Conditional Sparse ℓ_p-norm Regression With Optimal Probability

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Abstract

We consider the following *conditional linear regression* problem: the task is to identify both (i) a k-DNF[1] condition c and (ii) a linear rule f such that the probability of c is (approximately) at least some given bound μ, and f minimizes the ℓ_p loss of predicting the target z in the distribution of examples *conditioned on c*. Thus, the task is to identify a portion of the distribution on which a linear rule can provide a good fit. Algorithms for this task are useful in cases where simple, learnable rules only accurately model portions of the distribution. The prior state-of-the-art for such algorithms could only guarantee to find a condition of probability $\Omega(\mu/n^k)$ when a condition of probability μ exists, and achieved an $O(n^k)$-approximation to the target loss, where n is the number of Boolean attributes. Here, we give efficient algorithms for solving this task with a condition c that nearly matches the probability of the ideal condition, while also improving the approximation to the target loss. We also give an algorithm for finding a k-DNF *reference class* for prediction at a given query point, that obtains a sparse regression fit that has loss within $O(n^k)$ of optimal among all sparse regression parameters and sufficiently large k-DNF reference classes containing the query point.

[1]k-Disjunctive Normal Form (k-DNF): an OR of ANDs of at most k *literals*, where a literal is either a Boolean attribute or the negation of a Boolean attribute.

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1 Introduction

In areas such as advertising, it is common to break a population into *segments*, on account of a belief that the population as a whole is heterogeneous, and thus better modeled as separate subpopulations. A natural question is how we can identify such subpopulations. A related question arises in personalized medicine. Namely, we need existing cases to apply data-driven methods, but how can we identify cases that should be grouped together in order to obtain more accurate models?

In this work, we consider a problem of this sort recently formalized by Juba [2017]: we are given a data set where each example has both a real-valued vector together with a Boolean-valued vector associated with it. Let $\vec{Y}$ denote the real-valued vector for some example, and let $\vec{X}$ denote the vector of Boolean attributes. We are also given some target prediction Z of interest, and we wish to identify a k-DNF c such that $c(\vec{X})$ specifies a subpopulation on which we can find a linear predictor for Z, $\langle \vec{a}, \vec{Y} \rangle$, that achieves small loss. In particular, it may be that neither $\vec{a}$, nor any other linear predictor achieves small loss over the entire data distribution, and we simply wish to find the subset of the distribution, determined by our k-DNF c evaluated on the Boolean attributes of our data, on which accurate regression is possible. We refer to this subset of the distribution as a "segment." In real-world problems, identifying such "segments" sometimes can be extremely valuable. For example, after analyzing a medical data set, we could be able to detect an increased likelihood of cancer for those who happen to be over 65, on medication A, and of a certain ethnicity, etc. (described in form of a k-DNF c). Discovering these kinds of patterns would be difficult using standard statistical approaches.

Juba proposed two algorithms for different families of loss functions, one for the sup norm (that only applies to $O(1)$-sparse regression) and one for the usual ℓ_p norms. But, the algorithm for the ℓ_p norm has a major weakness, namely that it can only promise to identify a condition $c'(\vec{X})$ that has a probability polynomially smaller than optimal. This is especially problematic if the segment had relatively small probability to begin with: it may be that there is no such condition with enough examples in the data for adequate generalization. By contrast, the algorithm for the sup norm recovers a condition c' with probability essentially the same as the optimal condition, but of course the sup norm is an undesirable norm to use for regression, as it is maximally sensitive to outliers. In general, ℓ_p regression is more tolerant to outliers as p decreases, and the sup norm is roughly "$p = \infty$."

In this work, in Theorem 6, we show how to give an algorithm for $O(1)$-sparse ℓ_p regression with a two-fold improvement over the previous ℓ_p regression algorithm (mentioned above) when the coefficient vector is sparse:

1. The new algorithm recovers a condition c' with probability essentially matching the optimal condition.
2. We also obtain a smaller blow-up of the loss relative to the optimal regression parameters $\vec{a}$ and optimal condition c, reducing the degree of this polynomial factor.

More concretely, Juba's previous algorithm identifies k-DNF conditions, as does ours. This algorithm only identified a condition of probability $\Omega(\mu/n^k)$ when a condition with probability μ exists, and only achieved loss bounded by $O(\epsilon n^k)$ when it is possible to achieve loss ϵ. We improve the probability of the condition to $(1-\eta)\mu$ for any desired $\eta > 0$, and also reduce the bound on the loss to $\tilde{O}(\epsilon n^{k/2})$, or further to $\tilde{O}(\epsilon g \log\log(n^k))$ if the condition has only g "terms" (ANDs of literals), in Corollary 7. This latter algorithm furthermore features a smaller sample complexity, that only depends logarithmically on the number n of Boolean attributes. We include a synthetic data experiment demonstrating that the latter algorithm can successfully recover a planted solution under a small amount of noise, whereas Juba's sup norm algorithm completely fails. We also evaluate our algorithm against selective regression [El-Yaniv and Wiener, 2012] on some standard benchmarks.

We further give an algorithm for the closely related *reference class regression* problem in Theorem 9. In this problem, we are given a query point $\vec{x}^*$ (along with values for the real attributes $\vec{y}^*$), and we wish to estimate the corresponding z^*. In order to compute this estimate for a single point, we find some k-DNF c' such that $c'(\vec{x}^*) = 1$ so $\vec{x}^*$ is in the support of the distribution conditioned on $c'(\vec{X})$, and such that there are (sparse) regression parameters $\vec{a}'$ for c' that (nearly) match the optimal loss for sparse regression under any condition c with $c(\vec{x}^*) = 1$. In this case, the condition c' is a *reference class* containing $\vec{x}^*$, obtaining the tightest possible fit among k-DNF classes containing $\vec{x}^*$. We think of this as describing a collection of "similar cases" for use in estimating the target value z^* from the attributes $\vec{y}^*$. For this problem, we guarantee that we find a condition c' and sparse regression parameters $\vec{a}'$ that achieve loss within $O(n^k)$ of optimal.

Our algorithms are based on sparsifiers for linear systems, for example as obtained for the ℓ_2 norm by Batson et al. [2012], and as obtained for non-Euclidean ℓ_p norms by Cohen and Peng [2015] based on Lewis weights [Lewis, 1978]. The strategy is similar to Juba's sup norm algorithm: there, by enumerating the vertices of the polytopes obtained on various subsets of the data, he is able to obtain a list of all possible candidates for the optimal estimates of the sup-norm regression coefficients. Here, we can similarly obtain such a list of estimates for the ℓ_p norms by enumerating the possible sparsified linear systems. Sparsifiers are often used to accelerate algorithms, e.g., to run in time in terms of the number of nonzero entries rather than the size of the overall input. However, here we use the sparsified representation in a rather different way. Namely, the small representation allows for a feasible enumeration of possible candidates for the regression coefficients.

We note that this means that we also obtain an algorithm for $O(1)$-sparse ℓ_p-norm regression in the *list-learning* model [Balcan et al., 2008, Charikar et al., 2017]. Although Charikar et al. in particular are able to solve a large family of problems in the list-learning model, they observe that their technique can only obtain trivial estimates for regression problems. We show that once we have such a list, algorithms for the *conditional distribution search* problem [Juba, 2016, Zhang et al., 2017, Juba et al., 2018] can be generically used to extract a description of a good event c for the given candidate linear rule. Similar algorithms can also then be used to find a good reference class.

1.1 Relationship to other work

The conditional linear regression problem is most closely related to the problem of prediction with a "reject" option. In this model, a prediction algorithm has the option to abstain from making a prediction. In particular, El-Yaniv and Wiener [2012] considered linear regression in such a model. However, like most of the work in this area, their approach is based on scoring the "confidence" of a prediction function, and only making a prediction when the confidence is sufficiently high. This does not necessarily yield a nice description of the region in which the predictor will actually make predictions. On the other hand, Cortes et al. [2016] consider an alternative variant of learning with rejection that does obtain such nice descriptions; but, in their model, they assume that abstention comes at a fixed, known cost, and they simply optimize the overall loss when their space of possible prediction values includes this fixed-cost "abstain" option. By contrast, we can explicitly consider various rates of coverage or loss.

Our work is also related to the field of *robust statistics* [Huber, 1981, Rousseeuw and Leroy, 1987]: in this area, one seeks to mitigate the effect of outliers. That is, roughly, we suppose that a small fraction of the data is unusable (i.e. distant outliers), and we wish for our models and inferences to be relatively unaffected by these outliers. The difference between robust statistics and the conditional regression problem is that in conditional regression we are willing to ignore *most* of the data. We only wish to find a small fraction, described by some rule, on which we can obtain a good regression fit.

Another work that similarly focuses on learning for small subsets of the data is the the work by Charikar et al. [2017] in the list-learning model of Balcan et al. [2008]. In that work, one seeks to find a setting of some parameters that (nearly) minimizes a given loss function on an unknown small subset of the data. Since it is generally impossible to produce a single set of parameters to solve this task – many different small subsets could have different choices of optimal parameters – the objective is to produce a small list of parameters that contains a near-optimal solution for the unknown subset somewhere in it. As we mentioned above, our approach actually gives a list-learning algorithm for $O(1)$-sparse ℓ_p regression; but, we show furthermore how to identify a conditional distribution such that some regression fit in the list obtains a good fit, thus also solving the conditional linear regression task. Indeed, in general, we find that algorithms for solving the list-learning task for regression yield algorithms for conditional linear regression. The actual technique of Charikar et al. does not solve our problem since their approximation guarantee essentially always admits the zero vector as a valid approximation. Thus, they also do not obtain an algorithm for list-learning (sparse) regression either.

Another task similar to list-learning of linear regression is the problem of finding dense linear relationships, as solved by RANSAC [Fischler and Bolles, 1981]. But, these techniques only work in constant dimension. By contrast, although we are seeking sparse regression rules, this is a sparse fit in high dimension. As with list-learning, in contrast to the conditional regression problem, such algorithms do not provide a description of the points fitting the dense linear relationship.

Finally, yet another task similar to list-learning for regression is fitting linear mixed models [McCulloch and Searle, 2001, Jiang, 2007]. In this approach, one seeks to explain all (or almost all) of the data as a mixture of several linear rules. The guarantee here is incomparable to ours: in contrast to list-learning, the linear mixed model simply needs a list of linear rules that accounts for nearly all of the data; it does not need to find a list that accounts for all possible sufficiently large subsets of the data. So, there is no guarantee that any of the mixture components represent an approximation to the regression parameters corresponding to an event of interest. On the other hand, in linear mixed models, one does not need to give any description at all of which points should lie in which of the mixture components. In applications, one usually assigns points to the linear rule that gives it the smallest residual, but this may be less useful for predicting the values for new points.

2 Preliminaries

We now formally define the problems we consider in this work, and recall the relevant background.

2.1 The conditional regression and distribution search tasks

Formally, we focus on the following task:

Definition 1 (Conditional ℓ_p-norm Regression) Conditional ℓ_p-norm linear regression *is the following task. Suppose we are given access to i.i.d. examples drawn from a joint distribution over* $(\vec{X}, \vec{Y}, Z) \in \{0,1\}^n \times \{\vec{y} \in \mathbb{R}^d : \|y\|_2 \le b\} \times [-b, b]$ *such that for some k-DNF* c^* *and some coefficient vector* $\vec{a}^* \in \mathbb{R}^d$ *with* $\|\vec{a}^*\|_2 \le b$, $\mathbb{E}[|\langle \vec{a}^*, \vec{Y}\rangle - Z|^p | c^*(\vec{X})] \le \epsilon$ *and* $\Pr[c^*(\vec{X})] \ge (1+\eta)\mu$ *for some given* $b > 0$, $\epsilon > 0$, $\eta > 0$, *and* $\mu > 0$. *Then we wish to find* $\hat{c}$ *and* $\hat{\vec{a}}$ *such that* $\mathbb{E}[|\langle \hat{\vec{a}}, \vec{Y}\rangle - Z|^p | \hat{c}(\vec{X})]^{1/p} \le \alpha\epsilon$ *and* $\Pr[\hat{c}(\vec{X})] \ge (1-\eta)\mu$ *for an* approximation factor *function* α. *If* $\vec{a}^*$ *has at most s nonzero components, and we require* $\hat{\vec{a}}$ *likewise has at most s nonzero components, then this is the* conditional s-sparse ℓ_p-norm regression *task.*

As stressed by Juba [2017], the restriction to k-DNF conditions is not arbitrary. If we could identify conditions that capture arbitrary conjunctions, even for one-dimensional regression, this would yield PAC-learning algorithms for general DNFs. In addition to this being an unlikely breakthrough, recent work by Daniely and Shalev-Shwartz 2016 shows that such algorithms would imply new algorithms for random k-SAT, falsifying a slight strengthening of Feige's hypothesis [Feige, 2002]. We thus regard it as unlikely that any algorithm can find conditions of this kind. Of the classes of Boolean functions that do not contain all conjunctions, k-DNFs are the most natural large class, and hence are the focus for this model. We note though, that our algorithms could use disjunctions of any fixed polynomial-size families of formulas rather than ANDs of size k.

The difficulty of the problem lies in the fact that initially we are given neither the condition nor the linear predictor. Naturally, if we are told what the relevant subset is, we can just use standard methods for linear regression to obtain the linear predictor; conversely, if we are given the linear predictor, then we can use algorithms for the *conditional distribution search (learning abduction) task* (introduced by Juba [2016], recalled below) to identify a condition on which the linear predictor has small error. Our final algorithm actually uses this connection, by considering a list of candidates for the linear predictors to use for labeling the data, and choosing the linear predictor that yields a condition that selects a large subset. In particular, we will use algorithms for the following *weighted* variant of the *conditional distribution search* task:

Definition 2 (Conditional Distribution Search) Weighted conditional distribution search *is the following problem. Suppose we are given access to examples drawn i.i.d. from a distribution over* $(\vec{X}, W) \in \{0,1\}^n \times [0,b]$ *such that there exists a k-DNF condition* c^* *with* $\Pr[c^*(\vec{X})] \geq (1+\eta)\mu$ *and* $\mathbb{E}[W|c^*(\vec{X})] \leq \epsilon$ *for some given parameters* $\eta, \mu, b, \epsilon > 0$. *Then, find a k-DNF* $\hat{c}$ *such that* $\Pr[\hat{c}(\vec{X})] \geq (1-\eta)\mu$ *and* $\mathbb{E}[W|\hat{c}(\vec{X})] < \alpha \cdot \epsilon$ *for some* approximation factor *function* α *(or INFEASIBLE if no such* c^* *exists).*

This task is closely related to *agnostic* conditional distribution search, which is the special case where the weights only take values 0 or 1. The current state of the art for agnostic conditional distribution search is an algorithm given by Zhang et al. [2017], achieving an $\tilde{O}(\sqrt{n^k})$-approximation to the optimal error. That work built on an earlier algorithm due to Peleg [2007]. Peleg's work already showed how to extend his original algorithm to a weighted variant of the problem, and we observe that an analogous modification of the algorithm used by Zhang et al. will obtain an algorithm for the more general weighted conditional distribution search problem we are considering here:

Theorem 3 (Peleg 2007, Zhang et al. 2017) *There is a polynomial-time algorithm for weighted conditional distribution search achieving an* $\tilde{O}(n^{k/2}(\log b + \log 1/\eta + \log\log 1/\delta))$*-approximation with probability* $1-\delta$ *given* $m = \Theta\left(\frac{b^3}{\mu\epsilon\eta^2}(n^k + \log\frac{1}{\delta})\right)$ *examples.*

We note that it is possible to obtain much stronger guarantees when we are seeking a small formula. Juba et al. [2018] present an algorithm that, when there is a k-DNF with g terms achieving error ϵ, uses only $m = \tilde{O}(\frac{gk}{\mu\epsilon\eta^2}\log\frac{n}{\delta})$ examples and obtains a k-DNF with probability at least $(1-\eta)\mu$ and error $\tilde{O}(\epsilon g \log m)$. Although this is stated for the unweighted case (i.e., ϵ is a probability), it is easy to verify that since our loss is nonnegative and bounded, by rescaling the losses to lie in $[0,1]$, we can obtain an analogous guarantee for the weighted case:

Theorem 4 (Juba et al. 2018) *There is a polynomial-time algorithm for weighted conditional distribution search when the condition has g terms using* $m = \tilde{O}(\frac{bgk}{\mu\epsilon\eta^2}\log\frac{n}{\delta})$ *examples achieving an* $\tilde{O}(g\log m)$*-approximation using a k-DNF with* $\tilde{O}(g\log m)$ *terms with probability* $1-\delta$.

2.2 Reference class regression

Using a similar approach, we will also solve a related problem, selecting a best k-DNF "reference class" for regression. In this task, we are not merely seeking some k-DNF event of probability μ on which the conditional loss is small. Rather, we are given some specific observed Boolean attribute values $\vec{x}^*$, and we wish to find a k-DNF condition c *that is satisfied by* $\vec{x}^*$ solving the previous task. That is, c should have probability at least μ and our sparse regression fit has small conditional loss, conditioned on c. Naturally, the motivation here is that we have some specific point $(\vec{x}^*, \vec{y}^*)$ for which we are seeking to predict z^*, and so we are looking for a "reference class" c such that we can get the tightest possible regression estimate of z^* from $\vec{y}^*$; to do so, we need to take μ large enough that we have enough data to get a high-confidence estimate, and we need $\vec{x}^*$ to lie in the support of the conditional distribution for which we are computing this estimate.

Definition 5 (Reference Class ℓ_p-Regression) Reference class ℓ_p-norm regression *is the following task. We are given a* query point $\vec{x}^* \in \{0,1\}^n$, *target density* $\mu \in (0,1)$, *ideal loss bound* $\epsilon_0 > 0$ *approximation parameter* $\eta > 0$, *confidence parameter* $\delta > 0$, *and access to i.i.d. examples drawn from a joint distribution over* $(\vec{X}, \vec{Y}, Z) \in \{0,1\}^n \times \{\vec{y} \in \mathbb{R}^d : \|\vec{y}\|_2 \leq b\} \times [-b,b]$. *We wish to find* $\hat{\vec{a}} \in \mathbb{R}^d$ *with* $\|\hat{\vec{a}}\|_2 \leq b$ *and a* reference class *k-DNF* $\hat{c}$ *such that with probability* $1-\delta$, *(i)* $\hat{c}(\vec{x}^*) = 1$, *(ii)* $\Pr[\hat{c}(\vec{X})] \geq (1-\eta)\mu$, *and (iii) for a fixed* approximation factor $\alpha > 1$, $\mathbb{E}[|\langle\hat{\vec{a}}, \vec{Y}\rangle - Z|^p|\hat{c}(\vec{X})]^{1/p} \leq \alpha\max\{\epsilon^*, \epsilon_0\}$ *where* ϵ^* *is the optimal* ℓ_p *loss* $\mathbb{E}[|\langle\vec{a}^*, \vec{Y}\rangle - Z|^p|c^*(\vec{X})]^{1/p}$ *over* $\vec{a}^* \in \mathbb{R}^d$ *of* $\|\vec{a}^*\|_2 \leq b$ *and k-DNFs* c^* *such that* $c^*(\vec{x}^*) = 1$ *and* $\Pr[c^*(\vec{X})] \geq \mu$. *If we also require both* $\hat{\vec{a}}$ *and* $\vec{a}^*$ *to have at most s nonzero components, then this is the* reference class s-sparse ℓ_p-norm regression *task.*

The selection and use of such reference classes for estimation goes back to work by Reichenbach [1949]. Various refinements of this approach were proposed by

Table 1: Dimension needed for $(1 \pm \gamma)$-approximate ℓ_p sparsification of t-dimensional subspaces [Cohen and Peng, 2015]. $p = 2$ uses BSS weights [Batson et al., 2012, Boutsidis et al., 2014].

p	Required dimension r
$p = 1$	$\frac{t \log t}{\gamma^2}$
$1 < p < 2$	$\frac{1}{\gamma^2} t \log(t/\gamma) \log^2 \log(t/\gamma)$
$p = 2$	t/γ^2
$p > 2$	$\frac{\log 1/\gamma}{\gamma^5} t^{p/2} \log t$

Kyburg [1974] and Pollock [1990], e.g., to choose the estimate provided by the highest-accuracy reference class that is consistent with the most specific reference class containing the point of interest $\vec{x}^*$. Our approach is not compatible with these proposals, as they essentially disallow the use of the kind of *disjunctive* classes that are our exclusive focus. Along the lines we noted earlier, it is unlikely that there exist efficient algorithms for selecting reference classes that capture arbitrary conjunctions, so k-DNFs are essentially the most expressive class for which we can hope to solve this task. Bacchus et al. [1996] give a nice discussion of other unintended shortcomings of disallowing disjunctions. A concrete example discussed by Bacchus et al. is the genetic disease *Tay-Sachs*. Tay-Sachs only occurs in two very specific, distinct populations: Eastern European Jews and French Canadians. Thus, a study of Tay-Sachs should consider a reference class at least partially defined by a disjunction over membership in these two populations.

3 The weighting algorithms for conditional and reference class regression

The results from sparsification of linear systems (see supplemental material) tell us that we can estimate the loss on the subset of the data selected by the unknown condition by computing the loss on an appropriate "sketch," a weighted average of the losses on a small subset of the data. The dimensions in Table 1 give the size of these sparsified systems, i.e., the number of examples from the unknown subset we need to use to estimate the loss over the whole subset. The key point is that since the predictors are sparse, the dimension is small; and, since we are willing to accept a constant-factor approximation to the loss, a small number of points suffice. Therefore, it is feasible to enumerate these small tuples of points to obtain a list of candidate sets of points for use in the sketch. We also need to enumerate the weights for these points, but since we have also argued that the weights are bounded and we are (again) willing to tolerate a constant-factor approximation to the overall expression, there is also a small list of possible approximate weights for the points. The collection of points, together with the weights, gives a candidate for what might be an appropriate sketch for the empirical loss on the unknown subset. We can use each such candidate approximation for the loss to recover a candidate for the linear predictor. Thus, we obtain a list of candidate linear predictors that we can use to label our data as described above. More precisely, our algorithm is as shown in Algorithm 1. In the following, let $\Pi_{d_1,\ldots,d_s}$ denote the projection to coordinates $d_1, d_2, \ldots, d_s$.

Theorem 6 (Conditional sparse ℓ_p regression) *For any constant s and $\gamma > 0$, r as given in Table 1 for $t = (s+1)$, and $m = m_0 + \Theta\left(\frac{((1+\gamma)b)^3}{\mu\epsilon\eta^2}(n^k + s\log d + r\log\frac{m_0\log(\gamma^{1/p}s/r)}{\gamma} + \log\frac{1}{\delta})\right)$ examples, Algorithm 1 runs in polynomial time and solves the conditional s-sparse ℓ_p regression task with $\alpha = \tilde{O}((1+\gamma)\sqrt{n^k}(\log b + \log\frac{1}{\eta} + \log\log\frac{1}{\delta}))$.*

```
input       : Examples
              $(\vec{x}^{(1)}, \vec{y}^{(1)}, z^{(1)}), \ldots, (\vec{x}^{(m)}, \vec{y}^{(m)}, z^{(m)})$,
              target loss bound $\epsilon$ and fraction $\mu$.
output      : A $k$-DNF over $x_1, \ldots, x_n$ and linear
              predictor over $y_1, \ldots, y_d$, or
              INFEASIBLE if none exist.
subroutines : WtCond takes as inputs examples
              $(\vec{x}^{(1)}, \ldots, \vec{x}^{(m)})$, nonnegative weights
              $(w^{(1)}, \ldots, w^{(m)})$, and a bound $\mu$, and
              returns a $k$-DNF $\hat{c}$ over $x_1, \ldots, x_n$
              solving the weighted conditional
              distribution search task.
begin
    Let $m_0 = \lceil \frac{1}{\mu}(\frac{b^{2p}}{\gamma^{2p}\epsilon^{2p}}(2pb + \sqrt{2\ln(12/\delta)})^2 + \ln\frac{3}{\delta})\rceil$,
      $r$ is as given in Table 1.
    forall $(d_1, \ldots, d_s) \in \binom{[d]}{s}$, $(q_1, \ldots, q_r) \in$
      $\{-\lceil\frac{1}{\gamma}(\ln r - \frac{1}{p}\ln\gamma)\rceil, \ldots, 0, \ldots, \lceil\ln(s+1)/2\gamma\rceil\}$
      and $(j_1, \ldots, j_r) \in \binom{[m_0]}{r}$ do
        Let $\vec{a}$ be a solution to the following convex
          optimization problem: minimize
          $\sum_{\ell=1}^{r}((1+\gamma)^{q_\ell}(\langle\vec{a}, \Pi_{d_1,\ldots,d_s}\vec{y}^{(j_\ell)}\rangle - z^{(j_\ell)})^p$
          subject to $\|\vec{a}\|_p \leq b$.
        Put $c \leftarrow$ the output of WtCond on
          $(\vec{x}^{(1)}, \ldots, \vec{x}^{(m)})$ with the weights
          $w^{(i)} = |\langle\vec{a}, \Pi_{d_1,\ldots,d_s}\vec{y}^{(i)}\rangle - z^{(i)}|^p$ and bound $\mu$.
        if WtCond did not return INFEASIBLE and
          $\hat{\mathbb{E}}[(\langle\vec{a}, \Pi_{d_1,\ldots,d_s}\vec{Y}\rangle - Z)^p | c(\vec{X})]^{1/p} \leq \alpha\epsilon$ then
          return $\vec{a}$ and $c$.
    end
    return INFEASIBLE.
end
```

Algorithm 1: Weighted Sparse Regression

A proof is in the supplemental material. Taking the approximation parameters η and γ, the bound on the parameters b, norm p, sparsity s, term size k, and confidence δ to be constants, we find that we use $\sim \frac{1}{\mu\epsilon^{2p}} + \frac{1}{\mu\epsilon}(n^k + \log d + \log\frac{1}{\mu\epsilon})$ examples to obtain a $\tilde{O}(\sqrt{n^k})$-approximation to the ℓ_p loss with a condition of probability $\sim \mu$, whereas the previous algorithm only obtained a $\tilde{O}(n^k)$-approximation with a condition of probability $\sim \mu/n^k$. Furthermore, by simply plugging in the algorithm from Theorem 4 for WtCond, we can obtain the following improvement when the desired k-DNF condition is small:

Corollary 7 *For any constant s and $\gamma > 0$, r as given in Table 1 for $t = (s+1)$, and $m = m_0 + \Theta\left(\frac{(1+\gamma)bgk}{\mu\epsilon\eta^2}\left(\log\frac{n}{\delta} + s\log d + r\log\frac{m_0\log(\gamma^{1/p}s/r)}{\gamma}\right)\right)$ examples, if there is a g-term k-DNF solution to the conditional s-sparse ℓ_p-regression task, then the modified Algorithm 1 runs in polynomial time and solves the task with a $\tilde{O}(g\log m)$-term k-DNF with $\alpha = \tilde{O}((1+\gamma)g\log m)$.*

That is, using $\sim \frac{1}{\mu\epsilon^{2p}} + \frac{g}{\mu\epsilon}(\log n + \log d + \log\frac{1}{\mu\epsilon})$ examples, we get a $\tilde{O}(g(\log\frac{1}{\mu\epsilon} + \log\log n + \log\log d))$-approximation to the error obtained by the best g-term k-DNF (and the k-DNF we find itself only has $\tilde{O}(g(\log\frac{1}{\mu\epsilon} + \log\log n + \log\log d))$ terms). Note that this guarantee is particularly strong in the case where the k-DNF would be small enough to be reasonably interpretable by a human user.

The extension to reference class ℓ_p-norm regression proceeds by replacing the weighted condition search algorithm with a variant of the tolerant elimination algorithm from Juba [2016], given in Algorithm 2. (The analysis again is in the supplemental material.)

Lemma 8 *If $m \geq \Omega(\frac{b^3}{\eta^2(\epsilon_0+\epsilon^*)\mu_0}(k\log n + \log\frac{1}{\eta\delta} + \log\log\frac{1}{\mu_0} + \log\log\frac{b}{\epsilon_0}))$ where $W \in [0,b]$, then Algorithm 2 returns a k-DNF $\hat{c}$ such that with probability $1-\delta$, 1. $\hat{c}(\vec{x}^*) = 1$ 2. $\Pr[\hat{c}(\vec{X})] \geq \mu_0/(1+\eta)$ 3. $\mathbb{E}[W|\hat{c}(\vec{X})] \leq O((1+\eta)^4 n^k(\epsilon_0+\epsilon^*))$ where ϵ^* is the minimum $\mathbb{E}[W|c^*(\vec{X})]$ over k-DNF c^* such that $c^*(\vec{x}^*) = 1$ and $\Pr[c^*(\vec{X})] \geq (1+\eta)\mu_0$.*

Now, as noted above, our algorithm for reference class regression is obtained essentially by substituting Algorithm 2 for the subroutine WtCond in Algorithm 1; the analysis, similarly, substitutes the guarantee of Lemma 8 for Theorem 3. In summary, we find:

Theorem 9 (Reference class regression) *For any constant s and $\gamma > 0$, r as given in Table 1 for*

```
input  : Examples $(\vec{x}^{(1)}, w^{(1)}), \ldots, (\vec{x}^{(m)}, w^{(m)})$,
         query point $\vec{x}^*$, minimum fraction $\mu_0$,
         minimum loss target $\epsilon_0$, approximation
         parameter $\eta$.
output : A $k$-DNF over $x_1, \ldots, x_n$.
begin
    Initialize $\mu \leftarrow 1$, $\hat{c} \leftarrow \perp$, $\hat{\epsilon} \leftarrow \max_j w^{(j)}$
    while $\mu \geq \mu_0$ do
        Initialize $\epsilon \leftarrow \hat{\epsilon}$
        while $\epsilon \geq \epsilon_0/(1+\eta)$ do
            Initialize $c$ to be the empty disjunction
            forall Terms $T$ of at most $k$ literals do
                if $\sum_{j:T(\vec{x}^{(j)})=1} w^{(j)} \leq \epsilon\mu m$ then Add
                  $T$ to $c$.
            end
            Put $\epsilon \leftarrow \epsilon/(1+\eta)$
        end
        if $c(\vec{x}^*) = 1$, $\sum_{j=1}^m c(\vec{x}^{(j)}) \geq \mu m$, and $\epsilon < \hat{\epsilon}$
          then Put $\hat{c} \leftarrow c$, $\hat{\epsilon} \leftarrow \epsilon$
        Put $\mu \leftarrow \mu/(1+\eta)$
    end
    return $\hat{c}$
end
```

Algorithm 2: Reference Class Search

$t = (s+1)$, and

$$m \geq m_0 + \Omega\left(\frac{(1+\gamma)^3 b^3}{\eta^2(\epsilon_0+\epsilon^*)\mu_0}\left(r\log\frac{m_0\log(\gamma^{1/p}s/r)}{\gamma} + \log\left(\frac{n^k d^s}{\eta\delta}\log\frac{1}{\mu_0}\log\frac{(1+\gamma)b}{\epsilon_0}\right)\right)\right)$$

examples, our modified algorithm runs in polynomial time and solves the reference class s-sparse ℓ_p regression task with $\alpha = O((1+\gamma)(1+\eta)^4 n^k)$.

4 Experimental evaluation

To evaluate our algorithm's performance in practice, we performed two kinds of experiments: one using synthetic data with a planted solution, and another using standard benchmark data sets from the LIBSVM repository [Chang and Lin, 2011]. The aims of these experiments are to: 1) compare the performance of our algorithm to baseline algorithms; 2) demonstrate the algorithm's ability to scale to realistic data sets.

Synthetic data. To generate the synthetic data sets, we first chose a random 2-DNF over Boolean attributes by sampling terms uniformly at random. We also fixed two out of d coordinates at random and sampled parameters $\vec{a}$ from a mean 0, variance σ^2 Gaussian for our target regression fit. For the actual data, we then sampled Boolean attributes $\vec{X}$ that, with 25% probability are a uniformly random satisfying assignment

Table 2: Synthetic Data Set Properties

Size	#RealAtt	#BoolAtt	k-DNF Terms	Var σ^2
1000	6	10	4	0.01
5000	10	50	8	0.1
10000	20	50	16	0.1

to the DNF and with 75% probability are a uniformly random falsifying assignment. We sampled $\vec{Y}$ from a standard Gaussian in $\mathbb{R}^d$. For the examples where $\vec{X}$ was a satisfying assignment, we let Z be given by the linear rule $\langle \vec{a}, \vec{Y} \rangle + \nu$ where ν is a mean 0, variance σ^2 Gaussian; otherwise, if $\vec{X}$ was a falsifying assignment, Z was set to ν^* where ν^* is a standard mean 0, variance 1 Gaussian. (Parameters shown in Table 2.) Furthermore, for the 5000-example and 10000-example data sets, we also add 5% noise: with 5% probability, we instead set Z to a mean-0, variance-4 Gaussian. In general, conditioned on the planted formula, there is a small-error regression fit $\vec{a}$—the expected error of $\vec{a}$ is $|\nu|$. Off of this planted formula, the expected error for the optimal prediction $z = 0$ is $|\nu^*| = 1$.

For these simple data sets, we made several modifications to simplify and accelerate the algorithm. First, we simply fixed all of the weights (of the form $(1+\gamma)^{q_i}$) to 1, i.e., $q_i = 0$ for all i, since the leverage scores for Gaussian data (and indeed, most natural data sets) are usually small. Second, we used the version of the algorithm for small DNFs considered in Corollary 7, that uses algorithm of Juba et al. [2018] (recalled in Theorem 4) for WtCond. Finally, we set $m_0 = 100$ for all three data sets. Note that m_0 can be interpreted as the amount of data required to obtain a good linear regression fit in a constant number of dimensions. We observed that a reasonably small number suffices in practice without affecting the quality of results (namely, still obtaining the same k-DNF). The algorithm then searches over sketches for a subset of size m_0, which is in fact much smaller than the actual size m of the data set. It reduces the amount of computation in the search and thus, helps us to scale the algorithm to larger data sets.

As a baseline, we also used the Sup-Norm algorithm from Juba [2017] for conditional sparse sup-norm regression on our synthetic data. In order to obtain reasonable performance, we needed to make some modifications: we modified the algorithm along the lines of Algorithm 1, to only search over tuples of a random $m_0 = 100$ examples in producing its candidate regression parameters. We also chose to take the regression parameters that achieved the smallest residuals under the sup norm, among those that obtained a condition satisfied by the desired fraction of the data. We considered a range of possible values for ϵ; when $\epsilon \approx 1.5$ on the noiseless $\sigma^2 = 0.01$ data, the sup norm algorithm started adding terms outside the planted DNF, significantly increasing its error, but the algorithm is successful for the parameters shown here. Note that the sup norm of the Gaussian residuals is only bounded with high probability, and given m examples we expect it to grow approximately like $\sim \sigma \log m$. Parameters settings for both algorithms are given in Table 3.

Table 3: Parameter Settings for ℓ_2-Norm and Sup-Norm Algorithms

(a) ℓ_2-Norm

Dataset Size	s	μ	γ
1000	2	0.2250	1
5000	2	0.2465	1
10000	2	0.2495	1

(b) Sup-Norm

Dataset Size	s	μ	ϵ
1000	2	0.2450	1.2
5000	2	0.2450	3.3–3.8
10000	2	0.2450	3.3–3.8

For the 1000-example data set, both algorithms were able to identify the planted 2-DNF (note that once the planted 2-DNF has been identified, a good regression fit can be found by any number of methods.) However, for the 5000-example and 10000-example data sets, while the ℓ_2-norm algorithm (i.e. Algorithm 1) could still find the planted 2-DNF, the sup-norm algorithm failed: below $\epsilon = 3.3$ it returns an empty DNF, and at $\epsilon = 3.8$, it returned formulas that were satisfied by 96.86% of the examples for the 5000-example data set and 99.99% of the examples for the 10000-example data set, whereas the planted condition contains about 25%. In between this range, the algorithm finds a small (2–6 term) DNF that is satisfied by 25–29% of the data, but that contains at most one term from the planted DNF. On average, only 35% of the examples that satisfied the found DNF on the 5000 example data set were from the planted DNF, comprising 10–18% of the data overall, and only 40% were from the planted DNF on the 10000 example data set, comprising 15–19% of the data overall. Thus, the found DNF includes a substantial fraction of the data that is not from the planted distribution and thus has high error. The sup norm is maximally sensitive to extreme points. Thus, for these two data sets, the sup norm on each term was dominated by the outliers (i.e. noise, which have higher variance). So, the algorithm cannot distinguish good terms from bad, and it adds many terms outside the planted 2-DNF. Hence, it was not able to find the true condition under which the bulk of the data can be predicted accurately.

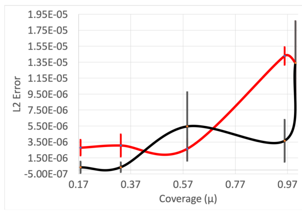
Real-world benchmarks. We also compared Algorithm 1 against l_2 *selective regressors* [El-Yaniv and Wiener, 2012] on some of the LIBSVM [Chang and Lin, 2011] regression data sets that were used to evaluate that work. We split each of those datasets into a training set (1/3 of the data) and test set (2/3 of the data). We generated Boolean attributes using binary splits at the median values on the numerical features, excluding the target attribute. We then ran Algorithm 1 on the training set to obtain a 2-DNF, using $m_0 = 100$ for the smaller data sets (Boston housing and Body fat) and $m_0 = 200$ for the larger data sets (Cpusmall and Space).[2] Next, we filtered both the training and test data satisfying the 2-DNF, using the first one to train a new regression fit on the selected subset and the other as a holdout to estimate the error of this resulting regression fit. We compared this to the test error for selective regression, averaged over ten runs. Risk-Coverage (RC) curves of the results are shown in Figure 1. We outperform the baseline on the Boston housing dataset, and we generally achieved lower error than the baseline for lower coverage (less than 0.5) on all data sets except Body fat and somewhat higher error for higher coverage on three data sets (Body fat, Space, and Cpusmall). Since selective regression first tries to fit the entire data set, and then chooses a subset where that fixed predictor does well, it is to be expected that it may miss a small subset where a different predictor can do better, but that its freedom to abstain on a somewhat arbitrary subset may give it an advantage at high coverage. However, note that in addition, we also obtained a 2-DNF that describes the subpopulation on all four data sets, which is the main advantage of our method. For example, for the Boston housing data set, at 25 % coverage, our algorithm detected a good regression fit on a subset of the data specified by the following condition (i.e. 2-DNF):

$$\begin{aligned}&(NOX > 0.54 \textit{ and } DIS > 3.2)\\ \textit{or } &(CRIM \leq 0.26 \textit{ and } RAD > 5)\\ \textit{or } &(DIS > 3.2 \textit{ and } PTRATIO > 19)\end{aligned}$$

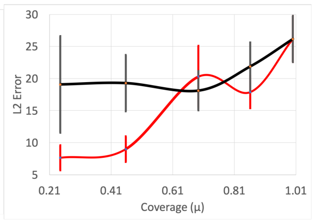
where NOX: nitric oxides concentration (parts per 10 million); DIS: weighted distances to five Boston employment centres; $CRIM$: per capita crime rate by town; RAD: index of accessibility to radial highways; $PTRATIO$: pupil-teacher ratio by town.

Given this condition and the regression fit, we can easily predict the target variable (in this case, is the median value of owner-occupied homes in $\$1000's$) for any new data that satisfies this condition.

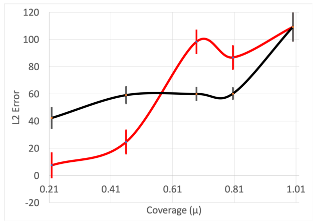
[2]Empirically, we observed on Boston housing that we could have taken m_0 as small as 50 for $\mu \geq 20\%$ and we still find the same DNFs; similarly, for Space, we could take m_0 as small as 120 for $\mu \geq 20\%$ and still find the same DNFs.



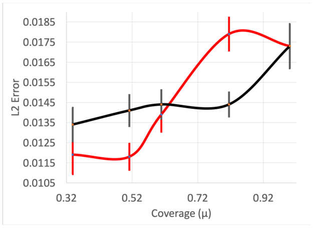
(a) Body fat



(b) Boston housing



(c) Cpusmall



(d) Space

Figure 1: RC curves of Algorithm 1 (red line) and the baseline method (black line) on different regression data sets from the LIBSVM repository for a single run of the algorithms. The horizontal axis represents the coverage (μ), while the vertical axis represents the error.

Scala code for these experiments that uses the Google Kubernetes engine is available at `https://github.com/johnhainline/sclr/`.

5 Directions for future work

There are several natural open problems. First, although sparsity is desirable, our exponential dependence of the running time (or list size) on the sparsity is problematic. (The sup norm regression algorithm [Juba, 2017] also suffered this deficiency.) Is it possible to avoid this? Second, our algorithm for reference class regression has an $O(n^k)$ blow-up of the loss, as compared to $\tilde{O}(n^{k/2})$ for conditional regression. Can we achieve a similar approximation factor for reference class regression? Finally, we still do not know how close to optimal this blow-up of the loss is; in particular, we do not have any lower bounds. Note that this is a computational and not a statistical issue, since we can obtain uniform convergence over all k-DNF conditions.

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