INVERSE PROBLEM FOR COMPACT FINSLER MANIFOLDS WITH THE BOUNDARY DISTANCE MAP

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Abstract. We prove that the boundary distance map of a smooth compact Finsler manifold with smooth boundary determines its topological and differential structures. We construct the optimal fiberwise open subset of its tangent bundle and show that the boundary distance map determines the Finsler function in this set but not in its exterior. If the Finsler function is fiberwise real analytic, it is determined uniquely. We also discuss the smoothness of the distance function between interior and boundary points.

1. Introduction

This paper is devoted to an inverse problem for smooth compact Finsler manifolds with smooth boundaries. We prove that the boundary distance map of such a manifold determines its topological and differential structures. In general, the boundary distance map is not sufficient to determine the Finsler function in those directions which correspond to geodesics that are either trapped or are not distance minimizers to terminal boundary points. To prove our result, we embed a Finsler manifold with boundary into a function space and use smooth boundary distance functions to give a coordinate structure and the Finsler function where possible.

This geometric problem arises from the propagation of singularities from a point source for the elastic wave equation. The point source can be natural (e.g. an earthquake as a source of seismic waves) or artificial (e.g. produced by the boundary control method or by a wave sent in scattering from a point scatterer). Due to polarization effects, there are singularities propagating at various speeds. We study the first arrivals and thus restrict our attention to the fastest singularities (corresponding to so-called qP-polarization, informally "pressure waves"). They follow the geodesic flow of a Finsler manifold, as we shall explain in more detail in section 2.

An elastic body — e.g. a planet — can be modeled as a manifold, where distance is measured in travel time: The distance between two points is the shortest time it takes for a wave to go from one point to the other. If the material is elliptically

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anisotropic, then this elastic geometry is Riemannian. However, this sets a very stringent assumption on the stiffness tensor describing the elastic system, and Riemannian geometry is therefore insufficient to describe the propagation of seismic waves in the Earth. We make no structural assumptions on the stiffness tensor apart from the physically necessary symmetry and positivity properties, and this leads necessarily to Finsler geometry.

The inverse problem introduced above can be rephrased as the following problem in geophysics. Imagine that earthquakes occur at known times but unknown locations within Earth's interior and arrival times are measured everywhere on the surface. Are such travel time measurements sufficient to determine the possibly anisotropic elastic wave speed everywhere in the interior and pinpoint the locations of the earthquakes? While earthquake times are not known in practice, this is a fundamental mathematical problem that underlies more elaborate geophysical scenarios. In the Riemannian realm the corresponding result [27, 29] is a crucial stepping stone towards the results of [7, 17, 19, 26, 30, 35]. We expect that solutions to inverse problems for the fully anisotropic elastic wave equation rely on geometrical results similar to the ones presented in this paper.

1.1. **Main results.** We let (M, F) be a smooth compact, connected Finsler manifold with smooth boundary ∂M (For the basic of theory of Finsler manifolds see the appendix (Section A) at the end of this paper.). We denote the tangent bundle of M by TM and use the notation (x, y) for points in TM, where x is a base point and y is a vector in the fiber T_xM. The notations T^*M and (x, p) are reserved for the cotangent bundle and its points respectively.

We write $d_F : M \times M \to \mathbb{R}$ for the non-symmetric distance function given by a Finsler function F. For a given $x \in M$ the boundary distance function related to x is

$$r_x \colon \partial M \to [0, \infty), \quad r_x(z) = d_F(x, z).$$

We denote $\mathcal{R}(M^{int}) = \{r_x : x \in M^{int}\}$ the collection of all boundary distance functions. In this paper, we study the inverse problem with *boundary distance data*

$$(\mathcal{R}(M^{int}), \partial M) \tag{1}$$

Inverse problem 1.1. *Do the boundary distance data* (1) *determine* (M, F) *up to isometry?*

We emphasize that we do not assume d_F to be symmetric and therefore data (1) contain only information where the distance is measured from the points of M^{int} to points in ∂M. We note that for any $x \in M^{int}$,

$$r_x(z) = d_{\overleftarrow{F}}(z, x), \quad z \in \partial M,$$

where $\overleftarrow{F}$ is the Finsler function

$$\overleftarrow{F}(x, y) := F(x, -y). \tag{2}$$

Therefore, data (1) are equivalent to the data

$$(\{d_{\overleftarrow{F}}(\cdot, x) \colon \partial M \to \mathbb{R} \mid x \in M^{int}\}, \partial M),$$

where the distance is measured from the boundary to the interior. In [27, 29] it is shown that the data (1) determine a Riemannian manifold (M, g) up to isometry. In the Finsler case this is not generally true. Next we explain what can be obtained from the Finslerian boundary distance data (1).

Notation 1.2. *For a Finsler manifold (M, F) with boundary, we denote by $G(M, F)$ the set of points $(x, y) \in TM \setminus 0$, $x \in M^{int}$ for which the geodesic starting at x in direction y reaches the boundary in finite time $t(x, y)$ and is minimizing between x and $z(x, y) := \gamma_{x,y}(t(x, y)) \in \partial M$, that is*

$$\gamma_{x,y}(0, t(x, y)) \subset M^{int}.$$

We emphasize that for any interior point $x \in M^{int}$ and for any $y \in T_xM$ it holds that $t(x, y) > 0$.

Since for any $(x, y) \in TM \setminus \{0\}$ and $a > 0$ it holds that $\gamma_{x,ay}(t) = \gamma_{x,y}(at)$, we notice that $G(M, F)$ is a conic set. Let $(x, y) \in G(M, F)$, then $t(x, ay) = a^{-1}t(x, y)$ and $z(x, y) = z(x, ay)$ for any $a > 0$. Moreover if $F(y) = 1$, then $t(x, y) = d_F(x, z(x, y))$.

We show that the data (1) determine the Finsler function in the closure of the set $G(M, F)$ and that the data (1) are not sufficient to recover the Finsler function F on $TM^{int} \setminus \overline{G(M, F)}$. The reason is that the data (1) do not provide any information about the geodesics that are trapped in M^{int} or do not minimize the distance between the point of origin and the terminal boundary point. Therefore, to recover the Finsler function F globally we assume that for every $x \in M$ the function $F(x, \cdot)\colon T_xM \setminus \{0\} \to \mathbb{R}$ is real analytic. We call such a Finsler function *fiberwise real analytic*. For instance Finsler functions $F(x, y) = \sqrt{g_x(y, y)}$, where g is a Riemannian metric, and Randers metrics are fiberwise real analytic. In Section 2 we show that also the Finsler metric related to the fastest polarization of elastic waves is fiberwise real analytic.

Now we formulate our main theorems. If (M_i, F_i), $i \in \{1, 2\}$ are smooth, connected, compact Finsler manifolds with smooth boundaries, we call a smooth map $\Phi\colon (M_1, F_1) \to (M_2, F_2)$ a *Finslerian isomorphism* if it is a diffeomorphism which satisfies

$$F_1(x, y) = F_2(\Phi(x), \Phi_* y), \quad (x, y) \in TM_1.$$

Here Φ_* is the pushforward by Φ. We say that the boundary distance data of manifolds (M_i, F_i), $i = 1, 2$ agree, if there exists a diffeomorphism $\phi\colon \partial M_1 \to \partial M_2$ such that

$$\{r_{x_1} : x_1 \in M_1^{int}\} = \{r_{x_2} \circ \phi : x_2 \in M_2^{int}\} \subset C(\partial M_1). \tag{3}$$

We emphasize that this is an equality of non-indexed sets and we do not know the point x_1 corresponding to the function r_{x_1}.

Our first main result shows that the boundary distance data (1) determine a manifold upto a diffeomorphism and a Finsler function in an optimal set.

Theorem 1.3. *Let (M_i, F_i), $i = 1, 2$ be smooth, connected, compact Finsler manifolds with smooth boundaries. We suppose that there exists a diffeomorphism*

$\phi\colon \partial M_1 \to \partial M_2$ so that (3) *holds. Then there is a diffeomorphism $\Psi\colon M_1 \to M_2$ so that $\Psi|_{\partial M_1} = \phi$. The sets $\overline{G(M_1, F_1)}$ and $\overline{G(M_1, \Psi^* F_2)}$ coincide and in this set $F_1 = \Psi^* F_2$, where the pullback $\Psi^* F_2 : TM_1 \to \mathbb{R}$ is the function*

$$\Psi^* F_2 : TM_1 \to \mathbb{R}, \quad \Psi^* F_2(x, y) = F_2(\Psi(x), \Psi_* y).$$

Moreover, for any $(x_0, y_0) \in TM_1^{int} \setminus \overline{G(M_1, F_1)}$ there exists a smooth Finsler function $F_3\colon TM_1 \to [0, \infty)$ so that $d_{F_1}(x, z) = d_{F_3}(x, z)$ for all $x \in M_1$ and $z \in \partial M_1$ but $F_1 \neq F_3$.

Remark 1.4. *The set $G(M, F)$ can be large or small as the following examples illustrate. If every geodesic of (M, F) is minimizing, then it holds that $\overline{G(M, F)} = TM$. This holds for instance on simple Riemannian manifolds. If M is any subset of S^2 larger than the hemisphere and if F is given by the round metric, then $TM^{int} \setminus \overline{G(M, F)}$ contains an open non-empty set U whose canonical projection to M is an open neighborhood of the equator.*

Our second main result shows that the boundary distance data (1) determine a fiberwise Finsler manifold upto isometry.

Theorem 1.5. *Let (M_i, F_i), $i = 1, 2$ be smooth, connected, compact Finsler manifolds with smooth boundary. We suppose that there exists a diffeomorphism $\phi\colon \partial M_1 \to \partial M_2$ such that* (3) *holds. If Finsler functions F_i are fiberwise real analytic, then there exists a Finslerian isometry $\Psi\colon (M_1, F_1) \to (M_2, F_2)$ so that $\Psi|_{\partial M_1} = \phi$.*

Remark 1.6. *In Theorems 1.3 and 1.5 we measure distances from the interior to the boundary. If we measure in the opposite direction, from boundary to the interior, this corresponds to the data* (1) *given with respect to the reversed Finsler function $\overleftarrow{F}(x, y)$. Our results give uniqueness for $\overleftarrow{F}$ and therefore F. That is, our main results hold no matter which way distances are measured.*

1.1.1. *Outline of the proofs of the main results.* Theorem 1.5 essentially follows from Theorem 1.3. We split the proof of Theorem 1.3 into four parts (subsections 3.1–3.4). In the first part, we show that the data (1) determine r_x for any $x \in M$. Then we study the properties of the map $\mathcal{R} : M \ni x \mapsto r_x \in L^\infty(\partial M)$ and show that this map is a topological embedding. We use the map $\mathcal{R}$ to construct a map $\Psi : (M_1, F_1) \to (M_2, F_2)$ that will be shown to be a homeomorphism. In the second part, we show that the map Ψ is a diffeomorphism. In the third part we connect the set $G(M, F)$ to smoothness of the distance functions of the form $d_F(\cdot, z)$, $z \in \partial M$. In the final section we use this to prove that the map Ψ is a Finslerian isometry.

We have included in this paper a supplemental Section 4 and the appendix (Section A), which contain necessary material for the proof of Theorem 1.3. We have also included some well-known results and properties in the Riemannian case while providing a detailed background of compact Finsler manifolds with and without boundary for the proof given in Section 3. To the best of our knowledge, most of this material cannot be found in the literature.

1.2. **Background and related work.**

1.2.1. *Geometric inverse problems.* The claim and proof of Theorem 1.3 are a modification of a similar result in a Riemannian case given in [27, 29]. The Riemannian version was first proven in [29]. In [27] also the construction of smooth structure is considered. The Riemannian version of Theorem 1.3 is related to many other geometric inverse problems. For instance, it is a crucial step in proving uniqueness for Gel'fand's inverse boundary spectral problem [27]. Gel'fand's problem concerns the question whether the data

$$\{\partial M, (\lambda_j, \partial_\nu \phi_j|_{\partial M})_{j=1}^\infty\}$$

determine (M, g) up to isometry. Above (λ_j, ϕ_j) are the Dirichlet eigenvalues and the corresponding L^2-orthonormal eigenfunctions of the Laplace-Beltrami operator. Belishev and Kurylev provide an affirmative answer to this problem in [7].

We recall that the Riemannian wave operator is a globally hyperbolic linear partial differential operator of real principal type. Therefore, the Riemannian distance function and the propagation of a singularity initiated by a point source in space time are related to one another. In other words, $r_x(z) = t(z) - s$, where $t(z)$ is the time when the singularity initiated by the point source $(s, x) \in (0, \infty) \times M$ hits $z \in \partial M$. If the initial time s is unknown, but the arrival times $t(z), z \in \partial M$ are known, then one obtains a boundary distance difference function $D_x(z_1, z_2) := r_x(z_1) - r_x(z_2),\ z_1, z_2 \in \partial M$. In [35] it is shown that if $U \subset N$ is a compact subset of a closed Riemannian manifold (N, g) and $U^{int} \neq \emptyset$, then *distance difference data* $\{(U, g|_U), \{D_x \colon U \times U \to \mathbb{R} \mid x \in N\}\}$ determine (N, g) up to isometry. This result was recently generalized to complete Riemannian manifolds [26] and for compact Riemannian manifolds with boundary [19].

If the sign in the definition of the distance difference functions is changed, we arrive at the distance sum functions,

$$D_x^+(z_1, z_2) = d(z_1, x) + d(z_2, x), \quad x \in M,\ z_1, z_2 \in \partial M. \tag{4}$$

These functions give the lengths of the broken geodesics, that is, the union of the shortest geodesics connecting z_1 to x and the shortest geodesics connecting x to z_2. Also, the gradients of $D_x^+(z_1, z_2)$ with respect to z_1 and z_2 give the velocity vectors of these geodesics. The inverse problem of determining the manifold (M, g) from the *broken geodesic data*, consisting of the initial and the final points and directions, and the total length, of the broken geodesics, has been considered in [30]. In [30] the authors show that broken geodesic data determine the boundary distance data and use then the results of [27, 29] to prove that the broken geodesic data determine the Riemannian manifold up to isometry.

We let u be the solution of the Riemannian wave equation with a point source at $(s, x) \in (0, \infty) \times M$. In [20, 22] it is shown that the image, Λ, of the wavefront set of u, under the canonical isomorphism $T^*M \ni (x, p) \mapsto g^{ij}(x)p_i \in TM$, coincides with the image of the unit sphere S_xM at x under the geodesic flow of g. Thus $\Lambda \cap \partial(SM)$, where SM is the unit sphere bundle of (M, g), coincides with the exit directions of geodesics emitted from p. In [36] the authors show that if (M, g) is a compact smooth non-trapping Riemannian manifold with smooth strictly convex

boundary, then generically the *scattering data of point sources* $\{\partial M, R_{\partial M}(M)\}$ determine (M, g) up to isometry. Here, $R_{\partial M}(x) \in R_{\partial M}(M)$, $x \in M$ stands for the collection of tangential components to boundary of exit directions of geodesics from x to ∂M.

A classical geometric inverse problem, that is closely related to the distance functions, asks: Does the Dirichlet-to-Neumann mapping of a Riemannian wave operator determine a Riemannian manifold up to isometry? For the full boundary data this problem was solved originally in [7] using the Boundary control method. Partial boundary data questions have been sudied for instance in [34, 41]. Recently [32] extended these results for connection Laplacians. Lately also inverse problems related to non-linear hyperbolic equations have been studied extensively [31, 37, 54]. For a review of inverse boundary value problems for partial differential equations see [33, 52].

Another well studied geometric inverse problem formulated with the distance functions is the Boundary rigidity problem. This problem asks: Does the *boundary distance function* $d_F \colon \partial M \times \partial M \to \mathbb{R}$, that gives a distance between any two boundary points, determine (M, F) up to isometry? For the best to our knowledge this problem has not been studied in Finsler geometry. For a general Riemannian manifold the problem is false: Suppose the manifold contains a domain with very slow wave speed, such that all the geodesics starting and ending at the boundary avoid this domain. Then in this domain one can perturb the metric in such a way that the boundary distance function does not change. It was conjectured in [40] that for all compact simple Riemannian manifolds the answer is affirmative. In two dimensions it was solved in [46]. For higher dimensional case the problem is still open, but different variations of it has been considered for instance in [16, 49, 50]. The Boundary distance data (1), studied in this paper, is much larger data than the knowledge of the boundary distance function. Therefore we can obtain the optimal determination of (M, F), as explained in theorems 1.3 and 1.5, even though we pose no geometric conditions on (M, F).

1.2.2. *Finsler and Riemannian geometry.* We refer to the monographs [5, 48] for the development of Finsler manifolds without boundaries. We point out that two major differences occur between the Riemannian and Finslerian realms that are related to the proof of Theorem 1.3. In Riemannian geometry the relation between TM and T^*M is simple; raising and lowering indices provides a fiberwise linear isomorphism. In Finsler geometry this is not possible, since the Legendre transform (see (58)) is not linear in the fibers. For this reason, we have to be more careful in the analysis of distance functions and the connection of their differential to the velocity fields of geodesics. Moreover the Finslerian gradient is not a linear operator.

The second issue arises from the lack of a natural linear connection compatible with F on vector bundle $\pi \colon TM \to M$, where $\pi(x, y) = x$ is the canonical projection to the base point. In Section 4 we consider properties of Chern connection ∇, which is a torsion free linear connection on the pullback bundle $\pi' \colon \pi^*TM \to TM$ (see [4, 14, 15]). We derive natural compatibility relations for ∇ and the fundamental tensor field g on π^*TM g (see [48, Section 5.2] and Lemma 4.2) in a special case. In

[47] Shen proved the general version of the compatibility relations. We use Lemma 4.2 to formulate the initial conditions for so-called transverse vector fields (see [13, Section III.6]) with respect to ∂M along boundary normal geodesics. After this we give a definition of an index form related to these vector fields and use it to prove results similar to classical theorems, originally by Jacobi, related to the minizing of geodesics after focal points (for the Riemannian case, see for instance [13, Section III.6]).

Inverse problems arising from elastic equations have been also extensively studied. See e.g. [1, 3, 6, 23, 25, 43, 44].

2. From Elasticity to Finsler geometry

The main physical motivation of this paper is to obtain a geometric and coordinate invariant point of view to the inverse problems related to the propagation of seismic waves. The seismic waves are modelled by the anisotropic elastic wave equation in $\mathbb{R}^{1+3}$. This elastic system can be microlocally decoupled to 3 different polarizations [51]. In this section, we introduce a connection between the fastest polarization (known as the quasi pressure polarization and denoted by qP) and the Finsler geometry. More over it turns out that the Finsler metric arising from elasticity is fiberwise real analytic. We use the typical notation and terminology of the seismological literature, see for instance [12]. We let $c_{ijk\ell}(x)$ be the smooth stiffness tensor on $\mathbb{R}^3$ which satisfies the symmetry

$$c_{ijk\ell}(x) = c_{jik\ell}(x) = c_{k\ell ij}(x), \quad x \in \mathbb{R}^3. \tag{5}$$

We also assume that the density $\rho(x)$ is a smooth function of x and define density–normalized elastic moduli

$$a_{ijk\ell}(x) = \frac{c_{ijk\ell}(x)}{\rho(x)}.$$

The elastic wave operator P, related to $a_{ijk\ell}$, is given by

$$P_{i\ell} = \delta_{i\ell}\frac{\partial^2}{\partial t^2} - a_{ijk\ell}(x)\frac{\partial}{\partial x^j}\frac{\partial}{\partial x^k} + \text{lower order terms.}$$

For every $(x,p) \in \mathbb{R}^3 \times \mathbb{R}^3$ we define a square matrix $\Gamma(x,p)$, by

$$\Gamma_{i\ell}(x,p) := a_{ijk\ell}(x)p^k p^j. \tag{6}$$

The matrix $\Gamma(x,p)$ is called the *Christoffel matrix*. Due to (5) the matrix $\Gamma(x,p)$ is symmetric. One also assumes that $\Gamma(x,p)$ is positive definite for every $(x,p) \in \mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\})$.

The principal symbol $\delta(t,x,\omega,p)$ of the operator P is then given by

$$\delta(t,x,\omega,p) = \omega^2 I - \Gamma(x,p), \quad (t,x,\omega,p) \in \mathbb{R}^{1+3} \times \mathbb{R}^{1+3}.$$

Since the matrix $\Gamma(x,p)$ is positive definite and symmetric, it has three positive eigenvalues $\lambda^m(x,p)$, $m \in \{1,2,3\}$.

We assume that

$$\lambda^1(x,p) > \lambda^m(x,p), \quad m \in \{2,3\},\ (x,p) \in \mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\}). \tag{7}$$

Then it follows from the Implicit Function Theorem that $\lambda^1(x,p)$ and a related unit eigenvector $q^1(x,p)$ are smooth with respect to (x,p). See for instance [21, Chapter 11, Theorem 2] for more details. Moreover the function $\lambda^1(x,p)$ is homogeneous of degree 2 with respect to p.

To keep the notation simple, we write from now on $\lambda := \lambda^1(x,p)$ and $q := q^1(x,p)$. We use $\Gamma q = \lambda q$ and (6) to compute the Hessian of $\lambda(x,p)$ with respect to p. We obtain

$$\mathrm{Hess}_p(\lambda(x,p)) = 2\Big(\Gamma(q(x,p)) + (Dq)^T(\lambda(x,p)I - \Gamma(x,p))Dq\Big), \tag{8}$$

where Dq is the Jacobian of $q(x,p)$ with respect to p and the superscript T stands for transpose. Since $\Gamma(q(x,p))$ is positive definite it follows from (7) and (8) that the Hessian of $\lambda(x,p)$ is also positive definite. We note that a similar result has been presented in [2] under the assumption the stiffness tensor is homogeneous and transversely isotropic.

We define a continuous function $f(x,p) := \sqrt{\lambda(x,p)}$, which is smooth outside $\mathbb{R}^3 \times \{0\}$. We conclude with summarizing the properties of the function f

- $f\colon \mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\}) \to (0,\infty)$ is smooth, real analytic on the fibers;
- for every $(x,p) \in \mathbb{R}^3 \times \mathbb{R}^3$ and $s \in \mathbb{R}$ it holds that $f(x,sp) = |s| f(x,p)$;
- for every $(x,p) \in \mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\})$ the Hessian of $\frac{1}{2}f^2$ is symmetric and positive definite with respect to p.

Therefore, f is a convex norm on the cotangent space. Finally, we define a Finsler function F to be the Legendre transform of f. Thus the bicharacteristic curves of Hamiltonian $\frac{1}{2}(\lambda(x,p)-1)$ are given by the co-geodesic flow of F. Moreover the qP group velocities are given by the Finsler structure.

Another geometrical inverse problem on Finsler manifolds, using exterior geodesic sphere data, is presented in [18], extending an earlier result on Riemannian manifolds [17].

3. Proof of theorem 1.3

In this section we provide a proof of Theorem 1.3. The proof is divided into four parts. In the first part, we consider the topology and introduce a homeomorphism Ψ from (M_1,F_1) onto (M_2,F_2). The second part is devoted to proving that homeomorphism Ψ is smooth and has a smooth inverse. In the third part, we study smoothness of a distance function $d_F(\cdot,z)$, $z \in \partial M$ in those interior points x where a distance minimizing curve from x to z is a geodesic contained in the interior. Then, in the final part, we use the result obtained in the third part to prove that the Finsler functions F_1, $\Psi^* F_2$ coincide in the set $\overline{G(M_1,F_1)}$ (recall Notation 1.2), but not necessarily in its exterior.

3.1. **Topology.** Here, we define a map $\Psi\colon (M_1, F_1) \to (M_2, F_2)$ that will be shown to satisfy the claim of Theorem 1.3. Whenever we do not need to distinguish manifolds M_1 and M_2 we drop the subindices.

We start with showing that data (1) determine the function $r_x\colon \partial M \to M$ for any $x \in \partial M$. By the triangle inequality and the continuity of distance function $d_F(\cdot, z)$ on M we have

$$r_x(z) := d_F(x,z) = \sup_{q \in M^{int}} (d_F(q,z) - d_F(q,x)) = \sup_{q \in M^{int}} (r_q(z) - r_q(x)) \tag{9}$$

for all $z \in \partial M$. Thus data (1) determine r_x, moreover (3) and (9) imply

$$\{r_{x_1} : x_1 \in M_1\} = \{r_{x_2} \circ \phi : x_2 \in M_2\} \subset C(\partial M_1). \tag{10}$$

Since ∂M is compact it holds that for any $x \in M$ the corresponding boundary distance function r_x belongs to $C(\partial M) \subset L^\infty(\partial M)$. By (1) and (9) we have recovered the mapping

$$\mathcal{R}\colon M \to C(\partial M), \quad \mathcal{R}(x) = r_x. \tag{11}$$

In the next proposition, we study the properties of this map.

Proposition 3.1. *Let (M, F) be a smooth compact Finsler manifold with smooth boundary. The map $\mathcal{R}$ given by* (11) *is a topological embedding.*

Proof. Since M is compact, d_F is a complete non-symmetric (path) metric, and by [11, Theorem 2.5.23] for any $x_1, x_2 \in M$ there exists a distance minimizing curve $\gamma\colon [0, d_F(x_1,x_2)] \to M$ from x_1 to x_2. Moreover, whenever $a, b \in [0, d_F(x_1,x_2)]$ are such that $\gamma((a,b)) \subset M^{int}$, then $\gamma\colon [a,b] \to M$ is a geodesic.

Since the unit sphere bundle $SM := F^{-1}\{1\}$ is compact there exists a universal constant $L > 1$, such that for all $x_1, x_2 \in M$ we have

$$\frac{1}{L} d_F(x_1,x_2) \le d_F(x_2,x_1) \le L d_F(x_1,x_2). \tag{12}$$

We let $x_1, x_2 \in M$ and $z \in \partial M$. By triangular inequality we have

$$|d_F(x_1,z) - d_F(x_2,z)| \le L d_F(x_1,x_2).$$

Thus $\|r_{x_1} - r_{x_2}\|_\infty \le L d_F(x_1,x_1)$, which proves that the map $\mathcal{R}$ is continuous.

We suppose then that $r_{x_1} = r_{x_2}$ for some $x_1, x_2 \in M$. We let z be one of the closest boundary points to x_1. Then z is also a closest boundary point to x_2. Denote $r_{x_1}(z) = h$. If $h = 0$ then $x_1 = z$ and thus $x_1 = x_2$. We suppose then that $h > 0$, which means that x_1 and x_2 are interior points of M. We let γ be a unit speed distance minimizing curve from x_1 to z. Then γ is a geodesic and $\dot\gamma(h)$ is an outward pointing normal vector to ∂M (see Lemma A.5 in the appendix for the details). Since γ is also a distance minimizer from x_2 to z, we have proved

$$x_1 = \overleftarrow{\gamma}\,(h) = x_2,$$

where $\overleftarrow{\gamma}\,(t) := \gamma(h-t)$.

The injectivity of $\mathcal{R}$ implies that it is a topological embedding, as any continuous one-to-one map from a compact space to a Hausdorff space is an embedding. □

Next we define maps

$$\Phi\colon C(\partial M_1) \to C(\partial M_2), \quad \Phi(f) = f \circ \phi^{-1}$$

and

$$\Psi\colon M_1 \to M_2, \quad \Psi = \mathcal{R}_2^{-1} \circ \Phi \circ \mathcal{R}_1 \tag{13}$$

Here $\mathcal{R}_i$ is defined as $\mathcal{R}$ in (11). The main theorem of the section is the following

Theorem 3.2. *Let (M_i, F_i), $i = 1, 2$ be as in Theorem 1.3. Then the map $\Psi\colon M_1 \to M_2$ given by* (13) *is a homeomoprhism. Moreover $\Psi|_{\partial M_1} = \phi$.*

Proof. By (10) and Proposition 3.1 it holds that Ψ is well defined. Clearly the map Φ is a homeomorphism and therefore Ψ is a homeomorphism.

We let $x_1 \in \partial M_1$. Then $(\Phi \circ \mathcal{R}_1)(x_1)$ is r_{x_2} for some $x_2 \in M_2$. Since

$$r_{x_2}(\phi(x_1)) = [(\Phi \circ \mathcal{R}_1)(x_1)](\phi(x_1)) = r_{x_1}(x_1) = 0.$$

This proves $\Psi(x_1) = \phi(x_1)$. □

3.2. **Differentiable structure.** Here, we show that the map $\Psi\colon M_1 \to M_2$ is a diffeomorphism. We split the study in two cases, near the boundary and far from the boundary. We begin with the former one.

We extend (M, F) to a closed Finsler manifold (N, H) to facilitate the study of boundary points.

We let $\overleftarrow{\nu_{in}}$ be the inward pointing unit normal vector field to ∂M with respect to reversed Finsler function $\overleftarrow{F}$. We define the normal exponential map $\exp^{\perp}\colon \partial M \times \mathbb{R} \to N$ so that

$$\exp^{\perp}(z, s) := \overleftarrow{\exp}_z\,(s\,\overleftarrow{\nu_{in}}\,(z)),$$

where $\overleftarrow{\exp}_z$ is the exponential map of the reversed Finsler function $\overleftarrow{H}$.

Lemma 3.3. *There exists $h > 0$ such that M is contained in the image of the normal map. Moreover there exists $r > 0$ such that $\exp^{\perp}\colon \partial M \times [0, r) \to M$ is a diffeomorphism onto its image.*

Proof. Define

$$h = \max\{d_F(x, \partial M) : x \in M\} + c,$$

for any $c > 0$. Since N is compact the map $\exp^{\perp}\colon \partial M \times [0, h) \to N$ is well defined. Moreover it holds that any interior point can be connected to any of its closest boundary points via distance minimizing geodesic that is normal to the boundary. Therefore we conclude that $M \subset \exp^{\perp}(\partial M \times [0, h))$.

Notice that

$$\exp^{\perp}(z, t) = \pi(\overleftarrow{\phi}_t\,(z, \overleftarrow{\nu_{in}}\,(z))),$$

where $\overleftarrow{\phi_t}$ is the geodesic flow of $\overleftarrow{H}$. Since $\overleftarrow{\nu_{in}}$ is a smooth unit length vector field, this proves that $\exp^{\perp}$ is smooth.

We let $z \in \partial M$. Give any local coordinates (z', f) near z such that $f|_{\partial M} = 0$ is a boundary defining function. Then with respect to coordinates (z', t) for $(\partial M \times (-h, h))$ we have

$$D\exp^{\perp}(z,0) = \begin{pmatrix} D_{z'}(z' \circ \exp^{\perp}) & \frac{\partial}{\partial t}(z' \circ \exp^{\perp}) \\ \\ D_{z'}(f \circ \exp^{\perp}) & \frac{\partial}{\partial t}(f \circ \exp^{\perp}) \end{pmatrix} = \begin{pmatrix} id_{n-1} & \overline{a} \\ \\ \overline{0}^T & df(\overleftarrow{\nu}_{in}) \end{pmatrix},$$

where $\overline{a}, \overline{0} \in \mathbb{R}^{n-1}$ and $df(\overleftarrow{\nu}_{in}) \neq 0$, since $\overleftarrow{\nu}_{in}$ is not tangential to ∂M and f is a boundary defining function. Thus the Jacobian $\exp^{\perp}(z,0)$ is invertible and by the Inverse Function Theorem $\exp^{\perp}$ is a local diffeomorphism.

Next we show that there exists $r \in (0,h)$ such that $\exp^{\perp}\colon \partial M \times [0,r) \to M$ is a diffeomorphism onto its image. If this does not hold, there exists a sequence $(x_j)_{j=1}^{\infty} \in M$ such that

$$\exp^{\perp}(z_j^1, s_j^1) = x_j = \exp^{\perp}(z_j^2, s_j^2)$$

for some $s_j^i \to 0$, $i \in \{1,2\}$ as $j \to \infty$ and for some boundary points z_j^1 and z_j^2 such that $(z_1, s_1) \neq (z_2, s_2)$. Then $d_F(x_j, \partial M) \to 0$ as $j \to \infty$ and by the compactness of N we may assume that $x_j \to x \in \partial M$. Let $\epsilon > 0$ and choose $j \in \mathbb{N}$ so that $d_F(x_j, x), s_j^i < \epsilon$. Then for $i \in \{1,2\}$ it holds that

$$d_F(x, z_j^i) \leq d_F(x, x_j) + d_F(x_j, z_j^i) < 2L\epsilon$$

where L is the constant of (12). Therefore, $z_j^i \to x$ as $j \to \infty$ for $i \in \{1,2\}$. This is a contradiction to the local diffeomorphism property of $\exp^{\perp}$. Thus there exists $r > 0$ that satisfies the claim of this lemma. □

We immediately obtain

Corollary 3.4. *Let (M, F) be compact Finsler manifold with smooth boundary that is isometrically embedded into a closed Finsler manifold (N, H). Let us denote*

$$U(\partial M, \epsilon) := \{x \in M : d_F(x, \partial M) < \epsilon\}.$$

There exists $\epsilon > 0$ and a diffeomorphism $U(\partial M, \epsilon) \ni x \mapsto (z(x), s(x)) \in (\partial M \times [0, \epsilon))$, such that

$$d_F(x, z(x)) = d_F(x, \partial M) = s(x).$$

Proof. The claim follows from Lemma, 3.3, if we denote $(z(x), s(x)) := (\exp^{\perp})^{-1}(x)$. □

We then consider points far from the boundary. Our goal is to show that for every $x_0 \in M^{int}$ there exists points $(z_i)_{i=1}^n \subset \partial M$ and a neighborhood U of x_0 such that the map

$$U \ni x \mapsto (d_F(x, z_i))_{i=1}^n$$

is a coordinate map. To do this we need to set up some notation.

Definition 3.5. *Let $z \in \partial M$. We say that*

$$\tau_{\partial M}(z) := \sup\{t > 0 : d_F(\exp^\perp(z,t), z) = d_F(\exp^\perp(z,t), \partial M) = t\},$$

is the boundary cut distance to z. Then we define the collection of boundary cut points $\sigma(\partial M)$ as follows

$$\sigma(\partial M) = \{\exp^\perp(z, \tau_{\partial M}(z)) : z \in \partial M\}.$$

The set $\sigma(\partial M)$ is not empty and the next lemma explains why we cannot use the coordinate structure given by Lemma 3.4 far from ∂M.

Lemma 3.6. *Let $z \in \partial M$ and $t_0 = \tau_{\partial M}(z)$. Then at least one of the following holds*

1. *The map $\exp^\perp$ is singular at (z, t_0).*
2. *There exists $q \in \partial M$, $q \neq z$ such that $\exp^\perp(z, t_0) = \exp^\perp(q, t_0)$.*

Moreover for any $t \in [0, t_0)$ the map $\exp^\perp$ is non-singular at (z, t).

Proof. The proof of the first claim is a modification of the proof of [53, Chapter 13, Propostion 2.2]. The proof of the last claim is long. It is considered in detail in Section 4. □

Lemma 3.7. *The function $\tau_{\partial M} \colon \partial M \to \mathbb{R}$ is continuous.*

Proof. The proof is a modification of the proofs of [28, Lemma 2.1.15] and [53, Chapter 13, Proposition 2.9]. □

Recall that the *cut distance function* of the extended manifold (N, H) is defined as

$$\tau(x,v) = \sup\{t > 0 : d_H(x, \gamma_{x,v}(t)) = t\}, \quad (x,v) \in TN, \quad F(x,v) = 1. \tag{14}$$

We call a point $\gamma_{x,v}(\tau(x,v))$ *an ordinary cut point* to x. In the next Lemma we show that a boundary cut point always occurs before an ordinary cut point.

Lemma 3.8. *For any $z \in \partial M$ it holds that*

$$\overleftarrow{\tau}\,(z, \overleftarrow{\nu}_{in}\,(z)) > \tau_{\partial M}(z),$$

where $\overleftarrow{\tau}$ is the cut distance function of the reversed Finsler metric $\overleftarrow{H}$.

Proof. The proof is a modification of the proof of [27, Lemma 2.13]. □

Corollary 3.9. *Let $x_1 \in M$ and $z_{x_1} \in \partial M$ be a closest boundary point to x. There exist neighborhoods $U \subset M$ of x_1 and V of z_{x_1} such that for every $(x,z) \in (U \times (\partial M \cap V))$ there exists the unique distance minimizing unit speed geodesic $\gamma_{x,z}$ from x to z and moreover $\gamma_{x,z}[0, d_F(x,y))) \subset M^{int}$.*

Proof. The claim follows from the Implicit Function Theorem, Lemmas A.2 and 3.8, and the fact that $\overleftarrow{\nu}_{in}$ is transversal to the boundary.

□

We let $z \in \partial M$ and define an evaluation function $E_z \colon \mathcal{R}(M) \to \mathbb{R}$ by $E_z(r) = r(z)$. We note that the functions E_z correspond to the distance function $d_F(\cdot, z) \colon M \to \mathbb{R}$ via the equation

$$d_F(x, z) = (E_z \circ \mathcal{R})(x). \tag{15}$$

Since $z \in \partial M$ was an arbitrary point we note that the function $d_F \colon M \times \partial M \to \mathbb{R}$ is determined by the data (1) in the sense of (15).

We define the *exit time function*

$$\tau_{exit} \colon SM^{int} \to [0, \infty], \quad \tau_{exit}(x, v) := \inf\{t > 0 : \gamma_{x,v}(t) \in \partial M\}. \tag{16}$$

Lemma 3.10. *If $(x, v) \in SM^{int}$ is such that $\tau_{exit}(x, v) < \infty$ and $\dot\gamma_{x,v}(\tau_{exit}(x, v))$ is transversal to ∂M then there exists a neighborhood $U \subset SM$ of (x, v) such that $\tau_{exit}|_U$ is well defined and C^∞-smooth.*

Proof. Since $\dot\gamma_{x,v}(t_0)$ is not tangential to ∂M the claim follows from the Implicit Function Theorem in boundary coordinates. □

Take an interior point $x \in M$ near which we want to construct a system of coordinates. We let $v \in S_xM$ be such that the geodesic $\gamma_{x,v}$ emanating from x to the direction v is the shortest curve between x and a terminal boundary point z_x. By Lemma 3.8 these two points are not conjugate along $\gamma_{x,v}$.

We let $U \subset SM$ be so small neighborhood of (x, v) that the exit time function $\tau_{exit} \colon U \to \mathbb{R}$ is defined and smooth. We have thus assumed that x and $z_x = \gamma_{x,v}(\tau_{exit}(x, v))$ are connected minimally and without conjugate points by $\gamma_{x,v}$.

We let $\ell_x \colon T_xM \to T_x^*M$ be the Legendre transform, (to recall the definition see (58) in the appendix). It and its inverse are smooth outside the origin. Thus the distance function $d_F(\cdot, z_x)$ is smooth near x and its differential at x is $\ell_x(v) \in T_x^*M$ (see Lemma A.4).

Pick any $u \in T_x^*M \setminus \{0\}$ with $\langle u, v\rangle = 0$. For $s \in \mathbb{R}$, denote

$$v_s = \frac{\ell_x^{-1}(\ell_x(v) + su)}{F^*(\ell_x(v) + su)}.$$

Here F^* is the dual of F, (see (60)). The map $s \mapsto v_s \in S_xM$ is smooth.

Consider the geodesics γ_{v_s} starting at x in the direction v_s. Since $\gamma_{x,v}(\tau_{exit}(x, v))$ is transversal to ∂M, then $s \mapsto \gamma_{v_s}(\tau_{exit}(x, v_s))$ is smooth near $s = 0$. Also since x is not an ordinary cut point to $\gamma_{v_s}(\tau_{exit}(x, v_s))$ at $s = 0$, it is not either an ordinary cut point to $\gamma_{v_s}(\tau_{exit}(x, v_s))$ when $|s|$ is small. Therefore, for s sufficiently close to zero the distance function to $\gamma_{v_s}(\tau_{exit}(x, v_s))$ is smooth near x.

The differential of the distance function at x amounts to

$$\ell_x(v_s) = \frac{\ell_x(v) + su}{F^*(\ell_x(v) + su))}.$$

Therefore, for any u with the required property there is a small non-zero s so that there is a distance function to a boundary point which is smooth near x and the differential at x is $\frac{\ell_x(v)+su}{F^*(\ell_x(v)+su)}$.

We take $n-1$ covectors $u_1, \dots, u_{n-1} \in T_x^*M$ so that the set

$$\{\ell_x(v), u_1, \dots, u_{n-1}\} \subset T_x^*M$$

is linearly independent and each $u_i \in T_x^*M$ is orthogonal to $v \in T_xM$. For each $i = 1, \dots, n-1$ we take $s_i \neq 0$ so that

$$\frac{\ell_x(v) + s_i u_i}{F^*(\ell_x(v) + s_i u_i)}$$

is the differential of a distance function to a boundary point as described above.

This gives rise to distance functions to n boundary points close to one another. These functions are smooth near x and the differentials are

$$\ell_x(v), \frac{\ell_x(v) + s_1 u_1}{F^*(\ell_x(v) + s_1 u_1)}, \dots, \frac{\ell_x(v) + s_{n-1} u_{n-1}}{F^*(\ell_x(v) + s_{n-1} u_{n-1})}.$$

This set is linearly independent, so the distance functions give a smooth system of coordinates in a neighborhood of x. Thus we obtain

Lemma 3.11. *Let $x_0 \in M^{int}$. There is a neighborhood U of x_0 and points $z_1, \dots, z_n \in \partial M$, where z_1 is a closest boundary point to x_0, so that the mapping $U \ni x \mapsto (d_F(x, z_i))_{i=1}^n$ is a smooth coordinate map.*

Moreover there exists an open neighborhood $V \subset \partial M$ of z_1 such that the distance function $d_F \colon U \times V \to \mathbb{R}$ is smooth and the set

$$\mathcal{V} := \Big\{ (z_i)_{i=2}^n \in V^{n-1} : det(f_{z_2,\dots,z_n}(x)) \Big|_{x=x_0} \neq 0 \Big\}.$$

is open and dense in $V^{n-1} := V \times \cdots \times V$. Where

$$f_{z_2,\dots,z_n}(x) := D\widetilde{f}_{z_2,\dots,z_n}(x), \tag{17}$$

and $D\widetilde{f}_{z_2,\dots,z_n}$ stands for the pushforward of the map

$$\widetilde{f}_{z_2,\dots,z_n}(x) := (d_F(x, z_i))_{i=1}^n) \in \mathbb{R}^n, \quad x \in U.$$

Proof. It remains to show that the set $\mathcal{V}$ is open and dense in V^{n-1}. Clearly the function

$$G : V^{n-1} \to \mathbb{R}, \quad G(z_2, \dots, z_n) = \det(f_{z_2,\dots,z_n}(x_0))$$

is continuous. Thus $\mathcal{V} = V^{n-1} \setminus G^{-1}\{0\}$ is open. Since the Legendre transform is an metric isometry between fibers, we have for every $z \in V$

$$d(d_F(\cdot, z))|_{x_0} \in S_{x_0}^*M := \{p \in T_{x_0}^*M : F^*(p) = 1\}$$

(For the details see Lemma A.4 in the appendix.). We let $(e_i)_{i=1}^n$ be a basis of $T_{x_0}^*M$ and define a map $T \colon (T_{x_0}^*M)^{n-1} \to \mathbb{R}$ by

$$T((u_i))_{i=2}^n = \det(M(\ell_{x_0}(v), u_2, \dots, u_n)),$$

where $\ell_{x_0}(v)$ is as in the discussion before this lemma and $M(\ell_{x_0}(v), u_1, \dots, u_{n-1})$ is a real $n \times n$ matrix with columns $\ell_{x_0}(v), u_1, \dots, u_{n-1}$, with respect to basis $(e_i)_{i=1}^n$ of $T_{x_0}^*M$. Notice that $(T_{x_0}^*M)^{n-1}$ is a real analytic manifold and T is a multivariable polynomial, and thus a real analytic function. Moreover by the discussion before this lemma we know that T is not identically zero. Therefore, it follows from [24, Lemma 4.3] that $T^{-1}\{0\} \subset (T_{x_0}^*M)^{n-1}$ is nowhere dense. Since determinant is

multilinear and the map $V \ni z \mapsto d(d_F(\cdot, z))|_{x_0} \in S^*_{x_0}M$ is a smooth embedding, it follows that $\mathcal{V} \subset V^{n-1}$ is dense. □

Now we are ready to prove the main theorem

Theorem 3.12. *The mapping* Ψ *given in* (13) *is a diffeomorphism.*

Proof. We let $x_0 \in M_1$. We suppose first that x_0 is an interior point. By the data (10) it holds that $z \in \partial M_1$ is a mimimizer of $d_{F_1}(x_0, \cdot)|_{\partial M_1}$ if and only if $\phi(z) \in \partial M_2$ is a minimizer of $d_{F_2}(\Psi(x_0), \cdot)|_{\partial M_2}$. We let z_1 be a minimizer of $d_{F_1}(x_0, \cdot)|_{\partial M_1}$. Since the map $\phi\colon \partial M_1 \to \partial M_2$ is a diffeomorphism it follows from the Lemma 3.11 that there exist points $z_2, \dots, z_n \in \partial M_1$ and a neighborhood $U \subset M_1$ of x_0 such that the maps $U \ni x_1 \mapsto (d_{F_1}(x_1, z_i))_{i=1}^n$ and

$$\Psi(U) \ni x_2 \mapsto (d_{F_2}(x_2, \phi(z_i)))_{i=1}^n = (d_{F_1}(\Psi^{-1}(x_2), z_i))_{i=1}^n$$

are smooth coordinate maps. Thus with respect to these coordinates it holds that the local representation of Ψ near x_0 is an indentity map of $\mathbb{R}^n$. This proves that Ψ is a local diffeomorphism near any interior point of M_1.

We consider next the boundary case. We denote

$$U = \{x_1 \in M_1 : \text{ there exists precisely one minimizer for } d_{F_1}(x_1, \cdot)|_{\partial M_1}\}^{int}.$$

By (10) and since Ψ is a homeomorphism, it holds that

(18)
$$\Psi(U) = \{x_2 \in M_2 : \text{ there exists precisely one minimizer for } d_{F_2}(x_2, \cdot)|_{\partial M_2}\}^{int}.$$

By Lemma 3.3 it follows that there exists $\epsilon > 0$ and open neighborhood $V \subset U \subset M_1$ of ∂M_1 such that the maps

$$\exp^{\perp}_{F_1} : \partial M_1 \times [0, \epsilon) \to V, \quad \text{and}$$
$$\exp^{\perp}_{F_2} : \partial M_2 \times [0, \epsilon) \to \Psi(V)$$

are diffeomorphisms. Moreover due to Lemma 3.4 for every $x \in V$ it holds that $x = \exp^{\perp}_{F_1}(z(x), s(x))$, where $z(x)$ is the minimizer of $d_{F_1}(x, \cdot)|_{\partial M_1}$ and $s(x) = d_{F_1}(x, z(x))$. Therefore, by (18) it holds that

$$(\exp^{\perp}_{F_2})^{-1}(\Psi(p)) = (\phi(z(p)), s(p)).$$

Thus we have proved that with respect to coordinates $(V, (\exp^{\perp}_{F_1})^{-1})$ and $(\Psi(V), (\exp^{\perp}_{F_2})^{-1})$ the local representation of Ψ is

$$(\partial M_1 \times [0, \epsilon)) \ni (z, s) \mapsto (\phi(z), s) \in (\partial M_2 \times [0, \epsilon)).$$

Since $\phi\colon \partial M_1 \to \partial M_2$ is a diffeomorphism we have proved that Ψ is a local diffeomorphism near ∂M_1.

By Theorem 3.2 the map Ψ is one-to-one and we have proved that Ψ is a diffeomorphism. □

3.3. **Smoothness of the boundary distance function.** Here we consider the smoothness of a boundary distance function and show that the closure of

(19)
$$\widehat{G}(M,F) := \{(x,y) \in G(M,F) : x \in M^{int},\ d_F(\cdot, z(x,y)) \text{ is } C^\infty \text{ at } x\} \cup \partial_{out}TM,$$

where $\partial_{out}TM \subset \partial(TM)$ is the collection of outward pointing vectors (excluding tangential ones), coincides with the set $\overline{G(M,F)}$ (see Definition 1.2). This will be used in the next subsection to reconstruct F in $G(M,F)$.

We note that if $(x,y) \in G(M,F)$, with $F(y) = 1$ then

$$\tau_{exit}(x,y) = t(x,y),$$

where $t(x,y)$ is as in Definition 1.2. We assume below in this section that all vectors are of unit length.

For those $(x,z) \in M^{int} \times \partial M$ for which $d_F(\cdot, z)$ is smooth at x we can use the differential of the distance function to determine the image of the distance minimizing geodesic from x to z. In this sense our problem is related to the Finslerian version of Hilbert's 4^{th} problem which is: To recover Finsler metric from the images of the geodesics. In this setting the problem has been studied for instance in [8, 9, 10, 39, 45].

The main result in this section is

Proposition 3.13. *For any smooth connected and compact Finsler manifold (M,F) with smooth boundary it holds that*

$$\overline{\widehat{G}(M,F)} = \overline{G(M,F)}.$$

We need a couple of auxiliary results to prove this proposition. We state these auxiliary results below and prove them after the proof of Proposition 3.13.

Lemma 3.14. *Let $x_1, x_2 \in M$ and let $c\colon [0,1] \to M$ be a rectifiable curve from x_1 to x_2. Let $t_0 \in (0,1)$ be such that $c(t_0) \in \partial M$. If there exits $\delta > 0$ such that $c|_{[t_0-\delta,t_0]}$ is a geodesic and $\lim_{t \nearrow t_0} \dot{c}(t)$ is transversal to ∂M then there exists a rectifiable curve $\alpha\colon [0,1] \to M$ from x_1 to x_2 such that*

$$\mathcal{L}(\alpha) < \mathcal{L}(c).$$

Lemma 3.15. *Suppose that $(x,v) \in G(M,F)$. If the exit direction is transversal to the boundary then for any $s \in (0, \tau_{exit}(x,v))$ the point $(x',v') := (\gamma_{x,v}(s), \dot{\gamma}_{x,v}(s)) \in \widehat{G}(M,F)$.*

Lemma 3.16. *Suppose that $(x,v) \in G(M,F)$, and the exit direction η is tangential to the boundary at z. Assume that there exists $h > 0$ such that for any $h' \in (0,h)$ the geodesic $\overleftarrow{\gamma}_{z,\xi_{h'}}\colon [0, \tau_{exit}(x,v)] \to M$ is well defined, where*

$$\xi_{h'} := \frac{-\eta + h' \overleftarrow{\nu}_{in}}{\overleftarrow{F}\,(z, -\eta + h' \overleftarrow{\nu}_{in})} \in T_zM.$$

Then there exist sequences $(h_j)_{j=1}^{\infty}, (\epsilon_j)_{j=1}^{\infty} \subset \mathbb{R}$ *such that* $h_j, \epsilon_j > 0$, $h_j, \epsilon_j \to 0$ *as* $j \to \infty$ *and moreover the geodesic* $\overleftarrow{\gamma}_{z,\xi_{h_j}}$ *is a distance minimizing curve of* $(M, \overleftarrow{F})$ *from* z *to* $\overleftarrow{\gamma}_{z,\xi_{h_j}}(\tau_{exit}(x,v) - \epsilon_j)$ *for any* $j \in \mathbb{N}$ *that is large enough.*

Proof of Proposition 3.13. Since the sets $\widehat{G}(M,F)$ and $G(M,F)$ are conical it suffices to prove that

$$\overline{\widehat{G}(M,F)} \cap SM = \overline{G(M,F)} \cap SM.$$

We first prove $\widehat{G}(M,F) \subset \overline{G(M,F)}$, which implies $\overline{\widehat{G}(M,F)} \subset \overline{G(M,F)}$. We let $(x,v) \in \widehat{G}(M,F)$. If $x \in M^{int}$, then clearly $(x,v) \in \overline{G(M,F)}$. If $(x,v) \in \partial_{out}TM$ then due to transversality of v and $T_x\partial M$ there exists $\epsilon > 0$ such that for every $t \in (0,\epsilon)$ we have

$$(\gamma_{x,v}(-t), \dot{\gamma}_{x,v}(-t)) \in G(M,F).$$

Thus $(x,v) \in \overline{G(M,F)}$.

Next we show that $G(M,F) \subset \overline{\widehat{G}(M,F)}$. We let $(x,v) \in G(M,F)$. Lemma 3.15 implies that for any $s \in (0, \tau_{exit}(x,v))$ we have $(\gamma_{x,v}(s), \dot{\gamma}_{x,v}(s)) \in \widehat{G}(M,F)$ if $\dot{\gamma}_{x,v}(\tau_{exit}(x,v))$ is transversal to ∂M. This implies, $(x,v) \in \widehat{G}(M,F)$.

Therefore, we assume that $(\gamma_{x,v}(\tau_{exit}(x,v)), \dot{\gamma}_{x,v}(\tau_{exit}(x,v))) := (z,\eta)$ is tangential to ∂M. We let (N,H) be a smooth complete Finsler manifold without boundary that extends $(M, \overleftarrow{F})$ and $\Pi \subset T_zN$ be the two dimensional vector subspace spanned by $\{\eta, \overleftarrow{\nu_{in}}\}$. If $a \in (0, \tau_{exit}(x,v))$ is small enough, then

$$S(a) := \{\overleftarrow{\exp}_z(w) \in N : w \in \Pi,\ \overleftarrow{F}(z,w) < a\}$$

is a C^1-smooth hyper surface of N with a coordinate system given by η and $\overleftarrow{\nu_{in}}$.

We note that possible after choosing smaller a the set $S(a) \cap \partial M$ is given by a C^1-smooth graph $(s, c(s)) \in S(a)$ such that for $s < 0$ we have $c(s) < 0$. This follows since ∂M is a smooth co-dimension 1 manifold and with respect to the coordinates $(\eta, \overleftarrow{\nu_{in}})$ of $S(a)$ we have $(-t, 0) = \gamma_{x,v}(\tau_{exit}(x,v) - t)$ and $\gamma_{x,v}(\tau_{exit}(x,v) - t)$ does not hit ∂M, if $t \in (0,a)$. Thus for any $h > 0$ and $t \in (0,a)$ the geodesic $\overleftarrow{\gamma}_{z,\xi_h}$,

$$\xi_h := \frac{-\eta + h\,\overleftarrow{\nu_{in}}}{\overleftarrow{F}(z, -\eta + h\,\overleftarrow{\nu_{in}})}$$

of the Finsler function H, satisfies $\overleftarrow{\gamma}_{z,\xi_h}(t) \in M^{int}$.

Since the interval $[0, \tau_{exit}(x,v) - \frac{a}{2}]$ is compact and $\gamma_{x,v}([0, \tau_{exit}(x,v) - \frac{a}{2}]) \subset M^{int}$. There exists $r > 0$ such that for all $t \in [0, \tau_{exit}(x,v) - \frac{a}{2}]$ we have $d_F(\gamma_{x,v}(t), \partial M) < r$. Therefore, the continuity of the exponential map implies that for any $h > 0$ small enough and $t \in (0, \tau_{exit}(x,v)]$ we have $\overleftarrow{\gamma}_{z,\xi_h}(t) \in M^{int}$.

We note that Lemma 3.16 implies that for any $h, \epsilon > 0$ that are small enough we have

$$(x', v') := \Big(\overleftarrow{\gamma}_{z,\xi_h}(\tau(x,v) - \epsilon), -\dot{\overleftarrow{\gamma}}_{z,\xi_h}(\tau(x,v) - \epsilon) \Big) \in G(M,F).$$

Then Lemma 3.15 implies $(x', v') \in \overline{\widehat{G}(M, F)}$. Taking h and ϵ to zero we finally obtain $(x, v) \in \overline{\widehat{G}(M, F)}$. □

Proof of Lemma 3.14. Since $\lim_{t \nearrow t_0} \dot{c}(t)$ is transversal to ∂M, there exists $\epsilon \in (0, \delta)$ such that $c|_{(t_0-\epsilon, t_0)}$ is a geodesic in M^{int}. We let $(\widetilde{M}, \widetilde{F})$ be any compact Finsler manifold that extends (M, F) and for which $c(t_0)$ is an interior point. Since $c_{|_{t_0-\epsilon, t_0}}$ is also a geodesic of the extended manifold $(\widetilde{M}, \widetilde{F})$, it follows from [48, Proposition 11.3.1] that there exists $t_1 \in (t_0 - \epsilon, t_0)$ such that for $x := c(t_1)$, $z := (c(t_0))$ we have

$$(t_0 - t_1) = d_{\widetilde{F}}(x, z) = d_F(x, z) = \tau_{exit}(x, \dot{c}(t_1)),$$

and the exponential map of $(\widetilde{M}, \widetilde{F})$ is a C^1-diffeomorphism from

$$\{y \in T_x M : F(y) < 2(t_0 - t_1)\}$$

onto a metric ball $B_{\widetilde{F}}(x, 2(t_0 - t_1))$ of $(\widetilde{M}, \widetilde{F})$.

Since $\lim_{t \nearrow t_0} \dot{c}(t)$ is transversal to ∂M it follows from the Implicit Function Theorem that there exists a neighborhood $U \subset S_x M$ of $v := \dot{c}(t_1)$ where the function τ_{exit} is smooth and $\tau_{exit}(x, w) < 2(t_0 - t_1)$, whenever $w \in U$. We let

$$C := \{rw \in T_x M : w \in U,\ r \in [0, \tau_{exit}(x, w)]\}.$$

Then $\exp_x(C)$ contains an open neighborhood of z in M. Since path c is continuous there exists $t_2 > t_0$ such that $\exp_x^{-1}(c(t_2)) \in C$, and moreover

$$F(x, \exp_x^{-1}(c(t_2))) = d_{\widetilde{F}}(x, c(t_2)) = d_F(x, c(t_2)),$$

since for any $w \in U$ the radial geodesic $\gamma_{x,w}$ of $(\widetilde{M}, \widetilde{F})$ has the minimal length among all curves connecting x to $\gamma_{x,w}(t), t \in (0, 2(t_0 - t_1))$.

If we denote by $\widetilde{c}$ the geodesic of $(\widetilde{M}, \widetilde{F})$ that satisfies the initial condition $(\widetilde{c}(0), \dot{\widetilde{c}}(0)) = (x, \dot{c}(t_1))$, then it leaves M at z. Therefore, there exits a geodesic γ of (M, F) connecting $x = c(t_1)$ to $c(t_2)$ which satisfies

$$\mathcal{L}(c|_{[t_1, t_2]}) > \mathcal{L}(\gamma).$$

This implies the claim. □

Proof of Lemma 3.15. We note first that it follows from the Implicit Function Theorem that there exists a neighborhood $U \subset SM$ of (x, v) such that function τ_{exit} is smooth in U. Therefore, the mapping

$$U \ni (\widetilde{x}, w) \mapsto (z(\widetilde{x}, w), \eta(\widetilde{x}, w)) := (\gamma_{\widetilde{x}, w}(\tau_{exit}(\widetilde{x}, w)), \dot{\gamma}_{\widetilde{x}, w}(\tau_{exit}(\widetilde{x}, w)))$$

is smooth and without loss of generality we may assume $\eta(\widetilde{x}, w)$ is transverse to ∂M for any $(\widetilde{x}, w) \in U$.

We let (N, H) be any compact Finsler manifold without boundary extending (M, F). We set $(x', v') := (\gamma_{x,v}(s), \dot{\gamma}_{x,v}(s))$ and it follows that the points x' and $z := z(x, v)$ are not conjugate along $\gamma_{x,v}$. Therefore, the exponential map $\exp_{x'}$, of Finsler function H is a diffeomorphism in a neighborhood $V \subset T_{x'} N$ of hv', for $h := (\tau_{exit}(x, v) - s)$ onto some neighborhood of z in N. Moreover

$$h = d_F(x', z) \geq d_H(x', z).$$

To finish the proof, we show that we can find a smooth Finsler manifold $(\widetilde{M}, \widetilde{F})$ so that $M \subset \widetilde{M} \subset N$, $\widetilde{F} = H|_{\widetilde{M}}$, $z \in \widetilde{M}^{int}$ and there exists a neighborhood $A \subset M^{int}$ of x' so that

$$d_F(\widetilde{x}, z) = \overset{\leftarrow}{\widetilde{F}}(z, \exp_z^{-1}(\widetilde{x})), \quad \widetilde{x} \in A.$$

Above the exponential map is given with respect to $\overset{\leftarrow}{\widetilde{F}}$. This implies $(x', v') \in \overline{\widehat{G}(M, F)}$ and since $s \in (0, \tau_{exit}(x, v))$ was arbitrary we have $(x, v) \in \overline{\widehat{G}(M, F)}$.

We let W_0 be the image of V under the orthogonal projection $y \mapsto \frac{y}{F(x', y)}$ on $S_{x'}N$. We let $r_0 \in (0, d_F(x', \partial M))$ be so small that for any $w \in S_{x'}M$ geodesic $\gamma_{x',w}|_{[0,r_0]}$ is a distance minimizer of H and contained in M^{int}. In addition we define

$$\Gamma := \{\exp_{x'}(r_0 w) \in M^{int} : w \in \overline{(S_{x'}N) \setminus W_0}\}.$$

Since this set is compact it follows from the triangle inequality that there exists $\epsilon_0 > 0$ which satisfies

$$r_0 + d_F(\Gamma, z) \geq d_F(x', z) + \epsilon_0, \tag{20}$$

as otherwise there would exist a $\overset{\leftarrow}{F}$-distance minimizing curve from z to x which is not C^1 at x'.

For $p \in N$ and $r > 0$ we define

$$\overset{\leftarrow}{B}_H(p, r) := \{q \in N : d_H(q, p) < r\}.$$

Since the points x' and z are not conjugate along $\gamma_{x,v}$ we can choose a neighborhood set $W_1 \subset W_0$ of v', and $2\epsilon_1 < \epsilon_0$, $\delta > 0$ such that

$$\overset{\leftarrow}{B}_H(z, 2\epsilon_1) \subset (\exp_{x'}((0, h + \delta) \times W_1)) \cap \overset{\leftarrow}{B}_F(z, \epsilon_0))$$

and the geodesic $\gamma_{x,v}$ is the shortest curve from x' to z contained in $\exp_{x'}([0, h + \delta) \times W_1))$.

We write $M_k := M \cup \overline{\overset{\leftarrow}{B}_H(z, k\epsilon_1)}$, for $k \in \{1, 2\}$. Finally, we let $(\widetilde{M}, \widetilde{F})$ be any smooth compact Finsler manifold with boundary such that

$$M_1 \subset \widetilde{M} \subset M_2 \quad \text{and} \quad \widetilde{F} = H|_{\widetilde{M}}.$$

If β is a distance minimizing curve of $(\widetilde{M}, \widetilde{F})$ from x' to z it is a geodesic of (M, F) for $t < r_0$. Therefore, we have that $\beta = \gamma_{x,v}$ if $\dot{\beta}(0) \in W_1$. If $\dot{\beta}(0) \in (W_0 \setminus W_1)$, then β hits $\partial\widetilde{M}$ transversally outside $\overset{\leftarrow}{B}_H(z, 2\epsilon_1)$, which cannot happen due to Lemma 3.14. If $\dot{\beta}(0) \in \overline{(S_{x'}N) \setminus W_0}$ then by (20) we have

$$r_0 + d_F(\Gamma, z) - \epsilon_0 \geq d_F(x', z) \geq d_{\widetilde{F}}(x', z) \geq r_0 + s_1 + 2\epsilon_1,$$

where $s_1 \geq 0$ is the time it takes to travel from Γ to $(\overset{\leftarrow}{B}_H(z, 2\epsilon_1) \cap \widetilde{M})$ along the curve β. We note that $\beta(r_0 + s_1)$ is contained in M which implies

$$s_1 + \epsilon_0 \geq d_F(\Gamma, z).$$

Thus we arrive at a contradiction $0 \geq 2\epsilon_1$, and we have proven that $\gamma_{x,v}$ is the unique distance minimizing curve of $(\widetilde{M}, \widetilde{F})$ connecting x' to z.

Since x' and z are not conjugate points along $\gamma_{x,v}$, the exponential map of the reversed Finsler function $\overleftarrow{\widetilde{F}}$ is a diffeomorphism from a neighborhood of $-h\eta \in T_z\widetilde{M}$, $\eta := \eta(x,v)$ to a neighborhood of x'. Thus the local distance function

$$q \mapsto \rho(q,z) := \overleftarrow{\widetilde{F}}\left(z, \left(\overleftarrow{\widetilde{\exp}}_z\right)^{-1}(q)\right)$$

is smooth near x' and due to earlier part of this proof it coincides with $d_{\widetilde{F}}(\cdot,z)$ at x'.

We suppose that there exists a sequence $(x_j)_{j=1}^{\infty} \subset M$ that converges to x' and for which it holds that

$$d_{\widetilde{F}}(x_j,z) < \rho(x_j,z). \tag{21}$$

We let β_j be a distance minimizing curve of $\overleftarrow{\widetilde{F}}$ from z to x_j. Since $(\widetilde{M},\widetilde{F})$ is a compact (non-symmetric) metric space it follows form [42] there exists a rectifiable curve β_∞ connecting z to x', that is a uniform limit of β_j and whose length is not greater than $d_{\widetilde{F}}(x',z)$. This implies $\beta_\infty(t) = \gamma_{x,v}(h-t)$, since $\gamma_{x,v}$ is the unique $\widetilde{F}$ distance minimizer from x' to z.

Since $z \in \widetilde{M}^{int}$, there exists $R > 0$ such that for every $j_k \in \mathbb{N}$ the curve $\beta_j(t)$ is a geodesic of $(\widetilde{M}, \overleftarrow{\widetilde{F}})$ if $t \in [0,R]$. Therefore,

$$\dot{\beta}_j(0) \to -\eta \in S_z\widetilde{M} \tag{22}$$

and the continuity of the exit time function τ_{exit} implies that there exists $J \in \mathbb{N}$ such that for every $j > J$ the curve β_j is a geodesic of $(M, \overleftarrow{F})$. Thus (21) and (22) contradict with the assumption that $\overleftarrow{\widetilde{\exp}}_z$ is a diffeomorphism near $-h\eta$. Therefore, (21) cannot hold and we have that $d_{\widetilde{F}}(\cdot,z)$ and the local distance function $\rho(\cdot,z)$ coincide near x'. Hence there exists a neighborhood $A \subset M^{int}$ of x' in which we have

$$d_F(\cdot,z) = d_{\widetilde{F}}(\cdot,z) = \rho(\cdot,z),$$

due to continuity of the exit time function. □

Proof of Lemma 3.16. We let (z,η) be the exit point and direction of $\gamma_{x,v}$. We let $(\widetilde{M},\widetilde{F})$ be any compact Finsler manifold for which $z \in M^{int}$, and choose $s \in (0, \tau_{exit}(x,v))$ and denote $\ell = \ell(s) := \tau_{exit}(x,v) - s$. Then for $x' := (\gamma_{x,v}(s))$ the point z is not a conjugate point along $\gamma_{x,v}$. Since the conjugate distance function is lower continuous [48, Section 12.1] there exist neighborhoods $V \subset T_zM$ of $-\ell\eta$ and $U \subset \widetilde{M}$ of $\gamma_{x,v}([s, \tau_{exit}(x,v)])$ such that for any $y \in V$ the shortest curve that is contained in U and connects z to the point $x(y) := \overleftarrow{\exp}_z(y) \in U$, is the geodesic $t \mapsto \overleftarrow{\exp}_z(ty)$, $t \in [0,1]$.

We let $(h_j)_{j=1}^{\infty}, (\epsilon_j)_{j=1}^{\infty} \subset \mathbb{R}$ be such that $h_j, \epsilon_j > 0$, $h_j, \epsilon_j \to 0$ as $j \to \infty$. Denote $x_j := \overleftarrow{\gamma}_{z,\xi_{h_j}}(\ell - \epsilon_j)$, where $\xi_{h_j} := \frac{-\eta + h_j \overleftarrow{\nu}_{in}}{\overleftarrow{F}(z, -\eta + h_j \overleftarrow{\nu}_{in})}$. Then $x_j \to x'$ as $j \to \infty$. We

let $c_j : [0, \ell] \to M$ be a distance minimizing curve of $(M, \overleftarrow{F})$ from z to x_j. Then we have

$$\mathcal{L}(c_j) \longrightarrow \ell \text{ as } j \longrightarrow \infty.$$

Due to [42] we can without loss of generality assume that curves c_j converge uniformly to a rectifiable curve c_∞, from z to x', that satisfies

$$\mathcal{L}(c_\infty) \leq \ell.$$

Then [48, Proposition 11.3.1] implies $c_\infty = \overleftarrow{\gamma}_{z,-\eta} \mid_{[0,\ell]}$, since otherwise there would exist a distance minimizing curve of $(M, \overleftarrow{F})$ from z to x that is not C^1-smooth at x'. Since $c_j \to \overleftarrow{\gamma}_{z,-\eta|_{[0,\ell]}}$ uniformly in $[0, \ell]$ there exists $J \in \mathbb{N}$ such that for all $j \geq J$ we have $(\overleftarrow{\exp}_z)^{-1}(x_j) \in V$ and the image of c_j is contained in U. Thus after unit speed reparametrization of c_j we have, $c_j(t) = \overleftarrow{\gamma}_{z,\xi_{h_j}}(t)$ for any $t \in [0, \ell - \epsilon_j]$.

We recall that above we had $\ell = \tau_{exit}(x, v) - s$. The claim of this lemma follows using a diagonal argument for sequence $(\epsilon_i^j)_{i=1}^\infty, (h_i^j)_{i=1}^\infty \subset (0, 1)$ which are chosen as above for $s_j \in (0, \tau_{exit}(x, v))$, $j \in \mathbb{N}$, such that $s_j \to 0$ when $j \to 0$. □

3.4. **Finsler structure.** In this section, we prove that the data (1) determine the set $\widehat{G}(M, F)$, (see (19)) and the Finsler function F on it. Again we deal separately with interior and boundary cases.

Lemma 3.17. *Let $x \in M^{int}$. The set $T_xM \cap \widehat{G}(M, F)$ contains an open non-empty set. Moreover the data* (1) *determine the set $T_xM \cap \widehat{G}(M, F)$ and F on it.*

Proof. We let $z_x \in \partial M$ be a closest boundary point to x. By Lemma 3.8 the function $d_F(\cdot, z_x)$ is smooth at x. Moreover

$$v := -\dot{\overleftarrow{\gamma}}_{z_x, \overleftarrow{\nu}_{in}(z_x)}(d_F(x, z_x)) \in S_xM \cap \widehat{G}(M, F).$$

Thus the set $S_xM \cap \widehat{G}(M, F)$ is not empty. By Lemma 3.8 there exists a neighborhood $U \subset S_xM$ of v that is contained in $\widehat{G}(M, F)$.

Next, we prove the latter claim. We let $z \in \partial M$ be such that the function $d_F(\cdot, z)$ is smooth at x. We let $v \in \overleftarrow{S_xM}$. Then

$$d(d_{\overleftarrow{F}}(z, \cdot))\Big|_x = \overleftarrow{g_v}(v, \cdot) = \overleftarrow{\ell_x}(v) \quad \text{if and only if} \quad \gamma_{x,-v}(d_F(x, z)) = z, \tag{23}$$

where $g_y(\cdot, \cdot)$ is the hessian of $\frac{1}{2}F^2(x, y)$ with respect to y variables and $\overleftarrow{\ell_x}$ is the Legendre transform of Finsler function $\overleftarrow{F}$ at x. The property (23) implies that the set

$$A(x) := \{d(d_{\overleftarrow{F}}(z, \cdot))\Big|_x : z \in \partial M,\ d_{\overleftarrow{F}}(z, \cdot) \text{ is } C^\infty \text{ at } x\} \tag{24}$$

satisfies

$$A(x) = \overleftarrow{\ell_x}(\overleftarrow{S_xM} \cap (-\widehat{G}(M, F))).$$

Since the Legendre transform is an isometry the dual map $\left(\overleftarrow{F}\right)^*$ is constant 1 on $A(x)$. As the function $\left(\overleftarrow{F}\right)^*$ is positively homogeneous of order 1 we have determined $\left(\overleftarrow{F}\right)^*$ on $\mathbb{R}_+A(x) := \{rp \in T_x^*M : p \in A(x),\ r > 0\}$. Recall that the components of the Legendre satisfy

$$\Big((\overleftarrow{\ell}_x)^{-1}(p))\Big)_j = \frac{1}{2}\Big(\frac{\partial}{\partial p^i}\frac{\partial}{\partial p^j}\Big(\Big(\overleftarrow{F}\Big)^*\Big)^2(x,p)\Big)p_i \tag{25}$$

for all $p \in T_x^*M$. Since $\left(\overleftarrow{F}\right)^*$ is recovered on $\mathbb{R}_+A(x)$ the equation (25) determines $(\overleftarrow{\ell}_x)^{-1}$ on $\mathbb{R}_+A(x)$. Therefore,

$$T_xM \cap (-\widehat{G}(M,F)) = (\overleftarrow{\ell}_x)^{-1}(\mathbb{R}_+A(x)) \text{ and}$$
$$\overleftarrow{F} = \Big(\Big(\overleftarrow{F}\Big)^*\Big)^* \text{ on } T_xM \cap (-\widehat{G}(M,F)).$$

Finally,

$$F(x,y) = \overleftarrow{F}\,(x,-y).$$

This concludes the proof. □

Lemma 3.18. *Let $x \in M_1^{int}$. Then*

$$\widehat{G}(M_1,F_1) \cap T_xM_1 = \widehat{G}(M_1,\Psi^*F_2) \cap T_xM_1$$

and

$$F_1(y) = F_2(\Psi_* y), \quad y \in (\widehat{G}(M_1,F_1) \cap T_xM_1).$$

Proof. The mapping Ψ is a diffeomorphism that satisfies

$$d_{F_2}(\Psi(\cdot),\phi(\cdot))|_{M_1\times\partial M_1} = d_{F_1}(\cdot,\cdot)|_{M_1\times\partial M_1}. \tag{26}$$

Thus for any $z \in \partial M_1$ the function $d_{F_2}(\Psi(\cdot),\phi(z))$ is smooth at $\Psi(x)$ if and only if $d_{F_1}(\cdot,z)$ is smooth at x. Therefore, the claims follow by applying the differential of M_1 to the both sides of (26) and using (23)–(25). □

Next, we consider the boundary case.

Lemma 3.19. *Let $x \in \partial M$. Then data* (1) *determine F on*

$$\partial_{out}TM \cap T_xM = \{y \in T_xM : g_{\nu_{in}}(\nu_{in},y) < 0\}.$$

Proof. We let $y \in T_xM \setminus \{0\}$ be an outward pointing vector that is not tangential to the boundary. We let $b > a \geq 0$ and choose any smooth curve $c\colon [a,b] \to M$ such that

$$c((a,b)) \subset M^{int}, \quad c(b) = x,\ \dot{c}(b) = y.$$

Recal that with respect to the geodesic coordinates at x we have $d_F(x,c(t)) = F(\exp_x^{-1}(c(t)))$. Since F is continuous we have

$$\lim_{t\to b}\frac{d_F(x,c(t))}{b-t} = F(\dot{c}(b)) = F(y). \tag{27}$$

Since $y \in T_xM \setminus \{0\}$ was an arbitrary outward pointing vector the data (1) and (27) determine F on the set $\partial_{out}TM \cap T_xM$. □

Lemma 3.20. *For any $(x, y) \in \partial_{out}TM_1$ it holds that*

$$F_1((x, y)) = F_2(\Psi_*(x, y)).$$

Proof. The claim follows from (26) and (27). □

Now we are ready to give a proof of Theorem 1.3.

Proof of Theorem 1.3. By theorems 3.2 and 3.12 the map $\Psi\colon M_1 \to M_2$ is a diffeomorphism, and the pullback Ψ^*F_2 of F_2 gives a Finsler function on M_1. By Lemmas 3.18 and 3.20 we have proved that $\overline{G(M_1, F_1)}$ and $\overline{G(M_1, \Psi^*F_2)}$ coincide and in this set $F_1 = \Psi^*F_2$.

We still have to show that the data (1) are not sufficient to guarantee that F_1 and F_2 coincide in $TM_1^{int} \setminus \overline{G(M_1, F)}$. We denote a manifold M_1 by M and a Finsler function F_1 by F. If $TM^{int} \setminus \overline{G(M, F)}$ is not empty, we choose $(x_0, v_0) \in TM^{int} \setminus \overline{G(M, F)}$, $F(v_0) = 1$ and a neighborhood $V \subset TM^{int} \setminus \overline{G(M, F)}$ of (x_0, v_0) such that

$$\operatorname{dist}_F(\pi(V), \partial M) > 0. \tag{28}$$

We denote the orthogonal projection of V to the unit sphere bundle of (M, F) by W. We let $\alpha \in C_0^\infty(W)$ be non-negative and define a function

$$H\colon \mathbb{R} \times TM \to \mathbb{R}, \quad H(s, y) = \Big(1 + s\alpha\Big(\frac{y}{F(y)}\Big)\Big)F(y). \tag{29}$$

We show that there exists $\epsilon > 0$ such that for any $s \in (-\epsilon, \epsilon)$ the function $H(s, \cdot)\colon TM \to \mathbb{R}$ is a Finsler function.

Since α is compactly supported it holds, for $|s|$ small enough, that $H(s, \cdot)$ is non-negative, continuous and $H(s, y) = 0$ if and only if $y = 0$. Moreover $H(s, \cdot)$ is smooth outside the zero section of M. Clearly also the scaling property $H(s, ty) = tH(s, y)$, $t > 0$ is valid.

We let (x, y) be a smooth coordinate system of TM near (x_0, v_0). To prove that $H(s, \cdot)$ is a Finsler function, we have to show that for every $(x, y) \in TM \setminus \{0\}$ the Hessian

$$\frac{1}{2}\frac{\partial}{\partial y^i}\frac{\partial}{\partial y^j}H^2(s, (x, y)) = \frac{1}{2}\frac{\partial}{\partial y^i}\frac{\partial}{\partial y^j}\Big[\Big(1 + s\alpha\Big(\frac{y}{F(y)}\Big)\Big)^2\Big(F(y)\Big)^2\Big]$$

is symmetric and positive definite. Since $H^2(s, (x, \cdot))\colon T_xM \to \mathbb{R}$ is smooth outside 0 it follows that the Hessian is symmetric, and $\alpha \in C_0^\infty(W)$ implies

$$\frac{1}{2}\frac{\partial}{\partial y^i}\frac{\partial}{\partial y^j}H^2(s, (x, y)) = g_{ij}(x, y) + \mathcal{O}(s)$$

where g_{ij} is the Hessian of $\frac{1}{2}F^2$. Therefore, for $|s|$ small enough $H(s, \cdot)\colon TM \to \mathbb{R}$ is a Finsler function.

We let $\epsilon > 0$ be so small that $H(s,\cdot)$ is a Finsler function for $s \in (0,\epsilon)$. We prove that for any $x \in M$ and $z \in \partial M$

$$d_{H(s,\cdot)}(x,z) = d_F(x,z). \tag{30}$$

This implies that the boundary distance data of F and $H(s,\cdot)$ coincide.

If $c\colon [0,1] \to M$ is any piecewise C^1-smooth curve, (29) implies

$$\mathcal{L}_F(c) \leq \mathcal{L}_{H(s,\cdot)}(c). \tag{31}$$

We let $x \in M$ and $z \in \partial M$. Since M is compact there exists a F-distance minimizing curve $c\colon [0, d_F(x,z)] \to M$ from x to z. We let $I, J \subset [0, d_F(x,z)]$ be a partition $[0, d_F(x,z)]$ such that

$$c(t) \in M^{int} \quad \text{if and only if} \quad t \in I.$$

Then I is open in $[0, d_F(x,z)]$ and J is closed. On set I the curve c is a union of distance minimizing geodesic segments of F which have end points in ∂M. Thus for any $t \in I$ we have $\dot{c}(t) \in G(M,F)$. This and (28) imply

$$d_F(x,z) = \mathcal{L}_F(c) = \mathcal{L}_{H(s,\cdot)}(c),$$

and the equation (30) follows from (31). □

4. The proof of Lemma 3.6

In this section, we denote by (N,F) a compact, connected smooth Finsler manifold without boundary. We present the second variation formula in the case when the variation curves start from a smooth submanifold S of N. We introduce the concept of a focal distance and connect it to the degeneracy of the normal exponential map $\exp^{\perp}$ of surface S. We use the results of this section to complete the proof of Lemma 3.6

We define a pullback vector bundle π^*TN over $TN \setminus \{0\}$ such that for every $(x,y) \in TN \setminus \{0\}$ the corresponding fiber is T_xN. Notice that π^*TN is then defined by the following equation

$$\pi^*TN = \{((x,y),(x,y')) \in (T_xN \setminus \{0\}) \times T_xN : x \in N\}.$$

We let (x,y) be a local coordinates for TN. We define a local frame $(\partial_i)_{i=1}^n$ for π^*TN by

$$\partial_i|_{(x,y)} := \Big((x,y), \frac{\partial}{\partial x^i}\Big). \tag{32}$$

and a local co-vector field on TN by

$$\delta y^i := dy^i + N^i_j dx^j, \quad N^i_j(x,y) := \frac{\partial}{\partial y^j} G^i(x,y).$$

Above the functions G^i are the geodesic coefficients of F in coordinates (x,y) (see (55) in A). Notice that $\frac{\partial}{\partial y^i}$ is a dual vector to δy^i and a dual vector $\frac{\delta}{\delta x^i}$ to dx^i is given by

$$\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j}. \tag{33}$$

Therefore, vectors dx^i and δy^j are linearly independet for all $i,j \in \{1,\dots,n\}$ and it holds that

$$T^*(TN\setminus\{0\}) = \operatorname{span}\{dx^i\}\oplus\operatorname{span}\{\delta y^i\} =: \mathcal{H}^*(TN)\oplus\mathcal{V}^*(TN). \tag{34}$$

We relate π^*TN locally to $\mathcal{H}^*(TN)$ and to $\mathcal{V}^*(TN)$ by mappings

$$\partial_i \mapsto dx^i \text{ and } \partial_i \mapsto \delta y^i,\ i\in\{1,\dots,n\}. \tag{35}$$

We denote the collection of smooth sections of π^*TN by $\mathcal{S}(\pi^*TN)$. The *Chern connection* is defined on π^*TN by

$$\nabla\colon \mathcal{T}(TN)\times\mathcal{S}(\pi^*TN)\to\mathcal{S}(\pi^*TN),\quad \nabla_X U = \Big\{dU^i(X)+U^j\omega^i_j(X)\Big\}\partial_i, \tag{36}$$

where $\mathcal{T}(TN)$ is the collection of all smooth vector fields on $TN\setminus\{0\}$ and the connection one forms ω^i_j on $TN\setminus\{0\}$ are given by

$$\omega^i_j(x,y) := \Gamma^i_{jk}(x,y)dx^k, \tag{37}$$

and functions $\Gamma^i_{jk}(x,y)$ are defined by [48, equation (5.25)]. They satisfy

$$y^k\Gamma^i_{jk}(x,y) = N^i_j(x,y) \quad \text{and} \quad \Gamma^i_{jk} = \Gamma^i_{kj}. \tag{38}$$

See [48, equations (5.24) and (5.25)].

Notice that any vector field X on TN, that is locally given by

$$X = X^i_x\frac{\delta}{\delta x^i} + X^i_y\frac{\partial}{\partial y^i},\quad X^i_x, X^i_y\in C^\infty(TN)$$

defines a section $\widetilde{X}\in\mathcal{S}(\pi^*TN)$ by

$$\widetilde{X}(x,y) = X^i_x(x,y)\partial_i.$$

Lemma 4.1. *Let X,Y,Z be vector fields on TN. Then*

$$\nabla_X\widetilde{Y} - \nabla_Y\widetilde{X} = \widetilde{[X,Y]}. \tag{39}$$

Proof. Equation (39) follows from the definition of the Chern connection and (38). □

The fundamental tensor g on π^*TN is defined by

$$g(U,V) := g_{ij}(x,y)U^i(x,y)V^i(x,y),\quad U,V\in\mathcal{S}(\pi^*TN),\ (x,y)\in TN,$$

where $g_{ij}(x,y) = g_y(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j})$.

Recall that if $X = X^i_x\frac{\delta}{\delta x^i} + X^i_y\frac{\partial}{\partial y^i}$ is in $T(TN)$, then $D\pi X = X^i_x\frac{\partial}{\partial x^i}\in TN$.

Lemma 4.2. *Let X,Y,Z be vector fields on TN. Then*

$$Yg(\widetilde{X},\widetilde{Z})\Big|_{D\pi X} = \Big[g(\nabla_Y\widetilde{X},\widetilde{Z}) + g(\widetilde{X},\nabla_Y\widetilde{Z})\Big]\Big|_{D\pi X}. \tag{40}$$

Proof. The proof is a direct evaluation in coordinates, using that g_{ij} is homogeneous of order zero with respect to directional variables. It is important to evaluate $Yg(\widetilde{X},\widetilde{Z})$ at $D\pi X$ as for an arbitrary direction, (40) does not hold, since the Cartan tensor does not vanish identically.

□

If $V(x) = V^i(x)\frac{\partial}{\partial x^i}$ is a vector field on N then $\widehat{V}(x,y) := V^i(x)\frac{\delta}{\delta x^i}$ is a horizontal vector field on TN. We call $\widehat{V}$ a horizontal lift of V. We define a covariant derivative D_t of smooth vector field V on geodesic γ as

$$D_tV(t) := \Big\{\dot{V}^i(t) + V^j(t)N^i_j(\dot{\gamma}(t))\Big\}\frac{\partial}{\partial x^i}\Big|_{\gamma(t)}. \tag{41}$$

In the next lemma we relate the covariant derivative to the Chern connection.

Lemma 4.3. *Let $t \mapsto c(t)$ be a geodesic on (N, F) and V be a smooth vector field on c that is extendible. Write $V(x) = V^i(x)\frac{\partial}{\partial x^i}$. Then $\widetilde{V}(x,y) := V^i(x)\partial_i|_{(x,y)}$ is a smooth section of $\mathcal{S}(\pi^*TN)$ and*

$$\widetilde{D_tV} = \nabla_{\widehat{c}}\widetilde{V}. \tag{42}$$

Proof. To prove the claim, we use (38) and do a direct evaluation in coordinates. □

We now consider variations of a geodesic $\gamma : [0,h] \to N$ normal to a hypersurface S so that one endpoint stays on S and the other one is fixed. We denote the starting point of γ by $z_0 \in S$. For a smooth curve σ on S we assume that a variation $\Gamma(s,t)$ satisfies $\Gamma(s,0) = \sigma(s)$, $\Gamma(s,h) = \gamma(h)$, and $\Gamma(0,t) = \gamma(t)$ for all values of the time $t \in [0,h]$ and the variation parameter s near zero. The variation field $J(t) := \frac{\partial}{\partial s}\Gamma(s,t)|_{s=0}$ is a vector field along γ and satisfies the boundary conditions $J(0) = \dot{\sigma}(0)$ and $J(h) = 0$. We additionally assume that the variation is normal: $g_{\dot{\gamma}}(\dot{\gamma}, J) \equiv 0$.

The second variation formula (see [48, Chapter 10]) and equations (40)–(42) imply that

$$\frac{\partial^2}{\partial s^2}\mathcal{L}(\Gamma(s,\cdot))\Big|_{s=0} = \int_0^h g_{\dot{\gamma}}(D_t^2J(t) - \mathbf{R}_{\dot{\gamma}}(J(t)),J(t))dt + g(\nabla_{\widehat{J}}\widetilde{\nu} - \nabla_{\widehat{\nu}}\widetilde{J},\widetilde{J})\Big|_{\nu},$$

where $\mathbf{R}_{(\cdot)}(\cdot)$ is the *Riemannian curvature tensor* (see [48, Chapter 6]). If J is a Jacobi field, the expression simplifies to

$$\frac{\partial^2}{\partial s^2}\mathcal{L}(\Gamma(s,\cdot))\Big|_{s=0} = g(\nabla_{\widehat{J}}\widetilde{\nu} - \nabla_{\widehat{\nu}}\widetilde{J},\widetilde{J})\Big|_{\nu}.$$

To discuss geodesic variations of γ, we consider normal Jacobi fields J along γ that satisfy

$$J(0) \in TS \text{ and } \Big(\nabla_{\widehat{J}}\widetilde{\nu} - \nabla_{\widehat{\nu}}\widetilde{J}\Big)\Big|_{\nu} = 0. \tag{43}$$

We let $\mathcal{J}$ be the collection of all normal Jacobi fields on γ that satisfy (43) and $\mathcal{J}_0 = \{J \in \mathcal{J} : J(h) = 0\}$. Following [13], we call $\mathcal{J}$ the space of *transverse Jacobi fields*.

Lemma 4.4. *A vector field J on γ is a transverse Jacobi field if and only if*

$$J(t) = D\exp^{\perp}\Big|_{(z_0,t)}t\eta \quad \textit{for some } \eta = J(0) \in T_{z_0}S. \tag{44}$$

Proof. We let $\epsilon > 0$ and $U \subset S$ be a neighborhood of z_0 be such that the normal exponential map $\exp^\perp \colon U \times (-\epsilon, \epsilon) \to N$ is a diffeomorphsim onto its image. We define a unit length vector field W, that is orthogonal to S, by

$$W(x) := \frac{\partial}{\partial t} \exp^\perp(z,t) = \dot\gamma_{z,\nu(z)}(t), \quad (z,t) \in U \times (-\epsilon, \epsilon),$$

where $x = \exp^\perp(z,t)$.

For any $z \in U$ the geodesic $\gamma_{z,\nu(z)}$ of F is also a geodesic of the local Riemannian metric

$$g_W(x) := \mathrm{Hess}_y\Big(\frac{1}{2}F(x,y)^2\Big)\Big|_{y=W(x)},$$

that is normal to S. This implies that the normal exponential maps of F and g_W coincide in $U \times (-\epsilon, \epsilon)$ and moreover $D_t = D_t^W$, where D_t is the covariant derivative of F (given in (41)) and D_t^W is the covariant derivative of Riemannian metric g_W on $\gamma_{z,\nu(z)}$.

Now we are ready to prove the claim of this lemma. We let $\sigma(s) \in S$ be a smooth curve with initial conditions $\sigma(0) = z_0$ and $\dot\sigma(0) = \eta = J(0)$. Define $\Gamma(s,t) = \exp^\perp(\sigma(s), t)$. Then $\Gamma(0,t) = \gamma(t)$ and all the variation curves $t \mapsto \Gamma(s,t)$ are geodesics. Therefore, the variation field

$$V(t) := \frac{\partial}{\partial s}\Gamma(s,t)\Big|_{s=0}, \quad t \in (-\epsilon, \epsilon)$$

is a Jacobi field of F that satisfies

$$V(0) = \frac{\partial}{\partial s}\Gamma(s,t)\Big|_{s=t=0} = D\exp^\perp\Big|_{(z_0,0)}\dot\sigma(0) = \eta.$$

Therefore, it suffices to show that $D_tJ(0) = D_tV(0)$. We note that since g_W is a Riemannian metric the following symmetry holds true,

$$D_t\frac{\partial}{\partial s} = D_s\frac{\partial}{\partial t}, \tag{45}$$

along any transverse curve $s \mapsto \Gamma(s,t)$. Above D_s is a covariant derivative of g_W along transverse curve $\Gamma(\cdot, t)$. We also assume that $\dot\sigma$ can be extended to a smooth vector field Y on N. Then the equation (45) implies

$$D_tV(0) = \overline{\nabla}_Y W\Big|_{z_0} = \overline{\nabla}_\eta W,$$

where $\overline{\nabla}$ is the Riemannian connection of g_W. To end the proof we still have to show the first equation of the following

$$\widetilde{\overline{\nabla}_\eta W} = \nabla_{\widehat\eta}\widetilde{W} = \nabla_{\widehat W}\widehat\eta = \widetilde{D_tJ(0)},$$

where ∇ is the Chern connection of F. The proof of this claim is a direct computations in local coordinates. □

We obtain the following lemma as a direct consequence

Lemma 4.5. *Set $\mathcal{J}$ is a real vector space of dimension $n-1$.*

Proof. The claim follows since the dimension of S is $n-1$ and the operator given by (44) is linear in $T_{z_0}S$ and onto at $t=0$. □

Similarly to the spaces $\mathcal{J}$ and $\mathcal{J}_0$ of Jacobi fields defined above, we denote by $\mathcal{V}$ the collection of piecewise smooth normal vector fields along γ satisfying (43) and by $\mathcal{V}_0$ the subspace vanishing at $\gamma(h)$. On $\mathcal{V}_0$ we define the index form

$$\begin{aligned} I(V,W) := \quad & \int_0^h g_{\dot\gamma}(D_tV(t), D_tW(t)) - g_{\dot\gamma}(\mathbf{R}_{\dot\gamma}(V(t)),W(t))dt \\ - \quad & g(\nabla_{\widehat{W}}\widetilde{V}\Big|_{\dot\gamma(0)}, \widetilde{\nu}). \end{aligned} \tag{46}$$

Lemma 4.6. *The index form I on $\mathcal{V}_0$ is a symmetric bilinear form.*

Proof. Clearly, I is bilinear. It is proven in [48, Section 8.1] that for all $x \in N$ and $y, v, w \in T_xN$, it holds that

$$g_y(\mathbf{R}_y(v), w) = g_y(v, \mathbf{R}_y(w)).$$

Since V, W are normal to $\dot\gamma$, the equation

$$g(\nabla_{\widehat{W}}\widetilde{V}\Big|_{\dot\gamma(0)}, \widetilde{\nu}) = g(\nabla_{\widehat{V}}\widetilde{W}\Big|_{\dot\gamma(0)}, \widetilde{\nu})$$

follows from (40) and the symmetry of the second fundamental form (see [48, Section 14.4]). □

Lemma 4.7. *Assume that γ is not self-intersecting on $[0,h]$. We let $V \in \mathcal{V}$. There exists $\delta > 0$ and a variation $\Gamma(s,t)\colon (\delta,\delta)\times[0,h] \to N$ of γ whose variation field $\frac{\partial}{\partial s}\Gamma(s,t)|_{s=0}$ is V and $\Gamma(s,0)$ is a smooth curve on S. Moreover if $t_1, \dots, t_k \in [0,h]$ are the points where V is not smooth then $\Gamma\colon (\delta,\delta)\times(t_i,t_{i+1}) \to N$ smooth.*

Proof. We let W be a smooth vector field that is an extension of $\dot\gamma(t)$ in a neighborhood of $\gamma([0,h])$. Using the Fermi coordinates of S, with respect to the local Riemannian metric g_W, we can construct a Riemannian metric $\widetilde{g}$ in some neighborhood of $\gamma([0,h])$ such that S is a geodesic submanifold of $\widetilde{g}$ and γ is a geodesic of $\widetilde{g}$ that is $\widetilde{g}$-normal to S. Then we use the following variation to prove the claim of this lemma.

We let $\delta > 0$ and define a variation of γ with

$$\Gamma(s,t) = \exp_{\widetilde{g}}(\gamma(t), sV(t));\ t \in [0,h], s \in (-\delta,\delta), \tag{47}$$

where $\exp_{\widetilde{g}}$ is the exponential map of metric tensor $\widetilde{g}$. Since S is a geodesic sub manifold with respect to $\widetilde{g}$ we have that

$$\Gamma(s,0) = \exp_{\widetilde{g}}(\gamma(0), sV(0)) \in S, \quad s \in (-\delta,\delta).$$

Moreover

$$\frac{\partial}{\partial s}\Gamma(s,t)\Big|_{s=0} = D((\exp_{\widetilde{g}})_{\gamma(t)})\Big|_0 V(t) = V(t).$$

The claim is proven. □

For a given vector field $V \in \mathcal{V}_0$ we call the variation of $\gamma(t)$ given by (47) the variation related to V.

Definition 4.8. *We say that $\gamma(h)$ is a focal point of S if the set $\mathcal{J}_0$ contains a non-zero Jacobi field.*

Lemma 4.9. *The point $\gamma(h)$ is a focal point of S if and only if $D\exp^{\perp}$ is singular at (z_0, h).*

Proof. The claim follows from Lemma 4.4. □

We define the quantities $\tau_S(z_0)$ and $\tau_f(z_0)$ as

$$\tau_S(z_0) = \sup\{t > 0 : t = d_F(z_0, \gamma_{z_0,\nu}(t)) = d_F(S, \gamma_{z_0,\nu}(t))\} \tag{48}$$

and

$$\tau_f(z_0) = \inf\{t > 0 : \gamma(t) \text{ is a focal point to } S\}. \tag{49}$$

We note that τ_S is analogous to $\tau_{\partial M}$ given in Definition 3.5. Our final goal is to show that $\tau_S(z_0) \leq \tau_f(z_0)$. This completes the proof of Lemma 3.6. To check the inequality, we still have to state one auxiliary result

Lemma 4.10. *If $\tau_f(z_0) > h$, then Index form I is positive definite on $\mathcal{V}_0$. If $\tau_f(z_0) = h$, then I is positive semidefinite on $\mathcal{V}_0$ and $I(V, V) = 0$ if and only if $V \in \mathcal{J}_0$.*

Proof. The proof is a modification of the proof of [13, Theorem II.5.4]. □

Lemma 4.11. *Suppose that $\tau_f(z_0) < h$. Then there exists $W \in \mathcal{V}_0$ such that*

$$I(W, W) < 0.$$

Moreover

$$\tau_S(z_0) \leq \tau_f(z_0). \tag{50}$$

Proof. Denote $\tau_f(z_0) := t_0 < h$. Choose a non-zero $J \in \mathcal{J}$ that vanishes at t_0. Define

$$V(t) = \begin{cases} J(t),\ t \leq t_0 \\ 0,\ t \in [t_0, h). \end{cases}$$

By previous Lemma it holds that $I(V, V) = 0$. Since $D_t J(t_0) \neq 0$ there exists a non-zero smooth vector field $X \in \mathcal{V}_0$ on γ that satisfies

$$\text{supp} X \subset (0, h) \text{ and } X(t_0) = -D_t J(t_0).$$

Therefore if $\epsilon > 0$ is small enough $I(V + \epsilon X, V + \epsilon X)$ is negative.

Finally, we prove (50). We denote $W := V + \epsilon X$. We let $\Gamma(s, t)$ be the variation of $\gamma(t)$ that is related to W. Since γ is a geodesic we have

$$\frac{d}{ds}\mathcal{L}(\Gamma(s, \cdot)) = 0 \text{ and } \frac{d^2}{ds^2}\mathcal{L}(\Gamma(s, \cdot)) = I(W, W) < 0.$$

Therefore, γ cannot minimize the lenght from S to $\gamma(h)$. Thus the inequality (50) is valid. □

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Appendix A. Basics of compact Finsler manifolds

In this appendix, we summarize some basic theory of compact Finsler manifolds. This section is intended for the readers having background in imaging methods and elasticity. We follow the notation of [48] and use it as a main reference. The main goal is to prove that if $x \in M^{int}$ and $z_x \in \partial M$ is a closest boundary point to x, that is the minimizer of $d_F(x,\cdot)|_{\partial M}$ or $d_F(\cdot,x)|_{\partial M}$, then the distance minimizing curve from x to z_x or from z_x to x respectfully is a geodesic that is perpendicular to the boundary. Readers who are not familiar with Finsler geometry are encouraged to read this section before embarking to the proof of Theorem 1.3 presented in Section 3.

Most of the claims and the proofs given in this section are modifications of similar theorems in Riemannian geometry. We refer to the classical material where the Riemannian version is presented.

We let N be a n-dimensional, compact, connected smooth manifold without boundary. We reserve the notation TN for the tangent bundle of N and say that a function $F\colon TN \to [0,\infty)$ is a *Finsler function* if

(1) $F\colon TN \setminus \{0\} \to [0,\infty)$ is smooth
(2) For each $x \in N$ the restriction $F\colon T_xN \to [0,\infty)$ is a *Minkowski norm.*

Recall that for a vector space V a function $F\colon V \to [0,\infty)$ is called a Minkowski norm if the following hold

- $F\colon V \setminus \{0\} \to \mathbb{R}$ is smooth.
- For every $y \in V$ and $s > 0$ it holds that $F(sy) = sF(y)$.
- For every $y \in V \setminus \{0\}$ the function $g_y\colon V \times V \to \mathbb{R}$ is a symmetric positive definite bilinear form, where

$$g_y(v,w) := \frac{1}{2}\frac{\partial}{\partial s}\frac{\partial}{\partial t}\Big[F^2(y+sv+tw)\Big]\Big|_{s=t=0}. \tag{51}$$

We call the pair (N,F) a Finsler manifold.

The length of a piecewise smooth curve $c\colon I \to N$, I is an interval, is defined as

$$\mathcal{L}(c) := \int_I F(\dot{c}(t))dt. \tag{52}$$

For every $x_1, x_2 \in N$ we define

$$d_F(x_1,x_2) := \inf_{c \in C_{x_1,x_2}} \mathcal{L}(c),$$

where C_{x_1,x_2} is the collection of piecewise smooth curves starting at x_1 and ending at x_2. The function $d_F\colon N \times N \to [0,\infty)$ is a non-symmetric path metric related to F, meaning that for some $x_1, x_2 \in N$ the distance $d_F(x_1,x_2)$ need not coincide with $d_F(x_2,x_1)$ (see [5, Section 6.2]).

We note that for all $x_1, x_2 \in N$ it holds that

$$d_F(x_1,x_2) = d_{\overleftarrow{F}}(x_2,x_1), \tag{53}$$

where $\overleftarrow{F}$ is the reversed Finsler function $\overleftarrow{F}(x,y) = F(x,-y)$.

We use the notation $g_{ij}(x,y)$ for the component functions of the Hessian of $\frac{1}{2}F^2$ as in (51). A C^1 curve $\gamma\colon I \to N$, with a constant speed $F(\dot\gamma(t)) \equiv c \geq 0$, is a geodesic of Finsler manifold (N,F) if $\gamma(t)$ solves the system of *geodesic equations*

$$\ddot\gamma^i(t) + 2G^i(\dot\gamma(t)) = 0, \quad i \in \{1,\dots,m\}. \tag{54}$$

Here, $G^i\colon TN \to \mathbb{R}$ is given in local coordinates (x,y) by

$$G^i(x,y) = \frac{1}{4}g^{il}(x,y)\Big\{2\frac{\partial g_{jl}(x,y)}{\partial x^k} - \frac{\partial g_{jk}(x,y)}{\partial x^l}\Big\}y^j y^k. \tag{55}$$

Since $F^2(x,y)$ is positively homogeneous of degree two with respect to y variables, it follows from (55) that G^i is positively homogeneous of degree two with respect to y, but not necessarily quadratic in y. Therefore, the geodesic equation (54) is not preserved if the orientation of the curve γ is reversed.

We define a vector field $\mathbf{G}$, by

$$\mathbf{G}(x,y) := y^i\frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}. \tag{56}$$

A curve γ is a geodesic of F if and only if $\gamma = \pi(c)$, where c is an integral curve of $\mathbf{G}$. Due to ODE theory for a given initial conditions $(x,y) \in TN$ there exists the unique solution $\gamma_{x,y}$ of (54), defined on maximal interval containing 0. Thus by defining $\mathbf{G}$ locally with (56), it extends to a global vector field on TN. We call $\mathbf{G}$ the *geodesic vector field*.

Lemma A.1. *Let c be an integral curve of geodesic vector field $\mathbf{G}$, then $F(c(t))$ is a constant.*

Proof. For the proof see [48, Section 5.4]. □

We use the notations ϕ_t for the geodesic flow of F on TN and (x,v) for points in SN. By Lemma A.1 we know that for $(x,v) \in SN$ and for any $t \in \mathbb{R}$ in the flow domain of (x,v) it holds that $\phi_t(x,v) \in SN$. Since SN is compact we have proven that ϕ on SN is a global flow (see for instance [38, Theorem 17.11]), which means that the map

$$\phi\colon \mathbb{R} \times SN \to SN$$

is well defined. Therefore, we can define the exponential mapping $\exp_x$, $x \in N$ by

$$\exp_x(y) := \pi(\phi_1(x,y)) = \gamma_{x,y}(1), \quad y \in T_xN. \tag{57}$$

Moreover in [48, Section 11.4], it is shown that for any points $x_1, x_2 \in N$ there exists a globally minimizing geodesic from x_1 to x_2.

In the following, we relate the smoothness of a distance function to distance minimizing property of geodesics. This is done via the cut distance function $\tau\colon SN \to \mathbb{R}$, which is defined by

$$\tau(x,v) = \sup\{t > 0 : d_F(x, \gamma_{x,v}(t)) = t\}.$$

In the next lemma, we collect properties of the cut distance function.

Lemma A.2. *Let $(x,v) \in SN$ and $t_0 = \tau(x,v)$. At least one of the following holds:*

(1) *The exponential map $\exp_x$, of F, is singular at $t_0 v$.*
(2) *There exists $\eta \in S_xN$, $\eta \neq v$ such that $\exp_x(t_0 v) = \exp_x(t_0\eta)$.*

Moreover for any $t \in [0, t_0)$ the map $\exp_x$ is non-singular at tv. Also the map $\tau\colon SN \to \mathbb{R}$ is continuous.

Proof. See [48, Chapter 12] or [5, Chapter 8]. □

In the next lemma, we consider the regularity of the function d_F

Lemma A.3. *Let $(x_1,v_1) \in SN$, $0 < t_1 < \tau(x_1,v_1)$ and $x_2 = \gamma_{x_1,v_1}(t_1)$. Then there exists neighborhoods U of x_1 and V of x_2 respectively such that the distance function $d_F\colon U \times V \to \mathbb{R}$ is smooth.*

Proof. Since the cut distance function τ is continuous, there exist a neighborhood $U' \subset SM$ of (x_1,v_1) and $\epsilon > 0$ such that for any $t \in (t_1-\epsilon, t_1+\epsilon)$ and $(x,v) \in U'$ holds $t < \tau(x,v)$.

Consider a smooth function

$$E\colon U' \times (t_1-\epsilon, t_1+\epsilon) \ni ((x,v),t) \to (x, \exp_x tv) \in N \times N.$$

Since for every $((x,v),t) \in U' \times (t_1-\epsilon, t_1+\epsilon)$ we have that the exponential map $\exp_x$ is not singular at $vt \in T_xN$, the Jacobian of E is invertible in $U' \times (t_1-\epsilon, t_1+\epsilon)$. Thus the Inverse Function Theorem implies the existence of the neighborhood $U \times V \subset N \times N$ of (x_1,x_2) such that E is a diffeomorphism onto $U \times V$. Therefore the map

$$U \times V \ni (x,y) \mapsto \exp_x^{-1} y \in TN$$

is smooth.

By the definition of the cut distance function and [48, Section 11.4], the following equation holds for any $(x,y) \in U \times V$,

$$d_F(x,y) = F(x, \exp_x^{-1} y).$$

This implies the claim as F is smooth outside the zero section. □

The duality map between the tangent bundle and the cotangent bundle is given by the *Legendre transform* $\ell\colon TN \setminus \{0\} \to T^*N \setminus \{0\}$ which is defined by

$$\ell(x,y) = \ell_x(y) := g_y(y,\cdot) \in T_x^*N, \quad y \in T_xN. \tag{58}$$

The Legendre transform is a diffeomorphism and for all $a > 0$ and $(x,y) \in TN \setminus \{0\}$ we have

$$\ell(x,ay) = a\ell(x,y). \tag{59}$$

(see [48, Section 3.1]). The dual F^* of the Finsler function F, which is given by

$$F^*(x,p) := \sup_{v\in S_xN} p(v), \quad (x,p)\in T^*N, \tag{60}$$

is a Finsler function on T^*N and the Legendre transform ℓ_x satisfies

$$F(x,v) = F^*(x,\ell_x(v)).$$

We let $S\subset N$ be a smooth submanifold of co-dimension 1. It is shown in [48, Section 2.3] that for every $z\in S$ there exists precisely two unit vectors $\nu_1,\nu_2\in S_zN$ such that

$$T_zS = \{y\in T_zN : g_{\nu_i}(\nu_i,y)=0\},\ i\in\{1,2\}.$$

Vectors $\nu_1,\nu_2\in S_pN$ are called the unit normals of S. Notice that generally $\nu_1\neq -\nu_2$.

In the next lemma, we relate the Legendre transform of the velocity field of a distance minimizing geodesic to the differential of the distance function.

Lemma A.4. *Let $x_1\in N$ and $x_2\in N$ be such that $d_F(x_1,\cdot)$ is smooth at x_2. Then*

$$d(d_F(x_1,\cdot))\Big|_{x_2} = g_{\dot\gamma_{x_1,v}(t)}(\dot\gamma_{x_1,v}(t),\cdot)\Big|_{t=d_F(x_1,x_2)} \in T^*_{x_2}N, \tag{61}$$

where $\gamma_{x_1,v}$ is the unique distance minimizing unit speed geodesic from x_1 to x_2.

Proof. Denote $t_0 = d_F(x_1,x_2)$ and

$$S(x_1,t_0) = \exp_{x_1}\{w\in T_{x_1}N : F(w)=t_0\}.$$

Recall that

$$d_F(x_1,\exp_{x_1}(tw)) = F(tw) = t, \quad t>0,\ w\in S_{x_1}N \tag{62}$$

if tw is close to t_0v. We use a shorthand notation d_F for the function $d_F(x_1,\cdot)$. We take a t-derivative from the both sides of (62) to obtain

$$d(d_F)\Big|_{\exp_{x_1}(tw)}(D\exp_{x_1}|_{tw}w) = d(d_F)\Big|_{\exp_{x_1}(tw)}(\dot\gamma_{x_1,w}(t)) = 1. \tag{63}$$

Due to (63) the set $S(x_1,t_0)$ is a regular level set of d_F near x_2, and moreover (62) implies

$$T_{x_2}S(x_1,t_0) = \ker d(d_F)\Big|_{x_2}.$$

Thus it suffices to prove that

$$\ker g_{\dot\gamma_{x_1,v}(t_0)}(\dot\gamma_{x_1,v}(t_0),\cdot) = T_{x_2}S(x_1,t_0).$$

Notice that for any $w\in T_{x_1}M$, such that $g_v(v,w)=0$ holds

$$0 = g_{tv}(tv,tw) = \frac{1}{2}\frac{d}{ds}[F^2](t(v+sw))\Big|_{s=0} = t_0 d(d_F)\Big|_{\exp_{x_1}(tv)}(D\exp_{x_1}|_{tv}tw). \tag{64}$$

Therefore, $d(d_F)|_{\exp_{x_1}(tv)}(D\exp_{x_1}|_{tv}tw)=0$ and $(D\exp_{x_1}|_{t_0v}t_0w)\in T_{x_2}S(x_1,t_0)$. Recall that $J(t) := D\exp_{x_1}|_{tv}tw$ is the unique Jacobi field with initial conditions $J(0)=0,\ D_tJ(0)=w$. By Gauss' Lemma [48, Lemma 11.2.1] we have

$$0 = g_v(v,w) = g_{\dot\gamma_{x_1,v}(t_0)}(\dot\gamma_{x_1,v}(t_0), D\exp_{x_1}|_{t_0v}w)).$$

In the above, we used the identity

$$g_{tv_1}(tv_1, tv_2) = t^2 g_{v_1}(v_1, v_2), \quad t > 0;\ v_1, v_2 \in T_x N.$$

This implies that

$$\ker g_{\dot\gamma_{x_1,v}(t_0)}(\dot\gamma_{x_1,v}(t_0), \cdot) = \{D\exp_{x_1}|_{t_0 v} w : g_v(v, w) = 0\} = T_{x_2} S(x_1, t_0),$$

since $\dim v^\perp = \dim T_{x_2} S(x_1, t_0)$ and $D\exp_{x_1}|_{t_0 v}$ is not degenerate. □

Lemma A.5. *Let $S \subset N$ be a smooth closed submanifold of co-dimension 1. Let $x \in N$. A distance minimizing curve from S to x (from x to S) is a geodesic that is orthogonal to S at the initial (terminal) point.*

Proof. Since S is compact there exists a closest point $z_x \in S$ to x. We denote $h = d_F(x, z_x)$. Since (N, F) is complete there exists a distance minimizing geodesic γ from x to z_x.

We suppose first that $d_F(x, \cdot)$ is smooth at z_x. We denote $r(z) = d_F(x, z)$ for $z \in S$. Since z_x is a minimal point of r we have $d_S r(z_x) = 0$. Here d_S is the differrential operator of smooth manifold S. Then $d_S r = \iota^* d(d_F(x, \cdot))$, where $\iota : S \hookrightarrow N$. Thus $d(d_F(x, \cdot))$ vanishes on $T_{z_x} S$. By (61) it holds that

$$d(d_F(x, \cdot))|_{z_x} = g_{\dot\gamma(h)}(\dot\gamma(h), \cdot) \neq 0.$$

Thus $\dot\gamma(h)$ is normal to S at z_x.

If $d_F(x, \cdot)$ is not smooth at z_x there exists $\epsilon > 0$ such that for any $t \in (\epsilon, h)$ $d_F(\gamma(t), \cdot)$ is smooth at z_x. By the first part of the proof it follows that $\dot\gamma(h)$ is perpendicular to S.

Due to (53) the second claim for the reversed distance function can be proven in the same way, upon replacing F by $\overleftarrow{F}$. □

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