A NOTE ON THE LOCALIZATION NUMBER OF RANDOM GRAPHS: DIAMETER TWO CASE

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Abstract. We study the localization game on dense random graphs. In this game, a *cop* x tries to locate a *robber* y by asking for the graph distance of y from every vertex in a sequence of sets $W_1, W_2, \ldots, W_\ell$. We prove high probability upper and lower bounds for the minimum size of each W_i that will guarantee that x will be able to locate y.

1. Introduction

In this paper we consider the following *Localization Game* related to the well studied *Cops and Robbers* game; see Bonato and Nowakowski [2] for a survey on this game. A robber is located at a vertex v of a graph G. In each round, a cop can ask for the graph distance between v and vertices $W = \{w_1, w_2, \ldots, w_k\}$, where a new set of vertices W can be chosen at the start of each round. The cop wins immediately if *the W-signature* of v, i.e. the set of distances, $\mathrm{dist}(v, w_i)$, $i = 1, 2, \ldots, k$ is sufficient to determine v. Otherwise, the robber will move to a neighbor of v and the cop will try again with a (possibly) different *test set* W. Given G, the *localization number* $\zeta(G)$ is the minimum k so that the cop can eventually locate the robber, that means, the cop determines the exact location of the robber from the test sets of size k. This game was introduced by Bosek et al. [3], who studied the localization game on geometric and planar graphs, and also independently, by Haslegrave et al. [6]. For some other related results see [4, 8, 9].

2. Results

The localization number is closely related to the *metric dimension* $\beta(G)$. This is the smallest integer k such that the cop can always win the game in *one* round. Clearly, $\zeta(G) \leq \beta(G)$.

In this note we will study the localization number of the random graph $G_{n,p}$ with diameter two. Here and throughout the whole paper $\omega = \omega(n) = o(\log n)$ denotes a function tending arbitrarily slowly to infinity with n. We will also use the notation

$$q = 1 - p \text{ and } \rho = p^2 + q^2.$$

We write $A_n \lesssim B_n$ to mean that $A_n \leq (1 + o(1))B_n$ as n tends to infinity. We further write $A_n \approx B_n$ if $A_n = (1 + o(1))B_n$ as n tends to infinity. Finally, we say that an event $\mathcal{E}_n$ occurs *asymptotically almost surely*, or a.a.s. for brevity, if $\lim_{n\to\infty} \mathbf{Pr}(\mathcal{E}_n) = 1$.

The metric dimension of $G_{n,p}$ was studied by Bollobás et al. [1]. If we specialize their result to large p then it can be expressed as:

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Theorem 2.1 ([1]). *Suppose that*

$$\left(\frac{2\log n+\omega}{n}\right)^{1/2} \le p \le 1-\frac{3\log\log n}{\log n}.$$

Then,

$$\frac{2\log np}{\log 1/\rho} \lesssim \beta(G_{n,p}) \lesssim \frac{2\log n}{\log 1/\rho} \ \text{a.a.s.}. \tag{1}$$

Note that the upper and lower bounds in (1) are asymptotically equal if $p \ge n^{-o(1)}$.
It is well-known (see, e.g., [5]) that if $np^2 \ge 2\log n+\omega$, then a.a.s. $\operatorname{diam}(G_{n,p}) \le 2$. We will condition on the diameter satisfying this. Graphs with diameter 2 enable some simplifications. Indeed, if a vertex v has W-signature $\{d_1,\dots,d_k\}$, where $W=\{w_1,\dots,w_k\}$, where $d_i = \operatorname{dist}(v,w_i)$, then

$$d_i = \begin{cases} 1 & \text{iff } \{v,w_i\}\in E \\ 2 & \text{iff } \{v,w_i\}\notin E. \end{cases}$$

Consequently, the probability that two vertices u and v in $G_{n,p}$ have the same W-signature, $W=\{w_1,\dots,w_k\}$, such that $u,v\notin W$ is equal to

$$\prod_{i=1}^{k} \mathbf{Pr}(u,v\in N(w_i) \ \text{ or } \ u,v\notin N(w_i)) = \rho^k.$$

The upper bound on p in the below theorem is determined by a result of [1] about the metric dimension of $G_{n,p}$.

Theorem 2.2. *Let*

$$\left(\frac{2\log n+\omega}{n}\right)^{1/2} \le p \le 1-\frac{3\log\log n}{\log n} \quad \text{and} \quad \eta=\frac{\log(1/p)}{\log n}$$

and let c be a positive constant such that

$$0<c<\min\left\{\frac{1}{2}\left(\frac{\log n-3\log\log n}{\log 1/p}-1\right),1\right\}.$$

Then, a.a.s.

$$\left(1-2\eta-\frac{4\log\log n}{\log n}\right)\frac{2\log n}{\log 1/\rho} \le \zeta(G_{n,p}) \le (1-c\eta)\frac{2\log n}{\log 1/\rho}.$$

2.1. **Observations about Theorem 2.2.**
First observe that if $p \ge \frac{\log n}{n^{1/3}}$, then

$$\frac{1}{2}\left(\frac{\log n-3\log\log n}{\log 1/p}-1\right)\ge 1$$

and so c can be any positive constant less than 1. Furthermore, for any $p \ge \left(\frac{2\log n+\omega}{n}\right)^{1/2}$ we have

$$\frac{1}{2}\left(\frac{\log n-3\log\log n}{\log 1/p}-1\right)\ge \frac{1}{2}\left(\frac{\log n-3\log\log n}{\frac{1}{2}(\log n-\log(2\log n+\omega))}-1\right)=\frac{1}{2}-o(1).$$

Hence, we can always take $c \ge \frac{1}{2}-o(1)$.

If $p = 1/n^{\alpha}$ for some constant $0 < \alpha < 1/2$, then,

$$\eta = \alpha \quad \text{and} \quad c \le \begin{cases} 1 - o(1) & \text{if } 0 < \alpha < \frac{1}{3} \\ \frac{1}{2\alpha} - \frac{1}{2} - o(1) & \text{otherwise.} \end{cases}$$

Moreover,

$$\rho = 1 - 2p + 2p^2 \text{ and so } \log 1/\rho = 2p + O(p^2) \approx \frac{2}{n^{\alpha}}.$$

Hence, Theorem 2.2 implies the following corollary.

Corollary 2.3. *Let $p = 1/n^{\alpha}$, where $0 < \alpha < 1/2$ is constant. Then, a.a.s.*

$$(1 - 2\alpha) n^{\alpha} \log n \lesssim \zeta(G_{n,p}) \lesssim \begin{cases} (1-\alpha) n^{\alpha} \log n & \text{if } 0 < \alpha < \frac{1}{3} \\ \left(\frac{1+\alpha}{2}\right) n^{\alpha} \log n & \text{otherwise.} \end{cases}$$

Notice that for $0 < \alpha < \frac{1}{3}$ the upper bound on $\zeta(G_{n,p})$ equals the lower bound from Theorem 2.1. Therefore, it is plausible to conjecture that $\zeta(G_{n,p}) < \beta(G_{n,p})$.
Now observe that if $p = n^{-1/\omega}$, then

$$2\eta = \frac{2\log(1/p)}{\log n} = \frac{2}{\omega} = o(1).$$

Thus, Theorem 2.2 implies:

Corollary 2.4. *Let $p = n^{-1/\omega}$. Then,*

$$\zeta(G_{n,p}) \approx \frac{2\log n}{\log 1/\rho}.$$

Clearly, this also holds for any constant p. In particular, for $p = 1/2$, we get:

Corollary 2.5. *For almost all graphs G we have*

$$\zeta(G) \approx \frac{2\log n}{\log 2} = 2\log_2(n).$$

2.2. **Proof of Theorem 2.2 – lower bound.**

Since we will deal with "mostly independent" random variables, we will use the following form of Suen's inequality (see, e.g. [7]).

Theorem 2.6 (Suen's Inequality)**.** *Let $\theta_i, i \in I$ be indicator random variables which take value 1 with probability p_i. Let L be a dependency graph. Let $X = \sum_{i\in I} \theta_i$, and $\mu = \mathbf{E}(X) = \sum_{i \in I} p_i$. Moreover, write $i \sim j$ if $ij \in E(L)$, and let $\Delta = \frac{1}{2}\sum_{i\sim j} \mathbf{E}(\theta_i\theta_j)$ and $\delta = \max_i \sum_{j\sim i} p_j$. Then,*

$$\mathbf{Pr}(X = 0) \le \exp\left\{ -\min\left\{ \frac{\mu^2}{8\Delta}, \frac{\mu}{2}, \frac{\mu}{6\delta} \right\} \right\}.$$

We will also use the following simple fact.

Lemma 2.7. *Let $0 < p < 1$ and $p + q = 1$. Then,*

$$\frac{\log(p^3 + q^3)}{\log \rho} \ge \frac{3}{2}.$$

Proof. This inequality is equivalent to

$$\log(p^3+q^3)^2 \le \log(p^2+q^2)^3$$

and so to

$$(p^3+q^3)^2 \le (p^2+q^2)^3.$$

The latter is equivalent to

$$2p^3q^3 \le 3p^4q^2 + 3p^2q^4 = 3p^2q^2(p^2+q^2) = 3p^2q^2(1-2pq)$$

and consequently to

$$2pq \le 3(1-2pq)$$

which is equivalent to

$$pq \le \frac{3}{8}.$$

But this is always true since $pq \le \frac{1}{4}$. □

The lower bound in Theorem 2.2 will follow from the following result.

Lemma 2.8. *Let*

$$\frac{\log^2 n}{n^{1/2}} < p \le 1 - \frac{1}{\log n} \qquad \textit{and} \qquad \varepsilon = \frac{2\log\left(\frac{\log^2 n}{p}\right)}{\log n} \qquad \textit{and} \qquad k = \frac{2(1-\varepsilon)\log n}{\log 1/\rho}.$$

Then a.a.s.,

$$\zeta(G_{n,p}) \ge k.$$

First observe that $\varepsilon = 2\eta + \frac{4\log\log n}{\log n}$ and so the lower bound in Theorem 2.2 holds.

Proof. For a fixed vertex u and k-set S let $X_{u,S}$ count the number of unordered pairs $w, v \in N(u)$ with the same signature induced by S. We prove that the probability that there is a vertex u and a k-set S such that $X_{u,S} = 0$ is $o(1)$. Consequently, this will imply that a.a.s. for every vertex u and k-set S there are at least two neighbors of u with the same signature in S. Hence, a.a.s. the localization number is at least k.
Clearly,

$$\mu = \mathbf{E}(X_{u,S}) = \binom{n-k-1}{2}\rho^k p^2 \ge \frac{p^2}{4}\exp\{k\log\rho + 2\log n\}$$
$$= \frac{p^2}{4}\exp\{-2(1-\varepsilon)\log n + 2\log n\} = \frac{p^2}{4}n^{2\varepsilon}.$$

Furthermore, since every triple of vertices in $N(u)$ with the same signature contributes three unordered pairs of variables to Δ, we get

$$\begin{aligned}\Delta &\le 3\binom{n}{3}(p^3+q^3)^k p^3\\ &\le \frac{p^3}{2}\exp\left\{k\log(p^3+q^3) + 3\log n\right\}\\ &= \frac{p^3}{2}\exp\left\{-2(1-\varepsilon)(\log n)\frac{\log(p^3+q^3)}{\log\rho} + 3\log n\right\}.\end{aligned}$$

Now, by Lemma 2.7,

$$\Delta \le \frac{p^3}{2} \exp\left\{-2(1-\varepsilon)(\log n)\cdot\frac{3}{2} + 3\log n\right\} = \frac{p^3}{2} n^{3\varepsilon}.$$

Similarly

$$\delta \le 2n\rho^k p^2 = 2p^2 \exp\left(k\log\rho + \log n\right) = 2p^2 n^{-1+2\varepsilon}.$$

Thus,

$$\frac{\mu^2}{8\Delta} \ge \frac{1}{64} pn^{\varepsilon}, \quad \frac{\mu}{2} \ge \frac{1}{8}(pn^{\varepsilon})^2 \quad \text{and} \quad \frac{\mu}{6\delta} \ge \frac{1}{48} n.$$

Since $0 < \varepsilon < 1$ and $pn^{\varepsilon} \to \infty$ (due to our choice of ε) the lower bound in the first inequality is the smallest. Hence, by Theorem 2.6,

$$\mathbf{Pr}(X_{u,S} = 0) \le \exp\left\{-\frac{1}{64} pn^{\varepsilon}\right\}.$$

Now we use the union bound to show that the probability that there is a vertex u and a k-set S such that $X_{u,S} = 0$ is $o(1)$. Indeed, this probability is at most

$$n\binom{n}{k} \exp\left\{-\frac{1}{64} pn^{\varepsilon}\right\} \le \exp\left\{(k+1)\log n - \frac{1}{64} pn^{\varepsilon}\right\}. \tag{2}$$

Now observe that $\rho = (p+q)^2 - 2pq = 1 - 2pq$ and so

$$k = \frac{2(1-\varepsilon)\log n}{\log 1/\rho} = -\frac{2(1-\varepsilon)\log n}{\log(1-2pq)} \le -\frac{2\log n}{\log(1-2pq)}.$$

Since $1 - x \le e^{-x}$ and $2pq < 1$, we get that

$$k \log n \le \frac{(\log n)^2}{pq}.$$

Furthermore, since by assumption $p \le 1 - \frac{1}{\log n}$, we obtain $q \ge \frac{1}{\log n}$ and so

$$k \log n \le \frac{(\log n)^3}{p}.$$

Also

$$pn^{\varepsilon} = pe^{\varepsilon \log n} = \frac{(\log n)^4}{p}.$$

Thus, the exponent in (2) tends to $-\infty$. This completes the proof of Lemma 2.8. □

2.3. **Proof of Theorem 2.2 – upper bound.**

Let $\deg(v)$ denote the degree of vertex v in $G_{n,p}$ and let $\mathrm{codeg}(v,w)$ denote the co-degree of vertices v, w in $G_{n,p}$. We observe next that the Chernoff bounds imply that a.a.s.

$$\deg(v) = np + O((np\log n)^{1/2}) \text{ for all } v \in [n]. \tag{3}$$

$$\mathrm{codeg}(v,w) = np^2 + O((np^2\log n)^{1/2}) \text{ for all } v \in [n]. \tag{4}$$

Lemma 2.9.

(i) Let

$$e^{-\frac{\log n}{\omega}} \le p \le 1 - \frac{3\log\log n}{\log n}.$$

Then, a.a.s.

$$\zeta(G_{n,p}) \lesssim \frac{2\log n}{\log 1/\rho}.$$

(ii) Let

$$\left(\frac{2\log n + \omega}{n}\right)^{1/2} \le p \le e^{-\Omega(\log n)} \quad \text{and} \quad \eta = \frac{\log 1/p}{\log n} \quad \text{and} \quad k = \frac{2(1-c\eta)\log n}{\log 1/\rho},$$

where

$$0 < c < \min\left\{\frac{1}{2}\left(\frac{\log n - 3\log\log n}{\log 1/p} - 1\right), 1\right\}.$$

Then, a.a.s.

$$\zeta(G_{n,p}) \le k.$$

Proof. Part (i) follows immediately from Theorem 2.1.

We now prove (ii). Equations (3) and (4) plus our bound of two on the diameter are all we need for this. So the analysis works for any graph satisfying these conditions. Let S_1 be a randomly chosen k-subset of V and let X_1 be the number of pairs with the same signature in S_1. Then, if

$$D(v,w) = (N(v) \setminus N(w)) \cup (N(w) \setminus N(v))$$

for $v, w \in [n]$ then

$$\begin{aligned}
\mathbf{E}(X_1) &= \sum_{v \ne w} \mathbf{Pr}((N(v) \cap S_1) = (N(w) \cap S_1)) \\
&= \sum_{v \ne w} \mathbf{Pr}(S_1 \cap D(v,w) = \emptyset) \\
&\le n^2 \left(1 - 2p(1-p)\left(1 + O\left(\frac{\log^{1/2} n}{n^{1/2}}\right)\right)\right)^k \qquad (5) \\
&= n^2 \rho^k \left(1 + O\left(\frac{k\log^{1/2} n}{n^{1/2}}\right)\right) \qquad (6) \\
&= (1+o(1))n^{2c\eta}. \qquad (7)
\end{aligned}$$

and by the Markov inequality we have $X_1 \le \omega n^{2c\eta}$ a.a.s.. (Going from (5) to (6) uses the trivial identity $1 - a(1-\varepsilon) = (1-a)\left(1 + \frac{a\varepsilon}{1-a}\right)$.) Thus, the set R of vertices with exactly the same signature in S as the robber is a.a.s. of size at most $\omega^{1/2}n^{c\eta}$. Let T_2 consist of R and the set of neighbors of R. The robber can move to somewhere in T_2. Clearly, $|T_2| \le 2\omega^{1/2}n^{c\eta}pn$ a.a.s..

Now let S_2 be another random k-subset of V, chosen independently of S_1. Let X_2 be the number of pairs of vertices from T_2 with the same signature in S_2. Arguing as for (7), we see that

$$\begin{aligned}\mathbf{E}(X_2) &\leq (2\omega^{1/2}n^{c\eta}pn)^2\rho^k\left(1+O\left(\frac{k\log^{1/2}n}{n^{1/2}p}\right)\right)\\ &= (1+o(1))(2\omega^{1/2}p)^2\exp((2+2c\eta)(\log n)+k\log\rho) = (4+o(1))\omega p^2 n^{4c\eta}\end{aligned}$$

and by the Markov inequality we get that a.a.s we have $X_2 \leq \omega^2p^2n^{4c\eta}$. Thus, the number of vertices with exactly the same signature as the robber in S_2 is at most $\omega pn^{2c\eta}$. Let T_3 consist of these vertices together with their neighbors. Clearly, $|T_3| \leq 2\omega p^2 n^{2c\eta+1}$.
We proceed inductively. Assume that $|T_i| \leq 2(\omega^{1/2}p)^{i-1}n^{(i-1)c\eta+1}$. Now, arguing as above with another independently chosen k-set S_{i+1}, we have

$$\mathbf{E}(X_{i+1}) \leq (2+o(1))((\omega^{1/2}p)^{i-1}n^{(i-1)c\eta+1})^2\rho^k = (2+o(1))(\omega^{1/2}p)^{2(i-1)}n^{2ic\eta}$$

and so by the Markov inequality,

$$X_{i+1} \leq \omega(\omega^{1/2}p)^{2(i-1)}n^{2ic\eta} \text{ a.a.s..} \tag{8}$$

Thus, the number of vertices with exactly the same signature in S_{i+1} is at most $\omega^{1/2}(\omega^{1/2}p)^{i-1}n^{ic\eta}$. Hence,

$$|T_{i+1}| \leq 2\omega^{1/2}(\omega^{1/2}p)^{i-1}n^{ic\eta}pn = 2(\omega^{1/2}p)^i n^{ic\eta+1},$$

completing the induction.
After ℓ rounds we get that with probability at least $1-\ell\omega^{-1}$ we have, using (8),

$$\begin{aligned}|X_\ell| &\leq \omega(\omega^{1/2}p)^{2(\ell-2)}n^{2(\ell-1)c\eta} = \omega^{\ell-1}\exp\left\{2(\ell-2)\log p + 2(\ell-1)c\eta\log n)\right\}\\ &= \omega^{\ell-1}\exp\left\{-2(\ell-2-c(\ell-1))\log(1/p)\right\}.\end{aligned} \tag{9}$$

Clearly, (9) is $o(1)$ for sufficiently large constant ℓ, since by assumption $\log(1/p) = \Omega(\log n)$. □

3. Summary

We have separated the localization value $\zeta(G_{n,p})$ from the metric dimension $\beta(G_{n,p})$ in the range where the diameter of $G_{n,p}$ is two a.a.s.. It would be interesting to continue the analysis in the range of p for which the diameter of $G_{n,p}$ is at least 3. It would also be of interest to examine the localization game on random regular graphs.

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References

[1] B. Bollobás, P. Prałat and D. Mitsche, Metric dimension for random graphs, *The Electronic Journal of Combinatorics* **20** (2013).
[2] A. Bonato and R. Nowakowski. The game of cops and robbers on graphs, American Mathematical Society, 2011.
[3] B. Bosek, P. Gordinowicz, J. Grytczuk, N. Nisse, J. Sokół and M. Śleszyńska-Nowak, Localization game on geometric and planar graphs, `arXiv:1709.05904`.
[4] J. Carraher, I. Choi, M. Delcourt, L. H. Erickson, and D. B. West, Locating a robber on a graph via distance queries, *Theoretical Computer Science* **463**, pp. 54–61 (2012).
[5] A.M. Frieze and M. Karoński, Introduction to Random Graphs, *Cambridge University Press*, 2015.

[6] J. Haslegrave, R. Johnson and S. Koch, Locating a robber with multiple probes, *Discrete Mathematics* **341** (2018), no. 1, 184–193.
[7] S. Janson, T. Łuczak and A. Ruciński, Random Graphs, *Wiley*, 2000.
[8] S. Seager, Locating a robber on a graph, *Discrete Mathematics* **312**, pp. 3265–3269 (2012).
[9] S. Seager, Locating a backtracking robber on a tree, *Theoretical Computer Science* **539**, pp. 28–37 (2014).

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