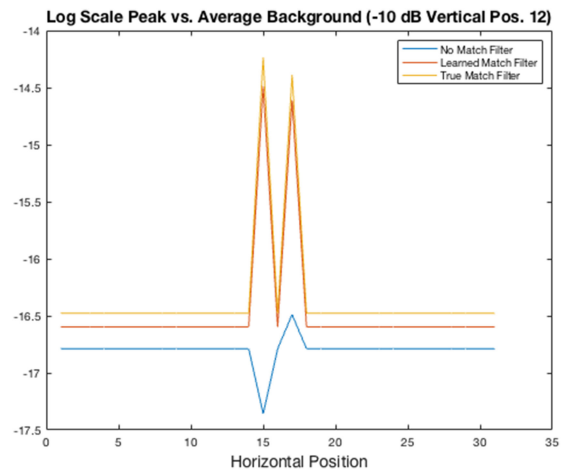


Fig. 13 Cross section of linearly reconstructed images at -10 dB SNR by matched filtering with true, initial (no matched filter), and learned waveform coefficients at vertical position 12. The background pixels at the cross section are averaged to depict the noise level with respect to peak values. All values displayed in logarithmic scale

scene reflectivity at forward propagation, and waveform coefficients become parameters that can be estimated by backpropagation.

We then cascade the RNN with a decoding layer consisting of normalisation and a linear forward map that synthesises SAR measurements from the reconstructed scene reflectivities resulting in a recurrent auto-encoder architecture. Thereby, we learn the transmitted waveform in an unsupervised manner by minimising the mismatch between a set of received SAR signals and corresponding SAR measurements synthesised by the network. At backpropagation, we employ a flat spectrum constraint on the waveform by performing updates via projected SGD. Our formulation has applicability to a wide range of spread spectrum signals that are common to transmitters of opportunity. The main advantage of our method is that the waveform estimation is performed in a task driven manner. The DL-based model ultimately estimates waveform coefficients with the goal of producing accurate imagery by forward propagation. Moreover, the structural form for the transmitted signals is merely used as a prior by the means of a constraint set in our framework, and lack thereof does not limit our framework.

We demonstrate the performance of our DL approach with numerical simulations, showing that with a limited number of training samples collected at realistic SNR levels, the model estimates QPSK modulated signals in a manner that produces

accurate SAR imagery. Furthermore, we show that our DL-based method is robust to estimation errors, as it reconstructs images highly consistent with the ones obtained by the true underlying waveform even in the presence of a non-negligible mismatch in learned and true waveform coefficients. In the future, we will pursue bypassing the slow-time stationarity assumption of transmitted waveform in training data collection, and explore decoding changing waveforms by our DL framework. Furthermore, we will further pursue improving the performance of our method in low SNR scenarios and test the estimation quality with other waveforms such as OFDM signals.

8 Acknowledgments

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10 Appendix

10.1 Waveform derivative

Due to having real-valued representations such that $\rho^k \in \mathbb{R}^N$, and $\bar{\rho}^* = \rho^*$ the first component of the complex backpropagation equation becomes

$$\frac{\partial \mathcal{L}(\mathbf{d}^*, \mathbf{d})}{\partial \rho^*} = \mathbf{F}^T(\bar{\mathbf{d}}^* - \bar{\mathbf{d}}) + \mathbf{F}^H(\mathbf{d}^* - \mathbf{d}) = 2 \operatorname{Re}(\mathbf{F}^H(\mathbf{d}^* - \mathbf{d})), \quad (42)$$

which is purely real valued as expected. From the chain rule with (42) and the normalisation derivative $\partial \rho^* / \partial \rho^L$, we obtain the non-layer dependent component of the network derivative in (39) by multiplying (42) with

$$\frac{\partial \rho^*}{\partial \rho^L} = \left(-\frac{1}{\|\rho^L\|_\infty} \frac{\partial \|\rho^L\|_\infty}{\partial \rho^L} (\rho^{LT}) + \frac{1}{\|\rho^L\|_\infty} \mathbf{I}_{N \times N} \right), \quad (43)$$

where the partial of the infinity norm of ρ^L is simply a column vector with entry 1 at the index of the maximal element of ρ^L , and 0's elsewhere.

First, we consider the second term in the brackets of (39). $\partial \mathbf{F} / \partial \mathbf{F}$ tensor is an $M \times N$ array of $M \times N$ matrices. The (m, n) th matrix in the array, $\mathbf{I}_{mn}$, has all entries $\mathbf{I}_{mn}(i, j) = 0$ except for 1 at $i = m, j = n$. From the definition of the tensor–vector multiplication, $(\partial \mathbf{F} / \partial \mathbf{F}) \rho^* = \sum_{i=1}^M \mathbf{I}_{i \times N \times M} \rho_i^*$ yields an $M \times N \times M$ tensor $\tilde{\mathbf{I}} = [\tilde{\mathbf{I}}_1, \tilde{\mathbf{I}}_2, \dots, \tilde{\mathbf{I}}_M]$ where the m th row of $\tilde{\mathbf{I}}_m$ equals ρ^{*T} , 0's otherwise. After another tensor–vector multiplication with conjugate error term, the second expression yields

$$\left(\frac{\partial \mathbf{F}}{\partial \mathbf{F}} \rho^* \right) (\bar{\mathbf{d}}^* - \bar{\mathbf{d}}) = (\bar{\mathbf{d}}^* - \bar{\mathbf{d}}) \rho^{*T}. \quad (44)$$

Taking the first component inside the brackets of (39), we denote

$$\partial_{\mathbf{F}}^k \mathcal{L} = \sum_{i=1}^L \frac{\partial \rho^k}{\partial \mathbf{F}} \frac{\partial \rho^L}{\partial \rho^k} \frac{\partial \rho^*}{\partial \rho^L} \frac{\partial \mathcal{L}(\mathbf{d}^*, \mathbf{d})}{\partial \rho^*}. \quad (45)$$

For each partial derivative of ρ^L with respect to other representations in the network, we can write the chain rule as $\partial \rho^L / \partial \rho^k = (\partial \rho^{k+1} / \partial \rho^k) (\partial \rho^L / \partial \rho^{k+1})$. Moving down the network beginning from layer L , this derivative can be evaluated by multiplying the $\partial \rho^{k+1} / \partial \rho^k$ term repeatedly for

$k = L-1, L-2, \dots, 1$. Denote $\mathbf{y}^k = |\mathbf{z}^k|$, where $\mathbf{z}^k = \mathbf{Q} \rho^{k-1} + \alpha \mathbf{F}^H \mathbf{d}$. The partial of ρ^k with respect to ρ^{k-1} can be evaluated as

$$\frac{\partial \rho^k}{\partial \rho^{k-1}} = \left(\frac{\partial \mathbf{z}^k}{\partial \rho^{k-1}} \frac{\partial \mathbf{y}^k}{\partial \mathbf{z}^k} + \frac{\partial \mathbf{z}^k}{\partial \rho^{k-1}} \frac{\partial \mathbf{y}^k}{\partial \mathbf{z}^k} \right) \frac{\partial \rho^k}{\partial \mathbf{y}^k}. \quad (46)$$

$\partial \rho^k / \partial \mathbf{y}^k$ is merely the derivative of the thresholding function $\mathcal{P}_{\tau \ell_1}(\cdot)$, which is a diagonal matrix with entries 1 at indexes where $\mathbf{y}_i^k > \tau$, and 0 otherwise. Similarly, $\partial \mathbf{y}^k / \partial \mathbf{z}^k$ and $\partial \mathbf{y}^k / \partial \mathbf{z}^k$ yield diagonal matrices with entries, for the i th diagonal term, $\mathbf{z}_i^k / (2|\mathbf{z}_i^k|)$ and $\bar{\mathbf{z}}_i^k / (2|\mathbf{z}_i^k|)$, respectively. Finally, the partial derivatives of $\mathbf{z}^k$ and $\bar{\mathbf{z}}^k$ with respect to ρ^{k-1} yield $\mathbf{Q}^H$ and $\mathbf{Q}^T$, respectively. Since $\mathbf{Q}$ is Hermitian symmetric, we have

$$\frac{\partial \rho^k}{\partial \rho^{k-1}} = \operatorname{Re} \left(\mathbf{Q} \operatorname{diag} \left(\frac{\mathbf{z}_i^k}{|\mathbf{z}_i^k|} \right) \right) \operatorname{diag}(\mathcal{J}_{\mathbf{y}^k > \tau}), \quad (47)$$

with $\mathcal{J}_{\mathbf{y}^k > \tau}$ denoting the set of indexes i such that $\mathbf{y}_i^k > \tau$. For the $\mathbf{F}$ derivative of the network representations, following the same notation, we have

$$\frac{\partial \rho^k}{\partial \mathbf{F}} = \left(\frac{\partial \mathbf{z}^k}{\partial \mathbf{F}} \frac{\partial \mathbf{y}^k}{\partial \mathbf{z}^k} + \frac{\partial \bar{\mathbf{z}}^k}{\partial \mathbf{F}} \frac{\partial \mathbf{y}^k}{\partial \bar{\mathbf{z}}^k} \right) \frac{\partial \rho^k}{\partial \mathbf{y}^k}. \quad (48)$$

The first component in the parenthesis vanishes because $\mathbf{z}^k$ only depends on $\mathbf{F}^H$. This results from the property of Wirtinger derivatives, such that $(\partial \bar{c} / \partial c) = 0$, for a complex variable $c \in \mathbb{C}$. From $(\partial \mathbf{z}^k / \partial \mathbf{F}) = \alpha (\partial (\mathbf{F}^T \bar{\mathbf{d}}) / \partial \mathbf{F})$, indexing the third dimension of the resulting $M \times N \times N$ tensor with i , at each i , with subscript $:$, j denoting the j th column, we have

$$\left(\frac{\partial (\mathbf{F}^T \bar{\mathbf{d}})_i}{\partial \mathbf{F}} \right)_{:,j} = \begin{cases} \bar{\mathbf{d}} & \text{if } j = i, \\ \mathbf{0} & \text{else.} \end{cases}$$

Denoting previous terms computed as $\partial \mathcal{L}_{\rho^k}$, the $\mathbf{F}$ -derivative at the layer k becomes

$$(\partial_{\mathbf{F}}^k \mathcal{L})_{:,i} = \frac{\alpha (\partial \mathcal{L}_{\rho^k})_i}{2} \frac{(\mathbf{Q} \rho^k + \alpha \mathbf{F}^H \mathbf{d})_i}{|\mathbf{Q} \rho^k + \alpha \mathbf{F}^H \mathbf{d}|_i} \bar{\mathbf{d}}, \quad (49)$$

if $|\mathbf{Q} \rho^k + \alpha \mathbf{F}^H \mathbf{d}|_i = \mathbf{y}_i^k > \tau$ and 0 everywhere else.

Finally, multiplying with the partial derivative of $\mathbf{F} = \operatorname{diag}(\mathbf{w}) \bar{\mathbf{F}}$, we obtain the derivative with respect to $\mathbf{w}$ by, for index $i = 1, 2, \dots, M$, and subscript i , denoting the i th $1 \times N$ row of the corresponding matrix

$$\left(\frac{\partial \mathcal{L}}{\partial \mathbf{w}} \right)_i = \bar{\mathbf{F}}_{i,:} \left(\frac{\partial \mathcal{L}}{\partial \mathbf{F}} \right)_{i,:}^T. \quad (50)$$

10.2 Threshold derivative

Since $\rho_i^k = \max(0, \mathbf{y}_i^k - \tau)$, the derivative $(\partial \rho^k / \partial \tau)_{1 \times N}$ will equal -1 at indexes $\mathbf{y}_i^k > \tau$ and 0 otherwise. Then, the k th layer derivative becomes

$$\frac{\partial \mathcal{L}}{\partial \tau} = \sum_{k=1}^L \sum_{i \in \mathcal{J}_{\mathbf{y}^k > \tau}} - \left(\frac{\partial \rho^L}{\partial \rho^k} \frac{\partial \rho^*}{\partial \rho^L} \frac{\partial \mathcal{L}(\mathbf{d}^*, \mathbf{d})}{\partial \rho^*} \right)_i, \quad (51)$$

where $\mathcal{J}_{\mathbf{y}^k > \tau}$ is again the set of indexes where $\mathbf{y}^k = |\mathbf{Q} \rho^k + \alpha \mathbf{F}^H \mathbf{d}| > \tau$.