Disorder in large-scale networks with uni-directional feedback

H. Giray Oral and Dennice F. Gayme

Abstract— **This work investigates local and global measures of disorder in large-scale directed networks of double-integrator systems connected over a multi-dimensional torus. We quantify these performance measures in systems subjected to distributed disturbances using an $\mathcal{H}_2$ norm with outputs corresponding to local state errors or deviations from the global average. We consider two directed uni-directional state feedback interconnections that correspond to relative position and relative velocity feedback in vehicle network applications. Our main result reveals that absolute state feedback plays a critical role in system robustness when local state measurements are uni-directional. Specifically, if absolute measurements of either state variable are available, then systems with uni-directional relative feedback perform as well as their symmetric bi-directional counterparts but have the advantage of reduced communication requirements. However in the absence of absolute feedback their performance is worse; in fact, it is impossible to maintain stability (i.e. a finite $\mathcal{H}_2$ norm) with uni-directional state measurements for arbitrarily large networks. Numerical examples illustrate the theory.**

I. Introduction

Synchronization performance of networked systems can describe system attributes such as efficiency and robustness. Measures of such performance quantifying coherence (e.g. state deviation from the average) and degrees of disorder (e.g. state deviations from those of the nearest and farthest agents) [1] have been widely studied in the context of coupled linear oscillators (double-integrator dynamics) and first order consensus networks, see e.g. [1]–[7].

Disorder is a measure of system robustness that has been shown to depend on the network interconnection structure and network size, which can be measured either in terms of the number of agents [1], [8]–[14] or the spatial dimension of agent interactions [1], [9], [12]–[14]. For networks of second order integrators with undirected, static feedback interconnections, both long and short range disorder can grow unboundedly with network size without absolute measurements of both the position and velocity states [1]. Dynamic feedback with undirected interconnections and at least one type of absolute state measurement can uniformly bound the state deviation from the average with respect to network size [9], [10]. Scaling of measures of disorder with network size has been investigated for directed first order consensus networks controlled by static feedback in one [8], [11] and multiple [12]–[14] spatial dimensions as well as in directed 1-dimensional double-integrator networks [8], [15].

H. G. Oral and D. F. Gayme are with the Dept. of Mechanical Engineering at the Johns Hopkins University, Baltimore, MD, USA, 21218, giray@jhu.edu, dennice@jhu.edu. Partial support by the NSF (CNS 1544771) is gratefully acknowledged.

Improved scaling of these measures were demonstrated in 1-dimensional vehicle strings with directed nearest-neighbor interactions [8], and recent work has shown that more general directed feedback interconnections can reduce disorder in double integrator networks [3]. However, as in the undirected case, coherence cannot be achieved without absolute state measurements [16]. The scaling of disorder measures in double-integrator systems over multiple spatial dimensions with directed feedback interconnection has yet to be investigated.

In this paper, we take a step toward determining whether or not a directed feedback structure can improve how performance scales with respect to network size by considering uni-directional feedback in networks with arbitrary but finite spatial dimension. More precisely, we compare the performance of a network of agents with double-integrator dynamics and directed uni-directional local state measurements defined over a multi-dimensional torus to that of the network with symmetric bi-directional local state measurements studied in [1]. These models represent, for example, spatially invariant vehicle networks where comparible performance in systems with directed uni-directional feedback structures would be desirable due to the reduced sensing/communication requirements. Performance of the network is evaluated in terms of both a local measure quantifying the variance of an agent's position error with respect to its nearest predecessor and a global measure describing the variance of each agent's position deviation from the network average (dispersion of consensus error). We compute these measures using an $\mathcal{H}_2$ norm of the system subjected to distributed stochastic disturbances for system outputs defined to yield the desired measures. We then extend the scaling laws in [1] to the case of directed uni-directional feedback.

We exploit the spatial invariance of the interconnection structure to define the feedback laws and the performance outputs in terms of circular convolution operators based on the machinary used in [1]. After establishing the necessary and sufficient condition for input-output stability, we provide the closed-form solution for the $\mathcal{H}_2$ norm of the system for general feedback laws and performance outputs. Then we focus on the special case of directed uni-directional feedback which employs 'look-ahead' state measurements. For networks with absolute measurements of velocity, we provide a sufficient condition under which the $\mathcal{H}_2$ norm (performance measure) for a system with uni-directional feedback lower bounds the $\mathcal{H}_2$ norm of systems with symmetric bi-directional ('look-ahead / look-behind') feedback for any finite network size and spatial dimension. This bound generalizes recent results suggesting that directed feedback can improve performance [3]. We then show that local and global measures of disorder scale identically in systems with uni-directional and symmetric bi-directional feedback if at least one type of absolute state (position or velocity) feedback is employed. Conversely, in the absence of absolute velocity measurements we prove that it is impossible to maintain the input-output stability with uni-directional relative position and velocity feedback as the network size increases for arbitrary spatial dimension, which is consistent with the observations for 1-dimensional cyclic networks [17]–[20]. We note that a similar result more recently appeared in [21]. This condition represents an important difference between the uni-directional and the symmetric bi-directional feedback structures, as the latter maintains the stability for arbitrarily large spatially invariant networks [1].

Our results highlight a trade-off between performance and stability in large-scale networks with uni-directional feedback; while achieving comparable performance with reduced communication can be favorable, it comes at the cost of degradation of stability for certain feedback interconnection structures. Numerical examples confirm the theoretical results regarding performance scaling with respect to network size and the loss of stability for arbitrarily large networks without absolute velocity feedback.

The remainder of this paper is organized as follows. Section II-A defines the notation and provides the mathematical background used throughout the paper. Section II-B presents the system models and Section II-C presents the feedback policies. Section II-D describes the performance measures. Sections III provides the conditions for the input-output stability of the system, the closed-form solution for the $\mathcal{H}_2$ norm as well as a description of how performance scales with respect to network size and dimension. Section IV presents numerical examples supporting the theoretical results. Section V concludes the paper.

II. Problem Formulation

A. Preliminaries and Notation

We consider systems connected over the d-dimensional torus $\mathbb{Z}_N^d = \mathbb{Z}_N \times \cdots \times \mathbb{Z}_N$ defined as the d-fold cartesian product of the 1-dimensional torus $\mathbb{Z}_N = \{0, 1, \ldots, N-1\}$. An array A is defined as the mapping $A : \mathbb{Z}_N^d \mapsto \mathbb{C}^{p\times q}$ where p and q are scalars and A_k denotes each of the array elements corresponding to the spatial multi-index $k = (k_1, \ldots, k_d) \in \mathbb{Z}_N^d$. We denote vector-valued arrays ($q = 1$) with a lower-case letter. For example, the position state $x(t)$ is an array whose elements $x_k(t) \in \mathbb{R}^d$ represent the position of the k^{th} system in d spatial dimensions. Addition is performed modulo N for indices $k, l \in \mathbb{Z}_N^d$, i.e. $m = k + l$ with $m_i = (k_i + l_i)_N$ for $i = 1, \ldots, d$.

The multi-dimensional circular convolution of the arrays A and h yields an array z with elements given by

$$z_k = \sum_{l\in\mathbb{Z}_N^d} A_{k-l} h_l. \tag{1}$$

We equivalently write (1) as $z = \mathcal{A}h$, where $\mathcal{A}$ denotes the circular convolution operator associated with array A acting on array h. The multi-dimensional Discrete Fourier Transform (DFT) of A is defined by

$$\hat{A}_n := \sum_{k\in\mathbb{Z}_N^d} A_k e^{-j\frac{2\pi}{N} n\cdot k}, \tag{2}$$

where $(\cdot)$ denotes the scalar product, $n \in \mathbb{Z}_N^d$ is the wavenumber and $\hat{A}_n$ is the Fourier symbol of $\mathcal{A}$. It is a well-known fact that the DFT diagonalizes a circular convolution operator [1], so

$$\hat{z}_n = \hat{A}_n \hat{h}_n \quad \forall n \in \mathbb{Z}_N^d.$$

If $\hat{A}_n$ is a square matrix, then the eigenvalues of the circular convolution operator $\mathcal{A}$ are the union of the eigenvalues of all $\hat{A}_n$, i.e. $\sigma(\mathcal{A}) = \cup_{n\in\mathbb{Z}_N^d} \sigma(\hat{A}_n)$, where $\sigma(\cdot)$ denotes the spectrum of its argument.

The adjoint (conjugate transpose) of an operator (matrix) Q is denoted by Q^*. $E\{\cdot\}$ denotes the expected value of a random variable and $\|\cdot\|_{\mathcal{H}_2}$ denotes the $\mathcal{H}_2$ norm of a linear system. The zero and identity operators (matrices) are denoted by $\mathcal{O}$ (0) and $\mathcal{I}$ (I), respectively. T denotes an array with identical non-zero elements, i.e. $T_k = T_l \neq 0$ for all $k, l \in \mathbb{Z}_N^d$, and $\mathbf{1}$ denotes the array with elements $\mathbf{1}_k = I$ for all $k \in \mathbb{Z}_N^d$. The arrows $\nearrow$ and $\searrow$ repsectively denote the left and right limits to a real number. $O(\cdot)$ denotes the approximation order.

B. Double-Integrator Systems over the d-Dimensional Torus

We consider $M := N^d$ identical systems defined over $\mathbb{Z}_N^d$ each having double-integrator dynamics given by

$$\begin{aligned} \dot{v}_k &= -u_k + w_k, \\ v_k &= \dot{x}_k \quad \forall k \in \mathbb{Z}_N^d, \end{aligned} \tag{3}$$

where $x_k \in \mathbb{R}^d$, $v_k \in \mathbb{R}^d$, $u_k \in \mathbb{R}^d$ and $w_k \in \mathbb{R}^d$ respectively denote the position, velocity, control input and an exogenous local disturbance. The control input is of the form

$$u_k = g_o x_k + f_o v_k + \sum_{l\in\mathbb{Z}_N^d} G_{k-l} x_l + \sum_{l\in\mathbb{Z}_N^d} F_{k-l} v_l, \tag{4}$$

where $g_o, f_o \geq 0$ are the feedback gains associated with the measurements of states with respect to an absolute reference frame (absolute feedback). The circular convolutions of the states with the feedback arrays $G : \mathbb{Z}_N^d \mapsto \mathbb{R}^{d\times d}$ and $F : \mathbb{Z}_N^d \mapsto \mathbb{R}^{d\times d}$ define feedback laws based on relative state measurements (relative feedback).

Combining (3) and (4) yields

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} \mathcal{O} & \mathcal{I} \\ -\mathcal{A} & -\mathcal{B} \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} \mathcal{O} \\ \mathcal{I} \end{bmatrix} w, \tag{5}$$

where $\mathcal{A} = g_o\mathcal{I} + \mathcal{G}$ and $\mathcal{B} = f_o\mathcal{I} + \mathcal{F}$. Here, $\mathcal{G}$ and $\mathcal{F}$ are the circular convolution operators associated with the arrays G and F, respectively.

Remark 1: Since the feedback laws in (5) are determined by circulant operators, the feedback laws are invariant to the specific location $k \in \mathbb{Z}_N^d$, i.e. (5) describes a *spatially invariant* system [1], [22].

Assumptions: The following standard assumptions [1] will be imposed on $\mathcal{G}$ and $\mathcal{F}$ throughout the paper. Note that for clarity of exposition we state them only in terms of $\mathcal{G}$.

(A1) The feedback laws satisfy the property

$$\sum_{k\in\mathbb{Z}_N^d} G_k = 0,$$

which implies that $T \in \ker(\mathcal{G})$.

(A2) If $d \geq 2$, the feedback laws are decoupled in spatial coordinates, i.e. the interactions in the i^{th} spatial coordinate only depend on the state measurements in that spatial coordinate, for $i = 1, \dots, d$. This results in diagonal array elements G_k. In addition setting the non-zero entries of G_k to be equal leads to

$$G_k = g_k I, \quad g_k \in \mathbb{R}.$$

This condition also implies that the Fourier symbol of $\mathcal{G}$ is a scalar matrix

$$\hat{G}_n = \hat{g}_n I.$$

Therefore, by a slight abuse of notation we will refer to $\hat{g}_n$ as the Fourier symbol of $\mathcal{G}$.

(A3) If (A2) holds, the diagonal entries of each array element $G_k = g_k I$ satisfy the property

$$g_k \begin{cases} > 0, & k_1 = \cdots = k_d = 0 \\ \leq 0, & \text{otherwise.} \end{cases}$$

For spatially invariant systems, (A1) - (A3) generalize the properties of a circulant weighted graph Laplacian matrix to systems with arbitrary spatial dimension.

Under these assumptions, $\mathcal{G}$ and $\mathcal{F}$ can be specified to define feedback laws based on relative state measurements. In this setting, if $g_o \neq 0$ ($f_o \neq 0$), then the system is said to have absolute position (velocity) feedback. If $g_o = 0$ ($f_o = 0$), then we assume the system has relative position (velocity) feedback. If no relative position (velocity) feedback is used, then we assume $g_o \neq 0$ ($f_o \neq 0$).

C. Feedback Policies

We are interested in examining the effect of directed communication on the performance of large-scale networks by comparing systems with uni-directional and symmetric bi-directional feedback. In particular, we will investigate how the performance scales with network size. We next define the two feedback policies and then specify the peformance measures of interest in the subsequent subsection.

Bi-directional Feedback: In this communication structure, each agent employs a look-ahead / look-behind policy, in which the information flow in either direction is equally weighted. For example, if $d = 1$, this feedback interconnection is attained through the control input

$$\begin{aligned} u_k = g_o x_k + f_o v_k + \frac{1}{2}[&\gamma_g(x_k - x_{k+1}) + \gamma_g(x_k - x_{k-1}) \\ &+ \gamma_f(v_k - v_{k+1}) + \gamma_f(v_k - v_{k-1})], \end{aligned}$$

where $\gamma_g, \gamma_f \geq 0$ are control gains and the factor of $\frac{1}{2}$ provides a normalization of weights with respect to the uni-directional feedback described in the following subsection. For $d \geq 1$, the array associated with the corresponding local symmetric bi-directional feedback operator $\mathcal{Q}$ is given by

$$Q_k = \begin{cases} dI, & k_1 = \cdots = k_d = 0 \\ -\frac{1}{2}I, & k_i = \pm 1,\ k_j = 0 \text{ for } j \neq i \\ 0, & \text{otherwise,} \end{cases} \tag{6}$$

such that the operators $\mathcal{G}$ and $\mathcal{F}$ in (5) are given by

$$\mathcal{G} = \gamma_g \mathcal{Q}, \quad \mathcal{F} = \gamma_f \mathcal{Q}. \tag{7}$$

This feedback law was studied extensively in [1], [9].

Uni-directional Feedback: For directed communication, we consider uni-directional (look-ahead) feedback. For $d = 1$, the associated control input is given by

$$u_k = g_o x_k + f_o v_k + \gamma_g(x_k - x_{k+1}) + \gamma_f(v_k - v_{k+1}).$$

For $d \geq 1$, the array associated with the corresponding local uni-directional feedback operator $\mathcal{R}$ is given by

$$R_k = \begin{cases} dI, & k_1 = \cdots = k_d = 0 \\ -I, & k_i = -1,\ k_j = 0 \text{ for } j \neq i \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

In this case, the operators $\mathcal{G}$ and $\mathcal{F}$ in (5) are given by

$$\mathcal{G} = \gamma_g \mathcal{R}, \quad \mathcal{F} = \gamma_f \mathcal{R}. \tag{9}$$

The following proposition about the Fourier symbols of $\mathcal{Q}$ and $\mathcal{R}$ will be used in the subsequent results.

Proposition 1: The respective Fourier symbols $\hat{q}_n$ and $\hat{r}_n$ of the circular convolution operators $\mathcal{Q}$ and $\mathcal{R}$ defined by (6) and (8) are given by

$$\hat{r}_n = \sum_{i=1}^{d}\left(1 - e^{j\frac{2\pi}{N}n_i}\right), \quad \hat{q}_n = \sum_{i=1}^{d}\left(1 - \cos\left(\frac{2\pi}{N}n_i\right)\right). \tag{10}$$

Proof: Since $\mathcal{Q}$ can be decomposed as $\mathcal{Q} = \frac{\mathcal{R}+\mathcal{R}^*}{2}$, it holds that $\hat{q}_n = \mathrm{Re}(\hat{r}_n)$ for $n \in \mathbb{Z}_N^d$. Using the definition of the DFT given in (2) leads to $\hat{R}_n = \left(d - \sum_{i=1}^{d} e^{j\frac{2\pi}{N}n_i}\right) I$. The result is then obtained by invoking (A2), i.e. $\hat{R}_n = \hat{r}_n I$. ∎

D. Performance Measures

We now define system outputs that allow us to quantify local and global measures of system disorder through the input-output $\mathcal{H}_2$ norm of a system of the form (5) for the two feedback interconnection structures (7) and (9). These measures were detailed in [1] for systems with the feedback interconnection structure (7) but we repeat their definitions here for completeness.

Since we focus on spatially invariant systems, it is convenient to define a nodal performance measure of the form

$$P_k := \lim_{t\to\infty} E\{y_k^*(t) y_k(t)\}, \tag{11}$$

where y_k is the performance output given by the circular convolution

$$y_k = \sum_{l\in\mathbb{Z}_N^d} C_{k-l} x_l \quad \forall k \in \mathbb{Z}_N^d. \tag{12}$$

Here, we assume that C_k satisfies assumptions (A1) and (A2). Due to (A1), the consensus modes of (5) will be unobservable from the system output

$$y = \begin{bmatrix} \mathcal{C} & \mathcal{O} \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}, \tag{13}$$

where $\mathcal{C}$ represents the respective circular convolution operator associated with the operation in (12). We denote the input-output system defined by (5) and (13) by H. In this work we limit the analysis to performance measures based solely on the position, which is common for coordination [1], [9] and phase synchronization [4], [5] applications.

For white noise disturbance inputs w with unit covariance, the squared $\mathcal{H}_2$ norm of H quantifies the steady-state variance of the output [4]

$$\|H\|_{\mathcal{H}_2}^2 = \lim_{t\to\infty} E\{y^*(t)y(t)\}, \tag{14}$$

whenever H is input-output stable. Since the performance output of each system y_k is also spatially invariant, it is sufficient to divide (14) by the network size to recover each system's measure P_k, i.e.

$$P_k = \frac{1}{M}\|H\|_{\mathcal{H}_2}^2,$$

where we recall that $M = N^d$.

1) Local Error: This measure quantifies the steady-state variance of the deviation of each agent's position from that of its predecessor. For $d = 1$, the corresponding output for each system is

$$y_k = x_k - x_{k+1}.$$

The system output (13) for $d \geq 1$ can be obtained using the right shift operator along dimension i, namely $\mathcal{D}^i$ such that

$$(\mathcal{D}^i x)_{(k_1,\ldots,k_i,\ldots,k_d)} := x_{(k_1,\ldots,k_i+1,\ldots,k_d)},$$

and specifying

$$\mathcal{C} = \begin{bmatrix} \mathcal{I} - \mathcal{D}^1 \\ \vdots \\ \mathcal{I} - \mathcal{D}^d \end{bmatrix} \Rightarrow \begin{aligned} (y_k)_i = x&_{(k_1,\ldots,k_i,\ldots,k_d)} \\ &- x_{(k_1,\ldots,k_i+1,\ldots,k_d)}, \\ &\qquad i = 1,\ldots,d. \end{aligned} \tag{15}$$

The local measure of disorder for each system at location k is then given by

$$P_{loc} = \lim_{t\to\infty} E\left\{\sum_{i=1}^{d} (y_k)_i^*(y_k)_i\right\}. \tag{16}$$

2) Deviation from the Average: This measure quantifies the steady-state variance of the deviation of each system's position from the average position of all of the systems. Therefore each system's output gives the consensus error

$$y_k = x_k - \frac{1}{M}\sum_{l\in\mathbb{Z}_N^d} x_l. \tag{17}$$

In this case, the output operator $\mathcal{C}$ becomes

$$\mathcal{C} = \mathcal{I} - \frac{1}{M}\mathcal{J}, \tag{18}$$

where $\mathcal{J}$ denotes the circular convolution operator associated with the array $\mathbf{1}$. The corresponding performance measure of the form (11) quantifies the global degree of disorder in the network and will be denoted by P_{dav} for each system.

III. Disorder in Large-scale Uni-directional Networks

In this section, we first provide conditions for the input-output stability of H. We then derive the closed-form solution of its $\mathcal{H}_2$ norm, for the case in which the directed feedback operators $\mathcal{A}$ and $\mathcal{B}$ (satisfying (A1)-(A3)) in (5) and the directed output operator $\mathcal{C}$ (satisfying (A1) and (A2)) in (13) are circular convolution operators. These results for directed networks can be used to recover those in [1], which deal with the special case of undirected feedback.

Then we focus on the uni-directional feedback structure described in (9) and the specific performance measures P_{loc} and P_{dav} defined through the outputs in (15) and (17). We investigate these measures under various combinations of absolute and relative feedback and establish upper bounds on the $\mathcal{H}_2$ norm of H as a function of network size and spatial dimension. In particular, we provide sufficient conditions under which the uni-directional and the symmetric bi-directional feedback provide the same performance scaling.

Furthermore, for certain cases lacking absolute velocity feedback we show that uni-directional local measurements cannot maintain stability with finite control gains in any number of spatial dimensions if the network size is arbitrarily large.

A. Input-Output Stability

In this subsection, we derive conditions for the input-output stability of H. We first provide a condition for the case of any circulant output operator $\mathcal{C}$ satisfying assumptions (A1) and (A2), and then restate this condition for the specific cases of P_{loc} and P_{dav}.

We begin by stating a result from [23], which provides a generalization of the Routh-Hurwitz stability criterion to a second order polynomial with complex coefficients.

Proposition 2 (Lemma 4, [23]): The roots of a complex-coefficient polynomial $p(s) = s^2 + \beta s + \alpha$, where $\alpha, \beta \in \mathbb{C}$, satisfy $\mathrm{Re}(s) < 0$ if and only if the inequalities

$$\mathrm{Re}(\beta) > 0 \quad \text{and}$$

$$\mathrm{Re}(\alpha)\,\mathrm{Re}(\beta)^2 + \mathrm{Im}(\alpha)\,\mathrm{Im}(\beta)\,\mathrm{Re}(\beta) - \mathrm{Im}(\alpha)^2 > 0$$

simultaneously hold.

The following proposition provides the necessary and sufficient condition for the input-output stability of H. The proof builds upon (Corollary 3, [22]).

Proposition 3: System H defined by (5) and (13) is input-output stable if and only if the inequalities

$$\mathrm{Re}(\hat{b}_n) > 0 \quad \text{and} \tag{19a}$$

$$\begin{aligned}\Theta_n := &\,\mathrm{Re}(\hat{a}_n)\,\mathrm{Re}(\hat{b}_n)^2 \\ &+ \mathrm{Im}(\hat{a}_n)\,\mathrm{Im}(\hat{b}_n)\,\mathrm{Re}(\hat{b}_n) - \mathrm{Im}(\hat{a}_n)^2 > 0\end{aligned} \tag{19b}$$

simultaneously hold for all non-zero wavenumbers $n \neq 0,\ n \in \mathbb{Z}_N^d$ such that $\hat{c}_n \neq 0$.

Proof: Taking the DFT of the arrays on both sides of (5) and (13), one can obtain n subsystems of the form

$$\begin{bmatrix} \dot{\hat{x}}_n \\ \dot{\hat{v}}_n \end{bmatrix} = \begin{bmatrix} 0 & I \\ -\hat{A}_n & -\hat{B}_n \end{bmatrix} \begin{bmatrix} \hat{x}_n \\ \hat{v}_n \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} \hat{w}_n,$$
$$\hat{y}_n = \begin{bmatrix} \hat{C}_n & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_n \\ \hat{v}_n \end{bmatrix}, \quad n \in \mathbb{Z}_N^d. \tag{20}$$

Due to Assumption (A2), each subsystem can be decomposed into i identical subsystems

$$\begin{bmatrix} (\dot{\hat{x}}_n)_i \\ (\dot{\hat{v}}_n)_i \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\hat{a}_n & -\hat{b}_n \end{bmatrix} \begin{bmatrix} (\hat{x}_n)_i \\ (\hat{v}_n)_i \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (\hat{w}_n)_i,$$
$$(\hat{y}_n)_i = \begin{bmatrix} \hat{c}_n & 0 \end{bmatrix} \begin{bmatrix} (\hat{x}_n)_i \\ (\hat{v}_n)_i \end{bmatrix}, \quad i = 1, \ldots, d. \tag{21}$$

Denoting the transfer function of the realization in (20) by $\hat{H}_n(s)$ and that of the realization in (21) by $\hat{h}_n(s)$ leads to

$$\hat{H}_n(s) = \hat{h}_n(s) I \quad \text{and} \quad \hat{h}_n(s) = \frac{\hat{c}_n}{s^2 + \hat{b}_n s + \hat{a}_n}, \tag{22}$$

where we used the fact that (21) is in controllable canonical form. Since all of the modes associated with (20) are controllable, the poles of $H(s)$ are precisely given by the union of the poles of $\hat{H}_n(s)$ for all wavenumbers $n \in \mathbb{Z}_N^d$ such that $\hat{c}_n \neq 0$, i.e. they are determined by the observable modes. Since C_k satisfies Assumption (A1), we can use the definition of the DFT in (2) to obtain

$$\hat{C}_0 = \sum_{k \in \mathbb{Z}_N^d} C_k = 0,$$

which implies that $\hat{c}_0 = 0$ due to Assumption (A2), i.e. the output operator $\mathcal{C}$ has a zero Fourier symbol at $n = 0$. Therefore it is sufficient to consider only $n \neq 0$.

Then disregarding the multiplicities, the poles of $H(s)$ are precisely given by the poles of $\hat{h}_n(s)$ for all non-zero wavenumbers $n \neq 0,\ n \in \mathbb{Z}_N^d$ such that $\hat{c}_n \neq 0$. Invoking Proposition 2, the poles satisfy $\mathrm{Re}(s) < 0$ if and only if the inequalities in (19) simultaneously hold. ■

The interpretation of Proposition 3 is as follows. Since the output operator $\mathcal{C}$ satisfies (A1), the consensus modes of (5) associated with the wavenumber $n = 0$ (which are unstable in the absence of absolute feedback [8]) are unobservable from the output. Therefore, the input-output stability of H is equivalent to the stability of the observable modes associated with the non-zero wavenumbers. The next Lemma specializes this result to the cases of P_{loc} and P_{dav}.

Lemma 1: Consider the output matrices (15) and (18) associated with the performance measures P_{loc} and P_{dav}. System H defined by (5) and (13) is input-output stable if and only if the inequalities in (19) simultaneously hold for all $n \neq 0, n \in \mathbb{Z}_N^d$.

Proof: We first consider P_{loc}. Using (15) we get $\mathcal{Q} = \frac{1}{2}\mathcal{C}^*\mathcal{C}$ [1] therefore $|\hat{c}_n|^2 = 2\hat{q}_n$. Then for any n such that $n \neq 0$, we observe from (10) that $\hat{q}_n > 0$, which implies $\hat{c}_n \neq 0$. For P_{dav}, $\hat{c}_n = 1$ for any $n \neq 0$ [1]. In both cases $\hat{c}_n \neq 0$ for all $n \neq 0$, so Proposition 3 yields the result. ■

TABLE I: In systems with uni-directional feedback, asymptotic scalings of upper bounds on performance measures with respect to network size M in finite spatial dimension d. Quantities are up to a multiplicative factor that is independent of M,γ_g or γ_f.

	P_{loc}	P_{dav}
abs. pos. & abs. vel. $\left(f_o \geq \frac{\gamma_g}{\gamma_f}, \gamma_f \neq 0\right)$	$\frac{1}{\max\{\gamma_g,\gamma_f\}}$	1
rel. pos. & abs. vel. $\left(f_o \geq \frac{\gamma_g}{\gamma_f}, \gamma_f \neq 0\right)$	$1/\gamma_g$	$\frac{1}{\gamma_g}\begin{cases} M & d=1 \\ \ln(M) & d=2 \\ 1 & d\geq 3 \end{cases}$
abs. pos. & rel. vel. $(\gamma_g = 0)$	$1/\gamma_f$	$\frac{1}{\gamma_f}\begin{cases} M & d=1 \\ \ln(M) & d=2 \\ 1 & d\geq 3 \end{cases}$
abs. pos. & rel. vel. $(\gamma_g \neq 0)$	$+\infty$	$+\infty$
rel. pos. & rel. vel.	$+\infty$	$+\infty$

B. Performance Scaling with respect to Network Size

In this subsection we present the closed-form solution for the $\mathcal{H}_2$ norm of H. We then derive corresponding scaling bounds for the case of uni-directional feedback, in analogy with those reported in [1] for symmetric bi-directional feedback.

We first discuss the general setting with circulant directed feedback operators $\mathcal{A}$ and $\mathcal{B}$ (satisfying (A1)-(A3)) and a circulant directed output operator $\mathcal{C}$ (satisfying (A1) and (A2)).

Lemma 2: Suppose that system H defined by (5) and (13) is input-output stable. Then its $\mathcal{H}_2$ norm is given by

$$\|H\|_{\mathcal{H}_2}^2 = \frac{d}{2} \sum_{\substack{\hat{c}_n \neq 0, \\ n\neq 0, n \in \mathbb{Z}_N^d}} |\hat{c}_n|^2 \frac{\mathrm{Re}\,(\hat{b}_n)}{\Theta_n}, \tag{23}$$

where Θ_n is given by

$$\Theta_n = \mathrm{Re}\,(\hat{a}_n)\,\mathrm{Re}\,(\hat{b}_n)^2 + \mathrm{Im}\,(\hat{a}_n)\,\mathrm{Im}\,(\hat{b}_n)\,\mathrm{Re}\,(\hat{b}_n) - \mathrm{Im}\,(\hat{a}_n)^2.$$

Proof: Since the $\mathcal{H}_2$ norm of H is invariant to the change of basis that yields (20) [1], it is given by

$$\|H\|_{\mathcal{H}_2}^2 = \sum_{\substack{\hat{c}_n \neq 0, \\ n\neq 0, n \in \mathbb{Z}_N^d}} \|\hat{H}_n\|_{\mathcal{H}_2}^2 = d \sum_{\substack{\hat{c}_n \neq 0, \\ n\neq 0, n \in \mathbb{Z}_N^d}} \|\hat{h}_n\|_{\mathcal{H}_2}^2,$$

where we used (22) and the fact that unobservable modes have no contribution. Based on the realization of $\hat{h}_n$ given in (21), one can solve the associated Lyapunov equation

$$\begin{bmatrix} 0 & 1 \\ -\hat{a}_n & -\hat{b}_n \end{bmatrix}^* \begin{bmatrix} \hat{\phi}_{11} & \hat{\phi}_{12} \\ \hat{\phi}_{12}^* & \hat{\phi}_{22} \end{bmatrix} + \begin{bmatrix} \hat{\phi}_{11} & \hat{\phi}_{12} \\ \hat{\phi}_{12}^* & \hat{\phi}_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -\hat{a}_n & -\hat{b}_n \end{bmatrix} = \begin{bmatrix} -\hat{c}_n^*\hat{c}_n & 0 \\ 0 & 0 \end{bmatrix}$$

and use the fact that $\|\hat{h}_n\|_{\mathcal{H}_2}^2 = \hat{\phi}_{22}^{(n)}$. Solving the Lyapunov equation leads to $\hat{\phi}_{22}^{(n)} = |\hat{c}_n|^2 \frac{\mathrm{Re}\,(\hat{b}_n)}{2\Theta_n}$ and summing over all of the observable modes yields the result. ■

Lemma 2 indicates that the $\mathcal{H}_2$ norm depends on both the real and the imaginary parts of the Fourier symbols of $\mathcal{A}$ and $\mathcal{B}$. This is in contrast to the case in which the feedback structure is undirected, where the terms with the imaginary parts do not exist.

Remark 2: If the feedback operators $\mathcal{A}$ and $\mathcal{B}$ have even symmetry, i.e. if $A_k = A_{-k}$ and $B_k = B_{-k}$ for all the non-zero entries of the arrays A and B, then the feedback is undirected and Fourier symbols $\hat{a}_n$ and $\hat{b}_n$ are real. Then (23) reduces to the result in [1]

$$\|H\|_{\mathcal{H}_2}^2 = \frac{d}{2} \sum_{\substack{\hat{c}_n \neq 0, \\ n \neq 0, n \in \mathbb{Z}_N^d}} \frac{|\hat{c}_n|^2}{\hat{a}_n \hat{b}_n}. \tag{24}$$

The following lemma provides two sufficient conditions under which the $\mathcal{H}_2$ norm of the system with uni-directional feedback described by (9) respectively lower bounds or equals the $\mathcal{H}_2$ norm of the system with symmetric bi-directional feedback described by (7). At least one of these conditions can be satisfied for any finite network size in arbitrary spatial dimension given absolute measurements of at least one state variable (position or velocity).

Lemma 3: Consider the system H defined by (5) and (13). Let $H_{\mathcal{Q}}$ and $H_{\mathcal{R}}$ respectively denote the systems that have the feedback laws defined by (7) and (9). Then

1) $\|H_{\mathcal{R}}\|_{\mathcal{H}_2}^2 \leq \|H_{\mathcal{Q}}\|_{\mathcal{H}_2}^2$ if the following inequality holds

$$\gamma_f \left[f_o + \gamma_f \sum_{i=1}^{d} \left(1 - \cos\left(\frac{2\pi}{N} n_i \right) \right) \right] - \gamma_g \geq 0, \tag{25}$$

for all non-zero wavenumbers $n \neq 0$, $n \in \mathbb{Z}_N^d$ such that $\hat{c}_n \neq 0$.

2) $\|H_{\mathcal{R}}\|_{\mathcal{H}_2}^2 = \|H_{\mathcal{Q}}\|_{\mathcal{H}_2}^2$ if $\gamma_g = 0$.

Proof: We first consider the stability of $H_{\mathcal{R}}$, which has

$$\hat{a}_n = g_o + \gamma_g \hat{r}_n \quad \text{and} \quad \hat{b}_n = f_o + \gamma_f \hat{r}_n.$$

It holds that $\mathrm{Re}(\hat{r}_n) = \hat{q}_n = \sum_{i=1}^{d} \left(1 - \cos\left(\frac{2\pi}{N} n_i\right)\right)$ due to (10) in Proposition 1 and we see by inspection that $\hat{q}_n > 0$ for all $n \neq 0, n \in \mathbb{Z}_N^d$. Recalling that

$$\Theta_n = (g_o + \gamma_g \hat{q}_n)(f_o + \gamma_f \hat{q}_n)^2 + \gamma_g \,\mathrm{Im}\,(\hat{r}_n)^2 \left[\gamma_f (f_o + \gamma_f \hat{q}_n) - \gamma_g\right], \tag{26}$$

we observe that $\Theta_n > 0$ for all $n \neq 0, n \in \mathbb{Z}_N^d$ such that $\hat{c}_n \neq 0$ in either case of (25) or $\gamma_g = 0$ (since absolute or relative feedback is used for each state variable). Combining this with the fact that $\mathrm{Re}(\hat{b}_n) > 0$ for $n \neq 0$, we observe that (19) is satisfied for all $n \neq 0, n \in \mathbb{Z}_N^d$ such that $\hat{c}_n \neq 0$, hence $H_{\mathcal{R}}$ is input-output stable by Proposition 3. Setting $\mathrm{Im}\,(\hat{a}_n) = \mathrm{Im}\,(\hat{b}_n) = 0$ in (19) directly leads to the input-output stability of $H_{\mathcal{Q}}$, which has real $\hat{a}_n$ and $\hat{b}_n$.

Then one can rewrite (23) as

$$\|H_{\mathcal{R}}\|_{\mathcal{H}_2}^2 = \frac{d}{2} \sum_{\substack{\hat{c}_n \neq 0, \\ n \neq 0, n \in \mathbb{Z}_N^d}} \frac{|\hat{c}_n|^2 (f_o + \gamma_f \hat{q}_n)}{\Theta_n}.$$

Similarly, (24) reduces to

$$\|H_{\mathcal{Q}}\|_{\mathcal{H}_2}^2 = \frac{d}{2} \sum_{\substack{\hat{c}_n \neq 0, \\ n \neq 0, n \in \mathbb{Z}_N^d}} \frac{|\hat{c}_n|^2}{(g_o + \gamma_g \hat{q}_n)(f_o + \gamma_f \hat{q}_n)}.$$

Finally the inequality in (25) leads to

$$\frac{|\hat{c}_n|^2 (f_o + \gamma_f \hat{q}_n)}{\Theta_n} \leq \frac{|\hat{c}_n|^2}{(g_o + \gamma_g \hat{q}_n)(f_o + \gamma_f \hat{q}_n)}, \tag{27}$$

for all $n \neq 0, n \in \mathbb{Z}_N^d$ such that $\hat{c}_n \neq 0$. Summation over such n yields the first result. If $\gamma_g = 0$, equality holds in (27) due to (26). This leads to the second result. ∎

Lemma 3 provides a sufficient condition under which uni-directional feedback performs at least as well as symmetric bi-directional feedback in finite spatial dimension, for any circulant output operator $\mathcal{C}$ (satisfying (A1) and (A2)). Although achieving equal or better performance with a smaller number of relative state measurements is counterintuitive, this is possible through well tuned gains, for example using those that satisfy the inequality (25) in Lemma 3. However, in certain instances uni-directional feedback cannot perform better than symmetric bi-directional feedback, e.g. if the sign of this inequality is reversed. It must be emphasized that with appropriate gain selection, uni-directional feedback can be preferable due to only requiring single directional sensing.

We next employ this result to establish upper bounds on P_{loc} and P_{dav}, which we then invoke to specify how the performance scales with the network size M. The asymptotic scalings of the performance measures for the systems with uni-directional feedback are summarized in Table I.

Theorem 1: Consider the system with uni-directional feedback, namely $H_{\mathcal{R}}$. Then, the upper bounds on the performance measures have the following asymptotic scalings in finite spatial dimension d as $N \to \infty$.

1) Suppose that absolute velocity feedback is present, i.e. $f_o \neq 0$. Then for $\gamma_f \neq 0$ and $f_o \geq \frac{\gamma_g}{\gamma_f}$,

a) Absolute Position and Absolute Velocity Feedback

$$P_{loc} \sim \frac{1}{\max\{\gamma_g, \gamma_f\}}, \qquad P_{dav} \sim 1,$$

b) Relative Position and Absolute Velocity Feedback

$$P_{loc} \sim 1/\gamma_g, \qquad P_{dav} \sim \frac{1}{\gamma_g} \begin{cases} M & d = 1 \\ \ln(M) & d = 2 \\ 1 & d \geq 3 \end{cases}.$$

2) Absolute (but no relative) Position ($g_o \neq 0$ and $\gamma_g = 0$) and Relative Velocity Feedback

$$P_{loc} \sim 1/\gamma_f, \qquad P_{dav} \sim \frac{1}{\gamma_f} \begin{cases} M & d = 1 \\ \ln(M) & d = 2 \\ 1 & d \geq 3 \end{cases}.$$

Here the quantities are given up to a multiplicative factor that is independent of M, γ_g or γ_f.

Proof: It is shown in [1] that the upper bounds given above hold for $H_{\mathcal{Q}}$, i.e. the system with symmetric bi-directional feedback given by (7). We start by proving the first result. Recall from the proof of Lemma 1 that $\hat{c}_n \neq 0$ for all $n \neq 0$ in the case of P_{loc} and P_{dav}. Therefore, we invoke the first result of Lemma 3 for all $n \neq 0$. By assumption $f_o \geq \frac{\gamma_g}{\gamma_f}$, which implies that (25) is satisfied for all $n \neq 0, n \in \mathbb{Z}_N^d$ because the sum term is positive for such n. This yields $\|H_{\mathcal{R}}\|^2_{\mathcal{H}_2} \leq \|H_{\mathcal{Q}}\|^2_{\mathcal{H}_2}$, so the upper bounds on P_{loc} and P_{dav} also hold for $H_{\mathcal{R}}$. The second result follows from a similar argument and the second result of Lemma 3, since $\|H_{\mathcal{R}}\|^2_{\mathcal{H}_2} = \|H_{\mathcal{Q}}\|^2_{\mathcal{H}_2}$ if $\gamma_g = 0$. ■

Remark 3: In the absence of absolute velocity feedback, i.e. if $f_o = 0$, satisfying (25) for given $\gamma_g \neq 0$ and large wavenumbers n requires that $\gamma_f \to \infty$ as $N \to \infty$. In this case, the scaling laws of Theorem 1 do not necessarily hold.

As we demonstrate next for the system with uni-directional feedback, lack of absolute velocity measurements in systems with relative position and velocity feedback leads to instability (i.e. infinite $\mathcal{H}_2$ norm) in an arbitrarily large network connected over a multi-dimensional torus.

Theorem 2: Consider the system with uni-directional feedback, namely $H_{\mathcal{R}}$ and the performance measures P_{loc} and P_{dav}. Suppose that $f_o = 0$ and $\gamma_g \neq 0$, i.e. the system either has

1) Absolute (with relative) Position and Relative Velocity Feedback, or
2) Relative Position and Relative Velocity Feedback.

In finite spatial dimension d, if g_o, γ_g and γ_f are finite, then there exists a finite $\bar{N} > 0$ such that for all $N > \bar{N}$, $H_{\mathcal{R}}$ is unstable, i.e. does not have a finite $\mathcal{H}_2$ norm.

Proof: For absolute (with relative) position and relative velocity feedback, using (10) one can write (26) as

$$\Theta_n = g_o\gamma_f^2\left(\sum_{i=1}^{d} 1 - \cos\frac{2\pi}{N}n_i\right)^2 + \gamma_g\gamma_f^2\left(\sum_{i=1}^{d} 1 - \cos\frac{2\pi}{N}n_i\right)^3 + \gamma_g\left(\sum_{i=1}^{d}\sin\frac{2\pi}{N}n_i\right)^2\left[\gamma_f^2\left(\sum_{i=1}^{d} 1 - \cos\frac{2\pi}{N}n_i\right) - \gamma_g\right]. \quad (28)$$

Consider the wavenumber $n = (N-1,\ldots,N-1)$. Then $\frac{2\pi}{N}n_i \nearrow 2\pi$ as $N \to \infty$. Therefore, if we approximate $\cos(\cdot)$ and $\sin(\cdot)$ around 2π using the first three terms in the Taylor series expansion, we obtain

$$\cos(2\pi - \delta) \approx 1 - \frac{\delta^2}{2} \quad \text{and} \quad \sin(2\pi-\delta) \approx -\delta, \quad \delta > 0.$$

Using these expressions one can re-write $\Theta_{(N-1,\ldots,N-1)}$ as

$$\Theta_{(N-1,\ldots,N-1)} \approx \frac{\gamma_g\gamma_f^2 d^3}{8}\delta^6 + \gamma_f^2 d^2\left(\frac{g_o}{4} + \frac{\gamma_g d}{2}\right)\delta^4 - \gamma_g^2 d^2 \delta^2.$$

As $N \to \infty$, $\delta \searrow 0$ which leads to

$$\Theta_{(N-1,\ldots,N-1)} \approx -O(\delta^2).$$

Thus for any finite g_o, γ_g and γ_f, there exists a finite $\bar{N} > 0$ such that for all $N > \bar{N}$, it holds that $\Theta_{(N-1,\ldots,N-1)} < 0$, i.e. the second inequality in (19) is violated for $n = (N-1,\ldots,N-1)$. Then by Lemma 1 $H_{\mathcal{R}}$ is unstable, i.e. does not have a finite $\mathcal{H}_2$ norm. For relative position and velocity feedback, we have $g_o = 0$ and the same argument holds. ■

Remark 4: Due to Proposition 3, we note that the proof of Theorem 2 holds for any output of the form (13) such that $\hat{c}_{(N-1,\ldots,N-1)} \neq 0$, i.e. the modes which become unstable as the network size grows are observable from the output.

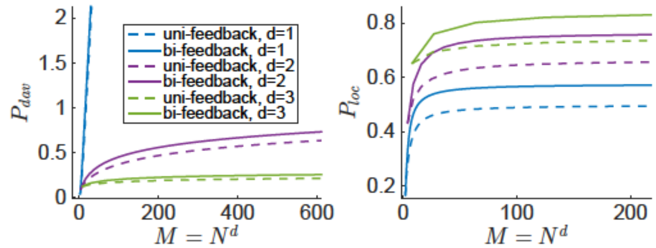

Fig. 1: P_{dav} and P_{loc} as a function of the network size M for relative position and absolute velocity feedback ($g_o = 0$, $f_o = 1$, $\gamma_g = 1$ and $\gamma_f = 1$). Performance scales as the laws given in Theorem 1.

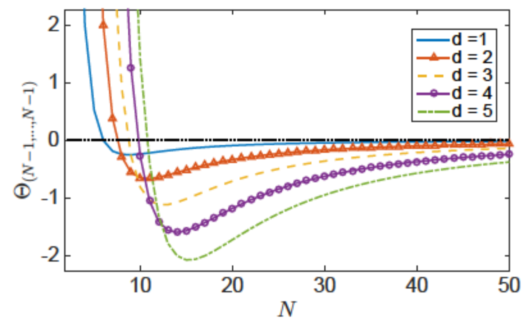

Fig. 2: With uni-directional relative position and velocity, but no absolute velocity feedback ($g_o = 1$, $f_o = 0$, $\gamma_g = 1$ and $\gamma_f = 1$), Θ_n in (28) cannot remain positive for $n = (N-1,\ldots,N-1)$ and finite N, which leads to instability due to Proposition 3.

Systems with double-integrator [17] or more general linear dynamics [18]–[20], which have directed relative feedback defined over the 1-dimensional torus, have been shown to exhibit similar instability behavior. While our result provides a generalization to the case of uni-directional feedback over a multi-dimensional torus, more recently a similar result appeared for directed bi-directional multi-neighbor interactions over the same lattice structure [21].

Theorem 2 highlights the limitation of uni-directional relative feedback. If relative position feedback is used, it is not possible to find a set of finite control gains that stabilizes arbitrarily large networks in any finite number of spatial dimension unless the agents have access to their absolute velocity. However, networks with uni-directional relative feedback can not only be stabilized but also provide the same performance scaling as that of the symmetric bi-directional feedback using at least one type (position or velocity) of absolute state information, as stated in Theorem 1. Namely, either by adding absolute velocity feedback to the cases with relative position feedback, or by eliminating relative position feedback given that absolute position measurements are available. While uni-directional feedback can lead to instability in arbitrarily large networks without absolute state information, access to it in position or velocity combined with well-tuned gains can lead to a favorable scheme, since the same performance scaling can be achieved with reduced sensing/communication requirements.

The next section provides numerical examples that illustrate the results in theorems 1 and 2.

IV. NUMERICAL EXAMPLES

In this section, we provide two numerical examples that confirm the theory presented in the previous section. The first one shows the performance scalings in the case of relative position and absolute velocity feedback. The second one demonstrates that stability is lost for finite network size if uni-directional relative position feedback is used in the absence of absolute velocity feedback.

In Figure 1, performance measures P_{loc} and P_{dav} are plotted as a function of the network size M. For relative position and absolute velocity feedback with the gains $f_o = 1$ and $\gamma_g = \gamma_f = 1$, the performance scaling obeys the laws presented in Theorem 1. It is also observed that the uni-directional feedback provides better performance compared to that of the symmetric bi-directional feedback, which is expected based on the result of Lemma 3, i.e. since the control gains strictly satisfy the inequality in (25).

We also confirm the result of Theorem 2 by plotting Θ_n in (28) for $n = (N-1, \ldots, N-1)$ as a function of N with uni-directional relative feedback and no absolute velocity information (i.e. $f_o = 0$) in Figure 2. Specifically, for absolute (with relative) position and relative velocity feedback with the gains $g_o = 1$ and $\gamma_g = \gamma_f = 1$ and spatial dimensions $d = 1, \ldots, 5$, $\Theta_{(N-1,\ldots,N-1)}$ cannot remain positive as N is increased. This leads to instability for finite N due to Proposition 3. As expected, $\Theta_{(N-1,\ldots,N-1)}$ goes to zero as $N \to \infty$.

V. CONCLUSIONS AND DIRECTIONS FOR FUTURE WORK

We have studied the asymptotic scaling of local and global measures of disorder in a large-scale directed network defined over a multi-dimensional torus. We have considered absolute as well as relative uni-directional state measurements. Our main results show that absolute state information plays a critical role in the performance and the stability of large-scale networks if the relative state measurements are uni-directional. Additionally a well-tuned uni-directional feedback structure can provide the same performance scaling with network size as the symmetric bi-directional feedback, with the advantage of requiring less sensing/communication. As a direction of future work, we will consider the performance scaling of bi-directional interactions with non-equal weights (a directed feedback structure), which has been shown to improve the transient behavior [24] but degrade string stability [20] in vehicle platoons.

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