CONSTRUCTION OF 2-PEAKON SOLUTIONS AND ILL-POSEDNESS FOR THE NOVIKOV EQUATION[*]

A. ALEXANDROU HIMONAS[†], CURTIS HOLLIMAN[‡], AND CARLOS KENIG[§]

Abstract. For the Novikov equation, on both the line and the circle, we construct a 2-peakon solution with an asymmetric antipeakon-peakon initial profile whose H^s-norm for $s < 3/2$ is arbitrarily small. Immediately after the initial time, both the antipeakon and peakon move in the positive direction, and a collision occurs in arbitrarily small time. Moreover, at the collision time the H^s-norm of the solution becomes arbitrarily large when $5/4 < s < 3/2$, thus resulting in norm-inflation and ill-posedness. However, when $s < 5/4$, the solution at the collision time coincides with a second solitary antipeakon solution. This scenario thus results in nonuniqueness and ill-posedness. Finally, when $s = 5/4$ ill-posedness follows either from a failure of convergence or from a failure of uniqueness. Considering that the Novikov equation is well-posed for $s > 3/2$, these results together establish $3/2$ as the critical index of well-posedness for this equation. The case $s = 3/2$ remains an open question.

Key words. Novikov equation, integrable equations, Cauchy problem, Sobolev spaces, ill-posedness, norm-inflation, Camassa–Holm-type equations, well-posedness, 2-peakon solutions, nonuniqueness

AMS subject classifications. 35Q53, 37K10

DOI. 10.1137/17M1151201

1. Introduction and results. We consider the Cauchy problem for the Novikov equation (NE) on the line and the circle

$$(1.1) \qquad u_t + u^2 u_x + \partial_x D^{-2}\Big[u^3 + \frac{3}{2}uu_x^2\Big] + D^{-2}\Big[\frac{1}{2}u_x^3\Big] = 0,$$

$$(1.2) \qquad u(x,0) = u_0(x), \quad x \in \mathbb{R} \text{ or } \mathbb{T}, \quad t \in \mathbb{R},$$

where D^{-2} is the Bessel potential $D^{-2} = (1 - \partial_x^2)^{-1}$, and construct specific 2-peakon solutions $u(t)$ that collide at a finite time T in such a way as to give rise to the phenomenon of norm-inflation. In particular, the norm-inflation generated by these 2-peakon collisions occurs in Sobolev spaces H^s with exponents between $5/4$ and $3/2$. As such, we will refer to $3/2$ as the *critical* exponent for well-posedness, as well-posedness has been proven for exponents greater than $3/2$ (see [HH2]). For exponents s less than $5/4$, the collision of the 2-peakons in fact converges to a single antipeakon $u(T)$, which can be thought of as a superposition of both peakons. This scenario allows us to demonstrate nonuniqueness. Taken together, these results prove that the NE is ill-posed in H^s for $s < 3/2$.

[†]Corresponding author. Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556 (himonas.1@nd.edu).

[‡]Department of Mathematics, The Catholic University of America, Washington, DC 20064 (holliman@cua.edu).

[§]Department of Mathematics, University of Chicago, Chicago, IL 60637-1514 (cek@math.uchicago.edu).

We recall that the NE is well-posed in the sense of Hadamard (see [H]) in Sobolev spaces H^s with exponents $s > 3/2$ (see [HH2]). More precisely, if u_0 belongs to the Sobolev space H^s on the circle or the line, then there exists $T_s = T_s(||u_0||_{H^s}) > 0$ and a unique solution $u \in C([0, T_s]; H^s)$ of the Cauchy problem for the NE (1.1)–(1.2) satisfying the estimate

$$(1.3) \qquad \|u(t)\|_{H^s} \le 2\|u_0\|_{H^s} \;\; \text{for} \;\; 0 \le t \le T_s, \quad \text{with} \quad T_s = \frac{1}{4c_s\|u_0\|_{H^s}^2},$$

where $c_s > 0$ is a constant depending on s. Furthermore, the data-to-solution map $u(0) \mapsto u(t)$ is continuous but not uniformly continuous.

The NE is an integrable equation, and its local form,

$$(1.4) \qquad\qquad (1 - \partial_x^2)u_t = u^2 u_{xxx} + 3uu_x u_{xx} - 4u^2 u_x,$$

was derived by Vladimir Novikov [N] in his attempt to classify all integrable Camassa–Holm–type equations with quadratic and cubic nonlinearities of the form $(1-\partial_x^2)u_t = P(u, u_x, u_{xx}, \ldots)$, where P is a polynomial of u and its derivatives. The Lax pair for NE was derived by Hone and Wang in [HW] and is given by the equations

$$(1.5a) \qquad \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}_x = U\,(m, \lambda) \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \quad \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}_t = V\,(m, u, \lambda) \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix},$$

where $m = u - u_{xx}$ and the matrices U and V are defined by
(1.5b)

$$U\,(m, \lambda) = \begin{pmatrix} 0 & \lambda m & 1 \\ 0 & 0 & \lambda m \\ 1 & 0 & 0 \end{pmatrix}, \quad V\,(m, u, \lambda) = \begin{pmatrix} \frac{1}{3\lambda^2} - uu_x & \frac{u_x}{\lambda} - \lambda mu^2 & u_x^2 \\ \frac{u}{\lambda} & -\frac{2}{3\lambda^2} & -\frac{u_x}{\lambda} - \lambda mu^2 \\ -u^2 & \frac{u}{\lambda} & \frac{1}{3\lambda^2} + uu_x \end{pmatrix}.$$

The NE possesses peakon traveling wave solutions [HM],[HLS], [GH], which on the real line are given by the formula

$$(1.6) \qquad\qquad u(x, t) = \pm\sqrt{c}\,e^{-|x-ct|},$$

where $c > 0$ is the wave speed. On the circle, the peakon solutions are given by the formula

$$(1.7) \;\; u(x, t) = \frac{\sqrt{c}}{\cosh(\pi)}\cosh([x - ct]_p - \pi), \quad \text{where} \quad [x - ct]_p \doteq x - ct - 2\pi\left[\frac{x - ct}{2\pi}\right].$$

In fact, the NE possesses multipeakon traveling wave solutions on both the line and the circle [HM], [HLS], [GH]. More precisely, on the line the n-peakon

$$(1.8) \qquad\qquad u(x, t) = \sum_{j=1}^{n} p_j(t)e^{-|x-q_j(t)|}$$

is a solution to NE if and only if the positions $(q_1, \ldots, q_n)$ and the momenta $(p_1, \ldots, p_n)$ satisfy the following system of $2n$ differential equations:

$$(1.9) \qquad\qquad \begin{cases} \dfrac{dq_j}{dt} = u^2(q_j), \\[2mm] \dfrac{dp_j}{dt} = -u(q_j)u_x(q_j)p_j. \end{cases}$$

The description of the periodic n-peakon is similar. Furthermore, NE solutions conserve the H^1-norm, that is,

$$(1.10) \qquad \int_{\mathbb{R} \text{ or } \mathbb{T}} \left[u^2(t) + u_x^2(t) \right] dx = \int_{\mathbb{R} \text{ or } \mathbb{T}} \left[u^2(0) + u_x^2(0) \right] dx.$$

Next, we state our first result that gives the basic properties of the 2-peakon solutions, which are constructed here and are needed for proving the ill-posedness of the NE below 3/2.

THEOREM 1. *For any $\varepsilon > 0$ there exists a $T > 0$ for which the NE Cauchy problem on the line and the circle (1.1)–(1.2) has a 2-peakon solution $u \in C([0, T]; H^s)$ such that its lifespan and its initial size satisfy the estimates*

$$(1.11) \qquad\qquad lifespan = T < \varepsilon,$$

$$(1.12) \qquad\qquad \|u_0\|_{H^s} < \varepsilon,$$

while as t approaches the lifespan T the H^s-norm of the solution $u(t)$ satisfies the estimates

$$(1.13) \qquad \lim_{t \to T} \|u(t)\|_{H^s} = \begin{cases} \infty & \text{(norm-inflation)}, & 5/4 < s < 3/2, \\ may\ not\ exist, & s = 5/4, \\ C_s & \text{for some } C_s > 0, & s < 5/4. \end{cases}$$

Moreover, when $s < 5/4$, then $u(t)$ converges to an antipeakon $u(T) = -\sqrt{c_T}\, e^{-|x - q_T|}$, for some $c_T > 0$ and $q_T > 0$, with $\|u(T)\|_{H^s} = C_s$.

This theorem is a very interesting result in its own right. Unlike the Camassa–Holm (CH) equation (see [CH], [FF], [L1], [MN])

$$(1.14) \qquad\qquad (1 - \partial_x^2)u_t = uu_{xxx} + 2u_x u_{xx} - 3uu_x$$

and the Degasperis–Procesi (DP) equation (see [DP], [HS], [LS], [L2], [DHH])

$$(1.15) \qquad\qquad (1 - \partial_x^2)u_t = uu_{xxx} + 3u_x u_{xx} - 4uu_x,$$

for which we can construct special symmetric 2-peakon solutions, called peakon-antipeakons, of the form

$$(1.16) \qquad\qquad u(x, t) = p(t)e^{-|x + q(t)|} - p(t)e^{-|x - q(t)|},$$

this is impossible for the NE. Peakon-antipeakon solutions, which are convenient to work with, are possible for CH and DP because these equations contain a symmetry that allows us to reduce the ODE system corresponding to (1.9) for the positions and the momenta via $p = p_1 = -p_2$ and $q = q_1 = -q_2$. This symmetry causes the peak and antipeak to move against each other and collide in finite time (see [HHG], [HGH], [By]). Such a construction is not possible for the NE because by (1.9) we have $\frac{dq_j}{dt} \geq 0$ for all positions q_j. Thus, we see that for the NE all the peaks and antipeaks move in the same direction. Therefore collision can occur only if the peakon that follows moves faster than and eventually overtakes the one ahead of it. For this scenario to occur we must break symmetry and solve the full system of the four highly nonlinear differential equations defined by system (1.9) for $n = 2$ with appropriate initial data. This procedure involves several novel ideas, which are described in sections 2 and 7. The results are summarized in Theorems 3 and 6.

Next, using Theorem 1 we obtain the following ill-posedness result for the NE.

THEOREM 2. *The Cauchy problem for the NE on the line and the circle* (1.1)–(1.2) *is ill-posed in Sobolev spaces H^s for $s < 3/2$. More precisely, if $5/4 < s < 3/2$, then the data-to-solution map is not continuous, while if $s < 5/4$, then the solution is not unique. When $s = 5/4$, either continuity or uniqueness fails.*

As we have mentioned before, this theorem combined with the well-posedness result for the NE in H^s, $s > 3/2$, proved in [HH2], completes the well-posedness picture of the NE in Sobolev spaces, except for $s = 3/2$, which remains an open question. It is worth comparing the ill-posedness of the NE, which has cubic nonlinearities, with that of CH and DP, the two integrable equations of the same type but with quadratic nonlinearities, which are both well-posed in H^s, $s > 3/2$. Defining the *inflation index* to be the Sobolev exponent s_i such that there is norm-inflation (which implies discontinuity of the data-to-solution map) for all $s_i < s < 3/2$, we have the following observations. For CH the inflation index $s_i = 1$ and coincides with the index of the H^1-norm, which is the most important conserved quantity of CH. For $s < 1$ the peakon-antipeakon traveling wave solution (1.16) for CH converges in H^s to $u(T) = 0$ as t approaches the collision time T, giving rise to another solution (namely, the trivial solution) demonstrating ill-posedness due to failure of uniqueness. However, for DP the inflation index $s_i = 1/2$. When $s < 1/2$, the corresponding peakon-antipeakon traveling wave solution for DP converges in H^s to a function, which gives rise to another kind of DP solution, called *shock peakon*, that results in failure of uniqueness (see [HHG]). From our results above we see that the inflation index for the NE is $5/4$, which is a very interesting number and follows from the limiting behavior of the momenta $p_1(t)$ and $p_2(t)$ as t approaches the collision time (see Theorem 3). For $s < 5/4$ it is shown that the 2-peakon solution (2.1) constructed in section 2 converges in H^s to an antipeakon, which gives rise to an antipeakon traveling wave solution demonstrating failure of uniqueness (see Proposition 5).

Finally, we mention that the method used here for proving ill-posedness for the NE is similar to that used by many authors for other nonlinear evolution equations. For example, Bourgain and Pavlovic in [BP] proved ill-posedness for the 3D Navier–Stokes equations in Besov spaces in the sense of norm-inflation. Similar methods for establishing ill-posedness for dispersive equations have been used by Kenig, Ponce, and Vega [KPV] and Christ, Colliander, and Tao [CCT]. The ill-posedness for the generalized KdV and nonlinear Schrödinger equations in Sobolev spaces has been tackled in [BKPSV]. The Euler equation in Sobolev spaces is examined in [BL1], where a norm-inflation result for the related vorticity equation provides the foundation for its ill-posedness. For the ill-posedness of the Burgers equation in $H^{3/2}$ we refer the reader to Linares, Pilod, and Saut [LPS]. For more results on traveling wave solutions, well-posedness, and other analytic and geometric properties of nonlinear evolution equations we refer the reader to the following works and the references therein: [BC], [BL2], [CHT], [CL], [CM], [DGH], [EEP], [ELY], [EY], [H], [HH1], [HK], [HKM], [HMP], [KL], [KT], [LO], [Mc], [MST], [Ti], [W].

This paper is organized as follows. In section 2, we construct the 2-peakon solutions on the line having the properties described in Theorem 1. We begin with the system of the four differential equations defined by (1.9) when $n = 2$, and after making the change of the dependent variables $q = q_2 - q_1$, $p = p_2 - p_1$, $w = p_2 + p_1$, and $z = p_1 p_2$, we solve the resulting system and find explicit formulas for p, w, and z in terms of q (see Proposition 1). For $q = q(t)$ we obtain a rather complicated autonomous differential equation, which can be dominated by a simpler one for which we can prove, by a comparison argument, that q becomes zero (collision) in finite time.

Also, a precise estimate of the collision time is derived. This is contained in Proposition 2. In section 3, we estimate the H^s-norm of the 2-peakon solutions constructed earlier (see Proposition 3), and in section 4 we choose the parameter appropriately so that both the lifespan (collision time) and the size of the 2-peakon solution at the initial time are small. In section 5, we prove norm-inflation and ill-posedness for $5/4 < s < 3/2$. Then, in section 6, we prove nonuniqueness for $s < 5/4$ by showing that our 2-peakon solution $u(t)$ converges in H^s to an antipeakon $u(T)$, which gives rise to a second solution for the NE having the same initial data. Also, we explain the ill-posedness of the NE for $s = 5/4$. Finally, in section 7 we prove our results on the circle. We use analogous arguments to those used on the line, with the necessary modifications to account for the periodic environment. A detailed outline of the periodic case can be found in subsection 7.1.

2. Construction of 2-peakon solutions. It can be shown (see [HW], [GH]) that the 2-peakon

$$(2.1) \qquad u(x,t) = p_1(t)e^{-|x-q_1(t)|} + p_2(t)e^{-|x-q_2(t)|}$$

is a solution of the NE if the positions q_1, q_2 and the momenta p_1, p_2 satisfy the following system of the four differential equations:

$$(2.2) \quad \begin{aligned} q_1' &= \left(p_1 + p_2 e^{-|q_1-q_2|} \right)^2, \\ q_2' &= \left(p_1 e^{-|q_2-q_1|} + p_2 \right)^2, \\ p_1' &= p_1 p_2 \left(p_1 + p_2 e^{-|q_1-q_2|} \right) \cdot \operatorname{sgn}(q_1 - q_2) e^{-|q_1-q_2|}, \\ p_2' &= p_1 p_2 \left(p_1 e^{-|q_2-q_1|} + p_2 \right) \cdot \operatorname{sgn}(q_2 - q_1) e^{-|q_2-q_1|}, \end{aligned}$$

where $\operatorname{sgn}(x)$ is the standard sign function defined to be 1 if $x > 0$, -1 if $x < 0$, and 0 if $x = 0$. At this point we make our first observation. Since $q_1' \geq 0$ and $q_2' \geq 0$, both positions are increasing with time. Therefore we cannot have the "typical" peakon-antipeakon collision which is created from the peakon traveling in the positive direction and antipeakon traveling in the negative direction as observed in the cases of the CH and DP equations. Also, we note that by translation we may assume that the initial positions q_1 and q_2 are symmetric, that is,
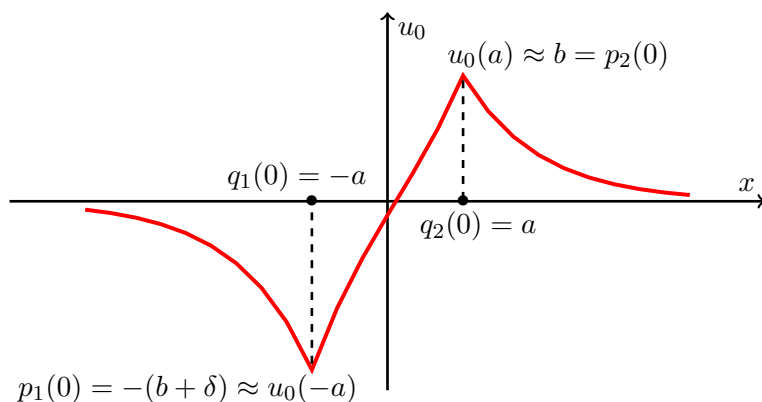
$$(2.3) \qquad q_1(0) = -a \quad \text{and} \quad q_2(0) = a \quad \text{for some } a > 0,$$

and, at least for a while, the difference between the positions is positive, that is,

$$(2.4) \qquad q(t) = q_2(t) - q_1(t) > 0.$$

Thus, the last system takes the simpler form

$$(2.5) \quad \begin{aligned} q_1' &= \left(p_1 + p_2 e^{-q} \right)^2, \\ q_2' &= \left(p_1 e^{-q} + p_2 \right)^2, \\ p_1' &= -p_1 p_2 \left(p_1 + p_2 e^{-q} \right) e^{-q}, \\ p_2' &= p_1 p_2 \left(p_1 e^{-q} + p_2 \right) e^{-q}. \end{aligned}$$

FIG. 1. *Initial profile $u_0(x)$.*

Furthermore, we shall assume that at time $t = 0$ the initial momenta are

$$(2.6) \qquad p_2(0) = b \gg 1, \qquad p_1(0) = -(b + \delta), \quad \delta > 0.$$

That is, the initial profile $u_0(x) = u(x, 0)$ is the asymmetric antipeakon-peakon

$$(2.7) \qquad u_0(x) = -(b + \delta)e^{-|x+a|} + be^{-|x-a|},$$

which is displayed in Figure 1.

Next, we shall solve the system of differential equations (2.5) with the antipeakon-peakon (2.7) as the initial data and prove that there is a collision in finite time. To demonstrate this claim, it is more convenient to work with the following new dependent variables:

$$(2.8) \quad \begin{aligned}
q(t) &= q_2(t) - q_1(t), & q(0) &= 2a > 0, \\
p(t) &= p_2(t) - p_1(t), & p(0) &= 2b + \delta > 0, \\
w(t) &= p_2(t) + p_1(t), & w(0) &= -\delta < 0, \\
z(t) &= p_2(t) \cdot p_1(t), & z(0) &= -b(b + \delta) < 0.
\end{aligned}$$

Deriving equations for q, p, w, and z. Subtracting the first equation of the system (2.5) from the second we have

$$\begin{aligned}
q' &= \left(p_1 e^{-q} + p_2 \right)^2 - \left(p_1 + p_2 e^{-q} \right)^2, \\
&= (p_2 - p_1)(p_2 + p_1)(1 - e^{-2q}), \\
(2.9) \qquad &= pw(1 - e^{-2q}).
\end{aligned}$$

Next, we shall try to form differential equations for p and w using the system (2.2). Assuming $p_1 < 0$ and $p_2 > 0$, at least for some time, for p we have

$$\begin{aligned}
p' &= p_1 p_2 \left(p_1 e^{-q} + p_2 \right) e^{-q} + p_1 p_2 \left(p_1 + p_2 e^{-q} \right) e^{-q}, \\
&= p_1 p_2 (p_2 + p_1) e^{-q} (1 + e^{-q}), \\
(2.10) \qquad &= zw e^{-q}(1 + e^{-q}).
\end{aligned}$$

For w we have

$$
\begin{aligned}
w' &= \Big(p_1 e^{-q} + p_2\Big)e^{-q} - p_1 p_2 \Big(p_1 + p_2 e^{-q}\Big)e^{-q}, \\
&= p_1 p_2 (p_2 - p_1)e^{-q}(1 - e^{-q}), \\
&= zpe^{-q}(1 - e^{-q}).
\end{aligned}
$$
(2.11)

Finally, for z we have

$$
\begin{aligned}
z' &= p_2' \cdot p_1 + p_2 \cdot p_1', \\
&= p_1^2 p_2 \Big(p_1 e^{-q} + p_2\Big)e^{-q} - p_1 p_2^2 \Big(p_1 + p_2 e^{-q}\Big)e^{-q}, \\
&= p_1 p_2 e^{-q}\Big[(p_1^2 - p_2^2)e^{-q}\Big], \\
&= -zwpe^{-2q}.
\end{aligned}
$$
(2.12)

To summarize, we have the following system for q, p, w, and z:

$$
\begin{aligned}
q' &= pw(1 - e^{-2q}), & q_0 = q(0) &= 2a > 0, \\
p' &= zwe^{-q}(1 + e^{-q}), & p_0 = p(0) &= 2b + \delta > 0, \\
w' &= zpe^{-q}(1 - e^{-q}), & w_0 = w(0) &= -\delta < 0, \\
z' &= -zwpe^{-2q}, & z_0 = z(0) &= -b(b + \delta) < 0.
\end{aligned}
$$
(2.13)

In the following result we derive explicit formulas for p, w, and z in terms of q. For q, we derive an autonomous differential equation, which in turn is dominated by a simpler such equation.

PROPOSITION 1 (solutions of transformed 2-peakon system). *The system of differential equations* (2.13) *has a unique smooth solution* $(q(t), p(t), w(t), z(t))$ *in an interval* $[0, T)$, *for some* $T > 0$, *such that* $z = z(t)$ *is decreasing and, in terms of* q, *is expressed by the formula*

$$
(2.14) \qquad z = \frac{-z_1}{\left(1 - e^{-2q}\right)^{1/2}} < 0, \quad where \quad z_1 = b(b + \delta)\left(1 - e^{-2q_0}\right)^{1/2},
$$

$p = p(t)$ *is decreasing and as a function of* q *is expressed by the formula*

$$
(2.15) \qquad p = \left(p_0^2 + 2z_1\left[\frac{1 + e^{-q}}{\sqrt{1 - e^{-2q}}} - \frac{1 + e^{-q_0}}{\sqrt{1 - e^{-2q_0}}}\right]\right)^{1/2} > 0,
$$

and $w = w(t)$ *is decreasing and as a function of* q *is expressed by the formula*

$$
(2.16) \qquad w(t) = -\left(w_0^2 + 2z_1\left[\frac{\sqrt{1 - e^{-2q_0}}}{1 + e^{-q_0}} - \frac{\sqrt{1 - e^{-2q}}}{1 + e^{-q}}\right]\right)^{1/2} < 0.
$$

The difference of the positions $q = q(t)$ *is decreasing and satisfies the initial value problem*

$$
\begin{aligned}
q' = -f(q) &\doteq -\left(w_0^2 + 2z_1\left[\frac{\sqrt{1 - e^{-2q_0}}}{1 + e^{-q_0}} - \frac{\sqrt{1 - e^{-2q}}}{1 + e^{-q}}\right]\right)^{\frac{1}{2}} \cdot \\
&\quad \cdot \left(p_0^2 + 2z_1\left[\frac{1 + e^{-q}}{\sqrt{1 - e^{-2q}}} - \frac{1 + e^{-q_0}}{\sqrt{1 - e^{-2q_0}}}\right]\right)^{\frac{1}{2}} \cdot (1 - e^{-2q}),
\end{aligned}
$$
(2.17)

$$
q(0) = q_0 = 2a > 0.
$$

Furthermore, the initial value problem (2.17) *for q is dominated by the simpler initial value problem*

$$(2.18) \qquad q' = -g(q) \doteq -q_1 \big(1 - e^{-2q} \big)^{3/4}, \quad 0 < q(0) = 2a < 1/2,$$

where

$$(2.19) \qquad q_1 = \delta \sqrt{2b(b+\delta)} \cdot q_0^{1/4}.$$

Proof. We begin by expressing z in terms of q. Using the equation for z' and q', we find

$$\frac{z'}{q'} = \frac{-zpwe^{-2q}}{pw(1 - e^{-2q})} \quad \text{or} \quad \frac{z'}{z} = \frac{-e^{-2q}q'}{(1 - e^{-2q})}.$$

Since $z(0) < 0$, we shall assume that $z(t)$ will remain negative. Therefore, from the last relation we have

$$\frac{d}{dt}[\ln(-z)] = -\frac{1}{2}\frac{d}{dt}[\ln(1 - e^{-2q})].$$

Integrating from 0 to t gives

$$\ln\Big[\frac{z(t)}{z_0}\Big] = -\frac{1}{2}\ln\Big[\frac{1 - e^{-2q}}{1 - e^{-2q_0}}\Big].$$

Finally, solving for z gives formula (2.14), which expresses z in terms of q.

Next we express p in terms of q. For this we divide the equation for p' by the equation for q' and get

$$\frac{p'}{q'} = \frac{zwe^{-q}(1 + e^{-q})}{pw(1 - e^{-2q})} \quad \text{or} \quad pp' = z \cdot \frac{e^{-q}(1 + e^{-q})q'}{(1 - e^{-2q})}.$$

Substituting into the above relation the formula for z given by (2.14), we have

$$(2.20) \qquad pp' = \frac{-z_1}{\big(1 - e^{-2q}\big)^{1/2}} \cdot \frac{e^{-q}(1 + e^{-q})q'}{(1 - e^{-2q})} = \frac{-z_1(1 + e^{-q})e^{-q}q'}{\big(1 - e^{-2q}\big)^{3/2}}.$$

Furthermore, by making the change of variables, $u = e^{-q(t)}$, we have $du = -e^{-q(t)}q'(t)dt$ and

$$\int \frac{-(1 + e^{-q})e^{-q}q'}{\big(1 - e^{-2q}\big)^{3/2}}dt = \int \frac{1 + u}{\big(1 - u^2\big)^{3/2}}du = \frac{1 + u}{\big(1 - u^2\big)^{1/2}} + C = \frac{1 + e^{-q(t)}}{\big(1 - e^{-2q(t)}\big)^{1/2}} + C.$$

Therefore, relation (2.20) reads as

$$(2.21) \qquad \frac{d}{dt}\Big[\frac{1}{2}p^2\Big] = z_1\frac{d}{dt}\Big[\frac{1 + e^{-q}}{\sqrt{1 - e^{-2q}}}\Big].$$

Integrating (2.21) from 0 to t gives

$$\frac{1}{2}\Big[p^2(t) - p_0^2\Big] = z_1\Big[\frac{1 + e^{-q(t)}}{\sqrt{1 - e^{-2q(t)}}} - \frac{1 + e^{-q_0}}{\sqrt{1 - e^{-2q_0}}}\Big],$$

which, when solved for p, gives formula (2.15), which expresses p in terms of q.

Finally, we express w in terms of q. Dividing the equation for w' by the equation for q' gives

$$\frac{w'}{q'} = \frac{zpe^{-q}(1 - e^{-q})}{pw(1 - e^{-2q})} \quad \text{or} \quad ww' = z \cdot \frac{e^{-q}(1 - e^{-q})q'}{(1 - e^{-2q})}.$$

Now, substituting the formula for z given by (2.14) into the above relation, we get

$$(2.22) \qquad ww' = \frac{-z_1}{\left(1 - e^{-2q}\right)^{1/2}} \cdot \frac{e^{-q}(1 - e^{-q})q'}{(1 - e^{-2q})} = \frac{-z_1(1 - e^{-q})e^{-q}q'}{\left(1 - e^{-2q}\right)^{3/2}}.$$

Furthermore, making again the change of variables $u = e^{-q(t)}$, we have

$$\int \frac{-(1 - e^{-q})e^{-q}q'}{\left(1 - e^{-2q}\right)^{3/2}} dt = \int \frac{1 - u}{\left(1 - u^2\right)^{3/2}} du = -\frac{\sqrt{1 - u^2}}{1 + u} + C = -\frac{\sqrt{1 - e^{-2q(t)}}}{1 + e^{-q(t)}} + C.$$

Therefore, relation (2.22) reads as

$$(2.23) \qquad \frac{d}{dt}\left[\frac{1}{2}w^2\right] = -z_1 \frac{d}{dt}\left[\frac{\sqrt{1 - e^{-2q(t)}}}{1 + e^{-q(t)}}\right].$$

Integrating (2.23) from 0 to t gives

$$\frac{1}{2}\left[w^2(t) - w_0^2\right] = z_1\left[\frac{\sqrt{1 - e^{-2q_0}}}{1 + e^{-q_0}} - \frac{\sqrt{1 - e^{-2q(t)}}}{1 + e^{-q(t)}}\right].$$

Solving for w while taking into consideration that $w(t) < 0$ in the choice of sign gives formula (2.16), which expresses w in terms of q.

Concerning the differential equation for q, we begin from its equation $q' = wp(1 - e^{-2q})$, and by substituting for w and p their expressions (2.16) and (2.15), we obtain the desired autonomous initial value problem (2.16). Next, we observe that

$$(2.24) \qquad \frac{\sqrt{1 - e^{-2q_0}}}{1 + e^{-q_0}} - \frac{\sqrt{1 - e^{-2q}}}{1 + e^{-q}} \geq 0, \quad 0 \leq q \leq q_0,$$

and also that

$$(2.25) \qquad p_0^2 - 2z_1 \frac{1 + e^{-q_0}}{\sqrt{1 - e^{-2q_0}}} \geq 0 \iff \frac{(2b + \delta)^2}{2b(b + \delta)} \geq 1 + e^{-q_0}.$$

In fact, (2.25) is implied by the stronger condition

$$\frac{(2b + \delta)^2}{2b(b + \delta)} \geq 2 \iff 4b^2 + 4b\delta + \delta^2 > 4b^2 + 4b\delta \iff \delta^2 > 0,$$

which is true. Now, using (2.24) and (2.25) we see that the function $f(q)$ in the right-hand side of the differential equation (2.17) can be bounded from below by

$$f(q) \geq \left(w_0^2\right)^{\frac{1}{2}} \cdot \left(2z_1\left[\frac{1 + e^{-q}}{\sqrt{1 - e^{-2q}}}\right]\right)^{\frac{1}{2}} \cdot (1 - e^{-2q})$$

$$= \delta \cdot \left(2b(b + \delta)\left(1 - e^{-2q_0}\right)^{1/2}\right)^{\frac{1}{2}}\left(\left[\frac{1 + e^{-q}}{\sqrt{1 - e^{-2q}}}\right]\right)^{\frac{1}{2}} \cdot (1 - e^{-2q}).$$

Using the bounds $1 + e^{-q} > 1$ and $1 - e^{-2q_0} \geq q_0$, for $0 \leq q_0 \leq 1/2$, which follow from the simple but useful approximation

$$\frac{x}{2} \leq 1 - e^{-x} \leq x \iff 1 - e^{-x} \simeq x \quad \text{if} \quad 0 \leq x \leq 1,$$

we have

$$f(q) \geq \delta \sqrt{2b(b+\delta)} \cdot q_0^{1/4} \cdot \left(1 - e^{-2q}\right)^{3/4} \doteq g(q).$$

Therefore, defining $q_1 \doteq \delta \sqrt{2b(b+\delta)} \cdot q_0^{1/4}$ we see that the complicated initial value problem for q given in (2.17) is dominated by the simpler one shown in (2.18). $\qquad \square$

Next we move our attention to the study of the solution $q(t)$ of the initial value problem stated in Proposition 1. From the formulas for p and w, we see that they blow up at a zero of q. Therefore, the lifespan of our 2-peakon solution is equal to the first such zero. The following result, which is applicable to the simpler dominant initial value problem (2.18), proves existence of a zero and provides an estimate for its size in terms of the initial data.

PROPOSITION 2 (zero of q). *If $r < 1$, then for given $q_0 \in (0, 1/2)$ and $q_1 > 0$ the solution to the initial value problem*

(2.26)
$$\frac{dq}{dt} = -g_r(q) \doteq -q_1 \left(1 - e^{-2q}\right)^r, \quad q(0) = q_0,$$

which begins positive and is decreasing, becomes zero in finite time T given by

(2.27)
$$T = \int_0^{q_0} \frac{dq}{g_r(q)} = \frac{1}{q_1} \int_0^{q_0} \frac{dq}{(1 - e^{-2q})^r} \simeq \frac{1}{1-r} \frac{q_0^{1-r}}{q_1}.$$

A key ingredient in proving Proposition 2 is the following elementary result that compares solutions of the initial value problem (2.26) for different values of r. It states that a bigger r corresponds to a bigger solution.

LEMMA 1 (comparison principle). *If r_1 and r_2 are two values of r such that $r_1 \leq r_2$, then the corresponding solutions $q_{r_1}(t)$ and $q_{r_2}(t)$ to the initial value problem (2.26) with the same initial data q_0 satisfy $q_{r_1}(t) \leq q_{r_2}(t)$. That is,*

$$r_1 \leq r_2 \quad \Longrightarrow \quad q_{r_1}(t) \leq q_{r_2}(t).$$

Proof. It follows from the fact that $r_1 \leq r_2$ implies

$$-q_1 \left(1 - e^{-2q}\right)^{r_1} \leq -q_1 \left(1 - e^{-2q}\right)^{r_2}. \qquad \square$$

Remark. We note that for $r \geq 1$ the solution to the initial value problem (2.26) has no zero. In fact, for $r = 1$ it reads as

$$\frac{dq}{dt} = -q_1 \left(1 - e^{-2q}\right), \quad q(0) = q_0.$$

Integrating this equation gives the explicit formula

$$q(t) = \frac{1}{2} \ln \left[1 + (e^{2q_0} - 1)e^{-q_1 t}\right] \doteq q_1(t).$$

From this formula we see that the solution $q(t)$ exists for all $t \geq 0$, is positive for all times, and decreases to zero as t goes to ∞. Thus when $r = 1$, $q_1(t)$ has no zero in finite time. Since by the comparison principle the solution $q_r(t)$ that corresponds to an $r > 1$ is greater than $q_1(t)$, we conclude that $q_r(t)$ has no zero in finite time if $r > 1$. Therefore, the lifespan T is equal to ∞ if $r \geq 1$.

Proof of Proposition 2. We begin with the case $r \leq 0$. When $r = 0$, our initial value problem (2.17) becomes the simple one $q'(t) = -q_1$, $q(0) = q_0$, whose solution is

$$q(t) = q_0 - q_1 t \doteq q_0(t),$$

which has a zero at $T = q_0/q_1$. Thus, by the comparison lemma, Lemma 1, the solution $q_r(t)$ that corresponds to an $r < 0$ is smaller than $q_0(t)$ and therefore has a zero in finite time. In fact, it is smaller than q_0/q_1. This proves existence of a zero for $q_r(t)$ when $r \leq 0$.

Existence of a zero for $q(t)$ if $0 < r < 1$. To prove existence of a zero of $q_r(t)$ for $0 < r < 1$, it suffices to do so under the additional condition

$$(2.28) \qquad r \neq \frac{n-1}{n} \quad \text{for all} \quad n = 1, 2, 3, \ldots.$$

In fact, if r were of the form $\frac{n-1}{n}$, then we would choose another $r_2 \in (0,1)$ which is not of this form and we have $r < r_2$. Then, by the comparison lemma, Lemma 1, proving the existence of a zero for $q_{r_2}(t)$ implies existence of a zero for $q_r(t)$. So, from now on we shall assume that r satisfies condition (2.28). Therefore, there is a positive integer $n \geq 2$ such that

$$(2.29) \qquad \frac{n-2}{n-1} < r < \frac{n-1}{n}.$$

It turns out that for proving existence of a zero for $q = q_r(t)$, we need its nth order Taylor polynomial approximation at $t = 0$. Differentiating (2.17) n times, we arrive at the formula

$$(2.30) \quad q^{(n)}(t) = q_1^n c_n(r) \left(1 - e^{-2q(t)}\right)^{nr-(n-1)} + q_1^n \sum_{j=1}^{n-1} c_j(r) \left(1 - e^{-2q(t)}\right)^{nr-(j-1)},$$

where

$$(2.31) \qquad c_n(r) = (-1)^n 2^{n-1} r(2r-1) \cdots \big([n-1]r - [n-2]\big)$$

and $c_j(r)$ for $j = 1, \ldots, n-1$ are coefficients depending on r. Also, we obtain the following formula for the $(n+1)$th derivative of q:

$$(2.32)$$

$$q^{(n+1)}(t) = q_1^{n+1} c_{n+1}(r) \left(1 - e^{-2q(t)}\right)^{(n+1)r-n} + q_1^{n+1} \sum_{j=1}^{n} c_j(r) \left(1 - e^{-2q(t)}\right)^{(n+1)r-(j-1)},$$

where

$$(2.33) \qquad c_{n+1}(r) = (-1)^{n+1} 2^n r(2r-1) \cdots \big(nr - [n-1]\big),$$

and again $c_j(r)$ for $j = 1, \ldots, n$ are coefficients depending on r. Therefore, the nth order Taylor polynomial approximation of $q(t)$ at $t = 0$ is given by

$$(2.34) \quad q(t) = q_0 + q'(0)t + \frac{q''(0)}{2!}t^2 + \frac{q^{(3)}(0)}{3!}t^3 + \cdots + \frac{q^{(n)}(0)}{n!}t^n + \frac{q^{(n+1)}(\tau)}{(n+1)!}t^{n+1},$$

where $0 \leq \tau \leq t$. Next, we shall show that the coefficients $c_n(r)$ and $c_{n+1}(r)$ defined by (2.31) and (2.33) have the same sign, which is the key ingredient for proving the existence of a zero for $q(t)$. We prove this claim by considering the two possible cases, n even and n odd. We begin with the case of n even. In this case, using the first part of inequality (2.29) that r satisfies, we see that $(n-1)r > n-2$, and this implies that $c_n(r)$ is a positive number. Also, using the second part of inequality (2.29) we see that $nr < n-1$, which implies that $c_{n+1}(r) > 0$ is a positive number too. Furthermore, in the expression of $q^{(n)}(0)$ the first term $q_1^n c_n(r) \left(1 - e^{-2q_0}\right)^{nr-(n-1)}$ is the dominant term for q_0 small enough since the exponent $nr - (n-1)$ is negative, while the exponents of $(1 - e^{-2q_0})$ appearing in all other terms of the sum (2.30) are positive. Thus we can conclude

$$(2.35) \qquad q^{(n)}(0) > 0 \quad \text{if } n \text{ is even and } q_0 \text{ is small enough.}$$

Similarly, in the expression of $q^{(n+1)}(t)$, the first term $q_1^{n+1} c_{n+1}(r) \left(1 - e^{-2q(t)}\right)^{(n+1)r-n}$ is the dominant term for q_0 small enough, since the exponent $(n+1)r - n$ is negative while all the exponents of $(1 - e^{-2q(t)})$ appearing in the sum (2.32) are positive, except the one that corresponds to $j = n$ which has exponent $(n+1)r - (n-1)$ which is negative. However, $\left(1 - e^{-2q(t)}\right)^{(n+1)r-(n-1)}$ is dominated by $\left(1 - e^{-2q(t)}\right)^{(n+1)r-n}$ for $q(t) \leq q_0$ small enough. Thus, we also have

$$(2.36) \qquad q^{(n+1)}(\tau) > 0 \quad \text{if } n \text{ is even and } q_0 \text{ is small enough.}$$

In the case that $n \geq 2$ is an odd positive integer, the signs change due to $(-1)^n = -1$, and using the same reasoning as in the even case we obtain that

$$(2.37) \qquad q^{(n)}(0) < 0 \quad \text{if } n \text{ is odd and } q_0 \text{ is small enough}$$

and

$$(2.38) \qquad q^{(n+1)}(\tau) < 0 \quad \text{if } n \text{ is odd and } q_0 \text{ is small enough.}$$

Now we are ready to prove the existence of a zero for $q(t)$. First we consider the case that n is an odd number. Then, using the nth order Taylor polynomial approximation (2.34) and the conditions (2.37), (2.38) we obtain that

$$q(t) \leq q_0 + q'(0)t + \frac{q''(0)}{2!}t^2 + \frac{q^{(3)}(0)}{3!}t^3 + \cdots + \frac{q^{(n)}(0)}{n!}t^n \quad \text{for all} \quad t \geq 0.$$

Furthermore, since for large t the term $\frac{q^{(n)}(0)}{n!}t^n$ dominates and $q^{(n)}(0) < 0$, we have that the nth order Taylor polynomial approximation of $q(t)$ becomes negative, thus crossing the t-axis. This forces $q(t)$ to have a zero at some positive time T, which is the desired conclusion.

Finally, we prove the existence of a zero for $q(t)$ in the even case. This is done by contradiction. In fact, if $q(t) > 0$ for all $t > 0$, then our differential equation $q'(t) = -q_1 \left(1 - e^{-2q(t)}\right)^r$ implies that $q(t)$ is decreasing for all $t > 0$ and therefore

$$(2.39) \qquad\qquad q(t) \leq q_0 \quad \text{for all} \quad t \geq 0.$$

However, if n is even, then using the nth order Taylor polynomial approximation of $q(t)$ at $t = 0$, which is given by (2.34), and conditions (2.35) and (2.36), we have that

$$(2.40) \quad q(t) \geq q_0 + q'(0)t + \frac{q''(0)}{2!}t^2 + \frac{q^{(3)}(0)}{3!}t^3 + \cdots + \frac{q^{(n)}(0)}{n!}t^n \quad \text{for all} \quad t \geq 0.$$

Inequality (2.40) leads to a contradiction because for large t the term $\frac{q^{(n)}(0)}{n!}t^n$ dominates the Taylor polynomial approximation. Thus, there is some large time $T > 0$ such that

$$(2.41) \qquad q(t) \geq q_0 + \frac{1}{2}\frac{q^{(n)}(0)}{n!}T^n > 2q_0, \quad 0 \leq t \leq T,$$

which contradicts inequality (2.39). This argument completes the proof of the existence of a zero for $q(t)$ when $0 < r < 1$.

Estimating the zero T of the position $q(t)$ when $r < 1$. Let T be the zero of the solution $q(t)$ of our initial value problem (2.17), which is $q'(t) = -q_1\left(1 - e^{-2q(t)}\right)^r$, $q(0) = q_0$. Integrating it from 0 to T we have

$$\int_0^T \frac{q'(t)}{\left(1 - e^{-2q(t)}\right)^r}\,dt = -q_1 T.$$

Then, making the substitution $q = q(t)$ and using the initial and terminal conditions $q(0) = q_0$ and $q(T) = 0$, we obtain the following formula for T:

$$(2.42) \qquad T = \frac{1}{q_1}\int_0^{q_0} \frac{dq}{(1 - e^{-2q})^r}T \simeq \frac{1}{q_1}\int_0^{q_0} \frac{dq}{q^r} = \frac{1}{1-r}\frac{q_0^{1-r}}{q_1}.$$

Above, we used the estimate $q \leq 1 - e^{-2q} \leq 2q$ if $0 \leq q \leq \frac{1}{2}$. This completes the proof of Proposition 2. □

Applying Proposition 2 with $r = 3/4$, we obtain the following result for the zero of the solution to the initial value problem (2.17) and the lifespan of our 2-peakon solution u.

COROLLARY 1 (zero of q and lifespan of u). *If $0 < q_0 < 1/2$ and b, δ satisfy condition (2.6), then the solution to the initial value problem (2.17) begins positive, is decreasing, and becomes zero in finite time T given by*

(2.43)

$$T = \int_0^{q_0} \frac{dq}{f(q)} \leq \frac{1}{q_1}\int_0^{q_0} \frac{dq}{(1 - e^{-2q})^{3/4}} \simeq \frac{q_0^{1/4}}{q_1} \simeq \frac{q_0^{1/4}}{\delta\sqrt{2b(b+\delta)}\cdot q_0^{1/4}} \simeq \frac{1}{\delta\sqrt{2b(b+\delta)}}.$$

Proof. The existence and uniqueness of the solution follows from the fundamental ODE theorem since $f(q)$ is a smooth function. That $q(t)$ is decreasing follows from the fact that $q' = -f(q) < 0$. Finally, that $q(t)$ becomes zero in finite time follows from the fact that our initial value problem (2.17) is dominated by the initial value problem (2.18) for which Proposition 2 is applicable with $r = 3/4$. Therefore, estimate (2.27) gives (2.43), and this completes the proof of the lemma. □

The properties of our special 2-peakon solutions are summarized in the following theorem and are a consequence of Proposition 1 and Corollary 1.

THEOREM 3 (construction of 2-peakon solutions). *For given $0 < a \leq 1/4$ and b, δ satisfying condition (2.6), the initial value problem for the positions q_1, q_2 and the*

momenta p_1, p_2,

$$(2.44) \quad \begin{aligned} q_1' &= \left(p_1 + p_2 e^{-q}\right)^2, & q_1(0) &= -a, \\ q_2' &= \left(p_1 e^{-q} + p_2\right)^2, & q_2(0) &= a > 0, \\ p_1' &= -p_1 p_2 \left(p_1 + p_2 e^{-q}\right) e^{-q}, & p_1(0) &= -b - \delta, \\ p_2' &= p_1 p_2 \left(p_1 e^{-q} + p_2\right) e^{-q}, & p_2(0) &= b, \end{aligned}$$

has a unique smooth solution $(q_1, q_2, p_1, p_2)(t)$ with a finite lifespan T, which is the zero of $q = q_2 - q_1$, and which satisfies the estimate

$$(2.45) \qquad\qquad T \lesssim \frac{1}{\delta\sqrt{2b(b+\delta)}}.$$

Furthermore, we have

$$p_1 = \frac{w - p}{2} < 0 \ decreasing,$$
$$\lim_{t \to T^-} p_1(t) = -\infty \ and \ -p_1 \simeq p \simeq q^{-1/4},$$

and

$$p_2 = \frac{w + p}{2} > 0 \ increasing,$$
$$\lim_{t \to T^-} p_2(t) = \infty \ and \ p_2 \simeq p \simeq q^{-1/4},$$

where p and w are given in Proposition 1. Also, $w = p_1 + p_2$ is decreasing from $w_0 < 0$ to w_T, where $w_T \doteq \lim_{t \to T^-} w(t)$, that is,

$$(2.46) \qquad\qquad w_T = -\left(\delta^2 + 2b(b+\delta)(1 - e^{-2a})\right)^{\frac{1}{2}}.$$

Finally, the 2-peakon

$$u(x, t) = p_1(t)e^{-|x - q_1(t)|} + p_2(t)e^{-|x - q_2(t)|}$$

is the NE solution for $x \in \mathbb{R}$, $0 < t < T$, with the following asymmetric antipeakon-peakon initial profile:
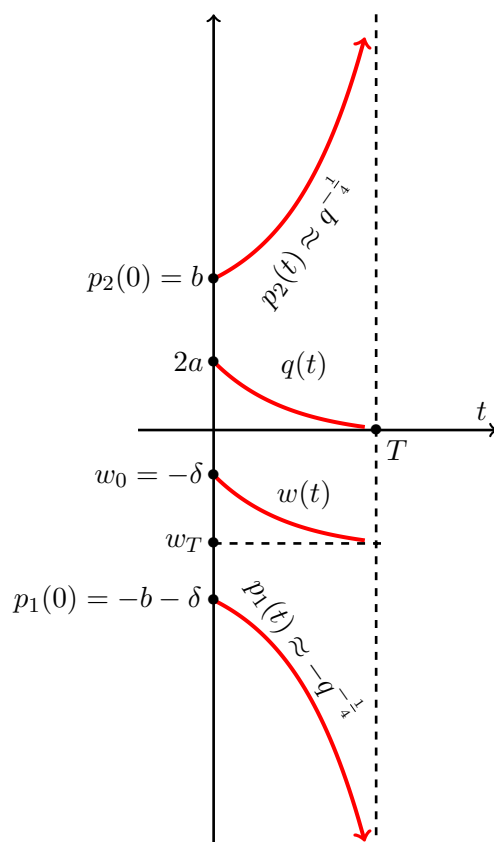
$$u(x, 0) = -(b + \delta)e^{-|x + a|} + be^{-|x - a|}, \ x \in \mathbb{R}.$$

For the visualization of p_1, p_2, q, and w we refer the reader to Figure 2.

3. Calculating the norm.

PROPOSITION 3. *Let $u(t)$ be the two-peakon solution to the NE. Then on $[0, T)$ we have*

$$(3.1) \quad \|u(t)\|_{H^s}^2 = 16r(t)p_1^2(t)Q_s(q) + 4c_s\left(1 - r(t)\right)^2 p_1^2(t), \quad with \quad r(t) \doteq -\frac{p_2(t)}{p_1(t)},$$

FIG. 2. *Graphs of p_1, p_2, q, and w.*

where $c_s = \int_{\mathbb{R}}(1+\xi^2)^{s-2}d\xi$ and $Q_s(q)$, which is given below, satisfies the estimates

(3.2) $\qquad Q_s(q) \doteq \int_{\mathbb{R}}(1+\xi^2)^{s-2}\sin^2\left(\frac{q\xi}{2}\right)d\xi \simeq \begin{cases} q^{3-2s}, & 1/2 < s < 3/2, \\ q^2 \cdot \ln(1/q), & s = 1/2, \\ q^2, & s < 1/2. \end{cases}$

Proof. Since $\widehat{e^{-|x|}}(\xi) = 2/(1+\xi^2)$, we have that the Fourier transform of

$$u(x,t) = p_1 e^{-|x-q_1|} + p_2 e^{-|x-q_2|}$$

is given by

$$\widehat{u}(\xi,t) = \frac{2p_1 e^{-i\xi q_1}}{1+\xi^2} + \frac{2p_2 e^{-i\xi q_2}}{1+\xi^2} = \frac{2}{1+\xi^2} \cdot p_1 e^{-i\xi q_1} \cdot \left(1 + \frac{p_2}{p_1}e^{-i\xi q}\right).$$

Taking the square of the H^s-norm of this quantity and factoring out p_1^2, we obtain

(3.3) $\qquad \|u(t)\|_{H^s}^2 = 4p_1^2 \int_{\mathbb{R}}(1+\xi^2)^{s-2}\left|1 + \frac{p_2}{p_1}e^{-i\xi q}\right|^2 d\xi.$

Using Proposition 1 we see that

$$r = r(t) \doteq -\frac{p_2(t)}{p_1(t)} = \frac{p+w}{p-w} < 1 \quad \text{and} \quad r(t) \nearrow 1 \text{ as } t \nearrow T.$$

Next, using r we write (3.3) as

$$(3.4) \qquad \|u(t)\|_{H^s}^2 = 4p_1^2 \int_{\mathbb{R}} (1+\xi^2)^{s-2} \left|1 - re^{-i\xi q}\right|^2 d\xi.$$

Expanding out the square under the integral in (3.4), we have

$$(3.5) \qquad |re^{iq\xi} - 1|^2 = (1-r)^2 + 4r\sin^2\left(\frac{q\xi}{2}\right).$$

Substituting (3.5) into (3.4), we have

$$\|u(t)\|_{H^s}^2 = 16rp_1^2 \int_{\mathbb{R}} (1+\xi^2)^{s-2} \sin^2\left(\frac{q\xi}{2}\right) d\xi + 4(1-r)^2 p_1^2 \int_{\mathbb{R}} (1+\xi^2)^{s-2} d\xi,$$

which gives

$$\|u(t)\|_{H^s}^2 = 16rp_1^2 Q_s(q) + 4c_s(1-r)^2 p_1^2,$$

where

$$(3.6) \qquad c_s = \int_{\mathbb{R}} (1+\xi^2)^{s-2} d\xi \quad \text{and} \quad Q_s(q) = \int_{\mathbb{R}} (1+\xi^2)^{s-2} \sin^2\left(\frac{q\xi}{2}\right) d\xi.$$

Now, we see that to prove Proposition 3 it suffices to show that for $0 < q < 1/8$ we have the estimate

$$Q_s(q) \simeq \begin{cases} q^{3-2s}, & 1/2 < s < 3/2, \\ q^2 \cdot \ln(1/q), & s = 1/2, \\ q^2, & s < 1/2. \end{cases}$$

Starting with the integrand for Q_s from (3.6) and making the change of variables $x = q\xi$, which gives $dx = q d\xi$, we can write $Q_s(q)$ as

$$(3.7) \qquad Q_s(q) = 2q^{3-2s} \int_0^\infty (q^2 + x^2)^{s-2} \sin^2(x/2) dx = 2q^{3-2s}[I_1 + I_2],$$

where

$$I_1 \doteq \int_0^1 \frac{x^2}{(q^2+x^2)^{2-s}} dx \quad \text{and} \quad I_2 \doteq \int_1^\infty \frac{\sin^2(x/2)}{(q^2+x^2)^{2-s}} dx.$$

If $s < 3/2$, then the integral I_2 is bounded since

$$(3.8) \qquad I_2 = \int_1^\infty \frac{\sin^2(x/2)}{(q^2+x^2)^{2-s}} dx \lesssim \int_1^\infty x^{2s-4} dx = \frac{1}{3-2s}.$$

Also, when $s > 1/2$ we have the following upper bound for I_1:

$$(3.9) \qquad I_1 \leq \int_0^1 x^{2s-2} dx = \frac{1}{2s-1}.$$

Furthermore, for any $s < 2$ we have

$$(3.10) \qquad I_1 \geq 2^{s-2} \int_0^1 x^2 dx = 2^{s-2} \cdot \frac{1}{3}.$$

Combining (3.10), (3.8), and (3.7) gives

$$\|f_q\|_{H^s} \simeq q^{\frac{3}{2}-s} \quad \text{if} \quad 1/2 < s < 3/2.$$

The case $s < 1/2$. Since $I_1 + I_2$ is bounded below by I_1 and $0 < y < 1$ we have

$$(3.11) \qquad I_1 + I_2 \gtrsim \int_0^q \frac{x^2}{(q^2 + x^2)^{2-s}} dx \gtrsim q^{2s-4} \int_0^q x^2 dx \simeq q^{2s-1}.$$

Combining (3.11) and (3.7) gives

$$(3.12) \qquad Q_s(q) \gtrsim q^{3-2s} \cdot q^{2s-1} = q^2 \quad \text{if} \quad s < 1/2.$$

To prove the reverse of inequality (3.12) we obtain an upper bound for I_1. For this argument, we let $z = x/y$ and get

$$I_1 \doteq \int_0^1 \frac{x^2}{(q^2 + x^2)^{2-s}} dx = q^{2s-1} \int_0^{1/q} \frac{z^2}{(1 + z^2)^{2-s}} dz \leq q^{2s-1} \int_0^\infty \frac{1}{(1 + z^2)^{1-s}} dz.$$

Since the last integral converges if $2(1 - s) > 1$, which is equivalent to $s < 1/2$, we see that it is equal to a finite constant c_s. Combining this fact with (3.7) and (3.8) we have

$$Q_s(q) \lesssim q^{3-2s}[q^{2s-1} + 1] \lesssim q^2 \quad \text{if} \quad s < 1/2,$$

which, together with (3.11), gives

$$Q_s(q) \simeq y^2 \quad \text{if} \quad s < 1/2.$$

The case $s = 1/2$. We observe that

$$I_1 = \int_0^1 \frac{x^2}{(q^2 + x^2)^{3/2}} dx = \ln\left(\sqrt{q^2 + 1} + 1\right) - \frac{1}{\sqrt{q^2 + 1}} + \ln(1/q).$$

Upper bound. From here we begin by removing the middle term and using the fact that $y < 1/4$ in the first term. We get

$$(3.13) \qquad I_1 \leq \ln(\sqrt{2} + 1) + \ln(1/q).$$

Substituting (3.13) back into (3.7) and taking into account estimate (3.8) for I_2 we have

$$(3.14) \qquad Q_{\frac{1}{2}}(s) \lesssim q^2 \ln(1/q).$$

Lower bound. Using the fact that $2\ln(1/q) > 1$ we have

$$I_1 \geq \ln(1/q) + (\ln(2) - 1)[2\ln(1/q)] = (2\ln(2) - 1)\ln(1/q).$$

We therefore arrive at

$$(3.15) \qquad Q_{\frac{1}{2}}(s) \gtrsim q^2 \cdot I_1 \gtrsim q^2 \ln(1/q).$$

Putting together the upper and lower bounds (3.14) and (3.15) and taking the square root of both sides of the equation gives the desired result of

$$Q_{\frac{1}{2}}(s) \simeq y^2 \cdot \ln(1/q). \qquad\qquad \square$$

4. Small lifespan and initial data. We begin by assuming that

$$p_2(0) = b \gg 1 \text{ and } -p_1(0) = b + \delta, \, \delta > 0,$$

so that the conditions for the existence of our 2-peakon with the lifespan estimate (2.43) hold. Then, we have the following.

Lifespan estimate. For given $\varepsilon > 0$, we need to find $b > 1$ such that $T < \varepsilon$. Since by Proposition 1 we have

$$T \lesssim \frac{1}{\delta\sqrt{2b(b+\delta)}} \leq \frac{1}{\delta b},$$

we must have

(4.1) $$\frac{1}{\delta b} \leq \varepsilon \iff b \geq \delta^{-1}\varepsilon^{-1}.$$

Initial data estimate. Now, for the same $\varepsilon > 0$ we need to find $q_0 < 1/8$ such that $\|u_0\|_{H^s} < \varepsilon$. For this argument we use Proposition 3, from which we have

$$\|u(0)\|_{H^s}^2 = 16r(0)p_1^2(0)Q_s(q_0) + 4c_s\big(1 - r(0)\big)^2 p_1^2(0), \quad r(t) \doteq -\frac{p_2(t)}{p_1(t)},$$

$$= 16b(b+\delta)Q_s(q_0) + 4c_s\delta^2,$$

which in turn gives

$$\|u(0)\|_{H^s}^2 \leq 32b^2 Q_s(q_0) + 4c_s\delta^2.$$

Case $1/2 < s < 3/2$. Then by Proposition 3 we have $Q_s(q_0) \lesssim q_0^{3-2s}$ and therefore

$$\|u(0)\|_{H^s}^2 \leq C_s b^2 q_0^{3-2s} + 4c_s\delta^2.$$

To demonstrate $\|u_0\|_{H^s} < \varepsilon$, it suffices to choose q_0 and δ such that $C_s b^2 q_0^{3-2s} + 4c_s\delta^2 \leq \varepsilon^2$ or

$$4c_s\delta^2 \leq \frac{\varepsilon^2}{2} \quad \text{and} \quad C_s b^2 q_0^{3-2s} \leq \frac{\varepsilon^2}{2}.$$

The first inequality holds if

(4.2) $$\delta \leq \frac{\varepsilon}{2\sqrt{2c_s}}.$$

Taking into consideration (4.2) and (4.1), the second inequality holds if

$$q_0^{3-2s} \leq \frac{\varepsilon^2}{2C_s b^2} \leq \frac{\varepsilon^2}{2C_s\delta^{-2}\varepsilon^{-2}} = \frac{\delta^2\varepsilon^4}{2C_s} \leq \frac{\varepsilon^2\varepsilon^4}{8c_s \cdot 2C_s}$$

or

$$q_0 \leq \left(\frac{\varepsilon^6}{16c_s C_s}\right)^{\frac{1}{3-2s}}.$$

Case $s \leq 1/2$. For such a Sobolev exponent s we have $\|u(0)\|_{H^s} \leq \|u(0)\|_{H^1}$. This combined with Proposition 3, which tells us that $Q_1(q_0) \lesssim q_0$, gives

$$\|u(0)\|_{H^s}^2 \leq \|u(0)\|_{H^1}^2 \leq C_1 b^2 q_0 + 4c_1 \delta^2.$$

Thus $\|u_0\|_{H^s} < \varepsilon$ if q_0 and δ satisfy the inequalities

$$4c_1 \delta^2 \leq \frac{\varepsilon^2}{2} \quad \text{and} \quad C_1 b^2 q_0 \leq \frac{\varepsilon^2}{2}.$$

These inequalities hold if

$$\delta \leq \frac{\varepsilon}{2\sqrt{2c_1}} \quad \text{and} \quad q_0 \leq \frac{\varepsilon^6}{16 c_s C_s}.$$

5. Norm-inflation and ill-posedness for $5/4 < s < 3/2$. From Proposition 3 we have

$$(5.1) \qquad \|u(t)\|_{H^s}^2 = 16 r(t) p_1^2(t) Q_s(q) + 4c_s p_1^2(t) \big(1 - r(t)\big)^2,$$

where the estimate for Q_s is given in (3.2). Also, using Theorem 3 we have

$$p_1^2(t) \simeq q^{-1/2}(t) \quad \text{and} \quad p_2^2(t) \simeq q^{-1/2}(t) \quad \text{for t close to T.}$$

Next, we see that

$$r = r(t) \doteq \frac{p_2(t)}{-p_1(t)} \simeq \frac{q^{-1/4}}{q^{-1/4}} \simeq 1 \quad \text{as } t \nearrow T,$$

and

$$p_1(t)\big(1 - r(t)\big) = p_1(t)\Big(1 + \frac{p_2(t)}{p_1(t)}\Big) = p_2(t) + p_1(t) = w(t).$$

Also, we have

$$(5.2) \qquad \lim_{t \to T} p_1^2(t)\big(1 - r(t)\big)^2 = \lim_{t \to T} w^2(t) = \delta^2 + 2b(b + \delta) \cdot (1 - e^{-q_0}).$$

Therefore, the first term of (5.1) can be estimated by

$$16 r(t) p_1^2(t) Q_s(q) \simeq \begin{cases} q^{\frac{5}{2} - 2s}, & 1/2 < s < 3/2, \\ q^{\frac{3}{2}} \cdot \ln(1/q), & s = 1/2, \\ q^{\frac{3}{2}}, & s < 1/2. \end{cases}$$

Combining the last estimate with the fact that $\frac{5}{2} - 2s = 0 \iff s = \frac{5}{4}$ we see that

$$(5.3) \qquad \lim_{t \to T} 16 r(t) p_1^2(t) Q_s(q) = \begin{cases} \infty \quad \text{(inflation)}, & 5/4 < s < 3/2, \\ \text{may not exist}, & s = 5/4, \\ 0, & s < 5/4. \end{cases}$$

Finally, using the limits (5.3) and (5.2) from formula (5.1) we conclude that

$$(5.4) \qquad \lim_{t \to T} \|u(t)\|_{H^s}^2 = \begin{cases} \infty \quad \text{(inflation)}, & 5/4 < s < 3/2, \\ \text{may not exist}, & s = 5/4, \\ 4c_s\Big[\delta^2 + 2b(b + \delta) \cdot (1 - e^{-q_0})\Big], & s < 5/4. \end{cases}$$

Therefore when $5/4 < s < 3/2$ we have norm-inflation and ill-posedness for the NE.

6. Nonuniqueness for $s < 5/4$. In this section, we prove that once we take the Sobolev exponent to be less than 5/4, the NE admits nonunique solutions.

THEOREM 4 (nonuniqueness). *For $s < 5/4$ the NE admits nonunique solutions.*

Our proof of nonuniqueness revolves around examining the behavior of the limit as $t \to T^-$ of the 2-peakon solution u with initial data given in (2.7). Once we take the Sobolev exponent to be $s < 5/4$, this limit exists, and it is a single antipeakon. The nonuniqueness then can be realized by taking a single antipeakon traveling wave that at time T has the same profile as $\lim_{t \to T^-} u(x,t)$. From this point, a change of variables can recast this scenario as two solutions arising from the same initial data. To proceed with this argument, we begin by examining the pointwise limit, then the L^r limit, and finally we use these results in addition to the generalized dominated convergence theorem (gDCT) to establish the H^s limit.

PROPOSITION 4 (pointwise limit). *For each $x \in \mathbb{R}$ we have*

$$\tag{6.1} \lim_{t \to T^-} u(x,t) = w_T e^{-|x-q_T|} \doteq v_T(x),$$

where w_t is given by (2.46), that is, $w_T = -\left(\delta^2 + 2b(b+\delta)(1-e^{-2a})\right)^{\frac{1}{2}} < 0$, and

$$q_T \doteq \lim_{t \to T^-} q_1(t) = \lim_{t \to T^-} q_2(t).$$

To prove Proposition 4 we need the following elementary result.

LEMMA 2. *Given our functions p_1, p_2 and $q = q_1 - q_1$ the following limits hold as $t \to T^-$:*

$$\tag{6.2} \lim_{t \to T^-} p_j(t)(1 - e^{-q(t)}) = 0, \quad j = 1, 2.$$

Proof. Using the estimates

$$p_1^2 \simeq q^{-1/2} \quad \text{and} \quad p_2^2 \simeq q^{-1/2}$$

and the inequality $1 - e^{-x} < x$ for $x \in [0,1]$ we have

$$\lim_{t \to T^-} |p_j(t)(1 - e^{-q(t)})| \leq \lim_{t \to T^-} |p_j(t)| \cdot |q(t)| \lesssim \lim_{t \to T^-} |q^{-1/4}(t)| \cdot |q(t)| = 0. \qquad \square$$

Proof of Proposition 4. As we are working with a pointwise limit, we consider the cases $x \geq q_T$ and $x < q_T$ separately so that we can evaluate the absolute values $|x - q_j|$ in the definition of the 2-peakon solution u.

Case $x \geq q_T$. Since $q_1 \leq q_2 \leq q_T$, we have $x - q_j \geq 0$ and therefore

$$u(x,t) = p_1(t)e^{-|x-q_1(t)|} + p_2(t)e^{-|x-q_2(t)|} = e^{-x} \cdot \left(p_1(t)e^{q_1(t)} + p_2(t)e^{q_2(t)}\right).$$

Next, we will rewrite u in such a way so as to utilize Lemma 2. We have

$$u(x,t) = e^{-x} \cdot e^{q_2(t)} \cdot \left(-p_1(t)(1 - e^{-q(t)}) + w(t)\right).$$

Finally, taking the limit as $t \to T^-$ of u and using (6.2) we get

$$\tag{6.3} \lim_{t \to T^-} u(x,t) = w_T \cdot e^{-x+q_T}, \quad x \geq q_T.$$

Case $x < q_T$. We follow essentially the same strategy as in the previous case, simply correcting for signs. Since x is fixed and $q_1 \le q_2 \le q_T$, we see that after some time t_0 we must have $x < q_1(t) \le q_2(t) \le q_T$. Therefore, for $t > t_0$ we have $x - q_j < 0$, and u can be written as

$$u(x,t) = e^x \cdot e^{-q_1(t)}\Big(w(t) - p_2(t)(1 - e^{-q(t)})\Big).$$

Thus taking the limit as $t \to T^-$ of u and using again Lemma 2 we obtain

$$(6.4) \qquad \lim_{t \to T^-} u(x,t) = w_T \cdot e^{x - q_T}, \qquad x < q_T.$$

Combining (6.3) and (6.4) we conclude that the 2-peakon solution $u(t)$ has a limit as $t \to T^-$, which is given by the antipeakon (6.1). □

We next examine the limit of u in L^r topology.

PROPOSITION 5 (convergence in L^r). *For our antipeakon-peakon solution u to the NE, we have*

$$\lim_{t \to T^-} \|u(x,t) - v_T(x)\|_{L^r} = 0 \quad for \quad 1 \le r < 4.$$

Proof. As we will need to evaluate the absolute values in the exponents, we note that the order of the peaks' positions of $u(x,t)$ and $v_T(x)$ is $q_1(t) < q_2(t) < q_T$. We now expand the L^r-norm as

$$\|u(x,t) - v_T(x)\|_{L^r}^r \doteq I_1(t) + I_2(t) + I_3(t) + I_4(t),$$

where the integrals $I_j(t)$ have their domains determined by $q_1 < q_2 < q_T$, that is,

$$I_1(t) \doteq \int_{-\infty}^{q_1(t)} |u(x,t) - v_T(x)|^r dx, \qquad I_3(t) \doteq \int_{q_2(t)}^{q_T} |u(x,t) - v_T(x)|^r dx,$$

$$I_2(t) \doteq \int_{q_1(t)}^{q_2(t)} |u(x,t) - v_T(x)|^r dx, \qquad I_4(t) \doteq \int_{q_T}^{\infty} |u(x,t) - v_T(x)|^r dx.$$

Evaluating I_1. Calculating the integral, we have

$$I_1(t) = \frac{e^{rq_1(t)}}{r} \cdot \Big|p_1 e^{-q_1(t)} + p_2 e^{-q_2(t)} - w_T e^{-q_T}\Big|^r.$$

In order to proceed with evaluating the limit, we observe the identity

$$p_1 e^{-q_1(t)} + p_2 e^{-q_2(t)} - w_T e^{-q_T} = e^{-q_2} p_1(e^q - 1) + w(t) e^{-q_2(t)} - w_T e^{-q_T}.$$

We can now evaluate the limit as

$$\lim_{t \to T^-} I_1(t) = \lim_{t \to T^-} \frac{e^{rq_1(t)}}{r} \cdot \Big|e^{-q_2} p_1(e^q - 1) + [w(t) e^{-q_2(t)} - w_T e^{-q_T}]\Big|^r = 0.$$

Evaluating I_2. Using Jensen's inequality $|a_1 + \cdots + a_n|^r \le n^r(|a_1|^r + \cdots + |a_n|^r)$ together with $e^{-|x - q_j(t)|} \le 1$, $e^{-|x - q_T(t)|} \le 1$, and $|p_j| \simeq q^{-1/4}$, we have

$$\lim_{t \to T^-} I_2(t) \lesssim \lim_{t \to T^-} (q^{1 - \frac{r}{4}}(t) + q(t)|w_T|^r) = 0 \quad (\text{assuming } r < 4).$$

Evaluating I_3. After evaluating the absolute values inside the exponential and using the identity

$$p_1 e^{-x+q_1} + p_2 e^{-x+q_2} - w_T e^{x-q_T} = e^{-x}e^{q_2}p_1(e^{-q}-1) + e^{-x}we^{q_2} - e^x w_T e^{-q_T},$$

an application of Jensen's inequality gives us

$$I_3(t) \lesssim \int_{q_2(t)}^{q_T} \left| e^{-x}e^{q_2}p_1(t)(e^{-q(t)}-1) \right|^r dx + \int_{q_2(t)}^{q_T} \left| e^{-x}w(t)e^{q_2(t)} - e^x w_T e^{-q_T} \right|^r dx.$$

We see that for the first term in this sum, we have

$$\int_{q_2(t)}^{q_T} \left| e^{-x}e^{q_2(t)}p_1(t)(e^{-q(t)}-1) \right|^r dx, \lesssim (q_T - q_2(t)) \cdot \left| e^{q_2(t)}p_1(t)(e^{-q(t)}-1) \right|^r.$$

For the second term of this sum, we use the fact that $|e^{-x}we^{q_2} - e^x w_T e^{-q_T}|^r \lesssim 1$ and Hölder's inequality to get

$$\int_{q_2(t)}^{q_T} \left| e^{-x}w(t)e^{q_2(t)} - e^x w_T e^{-q_T} \right|^r dx \lesssim (q_T - q_2(t)).$$

Putting these estimates together, we can now evaluate the limit of I_3 as $t \to T$ via

$$\lim_{t\to T} I_3 \lesssim \lim_{t\to T} \left((q_T - q_2(t)) \cdot \left| e^{q_2(t)}p_1(t)(e^{-q(t)}-1)) \right|^r + (q_T - q_2(t)) \right) = 0.$$

Evaluating I_4. This term is handled in precisely the same fashion as I_1. Performing the integration gives us

$$I_4(t) = \frac{e^{-rq_T}}{r} \cdot \left| p_1 e^{q_1(t)} + p_2 e^{q_2(t)} - w_T e^{q_T} \right|^r.$$

Rewriting the expression inside of the absolute value gives us

$$p_1 e^{q_1(t)} + p_2 e^{q_2(t)} - w_T e^{q_T} = e^{q_2(t)}p_1(e^{-q(t)}-1) + w(t)e^{q_2(t)} - w_T e^{q_T}.$$

Therefore, using the above identity along with the triangle inequality yields

$$\lim_{t\to T^-} I_4(t) \leq \lim_{t\to T^-} \frac{e^{-rq_T}}{r} \cdot \left(|e^{q_2(t)}p_1(e^{-q(t)}-1)| + |w(t)e^{q_2(t)} - w_T e^{q_T}| \right)^r = 0.$$

Summarizing the L^r convergence, $1 \leq r < 4$. As we have computed $\lim_{t\to T^-} I_j(t) = 0$ for $j = 1, 2, 3, 4$ it immediately follows that

$$\lim_{t\to T^-} \|u(x,t) - v_T(x)\|_{L^r}^r = \lim_{t\to T^-} \left(I_1(t) + I_2(t) + I_3(t) + I_4(t) \right) = 0. \qquad \square$$

COROLLARY 2. *As t goes to T our 2-peakon solution $u(t)$ converges in H^s, $s \leq 0$, to the antipeakon $v_T = w_T e^{-|x-q_T|}$.*

Now that we have successfully established pointwise and L^r convergence, we are ready to move on to a much stronger result that is of interest in itself. As $t \to T$, the antipeakon-peakon solution converges to a single solitary antipeakon in H^s for $s < 5/4$.

THEOREM 5 (convergence in H^s). *For $s < 5/4$, our 2-peakon solution $u(t)$ converges to the antipeakon v_T in H^s, i.e.,*

$$(6.5) \qquad \lim_{t \to T} \|u(t) - v_T\|_{H^s} = 0.$$

Proof. We will begin by simplifying the H^s-norm of $u(x,t) - v_T(x)$. We have

$$\|u(t) - v_T\|_{H^s}^2 = \int_{\mathbb{R}} (1 + \xi^2)^s |\widehat{u}(\xi,t) - \widehat{v_T}(\xi)|^2 d\xi.$$

Our objective, when taking the limit, will be to move the limit inside of the integral. Thus, the first thing we should verify is whether, pointwise, we have

$$(1 + \xi^2)^s |\widehat{u}(\xi,t) - \widehat{v_T}(\xi)|^2 \to 0 \quad \text{as} \quad t \to T^-.$$

We have

$$|\widehat{u}(\xi,t) - \widehat{u}(\xi,T)| \leq \|u(x,t) - u(x,T)\|_{L^1}.$$

As we have proved that $u(x,t) \to u(x,T)$ in L^1, we have

$$\lim_{t \to T^-} (1 + \xi^2)^s |\widehat{u}(\xi,t) - \widehat{v_T}(\xi)|^2 = 0.$$

Next, we will define the bounding functions that will allow us to apply the gDCT. We set

$$f_t(\xi) \doteq (1 + \xi^2)^s |\widehat{u}(\xi,t) - \widehat{v_T}(\xi)|^2 \leq 4(1 + \xi^2)^s \Big(|\widehat{u}(\xi,t)|^2 + |\widehat{v_T}(\xi)|^2 \Big) \doteq g_t(\xi).$$

Next, we need to establish that the g_t's have a pointwise limit g. The most obvious candidate for g is

$$g(\xi) = 8(1 + \xi^2)^s |\widehat{v_T}(\xi)^2|.$$

Indeed, using the laws of limits, we have that $\widehat{u}(\xi,t) \to \widehat{v_T}(\xi)$ pointwise in ξ implies $g_t \to g$ pointwise in ξ. To finish satisfying the hypotheses of the gDCT, we must now establish the integral properties of the g_t's. We have

$$\lim_{t \to T} \int_{\mathbb{R}} g_t(\xi) d\xi = \lim_{t \to T} 4 \int_{\mathbb{R}} (1 + \xi^2)^s |\widehat{u}(\xi,t)|^2 d\xi + \lim_{t \to T} 4 \int_{\mathbb{R}} (1 + \xi^2)^s |\widehat{v_T}(\xi)|^2 d\xi$$
$$\overset{(5.4)}{=} 32 c_s w_T^2,$$

where the left limit uses the hypothesis that $s < 5/4$. Furthermore, g is integrable and

$$\int_{\mathbb{R}} g(\xi) d\xi = \int_{\mathbb{R}} 8 |v_T(\xi)^2| d\xi = 8 \cdot \|v_T\|_{H^s}^2 = 8 \cdot 4 c_s w_T^2 = 32 c_s w_T^2.$$

Therefore, we have $\int g_t \to \int g$. We now see that the hypotheses for the gDCT are satisfied. Thus, we can conclude that

$$\lim_{t \to T} \int_{\mathbb{R}} f_t(\xi) d\xi = \int_{\mathbb{R}} f(x) d\xi,$$

which, written more explicitly, tells us that

$$\lim_{t \to T} \int_{\mathbb{R}} (1 + \xi^2)^s |\widehat{u}(\xi,t) - \widehat{v_T}(\xi)|^2 d\xi = 0.$$

Thus, we can conclude that as $t \to T$, we have $u(t) \to v_T$ in H^s. $\qquad \square$

Proof of Theorem 4. Translating the NE 1-peakon solution (1.6) by x_0 and choosing the minus sign we obtain the following antipeakon solution for the NE:

$$v(x,t) = -\sqrt{c}\, e^{-|(x-x_0)-ct|} \quad \text{for any} \quad c > 0 \quad \text{and} \quad x_0 \in \mathbb{R}.$$

Choosing

$$c = w_T^2 \quad \text{and} \quad x_0 = q_T - w_T^2 T$$

we obtain the NE antipeakon solution

$$v(x,t) = -\sqrt{w_T^2}\, e^{-|(x-q_T+w_T^2 T)-w_T^2\, t|}.$$

Since at $t = T$ we have

$$v(x,T) = w_T e^{-|x-q_T|} = u(x,T),$$

we see that we have constructed two different NE solutions, which belong in H^s, $s < 5/4$, and agree at $t = T$. From here, a change of variables can recast these two solutions as initial value problems at time $t = 0$. This proves failure of uniqueness in this range of Sobolev spaces. $\qquad\square$

The case $s = 5/4$. If $s = 5/4$, then there are two possibilities. Either our 2-peakon solution $u(t)$ does not converge in $H^{5/4}$, in which case we can prove (by a standard argument) that continuity of the solution map fails, or $u(t)$ converges in $H^{5/4}$ and has limit $u(T)$ (since this is the limit for lower Sobolev exponents). In the second case, we have nonuniqueness as in Theorem 4. This result completes the proof of both Theorem 1 and 2 in the nonperiodic case.

7. The periodic case.

7.1. Outline of the proofs in the periodic case. The proofs of Theorems 1 and 2 have been demonstrated on the line, and we now present these proofs on the circle, $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. The key ingredient is using a periodic version of the peakon. In subsection 7.2, we construct the 2-peakon solutions on the circle having the properties described in Theorem 1. In subsection 7.3, we estimate the H^s-norm of the 2-peakon solutions, and in subsection 7.4 we choose the parameters so that both the lifespan and the size of the 2-peakon solution at the initial time are simultaneously small. In subsection 7.5, we prove norm-inflation and ill-posedness for $5/4 < s < 3/2$. Finally, in subsection 7.6, we prove nonuniqueness for $s < 5/4$ and explain the ill-posedness of the NE for $s = 5/4$.

7.2. Construction of 2-peakon solutions on the circle. The 2-peakon solutions to the periodic version of the NE are similar to those on the real line, with the caveat that the peak is generated by periodizing the hyperbolic cosine rather than using the exponential of the negative absolute value. The following equations are taken from [GH] and [HM] and can also be derived in a straightforward fashion.

The periodic Novikov 2-peakon solutions are of the form

$$(7.1) \qquad u(x,t) = p_1(t)\cosh([x - q_1(t)]_p - \pi) + p_2(t)\cosh([x - q_2(t)]_p - \pi),$$

where $[\cdot]_p$ periodizes our function and is defined by the floor

$$(7.2) \qquad\qquad [x]_p = x - 2\pi\left\lfloor \frac{x}{2\pi} \right\rfloor.$$

We see that u solves the NE if the momenta p_1, p_2 and the positions q_1, q_2 satisfy the following system of ODEs, which can be obtained by using Theorem 1.2 of [HM] with the choice of parameters $a = 0$, $b = 3$. The 4×4 system we get is

$$q_1' = p_1^2[1 + \sinh^2 \pi] + 2p_1p_2 \cosh \pi \cosh([q_1 - q_2]_p - \pi) + p_2^2[1 + \sinh^2([q_1 - q_2]_p - \pi)],$$
$$q_2' = p_2^2[1 + \sinh^2 \pi] + 2p_1p_2 \cosh \pi \cosh([q_2 - q_1]_p - \pi) + p_1^2[1 + \sinh^2([q_2 - q_1]_p - \pi)],$$
$$p_1' = -p_1p_2 \sinh([q_1 - q_2]_p - \pi)[p_1 \cosh \pi + p_2 \cosh([q_1 - q_2]_p - \pi)],$$
$$p_2' = -p_1p_2 \sinh([q_2 - q_1]_p - \pi)[p_2 \cosh \pi + p_1 \cosh([q_2 - q_1]_p - \pi)].$$

Setting

$$E(x) \doteq \frac{1}{\cosh \pi} \cdot \cosh([x]_p - \pi), \quad E'(x) \doteq \frac{1}{\cosh \pi} \cdot \sinh([x]_p - \pi),$$

and using $q = q_2 - q_1$, our system can be written in the more compact form

(7.3)
$$\begin{aligned}
q_1' &= \cosh^2 \pi \cdot (p_1 + p_2 E(q))^2, \\
q_2' &= \cosh^2 \pi \cdot (p_1 E(q) + p_2)^2, \\
p_1' &= \cosh^2 \pi \cdot p_1 p_2 (p_1 + p_2 E(q)) E'(q), \\
p_2' &= -\cosh^2 \pi \cdot p_1 p_2 (p_1 E(q) + p_2) E'(q).
\end{aligned}$$

Initial data. From this point, we make the same initial data assumptions as in the real line case. We take the positions q_1 and q_2 at time $t = 0$ to be

$$q_1(0) = -a \quad \text{and} \quad q_2 = a \quad \text{for some } a > 0.$$

For the initial momenta, we shall assume that at time $t = 0$

(7.4)
$$p_2(0) = b \gg 1, \qquad p_1(0) = -(b + \delta), \quad \delta > 0.$$

With these assumptions, the initial profile $u_0(x) = u(x, 0)$ is the asymmetric periodic antipeakon-peakon

(7.5)
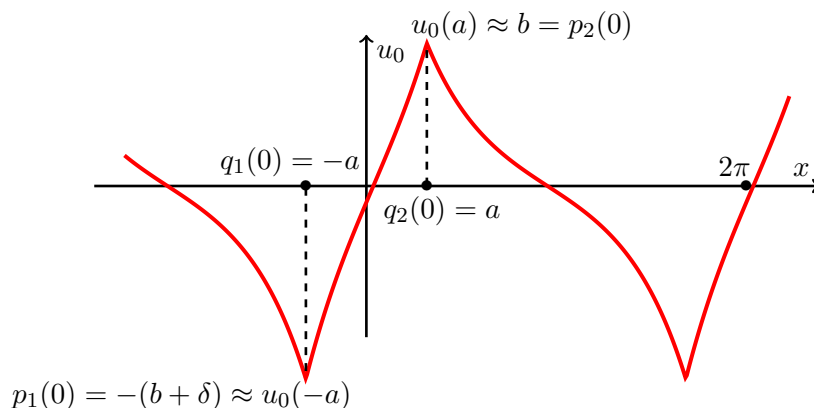$$u_0(x) = -(b + \delta) \cosh([x + a]_p - \pi) + b \cosh([x - a]_p - \pi).$$

This initial profile for u is displayed in Figure 3.

Following the intuition we developed in the real line case, we again will examine the ODE system (7.3) in the derived variables p, q, w, z given by

(7.6)
$$\begin{aligned}
q(t) &= q_2(t) - q_1(t), & q(0) &= 2a > 0, \\
p(t) &= p_2(t) - p_1(t), & p(0) &= 2b + \delta > 0, \\
w(t) &= p_2(t) + p_1(t), & w(0) &= -\delta < 0, \\
z(t) &= p_2(t) \cdot p_1(t), & z(0) &= -b(b + \delta) < 0.
\end{aligned}$$

Deriving equations for q, p, w, and z on the circle. Beginning with q, we follow the same strategy as in the nonperiodic case. We see that

(7.7)
$$\begin{aligned}
q' &= \cosh^2 \pi \cdot (p_1 E(q) + p_2)^2 - \cosh^2 \pi \cdot (p_1 E(q) + p_2)^2, \\
&= \cosh^2 \pi \cdot (p_2 - p_1)(p_2 + p_1)(1 - E^2(q)), \\
&= \cosh^2 \pi \cdot pw(1 - E^2(q)).
\end{aligned}$$

FIG. 3. *Initial profile $u_0(x)$.*

The computations for p, w, and z follow the same strategy, and we arrive at the 4×4 system

$$
\begin{aligned}
(7.8) \quad
q' &= \cosh^2 \pi \cdot pw(1 - E^2(q)), & q(0) &= 2a > 0, \\
p' &= -\cosh^2 \pi \cdot wz(1 + E(q))E'(q), & p(0) &= 2b + \delta > 0, \\
w' &= -\cosh^2 \pi \cdot zp(1 - E(q))E'(q), & w(0) &= -\delta < 0, \\
z' &= \cosh^2 \pi \cdot zwpE(q)E'(q), & z(0) &= -b(b + \delta) < 0.
\end{aligned}
$$

This derived system of ODEs is more easily manipulated than the original 4×4 system, and we are now ready to tackle Proposition 1 in the periodic setting.

PROPOSITION 6 (periodic version of Proposition 1). *The system of differential equations (7.8) has a unique smooth solution $(q(t), p(t), w(t), z(t))$ in an interval $[0, T)$, for some $T > 0$, such that $z = z(t)$ is decreasing and, in terms of q, is expressed by the formula*

$$
(7.9) \qquad z = \frac{-z_1}{\left(1 - E^2(q)\right)^{1/2}} < 0, \quad where \quad z_1 = b(b + \delta)\left(1 - E^2(q_0)\right)^{1/2} > 0,
$$

$p = p(t)$ is decreasing and as a function of q is expressed by the formula

$$
(7.10) \qquad p(t) = \left(p_0^2 + 2z_1\left[\frac{1 + E(q(t))}{\sqrt{1 - E^2(q(t))}} - \frac{1 + E(q_0)}{\sqrt{1 - E^2(q_0)}}\right]\right)^{1/2} > 0,
$$

and $w = w(t)$ is decreasing and as a function of q is expressed by the formula

$$
(7.11) \qquad w(t) = -\left(w_0^2 + 2z_1\left[\frac{\sqrt{1 - E^2(q_0)}}{1 + E(q_0)} - \frac{\sqrt{1 - E^2(q(t))}}{1 + E(q(t))}\right]\right)^{1/2} < 0.
$$

The difference between the positions $q = q(t)$ is decreasing and satisfies the initial

value problem

(7.12)

$$q' = -f(q) \doteq -\cosh^2 \pi \cdot \left(w_0^2 + 2z_1 \left[\frac{\sqrt{1 - E^2(q_0)}}{1 + E(q_0)} - \frac{\sqrt{1 - E^2(q(t))}}{1 + E(q(t))} \right] \right)^{1/2}$$

$$\cdot \left(p_0^2 + 2z_1 \left[\frac{1 + E(q(t))}{\sqrt{1 - E^2(q(t))}} - \frac{1 + E(q_0)}{\sqrt{1 - E^2(q_0)}} \right] \right)^{1/2} \cdot (1 - E^2(q)),$$

$$q(0) = q_0 = 2a > 0.$$

Furthermore, the initial value problem (7.12) *is dominated by the simpler initial value problem*

$$q' = -g(q) \doteq -q_1 \left(1 - e^{-2q} \right)^{3/4}, \quad 0 < q(0) = 2a < 1/2,$$

where

$$q_1 = \delta \sqrt{2b(b + \delta)} \cdot q_0^{1/4}.$$

Proof. We begin by solving for p, w, and z in terms of q. After this task is completed, we can form an autonomous equation for q by substituting in its defining ODE (7.8) the expressions for p, q, and w.

Expressing z in terms of q. Using the equation for z' and q' we find

$$\frac{z'}{q'} = \frac{\cosh^2 \pi \cdot zwpE(q)E'(q)}{\cosh^2 \pi \cdot pw(1 - E^2(q))} = \frac{zE(q)E'(q)}{1 - E^2(q)} \quad \text{or} \quad \frac{z'}{z} = \frac{E(q)E'(q)q'}{1 - E^2(q)}.$$

Since $z(0) < 0$, we assume that $z(t)$ will remain negative. Therefore, from the last relation we have

$$\frac{d}{dt}[\ln(-z)] = -\frac{1}{2}\frac{d}{dt}[\ln(1 - E^2(q))].$$

Integrating this equation from 0 to t gives

$$\ln\left[\frac{z(t)}{z_0}\right] = -\frac{1}{2}\ln\left[\frac{1 - E^2(q(t))}{1 - E^2(q_0)}\right].$$

Solving for $z(t)$, we find formula (7.9) for z in terms of q.

Expressing w in terms of q. Dividing the equation for w' by the equation for q' we have

(7.13)
$$\frac{w'}{q'} = \frac{-\cosh^2 \pi \cdot zp(1 - E(q))E'(q)}{\cosh^2 \pi \cdot pw(1 - E^2(q))} = \frac{-z(1 - E(q))E'(q)}{w(1 - E^2(q))}$$

(7.14)
$$\text{or} \quad ww' = -z \cdot \frac{(1 - E(q))E'(q)q'}{1 - E^2(q)}.$$

Substituting the formula for z given by (7.9) into the above equation gives us

(7.15)
$$ww' = \frac{z_1}{\left(1 - E^2(q)\right)^{1/2}} \cdot \frac{(1 - E(q))E'(q)q'}{1 - E^2(q)} = \frac{z_1(1 - E(q))E'(q)q'}{\left(1 - E^2(q)\right)^{3/2}}.$$

Making the change of variables $u = E(q(t))$, $du = E'(q(t))q'(t)dt$, we obtain

$$\int \frac{(1 - E(q))E'(q)q'}{\left(1 - E^2(q)\right)^{3/2}} dt = \int \frac{1 - u}{\left(1 - u^2\right)^{3/2}} du = -\frac{\sqrt{1 - u^2}}{1 + u} + C = -\frac{\sqrt{1 - E^2(q(t))}}{1 + E(q(t))} + C.$$

Therefore, relation (7.15) reads as

$$(7.16) \qquad \frac{d}{dt}\left[\frac{1}{2}w^2\right] = -z_1 \frac{d}{dt}\left[\frac{\sqrt{1 - E^2(q(t))}}{1 + E(q(t))}\right].$$

Integrating this equation from 0 to t gives

$$(7.17) \qquad \frac{1}{2}\left[w^2(t) - w_0^2\right] = z_1\left[\frac{\sqrt{1 - E^2(q_0)}}{1 + E(q_0)} - \frac{\sqrt{1 - E^2(q(t))}}{1 + E(q(t))}\right].$$

We are thus able to solve for $w(t)$ in terms of $q(t)$, which gives us formula (7.11).

Expressing p in terms of q. Dividing the equation for p' by the equation for q' we have

$$(7.18) \qquad \frac{p'}{q'} = \frac{-\cosh^2\pi \cdot wz(1 + E(q))E'(q)}{\cosh^2\pi \cdot pw(1 - E^2(q))} = \frac{-z(1 + E(q))E'(q)}{p(1 - E^2(q))},$$

$$(7.19) \qquad \text{or} \quad pp' = -z \cdot \frac{(1 + E(q))E'(q)q'}{1 - E^2(q)}.$$

Substituting in the above relation the formula for z given by (7.9) we have

$$(7.20) \qquad pp' = \frac{z_1}{\left(1 - E^2(q)\right)^{1/2}} \cdot \frac{(1 + E(q))E'(q)q'}{1 - E^2(q)} = \frac{z_1(1 + E(q))E'(q)q'}{\left(1 - E^2(q)\right)^{3/2}}.$$

Next, we make the change of variables $u = E(q(t))$, $du = E'(q(t))q'(t)dt$ and get

$$\int \frac{(1 + E(q))E'(q)q'}{\left(1 - E^2(q)\right)^{3/2}} dt = \int \frac{1 + u}{\left(1 - u^2\right)^{3/2}} du = \frac{1 + u}{\left(1 - u^2\right)^{1/2}} + C = \frac{1 + E(q)}{\left(1 - E^2(q)\right)^{1/2}} + C.$$

Therefore, relation (7.20) reads as

$$\frac{d}{dt}\left[\frac{1}{2}p^2\right] = z_1 \frac{d}{dt}\left[\frac{1 + E(q)}{\sqrt{1 - E^2(q)}}\right].$$

Integrating this equation from 0 to t gives us

$$\frac{1}{2}\left[p^2(t) - p_0^2\right] = z_1\left[\frac{1 + E(q(t))}{\sqrt{1 - E^2(q(t))}} - \frac{1 + E(q_0)}{\sqrt{1 - E^2(q_0)}}\right],$$

and we are able to solve for $p(t)$ and obtain formula (7.10).

Solving the q ODE. Starting with the differential equation for q, which is $q' = \cosh^2\pi \cdot pw(1 - E^2(q))$, we substitute for w and p their expressions (7.11) and (7.10), respectively. We consequently obtain the following autonomous differential equation for q:

(7.21)

$$q' = -f(q) \doteq \cosh^2\pi \cdot \left\{-\left(w_0^2 + 2z_1\left[\frac{\sqrt{1 - E^2(q_0)}}{1 + E(q_0)} - \frac{\sqrt{1 - E^2(q(t))}}{1 + E(q(t))}\right]\right)^{1/2}\right\}$$

$$\cdot \left(p_0^2 + 2z_1\left[\frac{1 + E(q(t))}{\sqrt{1 - E^2(q(t))}} - \frac{1 + E(q_0)}{\sqrt{1 - E^2(q_0)}}\right]\right)^{1/2} \cdot (1 - E^2(q)),$$

$$q(0) = q_0 = 2a > 0.$$

Next, we observe that

$$(7.22) \qquad \frac{\sqrt{1 - E^2(q_0)}}{1 + E(q_0)} - \frac{\sqrt{1 - E^2(q(t))}}{1 + E(q(t))} \geq 0, \quad 0 \leq q \leq q_0 < \pi.$$

This inequality follows from the fact that

$$\left(\frac{\sqrt{1 - E^2(x)}}{1 + E(x)} \right)' = \frac{-E'(x)}{2\sqrt{1 + E(x)}}.$$

Since the denominator is always positive, the sign of this derivative is controlled by the numerator, $-E'(x) = -\frac{1}{\cosh(\pi)} \sinh([x]_p - \pi)$, which is positive for $x \in [0, \pi)$. Next, we have

$$(7.23) \qquad p_0^2 - 2z_1 \frac{1 + E(q_0)}{\sqrt{1 - E^2(q_0)}} \geq 0 \iff \frac{(2b + \delta)^2}{2b(b + \delta)} \geq 1 + E(q_0).$$

Our choice of initial data allows for the inequality $1 + E(q_0) \leq 2$, and we have

$$\frac{(2b + \delta)^2}{2b(b + \delta)} \geq 2 \iff 4b^2 + 4b\delta + \delta^2 > 4b^2 + 4b\delta \iff \delta^2 > 0,$$

which is true.

Now, using (7.22) and (7.23) we see that the function $f(q)$ in the right-hand side of the differential equation (7.21) can be bounded from below as follows:

$$f(q) \geq \cosh^2 \pi \cdot \delta \left(2b(b + \delta)\left(1 - E^2(q_0) \right)^{1/2} \right)^{\frac{1}{2}} \left(\left[\frac{1 + E(q(t))}{\sqrt{1 - E^2(q(t))}} \right] \right)^{\frac{1}{2}} \cdot (1 - E^2(q)).$$

To continue our objective of finding a simpler dominating function for f, analogous to the strategy in the real line case of this proof, we use the fact that $E(q) \geq 0$ in conjunction with the following lemma.

LEMMA 3. *For $c \geq 2\cosh^2(\pi)/\sinh(2\pi - 1)$ and $x \in [0, 1/2]$,*

$$(7.24) \qquad c(1 - E^2(x)) \geq 1 - e^{-2x}.$$

Furthermore, we have the inequality

$$(7.25) \qquad 1 - E^2(q_0) \geq \frac{1}{3} q_0.$$

In particular, we will take $c = 3$ in later computations.

Proof. Define the function

$$f(x) \doteq \left(1 - e^{-2x} \right) - \left(c \cdot [1 - E^2(x)] \right).$$

Computing the derivative of $f(x)$ shows that it will be negative for $x \in (0, 1/2]$, and

$$c \geq \frac{2\cosh^2 \pi}{\sinh(2\pi - 1)}.$$

As (7.24) has been established, we now move onto proving (7.25). This inequality is obtained by applying our first inequality and then using the exponential inequality. We get

$$1 - E^2(q_0) \geq \frac{1}{3}(1 - e^{-2q_0}) \geq \frac{1}{3} q_0. \qquad \qquad \square$$

With the above lemma, we are now ready to return to the proof of the proposition.

Dominating equation (periodic version). Using the above inequalities, and following the same strategy as in the nonperiodic case, we obtain

$$f(q) \geq \left[\frac{\sqrt{2}\cosh^2 \pi}{3} \cdot \delta \cdot \sqrt{b(b+\delta)} \cdot q_0^{1/4} \right] \cdot \left(1 - e^{-2q} \right)^{3/4}.$$

Since $\frac{\sqrt{2}\cosh^2 \pi}{3} \geq 1$, we can remove this factor as we are bounding from below. Consequently, $f(q)$ has precisely the same lower bound as in the real line case given by

$$f(q) \geq \delta \cdot \sqrt{b(b+\delta)} \cdot q_0^{1/4} \cdot \left(1 - e^{-2q} \right)^{3/4} = q_1 \left(1 - e^{-2q} \right)^{3/4},$$

where the constant q_1 is given by

$$q_1 = \delta \cdot \sqrt{b(b+\delta)} \cdot q_0^{1/4}.$$

Thus, we see that the complicated initial value problem for q (7.21) is dominated by

$$(7.26) \qquad q' = -q_1 \left(1 - e^{-2q} \right)^{3/4}, \quad q(0) = q_0 = 2a > 0.$$

This ODE is precisely the same as the one derived in the real line case. Therefore we can immediately arrive at the same conclusions for q.

PROPOSITION 7 (periodic version of Proposition 2). *If $r < 1$, then for given $q_0 \in (0, 1/2)$ and $q_1 > 0$ the solution to the initial value problem*

$$(7.27) \qquad \frac{dq}{dt} = -g_r(q) \doteq -q_1 \left(1 - e^{-2q} \right)^r, \quad q(0) = q_0,$$

which begins positive and is decreasing, becomes zero in finite time T given by

$$(7.28) \qquad T = \int_0^{q_0} \frac{dq}{g_r(q)} = \frac{1}{q_1} \int_0^{q_0} \frac{dq}{(1 - e^{-2q})^r} \simeq \frac{1}{1-r} \frac{q_0^{1-r}}{q_1}.$$

COROLLARY 3 (periodic version of Corollary 1). *If $0 < q_0 < 1/2$ and $b > 1$, $\delta > 0$ satisfy condition (7.4), then the solution to the initial value problem (7.12) begins positive, is decreasing, and becomes zero in finite time T given by*

$$(7.29)$$
$$T = \int_0^{q_0} \frac{dq}{f(q)} \leq \frac{1}{q_1} \int_0^{q_0} \frac{dq}{(1 - e^{-2q})^{3/4}} \simeq \frac{q_0^{1/4}}{q_1} \simeq \frac{q_0^{1/4}}{\delta\sqrt{2b(b+\delta)} \cdot q_0^{1/4}} \simeq \frac{1}{\delta\sqrt{2b(b+\delta)}}.$$

We summarize the above results in the following theorem.

THEOREM 6 (periodic version of Theorem 3). *For given $0 < a \leq 1/4$, $b > 1$, and $\delta > 0$ satisfying condition (7.4), the initial value problem for the positions q_1, q_2 and the momenta p_1, p_2*

$$(7.30) \quad \begin{aligned} q_1' &= \cosh^2 \pi \cdot (p_1 + p_2 E(q))^2, & q_1(0) &= -a, \\ q_2' &= \cosh^2 \pi \cdot (p_1 E(q) + p_2)^2, & q_2(0) &= a, \\ p_1' &= \cosh^2 \pi \cdot p_1 p_2 (p_1 + p_2 E(q)) E'(q), & p_1(0) &= -(b+\delta), \\ p_2' &= -\cosh^2 \pi \cdot p_1 p_2 (p_1 E(q) + p_2) E'(q), & p_2(0) &= b, \end{aligned}$$

has a unique smooth solution $(q_1(t), q_2(t), p_1(t), p_2(t))$ *with a finite lifespan* T, *which is the zero of* $q = q_2 - q_1$, *satisfying the estimate* (7.29) *and such that*

$$p_1 = \frac{w - p}{2} < 0, \ decreasing, \ \lim_{t \to T^-} p_1(t) = -\infty, \ and \ -p_1 \simeq p \simeq q^{-1/4}, \ and$$

$$p_2 = \frac{w + p}{2} > 0, \ increasing, \ \lim_{t \to T^-} p_2(t) = \infty, \ and \ p_2 \simeq p \simeq q^{-1/4},$$

where p *and* w *are given in Proposition* 6. *Also,* $w = p_1 + p_2$ *is decreasing from* $w_0 < 0$ *to* w_T, *where*

$$w_T \doteq \lim_{t \to T^-} w(t) = -\Big(\delta^2 + 2b(b + \delta)(1 - E(2a))\Big)^{\frac{1}{2}}.$$

Finally, the 2-peakon

$$u(x, t) = p_1(t) \cosh([x - q_1(t)]_p - \pi) + p_2(t) \cosh([x - q_2(t)]_p - \pi), \ x \in \mathbb{T}, \ 0 < t < T,$$

is a solution to the NE with the following asymmetric antipeakon-peakon initial profile:

$$u(x, 0) = -(b + \delta) \cosh([x + a]_p - \pi) + b \cosh([x - a]_p - \pi),$$

The quantities p_1, p_2, q, *and* w *have properties similar to their analogues defined on the line, and we refer the reader to Figure* 2 *for a visualization of them.*

7.3. Calculating the norm on the circle. We begin with the following proposition which summarizes the calculation of the H^s-norm of u. This computation is nearly identical to the nonperiodic case with the exception of an extra factor of $\sinh^2 \pi$.

PROPOSITION 8 (periodic version of Proposition 3). *Let* $u(t)$ *be the 2-peakon solution* (7.1) *to the NE. Then on* $[0, T)$ *we have*

$$\|u(t)\|_{H^s}^2 = 16 \sinh^2 \pi \cdot r(t) p_1^2(t) Q_s(q) + 4 \sinh^2 \pi \cdot c_s \big(1 - r(t)\big)^2 p_1^2(t),$$

$$with \quad r(t) \doteq -\frac{p_2(t)}{p_1(t)},$$

where $c_s = \sum_{-\infty}^{\infty} (1 + n^2)^{s-2}$ *and* $Q_s(q)$, *which is given below, satisfies the estimates*

$$Q_s(q) \doteq \sum_{n=-\infty}^{\infty} (1 + n^2)^{s-2} \sin^2 \left(\frac{qn}{2}\right) \simeq \begin{cases} q^{3-2s}, & 1/2 < s < 3/2, \\ q^2 \cdot \ln(1/q), & s = 1/2, \\ q^2, & s < 1/2. \end{cases}$$

Proof. We begin by noting that the Fourier transform of E is calculated as

$$\widehat{E}(n) = \left(2 \cdot \frac{\sinh(\pi)}{\cosh(\pi)}\right) \cdot \frac{1}{1 + n^2}.$$

Recalling that the 2-peakon u can be written as

$$u(x, t) = \cosh \pi \cdot \Big(p_1(t) E(x - q_1(t)) + p_2(t) E(x - q_2(t))\Big),$$

we can express the Fourier transform of u as

$$\widehat{u}(n, t) = \frac{2 \sinh \pi}{1 + n^2} \cdot p_1 e^{-inq_1} \cdot \left(1 + \frac{p_2}{p_1} e^{-inq}\right).$$

Taking the square of the H^s-norm of this quantity, we obtain

(7.31)
$$\|u(t)\|_{H^s}^2 = \sum_{n=-\infty}^{\infty} (1+n^2)^s |\widehat{u}(n,t)|^2 = 4\sinh^2 \pi \cdot p_1^2 \sum_{n=-\infty}^{\infty} (1+n^2)^{s-2} \left|1 + \frac{p_2}{p_1}e^{-inq}\right|^2.$$

Using Proposition 6 we see that

(7.32) $$r = r(t) \doteq -\frac{p_2(t)}{p_1(t)} = \frac{p+w}{p-w} < 1 \quad \text{and} \quad r(t) \nearrow 1 \text{ as } t \nearrow T.$$

Using r we write (7.31) as

(7.33) $$\|u(t)\|_{H^s}^2 = 4\sinh^2 \pi \cdot p_1^2 \sum_{n=-\infty}^{\infty} (1+n^2)^{s-2} \left|1 - re^{-inq}\right|^2.$$

Expanding out the square of the absolute value inside of the sum (7.33), we have

$$|re^{iqn} - 1|^2 = (1-r)^2 + 4r\sin^2\left(\frac{qn}{2}\right).$$

We therefore obtain the formula

$$\|u(t)\|_{H^s}^2 = 16\sinh^2 \pi \cdot rp_1^2 Q_s(q) + 4\sinh^2 \pi \cdot c_s(1-r)^2 p_1^2,$$

where

$$c_s = \sum_{n=-\infty}^{\infty} (1+n^2)^{s-2} \quad \text{and} \quad Q_s(q) = \sum_{n=-\infty}^{\infty} (1+n^2)^{s-2} \sin^2\left(\frac{qn}{2}\right).$$

From this point, we note that Q_s has already been estimated in this periodic setting in [HHG]. Using equation (4.25) from [HHG], and also (4.28), where the norm is expanded into the sum of the squares of sines, we have

(7.34) $$Q_s \simeq \sum_{n=1}^{\infty} \sin^2\left(\frac{qn}{2}\right)(1+n^2)^{s-2} \simeq \begin{cases} q^{3/2-s}, & 1/2 < s < 3/2, \\ q\sqrt{\ln(1/q)}, & s = 1, 2, \\ q, & s < 1/2. \end{cases} \qquad \square$$

7.4. Small lifespan and initial data on the circle. This section follows the same argument as in the real line case, with the exception of an extra factor of $\sinh^2 \pi$ stemming from the periodic version of the norm-estimates. We begin by assuming that

$$p_2(0) = b \gg 1 \text{ and } -p_1(0) = b + \delta, \, \delta > 0,$$

so that the conditions for the existence of our 2-peakon with the lifespan estimate (7.29) hold.

Lifespan estimate. For given $\varepsilon > 0$, we need to find $b > 1$ such that $T < \varepsilon$. Since by Proposition 6 we have

$$T \lesssim \frac{1}{\delta\sqrt{2b(b+\delta)}} \leq \frac{1}{\delta b},$$

we must have

$$(7.35) \qquad \frac{1}{\delta b} \leq \varepsilon \iff b \geq \delta^{-1}\varepsilon^{-1}.$$

Initial data estimate. Now, for the same $\varepsilon > 0$ we need to find $q_0 < 1/8$ such that $\|u_0\|_{H^s} < \varepsilon$. For this we use Proposition 8, from which we have, recalling that $r(t) = -\frac{p_2(t)}{p_1(t)}$,

$$\|u(0)\|_{H^s}^2 = 16\sinh^2 \pi \cdot b(b+\delta)Q_s(q_0) + 4\sinh^2 \pi \cdot c_s\delta^2.$$

This identity implies

$$\|u(0)\|_{H^s}^2 \leq 32\sinh^2 \pi \cdot b^2 Q_s(q_0) + 4\sinh^2 \pi \cdot c_s\delta^2.$$

Case $1/2 < s < 3/2$. Now by Proposition 8 we have $Q_s(q_0) \lesssim q_0^{3-2s}$ and therefore

$$\|u(0)\|_{H^s}^2 \leq C_s\sinh^2 \pi \cdot b^2 q_0^{3-2s} + 4\sinh^2 \pi \cdot c_s\delta^2.$$

To have $\|u_0\|_{H^s} < \varepsilon$ it suffices to choose q_0 and δ such that $C_s\sinh^2 \pi \cdot b^2 q_0^{3-2s} + 4\sinh^2 \pi \cdot c_s\delta^2 \leq \varepsilon^2$ or

$$4\sinh^2 \pi \cdot c_s\delta^2 \leq \frac{\varepsilon^2}{2} \quad \text{and} \quad C_s\sinh^2 \pi \cdot b^2 q_0^{3-2s} \leq \frac{\varepsilon^2}{2}.$$

The first inequality holds if

$$(7.36) \qquad \delta \leq \frac{\varepsilon}{2\sinh \pi\sqrt{2c_s}}.$$

Taking into consideration (7.36) and (7.35), the second inequality holds if

$$q_0^{3-2s} \leq \frac{\varepsilon^2}{2C_s\sinh^2 \pi \cdot b^2} \leq \frac{\varepsilon^2}{2C_s\delta^{-2}\varepsilon^{-2}} = \frac{\delta^2\varepsilon^4}{2C_s} \leq \frac{\varepsilon^2\varepsilon^4}{8c_s \cdot 2C_s}$$

or

$$q_0 \leq \left(\frac{\varepsilon^6}{16c_sC_s}\right)^{\frac{1}{3-2s}}.$$

Case $s \leq 1/2$. For such a Sobolev exponent s we have $\|u(0)\|_{H^s} \leq \|u(0)\|_{H^1}$. This combined with Proposition 8, which tells us that $Q_1(q_0) \lesssim q_0$, gives

$$\|u(0)\|_{H^s}^2 \leq \|u(0)\|_{H^1}^2 \leq C_1 b^2 q_0 + 4c_1\delta^2.$$

Thus $\|u_0\|_{H^s} < \varepsilon$ if q_0 and δ satisfy the inequalities $4c_1\delta^2 \leq \frac{\varepsilon^2}{2}$ and $C_1 b^2 q_0 \leq \frac{\varepsilon^2}{2}$. These inequalities hold if

$$\delta \leq \frac{\varepsilon}{2\sqrt{2c_1}} \quad \text{and} \quad q_0 \leq \frac{\varepsilon^6}{16c_sC_s}.$$

7.5. Norm-inflation and ill-posedness on the circle. From Proposition 8 we have

$$(7.37) \qquad \|u(t)\|_{H^s}^2 = 16 \sinh^2 \pi \cdot r(t) p_1^2(t) Q_s(q) + 4 \sinh^2 \pi \cdot c_s p_1^2(t) \big(1 - r(t)\big)^2.$$

We see that the argument from section 5 holds, with the simple inclusion of a factor of $\sinh^2 \pi$. Following these arguments, we see that

$$(7.38) \quad \lim_{t \to T} \|u(t)\|_{H^s}^2 = \begin{cases} \infty \quad \text{(inflation)}, & 5/4 < s < 3/2, \\ \text{may not exist}, & s = 5/4, \\ 4 \sinh^2 \pi \cdot c_s \Big[\delta^2 + 2b(b + \delta) \cdot (1 - e^{-q_0}) \Big], & s < 5/4. \end{cases}$$

Therefore when $5/4 < s < 3/2$ we have norm-inflation and ill-posedness for the NE. $\square$

7.6. Nonuniqueness for $s < 5/4$ on the circle. As in the case on the line, the NE admits nonunique solutions once we take the Sobolev exponent $s < 5/4$. This is equally as interesting as the result in which periodic 2-peakons maintain the same collision properties as nonperiodic ones.

THEOREM 7 (nonuniqueness—periodic version of Theorem 4). *For $s < 5/4$ the NE admits nonunique solutions.*

Our proof of nonuniqueness in the periodic setting again follows the same strategy used in the real line case. We again examine the behavior of the limit as $t \to T^-$ of the 2-peakon solution u with initial data given in (7.5). Once this limit has been established in the desired ways, the same argument as in the real line case implies nonuniqueness.

PROPOSITION 9 (pointwise limit—periodic version of Proposition 4). *For each $x \in \mathbb{R}$ we have*

$$(7.39) \qquad\qquad \lim_{t \to T} u(x, t) = w_T \cosh([x - q_T]_p - \pi) \doteq v_T(x).$$

where

$$(7.40) \qquad\qquad q_T \doteq \lim_{t \to T^-} q_1(t) = \lim_{t \to T^-} q_2(t) \quad and \quad w_T \doteq \lim_{t \to T^-} w(t).$$

Remark. We can avoid the multiple cases needed in the real line version of this proof because we do not need to expand out an absolute value. Here, as we are using the hyperbolic cosine, we will have both e^x and e^{-x} present, thus avoiding the need to break into cases.

Proof. Our solution u is a 2π-periodic function, and we will restrict our attention to the interval $[0, 2\pi]$. As we know that the limits of q_1 and q_2 exist, we will further restrict our attention to a time $t_0 > 0$ when these position functions remain within a single period. This will avoid any complications of moving between periods which would require using the floor function in our definition. Using the exponential definition of the hyperbolic cosine, we get

$$u(x, t) = \frac{1}{2} \Big[e^{\pi - x} (p_1 e^{q_1} + p_2 e^{q_2}) + e^{-\pi + x} (p_1 e^{-q_1} + p_2 e^{-q_2}) \Big].$$

Rewriting this expression to generate terms containing w gives us

(7.41)
$$u(x, t) = \frac{1}{2} \Big[e^{\pi - x} \Big(e^{q_2} p_1 (e^{-q} - 1) + w e^{q_2} \Big) + e^{-\pi + x} \Big(w e^{-q_1} + e^{-q_1} p_2 (e^{-q} - 1) \Big) \Big].$$

Taking the limit as $t \to T^-$ of (7.41), and using the limit established in Lemma 2, we obtain

$$\lim_{t \to T^-} u(x,t) = w_T \cosh([x - q_T]_p - \pi). \qquad \square$$

We next demonstrate that u converges to v_T as $t \to T^-$ in L^r.

PROPOSITION 10 (convergence in L^r—periodic version of Proposition 5). *For our antipeakon-peakon solution u to the NE, we have*

$$(7.42) \qquad \lim_{t \to T^-} \|u(x,t) - v_T(x)\|_{L^r} = 0.$$

Proof. The same remarks that we made for the pointwise proof apply here to taking a $t_0 > 0$ such that q_1 and q_2 lie within a single 2π period after time t_0. Analogous to the pointwise limit, as we have both e^x and e^{-x} present in our hyperbolic cosines, we will not have to break our argument into cases in order to simplify the absolute values. This fact also allows us to bypass the restriction $1 \le r < 4$ as we do not cut the domain of the integration, creating the situation we saw in the real line case on the subintegral on $[q_1, q_2]$. After rewriting the hyperbolic cosines in their exponential form we get

$$\|u(x,t) - v_T(x)\|_{L^r}^r = \frac{1}{2} \int_0^{2\pi} \left| e^{x-\pi}\left(p_1 e^{-q_1} + p_2 e^{-q_2} - w_T e^{-q_T}\right) \right.$$
$$\left. + e^{\pi-x}\left(p_1 e^{q_1} + p_2 e^{q_2} - w_T e^{q_T}\right) \right|^r dx.$$

Using Jensen's inequality and evaluating the resulting integrals, we get

$$\frac{1}{2} \int_0^{2\pi} \left| e^{x-\pi}\left(p_1 e^{-q_1} + p_2 e^{-q_2} - w_T e^{-q_T}\right) + e^{\pi-x}\left(p_1 e^{q_1} + p_2 e^{q_2} - w_T e^{q_T}\right) \right|^r dx$$
$$\le \frac{e^{r\pi} - e^{-r\pi}}{r} \cdot \left| p_1 e^{-q_1} + p_2 e^{-q_2} - w_T e^{-q_T} \right|^r + \frac{e^{r\pi} - e^{-r\pi}}{r} \cdot \left| p_1 e^{q_1} + p_2 e^{q_2} - w_T e^{q_T} \right|^r.$$

Using Lemma 2, we have

$$\lim_{t \to T^-} \left| p_1 e^{q_1} + p_2 e^{q_2} - w_T e^{q_T} \right|^r = 0 \quad \text{and} \quad \lim_{t \to T^-} \left| p_1 e^{q_1} + p_2 e^{q_2} - w_T e^{q_T} \right|^r = 0.$$

Therefore applying the limit as $t \to T^-$ we get

$$\lim_{t \to T^-} \|u(x,t) - v_T(x)\|_{L^r}^r = 0. \qquad \square$$

Now that we have successfully established pointwise and L^r convergence, we will use these results to establish H^s by using the dominated convergence theorem.

THEOREM 8 (convergence in H^s—periodic version of Theorem 5). *For $s < 5/4$, our antipeakon-peakon solution u converges to v_T in H^s, i.e.,*

$$(7.43) \qquad \lim_{t \to T} \|u(x,t) - v_T(x)\|_{H^s} = 0.$$

Proof. From the definition of the H^s-norm of $u(x,t) - v_T(x)$, we have

$$\|u(x,t) - v_T(x)\|_{H^s}^2 = \sum_{n \in \mathbb{Z}} (1 + n^2)^s |\widehat{u}(n,t) - \widehat{v_T}(n)|^2.$$

Our objective, when taking the limit, will be to move the limit inside of the integral. Thus, we begin by examining the limit of the summand. As we have already established the convergence of u to v_T in L^1, via Proposition 10, we see that the inequality

$$|\widehat{u}(n, t) - \widehat{u}(n, T)| \leq \|u(x, t) - u(x, T)\|_{L^1}$$

implies that

$$\lim_{t \to T^-} (1 + n^2)^s |\widehat{u}(n, t) - \widehat{v_T}(n)|^2 = 0.$$

Next, we will define the bounding sequences that will allow us to apply the gDCT. We set

$$f_t(n) \doteq (1 + n^2)^s |\widehat{u}(n, t) - \widehat{v_T}(n)|^2 \leq 4(1 + n^2)^s \left(|\widehat{u}(n, t)|^2 + |\widehat{v_T}(n)|^2 \right) \doteq g_t(n).$$

We need to establish that the g_t's have a pointwise limit g, i.e., for each $n \in \mathbb{Z}$, $\lim_{t \to T^-} g_t(n) = g(n)$. The most obvious candidate for g is

$$g(n) = 8(1 + n^2)^s |\widehat{v}_T(n)|^2 \quad \text{where} \quad \sum_{n \in \mathbb{Z}} g(n) = 32 \cosh^2 \pi \cdot c_s w_T.$$

Indeed, using the laws of limits, we have that $\widehat{u}(n, t) \to \widehat{v}_T(n)$ for each n implies $g_t \to g$ for each n. To finish satisfying the hypotheses of the gDCT, we must now establish the sum properties of the g_t's. We have

$$\lim_{t \to T} \sum_{n \in \mathbb{Z}} g_t(n) = 32 \sinh^2 \pi \cdot c_s w_T,$$

where the left limit uses the 5/4-hypothesis (1.13). We now see that the hypotheses for the gDCT are satisfied. Thus, we can conclude that as $t \to T$, we have $u(x, t) \to v_T$ in H^s. □

Proof of Theorem 7. Translating the NE 1-peakon solution by x_0 and choosing the minus sign we obtain the following antipeakon solution for the NE:

$$v(x, t) = -\sqrt{c} \cosh([x - x_0 - ct]_p - \pi) \quad \text{for any} \quad c > 0 \quad \text{and} \quad x_0 \in \mathbb{T}.$$

As in the real line case, we choose $c = w_T^2$ and $x_0 = q_T - w_T^2 T$ and obtain the antipeakon solution $v(x, t) = w_T \cosh([x - x_0 - w_T^2 t] - \pi)$. Since at $t = T$ we have

$$v(x, T) = w_T \cosh([x - q_T]_p - \pi) = u(x, T),$$

we see that we have constructed two different NE solutions, which belong in H^s, $s < 5/4$, and agree at $t = T$. From here, a change of variables can recast these two solutions as stemming from the same initial data at time $t = 0$. This scenario proves failure of uniqueness in this range of Sobolev spaces. □

The case $s = 5/4$. The argument for ill-posedness in this case is precisely the same as that in the nonperiodic case. This result completes the proof of both Theorems 1 and 2 in the periodic case.

REFERENCES

[BKPSV] B. Birnir, C. Kenig, G. Ponce, N. Svanstedt, and L. Vega, *On the ill-posedness of the IVP for the generalized Korteweg-de Vries and nonlinear Schrödinger equations*, J. London Math. Soc., 53 (1996), pp. 551–559, https://doi.org/10.1112/jlms/53.3.551.

[BL1] J. Bourgain and D. Li, *Strong ill-posedness of the incompressible Euler equation in borderline Sobolev spaces*, Invent. Math., 201 (2015), pp. 97–157, https://doi.org/10.1007/s00222-014-0548-6.

[BL2] J. Bourgain and D. Li, *Strong illposedness of the incompressible Euler equation in integer C^m spaces*, Geom. Funct. Anal., 25 (2015), pp. 1–86, https://doi.org/10.1007/s00039-015-0311-1.

[BP] J. Bourgain and N. Pavlović, *Ill-posedness of the Navier-Stokes equations in a critical space in 3D*, J. Funct. Anal., 255 (2008), pp. 2233–2247, https://doi.org/10.1016/j.jfa.2008.07.008.

[BC] A. Bressan and A. Constantin, *Global conservative solutions of the Camassa-Holm equation*, Arch. Ration. Mech. Anal., 183 (2007), pp. 215–239, https://doi.org/10.1007/s00205-006-0010-z.

[By] P. Byers, *Existence time for the Camassa-Holm equation and the critical Sobolev index*, Indiana Univ. Math. J., 55 (2006), pp. 941–954, http://www.jstor.org/stable/24902427.

[CH] R. Camassa and D. Holm, *An integrable shallow water equation with peaked solitons*, Phys. Rev. Lett., 71 (1993), pp. 1661–1664, https://doi.org/10.1103/PhysRevLett.71.1661.

[CHT] C. Cao, D. Holm, and E. Titi, *Traveling wave solutions for a class of one-dimensional nonlinear shallow water wave models*, J. Dynam. Differential Equations, 16 (2004), pp. 167–178, https://doi.org/10.1023/B:JODY.0000041284.26400.d0.

[CCT] M. Christ, J. Colliander, and T. Tao, *Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations*, Amer. J. Math., 125 (2003), pp. 1235–1293, https://doi.org/10.1353/ajm.2003.0040.

[CL] A. Constantin and D. Lannes, *The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi equations*, Arch. Ration. Mech. Anal., 192 (2009), pp. 165–186, https://doi.org/10.1007/s00205-008-0128-2.

[CM] A. Constantin and H. McKean, *A shallow water equation on the circle*, Comm. Pure Appl. Math., 52 (1999), pp. 949–982, https://onlinelibrary.wiley.com/doi/10.1002/%28SICI%291097-0312%28199908%2952%3A8%3C949%3A%3AAID-CPA3%3E3.0.CO%3B2-D.

[DHH] A. Degasperis, D. D. Holm, and A. N. W. Hone, *A new integral equation with peakon solutions*, Theoret. Math. Phys., 133 (2002), pp. 1463–1474, https://doi.org/10.1023/A:1021186408422.

[DP] A. Degasperis and M. Procesi, *Asymptotic integrability*, in Symmetry and Perturbation Theory. (Rome, 1998), World Sci. Publ., 1999, pp. 23–37.

[DGH] H. Dullin, G. Gottwald, and D. Holm, *An integrable shallow water equation with linear and nonlinear dispersion*, Phys. Rev. Lett., 87 (2001), 194501, https://doi.org/10.1103/PhysRevLett.87.194501.

[EEP] M. Ehrnström, J. Escher, and L. Pei, *A note on the local well-posedness for the Whitham equation*, in Elliptic and Parabolic Equations, Springer Proc. Math. Stat. 119, Springer, 2015, pp. 63–75, https://doi.org/10.1007/978-3-319-12547-3_3.

[ELY] J. Escher, Y. Liu, and Z. Yin, *Shock waves and blow-up phenomena for the periodic Degasperis–Procesi equation*, Indiana Univ. Math. J., 56 (2007), pp. 87–117, http://www.jstor.org/stable/24902681.

[EY] J. Escher and Z. Yin, *Well-posedness, blow-up phenomena, and global solutions for the b-equation*, J. Reine Angew. Math., 624 (2008), pp. 51–80, https://doi.org/10.1515/CRELLE.2008.080.

[FF] B. Fuchssteiner and A. Fokas, *Symplectic structures, their Bäcklund transformations and hereditary symmetries*, Phys. D, 4 (1981), pp. 47–66, https://doi.org/10.1016/0167-2789(81)90004-X.

[GH]   K. GRAYSHAN AND A. HIMONAS, *Equations with peakon traveling wave solutions*, Adv. Dyn. Syst. Appl., 8 (2013), pp. 217–232.

[H]   J. HADAMARD, *Lectures on Cauchy's Problem in Linear Partial Differential Equations*, Dover, 1953.

[HGH]   A. HIMONAS, K. GRAYSHAN, AND C. HOLLIMAN, *Ill-posedness for the b-family of equations*, J. Nonlinear Sci., 26 (2016), pp. 1175–1190, https://doi.org/10.1007/s00332-016-9302-0.

[HHG]   A. HIMONAS, C. HOLLIMAN, AND K. GRAYSHAN, *Norm inflation and ill-posedness for the Degasperis-Procesi equation*, Comm. Partial Differential Equations, 39 (2014), pp. 2198–2215, https://doi.org/10.1080/03605302.2014.942737.

[HH1]   A. HIMONAS AND C. HOLLIMAN, *On well-posedness of the Degasperis-Procesi equation*, Discrete Contin. Dyn. Syst., 31 (2011), pp. 469–488, https://doi.org/10.3934/dcds.2011.31.469.

[HH2]   A. HIMONAS AND C. HOLLIMAN, *The Cauchy problem for the Novikov equation*, Nonlinearity, 25 (2012), pp. 449–479, https://doi.org/10.1088/0951-7715/25/2/449.

[HK]   A. HIMONAS AND C. KENIG, *Non-uniform dependence on initial data for the CH equation on the line*, Differential and Integral Equations, 22 (2009), pp. 201–224.

[HKM]   A. HIMONAS, C. KENIG, AND G. MISIOŁEK, *Non-uniform dependence for the periodic CH equation*, Comm. Partial Differential Equations, 35 (2010), pp. 1145–1162, https://doi.org/10.1080/03605300903436746.

[HM]   A. HIMONAS AND D. MATZAVINOS, *An ab-family of equations with peakon traveling waves*, Proc. Amer. Math. Soc., 144 (2016), pp. 3797–3811, https://doi.org/10.1090/proc/13011.

[HMP]   A. HIMONAS, G. MISIOŁEK, AND G. PONCE, *Non-uniform continuity in H^1 of the solution map of the CH equation*, Asian J. Math., 11 (2007), pp. 141–150.

[HS]   D. HOLM AND M. STALEY, *Nonlinear balance and exchange of stability in dynamics of solitons, peakons, ramps/cliffs and leftons in a $1 + 1$ nonlinear evolutionary PDE*, Phys. Lett. A, 308 (2003), pp. 437–444, https://doi.org/10.1016/S0375-9601(03)00114-2.

[HLS]   A. HONE, H. LUNDMARK, AND J. SZMIGIELSKI, *Explicit multipeakon solutions of Novikov's cubically nonlinear integrable Camassa-Holm type equation*, Dyn. Partial Differ. Equ., 6 (2009), pp. 253–289, https://doi.org/10.4310/DPDE.2009.v6.n3.a3.

[HW]   A. HONE AND J. WANG, *Integrable peakon equations with cubic nonlinearity*, J. Phys. A, 41 (2008), 372002, https://doi.org/10.1088/1751-8113/41/37/372002.

[KL]   H. KALISCH AND J. LENELLS, *Numerical study of traveling-wave solutions for the Camassa-Holm equation*, Chaos Solitons Fractals, 25 (2005), pp. 287–298, https://doi.org/10.1016/j.chaos.2004.11.024.

[KPV]   C. KENIG, G. PONCE, AND L. VEGA, *On the ill-posedness of some canonical dispersive equations*, Duke Math. J., 106 (2001), pp. 617–633, https://doi.org/10.1215/S0012-7094-01-10638-8.

[KT]   H. KOCH AND N. TZVETKOV, *Nonlinear wave interactions for the Benjamin-Ono equation*, Int. Math. Res. Not., 2005 (2005), pp. 1833–1847, https://doi.org/10.1155/IMRN.2005.1833.

[L1]   J. LENELLS, *Traveling wave solutions of the Camassa-Holm equation*, J. Differential Equations, 217 (2005), pp. 393–430, https://doi.org/10.1016/j.jde.2004.09.007.

[L2]   J. LENELLS, *Traveling wave solutions of the Degasperis-Procesi equation*, J. Math. Anal. Appl., 306 (2005), pp. 72–82, https://doi.org/10.1016/j.jmaa.2004.11.038.

[LO]   Y. LI AND P. OLVER, *Well-posedness and blow-up solutions for an integrable nonlinearly dispersive model wave equation*, J. Differential Equations, 162 (2000), pp. 27–63, https://doi.org/10.1006/jdeq.1999.3683.

[LPS]   F. LINARES, D. PILOD, AND J.-C. SAUT, *Dispersive perturbations of Burgers and hyperbolic equations* I: *Local theory*, SIAM J. Math. Anal., 46 (2014), pp. 1505–1537, https://doi.org/10.1137/130912001.

[LS]   H. LUNDMARK AND J. SZMIGIELSKI, *Multi-peakon solutions of the Degasperis-Procesi equation*, Inverse Problems, 19 (2003), pp. 1241–1245, https://doi.org/10.1088/0266-5611/19/6/001.

[Mc]   H. P. MCKEAN, *Breakdown of the Camassa-Holm equation*, Comm. Pure Appl. Math., 57 (2004), pp. 416–418, https://doi.org/10.1002/cpa.20003.

[MN]   A. MIKHAILOV AND V. NOVIKOV, *Perturbative symmetry approach*, J. Phys. A, 35 (2002), pp. 4775–4790, https://doi.org/10.1088/0305-4470/35/22/309.

[MST]   L. Molinet, J. C. Saut, and N. Tzvetkov, *Ill-posedness issues for the Benjamin–Ono and related equations*, SIAM J. Math. Anal., 33 (2001), pp. 982–988, https://doi.org/10.1137/S0036141001385307.

[N]   V. Novikov, *Generalizations of the Camassa-Holm equation*, J. Phys. A, 42 (2009), 342002, https://doi.org/10.1088/1751-8113/42/34/342002.

[Ti]   F. Tiğlay, *The periodic Cauchy problem for Novikov's equation*, Int. Math. Res. Not., 2011 (2011), pp. 4633–4648, https://doi.org/10.1093/imrn/rnq267.

[W]   G. Whitham, *Linear and Nonlinear Waves*, Wiley, 1974.