State Preparation and Measurement Tomography via Unitary Transformations

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Abstract: We describe a technique for simultaneously determining both the state of a quantum system and the positive value operator measure that describes a detector, while making a minimum of assumptions about each of them.
OCIS codes: (270.5585) Quantum information and processing; (270.0270) Quantum optics.

1. Introduction

Quantum tomography is an important tool for characterizing quantum systems and is useful for quantum information processing applications. Quantum-state tomography estimates the state of a quantum system, while quantum-detector tomography estimates the positive-operator-valued measure (POVM) that describes a detector. Here we present a technique for estimating both the state of a single qubit, and the parameters of a positive-operator-valued measure (POVM) that describes a detector, in a self-consistent manner [1, 2]. We accomplish this by performing a series of known, unitary transformations between the state preparation and measurement stages. We refer to this technique as state preparation and measurement tomography via unitary transformations (SPAMTUT).

SPAMTUT is similar to gate-set tomography (GST) in that both the state and the POVM are estimated [2]. SPAMTUT and GST differ in that for SPAMTUT the transformations performed between the state and the measurement are assumed to be known, in principle allowing for fewer measurements. The assumption that the transformations are known is valid if they can be calibrated using a bright, classical source and a classical detector, as is the case in our experiments. The only assumptions made about the state and the detector POVM are that we know their Hilbert-space dimensions.

In our experiments we use individual photons encoded as polarization qubits, so vectors that describe the state and the POVM reside within the Poincaré sphere and the unitary transformations act as rotations within this sphere. The vectors that describe the state and the POVM are determined by aligning the rotation axes of the transformations with these vectors, rather than by inverting a measured set of data. This allows us to estimate errors in the parameters that describe the state and the POVM in a straightforward manner. The directions of the state and the POVM are determined to within a four-fold ambiguity. This ambiguity is due to gauge degrees of freedom [2, 3]. If a second state preparation or POVM is available this can be reduced to a two-fold ambiguity, or, if some *a priori* information about the state or the POVM is known, that information can be used to distinguish between the four possibilities. Finally, there is an undetermined continuous degree of freedom (sometimes called a blame gauge [3]) that trades off between the purity of the state and the discrimination power of the detector. SPAMTUT determines a limit on the amount of this tradeoff.

Here we describe the theory of SPAMTUT, as well as an experiment we are performing to demonstrate it.

2. Theory

Suppose we have a qubit that is prepared in a state that is described by the density operator $\hat{\rho}$, which can be expressed in terms of the Pauli matrices $\hat{\sigma}_i$ (i = 1,2,3) as

$$\hat{\rho} = (1/2)\left(\hat{1} + \sum_{i=1}^{3} p_i \hat{\sigma}_i\right) . \tag{1}$$

The parameters that describe $\hat{\rho}$ can be arranged into a 3-component vector $\vec{p}$. Furthermore, we have a 2-outcome detector described by the POVM elements $\{\hat{\Pi}, \neg\hat{\Pi}\}$ (Π and NOT-Π). They are written in terms of the elements of a 3-component vector $\vec{w}$ and the detector bias parameter u as

$$\hat{\Pi} = (1/2)\left((1+u)\hat{1} + \sum_{i=1}^{3} w_i \hat{\sigma}_i \right) \quad , \quad \neg\hat{\Pi} = (1/2)\left((1-u)\hat{1} - \sum_{i=1}^{3} w_i \hat{\sigma}_i \right) . \tag{2}$$

Positivity is ensured by the constraints $|\vec{p}| \le 1$ and $|\vec{w}| + |u| \le 1$. The two-outcome POVM can be represented in terms of a single observable $\hat{\Sigma} = \hat{\Pi} - \neg\hat{\Pi}$, and the expectation value of a measurement is given by [3]

$$E = \mathrm{Tr}\left(\hat{\Sigma}\hat{\rho}\right) = \vec{w} \bullet \vec{p} + u \ . \tag{3}$$

The goal of SPAMTUT is to estimate $\vec{p}$, $\vec{w}$ and *u*, thereby fully characterizing the state and the POVM. This is done by performing known, unitary transformations $\hat{U}_j$ between the state preparation and the measurement. Since $\vec{p}$ and $\vec{w}$ are three-dimensional vectors whose magnitudes are bounded by 1, they reside within the Poincaré sphere (or, alternatively, the Bloch sphere). The transformation $\hat{U}_j = \hat{U}\left(\vec{k}_j, \varphi_j\right)$ rotates $\vec{p}$ (by an angle φ_j about the axis $\vec{k}_j$) into $\vec{p}_j$, so we measure

$$E_j = \mathrm{Tr}\left(\hat{\Sigma}\hat{U}_j\hat{\rho}\right) = \vec{w} \bullet \vec{p}_j + u \ . \tag{4}$$

We begin by searching for a transformation $\hat{U}_1$ that maximizes E_1; in this case $\vec{p}_1$ is aligned parallel to $\vec{w}$. Next, we find a transformation $\hat{U}_2$ that minimizes E_2. In this case $\vec{p}_2$ is aligned antiparallel to $\vec{w}$, so $\vec{w} \bullet \vec{p}_1 = -\vec{w} \bullet \vec{p}_2$. We find *u* from $u = \left(E_1 + E_2\right)/2$. Since *u* has been determined, we can now directly determine the dot product from our measurements: $E_j - u = \vec{w} \bullet \vec{p}_j$. We then find a transformation $\hat{U}'$ that makes the dot product 0, or at least minimizes its magnitude,

$$\mathrm{Tr}\left(\hat{\Sigma}\hat{U}'\hat{\rho}\right) - u = \vec{w} \bullet \vec{p}' = 0 \ , \tag{5}$$

so $\vec{w} \perp \vec{p}'$. Now that we have the transformation $\hat{U}'$, we can leave it in place and seek $\vec{p}'$. Once $\vec{p}'$ is determined we can find the original vector $\vec{p}$ by applying $\left(\hat{U}'\right)^{-1}$ to $\vec{p}'$.

Given $\hat{U}'$ and *u*, we perform measurements which yield

$$E_j' = \mathrm{Tr}\left(\hat{\Sigma}\hat{U}_j\hat{U}'\hat{\rho}\right) - u = \vec{w} \bullet \vec{p}_j' \ . \tag{6}$$

We now find a transformation $\hat{U}_3$ that minimizes $|E_3'|$ for all rotation angles φ_3 (in our simulations we find that minimizing $|E_3'|$ for the two angles $\varphi_3 = \pi/2, \pi$ is sufficient). This means that $\vec{w} \perp \vec{p}_3'$ for any φ_3. There are now two possibilities, which we will consider separately: the rotation axis $\vec{k}_3$ is parallel to either $\vec{w}$ or $\vec{p}_3'$.

Assuming $\vec{k}_3$ is parallel to $\vec{p}_3'$, we have found the direction of $\vec{p}'$, which is parallel to $\vec{p}_3'$ ($\hat{U}_3$ rotates about an axis parallel to $\vec{p}'$, so its direction does not change). Now we need to find $\vec{w}$. Recalling that $\vec{w} \perp \vec{p}'$, we can find a transformation $\hat{U}_4 = \hat{U}\left(\vec{k}_4, \pi/2\right)$ that maximizes E_4'. This makes $\vec{p}_4'$ parallel to $\vec{w}$. Knowing $\hat{U}_4$ and the direction of $\vec{p}'$ determines the direction of $\vec{w}$, as $\vec{k}_4 \times \vec{p}' \,||\, \vec{w}$. We also know $E_4' = wp' = wp$. While SPAMTUT determines the product of the magnitudes of $\vec{w}$ and $\vec{p}$, it cannot determine their individual magnitudes because of the blame gauge degrees of freedom described above. Given this fact, we will assume that *w* is maximized, so we have $w = 1 - |u|$, and then $p = w / E_4'$. All of the information we now have determines one solution for the state and the POVM, call this solution $\left(\vec{p}_a, \vec{w}_a, u\right)$.

As seen in Eq. (6), the expectation values that we measure are determined by a dot product. It might be the case that both $\vec{w}$ and $\vec{p}$ point in the opposite directions that we determined above, and the minus signs would cancel out. Thus, due to gauge degrees of freedom, the actual state and detector POVM might be given by the solution

$\left(-\vec{p}_a,-\vec{w}_a,u\right)$. This represents a choice of basis. In the case of polarization qubits it effectively corresponds to rotating both the state and the detector by 90º, i.e. interchanging what we mean by horizontal and vertical.

To find the next possible solution assume $\vec{k}_3$ is parallel to $\vec{w}$, so the direction of $\vec{w}$ is determined. The transformation $\hat{U}_4=\hat{U}\left(\vec{k}_4,\pi/2\right)$ that maximizes E_4' still ensures that $\vec{p}_4'$ is parallel to $\vec{w}$. Knowing $\hat{U}_4$ and the direction of $\vec{w}$ determines the direction of $\vec{p}'$, as $\vec{w}\times\vec{k}_4 \parallel \vec{p}'$. From the direction of $\vec{p}'$ we can find the direction of $\vec{p}$. Again, we can determine the product of the magnitudes of $\vec{w}$ and $\vec{p}$, but not their individual magnitudes. Call this solution $\left(\vec{p}_b,\vec{w}_b,u\right)$, and once again $\left(-\vec{p}_b,-\vec{w}_b,u\right)$ is also a possible solution. Effectively these solutions differ from the previously found solutions by swapping the roles of $\vec{w}$ and $\vec{p}$.

We have thus found four possible solutions that determine both the state and the detector POVM, to within the continuous blame gauge degree of freedom. If a second state preparation or detector POVM is available the whole process can be repeated. This will allow us to distinguish between solutions *a* and *b*, but not between the positive and negative solutions.

We have performed simulations of the SPAMTUT algorithm. We generate 1,000 random pairs of state preparations and detector POVMs. The directions of $\vec{w}$ and $\vec{p}$ are chosen randomly, and we assume $0.01\le p\le 1$, $-0.9\le u\le 0.9$ and $w=1-|u|$. In all trials, one of the four solutions returned by SPAMTUT correctly determined all of the parameters.

3. Experiment

To perform SPAMTUT we need to minimize or maximize certain measurements. Experimentally this can be done to within the uncertainties of the measurements, and these uncertainties bound our accuracy in determining both the state and the detector POVM. There are two types of uncertainties that are relevant: systematic errors in the calibration of the unitary transformations, and statistical uncertainties due to finite measurement times. Using classical sources and detectors to calibrate the transformations can, in principle, make systematic errors arbitrarily small. Longer counting times can be used to reduce statistical uncertainties.

We are in the process of implementing an experiment to demonstrate that we can use SPAMTUT to self-consistently determine the polarization state of a single photon, and the POVM of a detector that measures this polarization. To do this we need to perform arbitrary unitary transformations of a polarization qubit, which means that we need to be able to rotate through an arbitrary angle, about an arbitrary axis, in the Poincaré sphere. To implement these transformations we use a Soleil-Babinet compensator (SBC) placed between two quarter-wave plates [4]. The rotation of the axes of these components determine the axis of rotation, while the retardance of the SBC determines the rotation angle.

4. Conclusions

We have shown that SPAMTUT is capable of estimating both the quantum state of a single qubit, and the POVM that performs a measurement of this qubit, in a self-consistent manner. This is done by performing a series of known, unitary transformations between the state preparation and measurement stages. The only assumptions made about the state and the detector POVM are that we know their Hilbert-space dimensions.

Acknowledgements: This project was funded by NSF (1855174) and Reed College.

5. References

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