A NEW GOVERNING EQUATION FOR WEB TENSION BY EMPLOYING A NEO-HOOKEAN MATERIAL MODEL

A. R. Gomez-Constante, P. R. Pagilla[1], and K. R. Rajagopal

Department of Mechanical Engineering
Texas A&M University, College Station, TX
USA

ABSTRACT

In this work, we derive a governing equation for web tension in a span by employing a Neo-Hookean material model that is applicable for transport of web materials under both small and large strains. This governing equation may be employed to study the evolution of tension within a span as well as propagation of tension variations from span to span as the web is transported in the machine. First, we find the stretch in a web span and relate it to web tension via a Neo-Hookean material model; the Neo-Hookean model is linear for small strain and nonlinear otherwise. Second, we conduct a dimensional analysis by defining several key coefficients that aid in grouping the machine and web material parameters separately in order to obtain a compact system of governing equations; this representation may be utilized to efficiently study the impact of web and roller properties on transport behavior.

1 INTRODUCTION

In Roll-to-Roll (R2R) manufacturing, efficient transport of webs through various processes (for example, printing, coating, lamination, heat treatment, etc.) under controlled tension is critical to ensure process quality and performance. The two key web transport variables are the web speed and tension, and governing equations have been developed to predict their variations as the web is transported on rollers through processes under various conditions. A common approach for modeling is to apply mass balance to the web between two adjacent rollers (web span) to obtain a governing equation for web strain where the coupling between web strain and speed is evident. Then, a constitutive material law is assumed to relate web strain and tension based on the type of the web material. Various simplifying assumptions, such as transport under small strain, web wrap angle on the rollers being small when compared to the free web length in the span, etc., are made to obtain a governing equation for web tension that has been employed for system analysis as well as model-based controller design. Early work using this approach can be found in [1–3]. The work in [4] modeled the R2R

[1]Corresponding author email: ppagilla@tamu.edu

system as a large-scale system that can be simplified by using the notion of tension zones resulting in a dynamic subsystem for each tension zone; such subdivision aids in system analysis and model-based controller design.

A particular aspect of existing modeling approaches is the utilization of different governing equations based on the value of the elastic modulus of the transported material. Because of the traction requirements on rollers to facilitate transport, low modulus materials (such as non-wovens and certain polymers) are often transported under large strains, which leads to the materials exhibiting significant hyperelastic behavior during transport. Whereas, high modulus materials (metals and certain polymers) are transported under small strain which typically results in the material exhibiting linearly elastic behavior. In this sense, separate governing equations for web tension based on various linear and nonlinear relations between stress and strain were derived in [5]. In [5], a procedure to find dimensionless equations of motion by defining several dimensionless parameters was given; the dimensionless parameters were used to scale the response and controllers depending on the parameters of the web material and machine conditions. Although these strain conditions are applied in this manner to different materials, the small strain assumption is not particular to high modulus materials, as discussed for plastic films in [7]. Therefore, it is beneficial to obtain a single governing equation based on the stretch rate at which a material is being transported rather than on the value of its elastic modulus and transport conditions. In that sense, the fact that non-woven fabrics [8] and soft polymers exhibit nonlinear response motivates employing a Neo-Hookean material model to obtain a tension governing equation that may be utilized for both small and large strain transport.

The remainder of the paper is organized as follows. In Section 2, we first summarize the governing equations for web velocity on rollers and consider the kinematic constraint associated with the web span to define the time rate of stretch per unit stretch, which relates the web velocity over the rollers with the span length and stretch of the continuous web. Further, this relationship is used to derive a governing equation for tension for Neo-Hookean materials, which can be linearized to get a Hookean model for comparison. We perform the elastodynamic analysis presented in [9, 10] to determine a constitutive relation that is applicable to small and large strain deformation for the moving web. We show that the elastodynamic effects could be neglected for most materials when considering steady state transport conditions. In Section 3, we perform parametric analysis on the governing equations by defining scaling factors that isolate dynamic characteristics of the rollers, free spans and their corresponding equations, and obtain a set of convenient governing equations for the roll-to-roll (R2R) system. The defined parameters provide insights into the relationship between the different elements of the R2R system, and the implications in tension behaviour when changing the operating conditions, which can be used to qualitatively enhance the design process. Section 3 provides an analysis by linearizing the new representation of governing equations to show that for large stretch, the relative error between Neo-Hookean and Hookean models is considerable, which corroborates the necessity of a model that encompasses small and large strain deformation.

2 GOVERNING EQUATIONS FOR WEB TRANSPORT

We consider the transported web subjected to a reference tension t_{ref} and transport velocity v_{ref} as illustrated in Fig. 1. We analyze only uniaxial extension in the transport direction (or machine direction) and develop governing equations for web tension and web velocity in the transport direction. We also assume that the friction between the web and the roller is large enough that slip does not occur between the web and the roller.

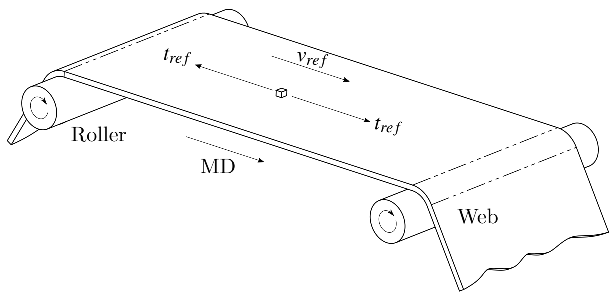

Figure 1 – We span under uniaxial extension

2.1 Governing equations for web velocity on the rollers

In general terms, a R2R system consist of a set of rollers of different characteristics and a continuous web that is transported from an unwind roll to rewind roll through several intermediate processes. An example R2R system is provided in Fig. 2. The definitions for the variables listed in the figure are the following: v_i denotes the web velocity on the i^{th} roller, u_i is the torque input to the i^{th} driven roller, and t_i and λ_i, respectively, are the web tension and stretch in the i^{th} web span.

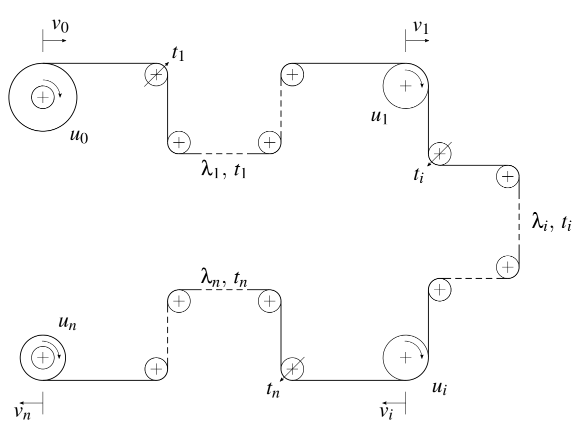

Figure 2 – An example R2R system

In any R2R system, there are driven rollers that facilitate moving the web through different processes and idle rollers that help shape the path of the web through processes. Under the assumption that the web does not slip on the rollers, the governing equations for the peripheral velocity of the rollers can be utilized to study the evolution of web velocity on the rollers. In this sense, we summarize the governing equations for web velocity on the unwind material roll, any intermediate roller and the rewind material roll as presented in [4]. First, the governing equation for the web velocity on the unwind roll is given by

$$\frac{J_0(t)}{R_0(t)}\dot{v}_0(t) = -\frac{b_{f0}}{R_0(t)}v_0(t) - \frac{h_w}{2\pi R_0(t)}\left(\frac{J_0(t)}{R_0^2(t)} - 2\pi\rho_w b_w R_0^2(t)\right)v_0^2(t) + t_1(t)R_0(t) + \eta_0 u_0(t) \qquad \{1\}$$

where $R_0(t)$ and $J_0(t)$ are radius and inertia of the unwind roll, respectively, ρ_w is the web density, h_w is the web thickness, b_w is the web width, b_{f_0} is the viscous friction coefficient in roller bearings, and η_0 is the gain that relates motor input to the torque applied to the roller. Since the unwind roll radius is time-varying as the material is expended into the process, an additional equation is necessary to estimate the time-varying radius which is given by

$$\dot{R}_0(t) = -\frac{h_w}{2\pi}\frac{v_0(t)}{R_0(t)}. \qquad \{2\}$$

Similarly, for any intermediate driven roller we have the following governing equation for its peripheral velocity:

$$\frac{J_i}{R_i}\dot{v}_i(t) + \frac{b_{fi}}{R_i}v_i(t) = \big(t_{i+1}(t) - t_i(t)\big)R_i + \eta_i u_i(t) \qquad i = 1, 2, \ldots, n-1 \qquad \{3\}$$

where R_i and J_i are the radius and inertia of the i^{th} roller, respectively, b_{fi} is the viscouse friction coefficient in the i^{th} roller bearkings, t_i and t_{i+1}, respectively, are the web tensions in the i^{th} span and the $(i+1)^{th}$ span, and η_i is the gain between the motor input and the torque applied to the i^{th} roller. Note that the governing equation for web velocity on the idle roller is simply obtained by setting $u_i(t) = 0$ in the above intermediate roller equation. Similar to the unwind roll (with corresponding variable definitions), the governing equation for web velocity on the rewind roll is given by

$$\frac{J_n(t)}{R_n(t)}\dot{v}_n(t) = -\frac{b_{fn}}{R_n(t)}v_n(t) + \frac{h_w}{2\pi R_n(t)}\left(\frac{J_n(t)}{R_n^2(t)} - 2\pi\rho_w b_w R_n^2(t)\right)v_n^2(t) - t_n(t)R_n(t) + \eta_n u_n(t). \qquad \{4\}$$

where an estimate of the time-varying radius is given by

$$\dot{R}_n(t) = \frac{h_w}{2\pi}\frac{v_n(t)}{R_n(t)} \qquad \{5\}$$

Based on the above governing equations for web velocity on rollers, one can write such governing equations for web velocity on driven and idle rollers in any R2R system.

2.2 Kinematic constraint associated with a web span

In this section, we analyze the kinematic constraint associated with a web span and derive a relationship for stretch in the web span due to the interaction of the rollers and the continuously transported web. First, consider the relative distance that tangentially connects two consecutive rollers, which is referred to as the free span length $\boldsymbol{L}_i$, which is illustrated in Fig. 3.

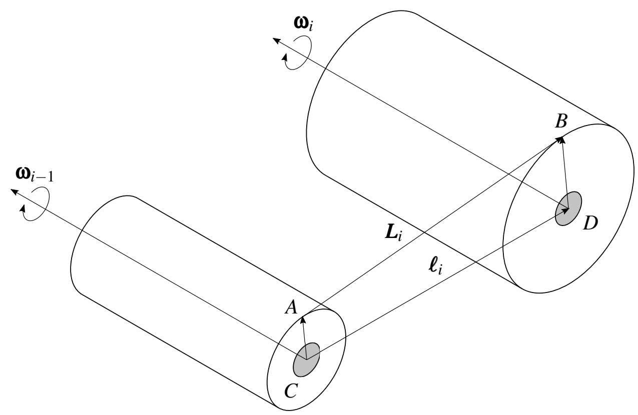

Figure 3 – Span length with respect of time.

In terms of the position vectors of A and B on the rollers the span length $\boldsymbol{L}_i = \boldsymbol{r}_B - \boldsymbol{r}_A$. The time derivative of this span length is given by

$$\frac{d}{dt}\boldsymbol{L}_i = \boldsymbol{v}_B - \boldsymbol{v}_A \tag{6}$$

where $\boldsymbol{v}_A$ and $\boldsymbol{v}_B$ are the velocities of a point passing through A and B, respectively, which are given by

$$\boldsymbol{v}_A = \boldsymbol{v}_C + \boldsymbol{\omega}_{i-1} \times \boldsymbol{r}_{CA} = \boldsymbol{v}_C + \boldsymbol{v}_{i-1}, \tag{7}$$

$$\boldsymbol{v}_B = \boldsymbol{v}_D + \boldsymbol{\omega}_i \times \boldsymbol{r}_{DB} = \boldsymbol{v}_D + \boldsymbol{v}_i \tag{8}$$

where $\boldsymbol{\omega}_{i-1}$, and $\boldsymbol{\omega}_i$ are the angular velocities of the $(i-1)^{th}$ and i^{th} rollers, respectively; and, $\boldsymbol{v}_{i-1}$, and $\boldsymbol{v}_i$ are the peripheral velocities of the rollers at the points A, and B, respectively. Substituting Eqs. {7} and {8} into {6}, we obtain

$$\frac{d}{dt}\boldsymbol{L}_i = \boldsymbol{v}_i - \boldsymbol{v}_{i-1} + \frac{d}{dt}\boldsymbol{\ell}_i \tag{9}$$

where $\frac{d}{dt}\boldsymbol{\ell}_i = \boldsymbol{v}_D - \boldsymbol{v}_C$ is the relative velocity of the point D with respect of C, which is the relative velocity between the axes of rotations of the i^{th} and $(i-1)^{th}$ rollers. Under the assumption that the web does not slip on the surface of the roller, we can define the stretch of the free span as

$$\lambda_i(t) = \frac{L_i(t)}{L_i^0} \tag{10}$$

where L_i and L_i^0, respectively, are the stretched and unstretched web lengths in the free span. Then, taking the time derivative of this relationship, we obtain

$$\frac{d}{dt}\lambda_i(t) = \frac{1}{L_i^0}\frac{d}{dt}L_i(t) \qquad \{11\}$$

Substituting Eqs. {9} and {10} into {11}, we find the following governing equation for stretch

$$\frac{\dot{\lambda}_i(t)}{\lambda_i(t)} = \frac{v_i(t) - v_{i-1}(t)}{L_i(t)} + \frac{\dot{\ell}_{p_i}(t)}{L_i(t)} \qquad i = 1, 2, \ldots, n \qquad \{12\}$$

where ℓ_{p_i} is the component of $\boldsymbol{\ell}_i$ parallel to the span $\boldsymbol{L}_i$. This governing equation represents the time rate of stretch per unit of stretch in the i^{th} span, which is a kinematic constraint that links the rollers as discrete elements with the web as a continuum.

2.3 Material models for web handling

Web tension behaviour has been mostly analyzed in the literature under the small strain assumption. This assumption restricts the model to the case of stretch rates that are very close to 1, which are suitable only for those materials that are not deformed considerably when subjected to a tension.

Bearing in mind that the elastic modulus E is meaningful only for linearized elasticity, we assume that for the case of nonlinear behaviour such modulus is $E = 3G$; there are three scenarios that can be defined considering the magnitude of the elastic modulus E and the strain ε. First, materials with high elastic modulus that are transported with low strain rates, such us metals, high impact polystyrene (HIPS), paper, etc.; these materials are depicted in the left side plot of Fig. 4. Second, materials with low elastic modulus that are transported with low strain rates like soft solids, rubber, plastic films, etc., which can also present viscoelastic behaviour under these transport conditions. Finally, we have the case of materials of low elastic modulus that are transported using high strain rates; these type of materials are mostly non-woven fabrics which exhibit nonlinear behaviour given their physical design, similar to rubber like materials; these materials are depicted in the right side plot of Fig. 4.

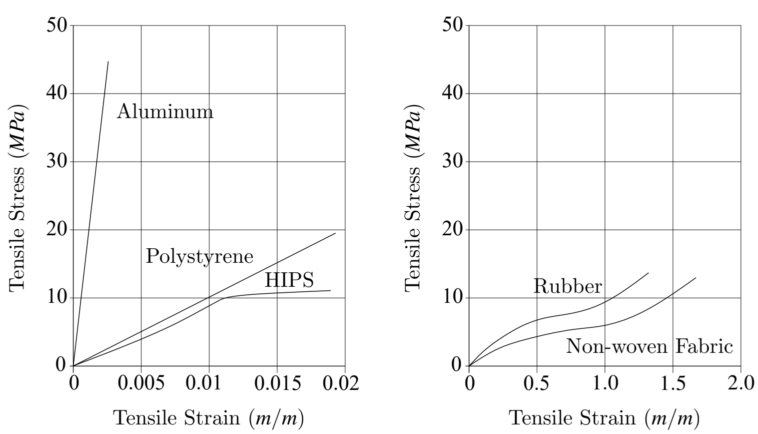

Figure 4 – Stress-Strain diagrams for common web materials [8].

Based on this understanding of materials transported through R2R processes, we want to consider a single model that can encompass many of these scenarios, that is, materials that may be subjected to small or large strains. Under the assumption that the density variations are negligible for an isothermal web handling process, we choose the following Neo-Hookean model:

$$t_i = GA\left(\lambda_i - \frac{1}{\lambda_i^2}\right) \quad \{13\}$$

where t_i and λ_i, respectively, denote the web tension and stretch in the i^{th} span; G is the shear modulus of the material; and A is its undeformed cross-sectional area. The advantage of this model is that for lower strain rates it has the same response as the commonly used Hookean model given by

$$t_i = EA\varepsilon_i \quad \{14\}$$

where $\varepsilon_i = \lambda_i - 1$ is the strain and E is the elastic modulus of the material which is equal to $3G$ for an incompressible material.

Now, given that the models in Eq. {13} and {14} are for the static case, it is important to analyze their validity for the case when the material is transported from span to span and passing through different tension zones. This can be achieved by performing an elastodynamic analysis as in [9,10] for the web transport situation. After performing this analysis, which is provided in Appendix A, we find the following expression for the tension in moving web:

$$t_i(t) = GA\left(\lambda_i(t) - \frac{1}{\lambda_i^2(t)}\right) - \frac{\rho}{12}\sqrt{\lambda_i(t)}\left(\frac{1}{\sqrt{\lambda_i(t)}}\right)_{tt}\frac{A(h_w^2+b_w^2)}{\lambda_i^3(t)} \quad \{15\}$$

where $(1/\sqrt{\lambda(t)})_{tt}$ is the second time derivative of the argument. Now, assuming small strain, we obtain the following linearized Hookean model,

$$t_i(t) = EA\varepsilon_i(t) - \frac{\rho}{12}\sqrt{1+\varepsilon_i(t)}\left(\frac{1}{\sqrt{1+\varepsilon_i(t)}}\right)_{tt} A(h_w^2+b_w^2). \quad \{16\}$$

Equations {15} and {16} indicate that both the Neo-Hookean and Hookean models should include a term that describes the motion of the media. However, in order to get a control oriented model, we assume that the frame of reference is at a steady operating condition. Thus, the second term in the right hand side of both models is small and may be ignored resulting in the following Neo-Hookean material model relating tension and stretch and the Hookean model relating tension and strain:

$$t_i(t) = GA\left(\lambda_i(t) - \frac{1}{\lambda_i^2(t)}\right), \quad \{17\}$$

$$t_i(t) = EA\varepsilon_i(t). \quad \{18\}$$

Additionally, the expression in Eq. {15} provides an opportunity for future research work where one can consider a model that takes into account the nature of a moving web and which better reflects the effects of web transport under transient conditions, i.e., when steady operating conditions are not imposed. Given this, taking the time derivative of Eq. {17} and substituting Eq. {12} and $E = 3G$, we have

$$\dot{t}_i(t) = \frac{EA}{3}\left(\lambda_i(t) + \frac{2}{\lambda_i^2(t)}\right)\left(\frac{v_i(t) - v_{i-1}(t)}{L_i(t)} + \frac{\dot{\ell}_{p_i}(t)}{L_i(t)}\right), \qquad i = 1, 2, \ldots, n \qquad \{19\}$$

which is the time rate of tension in the i^{th} span for a Neo-Hookean material. This is an implicit equation in $\lambda_i(t)$ where one can substitute for the stretch given in Eq. {17} to obtain the following:

$$\lambda_i(t) = \sqrt[3]{\frac{1}{2} + \left(\frac{t_i(t)}{EA}\right)^3 + \sqrt{\frac{1}{4} + \left(\frac{t_i(t)}{EA}\right)^3}} + \sqrt[3]{\frac{1}{2} + \left(\frac{t_i(t)}{EA}\right)^3 - \sqrt{\frac{1}{4} + \left(\frac{t_i(t)}{EA}\right)^3}} + \frac{t_i(t)}{EA} \qquad \{20\}$$

A similar analysis with the Hookean model of Eq. {18} gives the following evolution equation for web tension:

$$\dot{t}_i(t) = \Big(t_i(t) + EA\Big)\left(\frac{v_i(t) - v_{i-1}(t)}{L_i(t)} + \frac{\dot{\ell}_{p_i}(t)}{L_i(t)}\right), \qquad i = 1, 2, \ldots, n \qquad \{21\}$$

In summary, by applying the governing equations for the web velocity and tension developed in this section to rollers and spans of any specific R2R system, one can form the dynamic model for that system to study the transport behavior and develop model-based controller designs.

3 PARAMETRIC ANALYSIS

In this section, to understand how the physical and mechanical properties of the machine and the material affect the response of the system, we define a general setup of a R2R system and use a parametric analysis (similar to dimensional analysis) to determine scaled governing equations and scaling parameters. Given the structure of the R2R system, it is typical to decompose the system into subsystems which are specified by individual tension zones; a tension zone is the transport of web between any two consecutive driven rollers. For example, the unwind roller and the adjacent web spans until the next driven roller is defined as the unwind tension zone. Typically, the driven roller immediately after the unwind roller is under pure speed control and referred to as the master speed roller, and only web velocity on this roller is considered as the state of this subsystem. Subsequently, one can create process subsystems formed by governing equations for web tension and speed for the web between two driven rollers in the process section of the R2R machine. Finally, the last span with the rewind roller is called the rewind zone as illustrated in Fig. 2.

3.1 Scaled governing equations

The governing equations for web velocities given by Eq. {1}-{5} and web span tension given by Eq. {19} are utilized to form the model for the R2R system.

In order to perform a parametric analysis, we consider the following intuitive scaling of web velocity, web tension, and radii of the material rollers:

$$v_i(t) = v_{ref}\,\widetilde{v}_i(t) \qquad t_i(t) = t_{ref}\,\widetilde{t}_i(t) \qquad R_0(t) = r_0\,\widetilde{R}_0(t) \qquad R_n(t) = r_n\,\widetilde{R}_n(t) \tag*{\{22\}}$$

where v_{ref} and t_{ref} are the reference velocity and tension, respectively; $\widetilde{v}_i(t)$ and $\widetilde{t}_i(t)$ are dimensionless velocity and tension; and, r_0 and r_n are the core roller radii of the unwind and rewind rollers, respectively. Let J_{r_0} and J_{m_0} be the inertia of the core roller and motor, respectively. Substituting the definitions in Eq. {22} into the dimensional governing equations for the web velocity on the unwind roller, the unwind roller radius, the web velocity on any driven roller, and the web tension and simplifying we obtain the following scaled governing equations:

$$\tau_0\psi_0(t)\dot{\widetilde{v}}_0(t) = -\frac{\widetilde{v}_0(t)}{\widetilde{R}_0(t)} + \frac{\tau_0\,\beta_0}{\varphi_0}\xi_0(t)\widetilde{v}_0^2(t) + \alpha_0\,\widetilde{R}_0(t)\,\widetilde{t}_1(t) + \Omega_0 u_0(t) \tag*{\{23\}}$$

$$\varphi_0\,\dot{\widetilde{R}}_0(t) = -\beta_0\,\frac{\widetilde{v}_0(t)}{\widetilde{R}_0(t)} \tag*{\{24\}}$$

$$\tau_i\dot{\widetilde{v}}_i(t) + \widetilde{v}_i(t) = \alpha_i\big(\widetilde{t}_{i+1}(t) - \widetilde{t}_i(t)\big) + \Omega_i u_i(t), \qquad i = 2,3,\ldots,n-1 \tag*{\{25\}}$$

$$\gamma_i\,\dot{\widetilde{t}}_i(t) = \kappa_i(t)\big(\widetilde{v}_i(t) - \widetilde{v}_{i-1}(t)\big), \qquad i = 2,3,\ldots,n-1 \tag*{\{26\}}$$

where the scaled parameters for the unwind roller and its radius are given by

$$\tau_0 = \frac{J_{r_0} + J_{m_0}}{b_{f_0}},\ p_0 = (J_{r_0} + J_{m_0})\frac{v_{ref}}{r_0^2},\ f_0 = \frac{p_0}{\tau_0},\ \alpha_0 = \frac{t_{ref}}{f_0},\ \overline{\rho}_0 = \frac{\rho_w\,b_w}{\rho_{r_0}\,b_{r_0}},\ \overline{J}_0 = \frac{J_{r_0}}{J_{r_0} + J_{m_0}},$$

$$\psi_0(t) = \frac{(\overline{\rho}_0\overline{J}_0(\widetilde{R}_0^4(t) - 1) + 1)}{\widetilde{R}_0(t)},\ \xi_0(t) = \frac{(\overline{\rho}_0\overline{J}_0(3\widetilde{R}_0^4(t) + 1) - 1)}{\widetilde{R}_0^3(t)},\ \varphi_0 = \frac{2\pi\,r_0}{v_{ref}},\ \beta_0 = \frac{h_w}{r_0}.$$

The scaled parameters for the intermediate rollers are given by

$$\tau_i = \frac{J_i}{b_{f_i}},\ \alpha_i = \frac{t_{ref}}{f_i},\ f_i = \frac{p_i}{\tau_i},\ p_i = J_i\frac{v_{ref}}{R_i^2}.$$

The scaled parameters from the tension equation are given by

$$\gamma_i = \frac{L_i}{v_{ref}},\ \kappa_i(t) = \frac{\left(\lambda_i(t)\ + \frac{2}{\lambda_i^2(t)}\right)}{\left(\lambda_{ref}\ - \frac{1}{\lambda_{ref}^2}\right)}.$$

Notice that all of the scaled parameters are dimensionless except for the three time constants $(\tau_i, \gamma_i, \varphi_0)$. Further, all of them depend on the physical and mechanical parameters of the machine and the web, and their effect in terms of scaling is evident. Also, note that the time derivatives in the above equations are with respect to actual time since time is not scaled. One parameter that may deserve some discussion is the dimensionless stiffness $\kappa_i(t)$, which is dependent on both the current stretch $\lambda_i(t)$ and the reference stretch λ_{ref}. This is the parameter that appears in the tension governing equation {26} and depends on the amount of stretch, and does not directly depend on the magnitude of the elastic modulus.

Applying the same analysis to Eq. {21} for the Hookean material, the dimensionless stiffness is given by

$$\kappa_i^*(t) = \frac{\lambda_i(t)}{\lambda_{ref} - 1} \qquad \{27\}$$

We will compare these two quantities ($\kappa_i(t)$ and $\kappa_i^*(t)$) after linearization of the scaled equations which is given in the following.

3.2 Linearization of scaled equations

In this subsection we perform the linearization of the governing equations around the operating condition. To do this, we define the following state variables corresponding to different tension zones and the master speed roller:

$$z_0 = \begin{bmatrix} \delta\widetilde{t}_1(t) \\ \delta\widetilde{v}_0(t) \end{bmatrix}, \quad z_1 = \begin{bmatrix} \delta\widetilde{v}_1(t) \end{bmatrix}, \quad z_i = \begin{bmatrix} \delta\widetilde{t}_i(t) \\ \delta\widetilde{v}_i(t) \end{bmatrix}, \quad z_n = \begin{bmatrix} \delta\widetilde{t}_n(t) \\ \delta\widetilde{v}_n(t) \end{bmatrix}$$

where $\delta\widetilde{t}_i(t) = \widetilde{t}_i(t) - 1$ and $\delta\widetilde{v}_i(t) = \widetilde{v}_i(t) - 1$. Based on this definition of state variables, the linearized equations corresponding to each subsystem are given in the following:.

Unwind subsystem:

$$M_0(t)\,\dot{z}_0 = C_0(t)\,z_0 + A_{01}\,z_1 + B_0\,u_0(t) \qquad \{28\}$$

where

$$M_0(t) = \begin{bmatrix} \gamma_1 & 0 \\ 0 & \tau_0\psi_0(t) \end{bmatrix}, \; C_0(t) = \begin{bmatrix} 0 & -\overline{\kappa}_1 \\ \alpha_0\widetilde{R}_0(t) & -\dfrac{1}{\widetilde{R}_0(t)} \end{bmatrix}, \; A_{01} = \begin{bmatrix} \overline{\kappa}_1 \\ 0 \end{bmatrix}, \; B_0 = \begin{bmatrix} 0 \\ \Omega_0 \end{bmatrix}$$

Master speed subsystem:

$$M_1\,\dot{z}_1 = C_1\,z_1 + A_{10}\,z_0 + A_{12}\,z_2 + B_1\,u_1(t) \qquad \{29\}$$

where

$$M_1 = \begin{bmatrix} \tau_1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} -1 \end{bmatrix}, \quad A_{10} = \begin{bmatrix} -\alpha_1 & 0 \end{bmatrix} \quad A_{12} = \begin{bmatrix} \alpha_1 & 0 \end{bmatrix} \quad B_1 = \begin{bmatrix} \Omega_1 \end{bmatrix}$$

Process subsystems:

$$M_i\,\dot{z}_i = C_i\,z_i + A_{i,i-1}\,z_{i-1} + A_{i,i+1}\,z_{i+1} + B_i u_i \qquad \{30\}$$

where

$$M_i = \begin{bmatrix} \gamma_i & 0 \\ 0 & \tau_i \end{bmatrix}, \; C_i = \begin{bmatrix} 0 & \overline{\kappa}_i \\ -\alpha_i & -1 \end{bmatrix}, \; A_{i,i-1} = \begin{bmatrix} 0 & -\overline{\kappa}_i \\ 0 & 0 \end{bmatrix}, \; A_{i,i+1} = \begin{bmatrix} 0 & 0 \\ \alpha_i & 0 \end{bmatrix}, \; B_i = \begin{bmatrix} 0 \\ \Omega_i \end{bmatrix}$$

Rewind subsystem:

$$M_n(t)\,\dot{z}_n = C_n(t)\,z_n + A_{n,n-1}\,z_{n-1} + B_n\,u_n(t) \qquad \{31\}$$

where

$$M_n(t) = \begin{bmatrix} \gamma_n & 0 \\ 0 & \tau_n \psi_n(t) \end{bmatrix},\ C_n(t) = \begin{bmatrix} 0 & \overline{\kappa}_n \\ -\alpha_n \widetilde{R}_n(t) & -\dfrac{1}{\widetilde{R}_n(t)} \end{bmatrix},\ A_{n,n-1} = \begin{bmatrix} 0 & -\overline{\kappa}_n \\ 0 & 0 \end{bmatrix},\ B_n = \begin{bmatrix} 0 \\ \Omega_n \end{bmatrix}$$

In the above equations, $\overline{\kappa}_i$ is the linearized dimensionless stiffness of the Neo-Hookean material given by

$$\overline{\kappa}_i = \frac{\left(\lambda_{ref} + \frac{2}{\lambda_{ref}^2}\right)}{\left(\lambda_{ref} - \frac{1}{\lambda_{ref}^2}\right)}. \quad \{32\}$$

For the case of the Hookean model we have

$$\overline{\kappa}_i^* = \frac{\lambda_{ref}}{\left(\lambda_{ref} - 1\right)}. \quad \{33\}$$

From Eqs. {32} and {33}, if we consider a strain of 1%, we have $\overline{\kappa}_i = 100$ and $\overline{\kappa}_i^* = 101$, which is a relative error of 1% between the two. Whereas for a strain of 100%, we have $\overline{\kappa}_i = 1.43$ and $\overline{\kappa}_i^* = 2$, which is an error of about 40%. Given the fact that in most engineering applications only relative errors less than 10% are considered acceptable, the necessity of a model that can be employed for web transport irrespective of the strain value is reasonable, although the real impact of this difference needs to be ascertained by experimentation.

4 CONCLUDING REMARKS

In this paper, we have utilized the kinematic constraint associated with transport of webs in a web span to define the time rate of stretch per unit stretch of the web material; the time rate of stretch is related to the web transport velocity on the roller, span length, and stretch of the continuous web. Employing this definition of stretch for a web span we find a governing equation for web tension by considering a Neo-Hookean material model which can be employed for transport under a wide range of reference tensions. We have also provided a scaled form of the governing equations by considering an intuitive scaling of the web transport speed and tension. The scaled parameters and equations may be utilized to develop strategies for machine design as well as model-based control design. Based on the notion of separating the R2R system into tension zones, we provide a linearization of the scaled governing equations which also provides some insights into why the Neo-Hookean material model is relevant for web transport for different strain ranges. In this sense, the tension governing equation developed from the Neo-Hookean material model is useful in the analysis of web processing lines where the constant strain (small or large) assumption may not hold in the transient conditions; one particular example is the transport of web in a festoon or accumulator during an unwind or rewind roll change.

REFERENCES

1. Brandenburg, C. "The dynamics of elastic webs threading systems of driven rollers where the load is transmitted by Coulomb friction",Doctorate thesis, Technical University, Munich, 1971

2. J. J. Shelton, "Dynamics of Web Tension Control with Velocity or Torque Control," 1986 American Control Conference, Seattle, WA, USA, 1986, pp. 1423 - 1427.

3. Whitworth, D. P. D., and, Harrison, M.C., "Tension variations in pliable material in production machinery," Applied Mathematical Modelling, Volume 7, Issue 3, 1983.

4. Pagilla, P. R. , Siraskar, N. B., and Dwivedula, R. V. , "Decentralized control of web processing lines," IEEE Transactions on Control Systems Technology, vol. 15, pp. 106 - 116, January 2007.

5. Raul, P. R., and Pagilla, P. R., "Modeling of the Transport Behavior of Low Modulus Webs," Twelfth International Conference on Web Handling, Oklahoma State University, June 2013, Stillwater, OK.

6. Raul, P. R., and Pagilla, P. R., "Application of Dimensional Analysis to Roll-to-Roll manufacturing Systems," Twelfth International Conference on Web Handling, Oklahoma State University, June 2013, Stillwater, OK.

7. Hawkins, W. E., "The plastic film and foil web handling guide," CRC Press, 2002.

8. Tausif, M., Pliakas, A., O'Haire, T., Goswami, P., and Russell, S. J. (2017). "Mechanical Properties of Nonwoven Reinforced Thermoplastic Polyurethane Composites." Materials (Basel, Switzerland), 10(6), 618. doi:10.3390/ma10060618

9. K.R. Rajagopal, A.S. Wineman, "New exact solutions in non-linear elasticity," International Journal of Engineering Science, Volume 23, Issue 2, 1985, Pages 217-234, ISSN 0020-7225.

10. K.R. Rajagopal, "On a class of elastodynamic motions in a neo-Hookean elastic solid," International Journal of Non-Linear Mechanics, Volume 33, Issue 3, 1998, Pages 397-405, ISSN 0020-7462.

A APPENDIX

The web tension equation, Eq. {15}, is derived based on the elastodynamic analysis given in [9, 10]. This derivation is provided in the following. From the balance of linear momentum we have

$$\mathrm{div}(\boldsymbol{T}) + \rho\, \boldsymbol{b} = \rho\, \frac{d^2\boldsymbol{x}}{dt^2} \qquad \{34\}$$

where $\boldsymbol{T}$ is the stress tensor, ρ is the density, $\boldsymbol{b}$ is the vector of body forces, $\frac{d^2\boldsymbol{x}}{dt^2}$ is the acceleration, and $\mathrm{div}(\cdot)$ is the divergence operator. We perform this analysis for the Neo-Hookean model for an isotropic and incompressible solid given by

$$\boldsymbol{T} = -p\,\boldsymbol{I} + G\,\boldsymbol{B} \qquad \{35\}$$

where p is the indeterminate part of the stress due to the incompressiblity assumption, G is the shear modulus, $\boldsymbol{I}$ is the identity matrix, and $\boldsymbol{B}$ is the left Cauchy-Green tensor. Now, for the case of an axially extended moving web, we define the following mapping that describes an homogenoeus deformation:

$$x = \frac{X}{\sqrt{\lambda(t)}},\ y = \frac{Y}{\sqrt{\lambda(t)}},\ z = \lambda(t)Z + z_0(t). \qquad \{36\}$$

From this, we can calculate the deformation gradient $\boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}}$, and then the left Cauchy-Green strain tensor by $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T$. Thus, we obtain

$$\boldsymbol{F} = \begin{bmatrix} \frac{1}{\sqrt{\lambda(t)}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{\lambda(t)}} & 0 \\ 0 & 0 & \lambda(t) \end{bmatrix},\quad \boldsymbol{B} = \begin{bmatrix} \frac{1}{\lambda(t)} & 0 & 0 \\ 0 & \frac{1}{\lambda(t)} & 0 \\ 0 & 0 & \lambda^2(t) \end{bmatrix}. \qquad \{37\}$$

Then, substituting Eq. {35} into Eq. {34}, and after neglecting body forces $\boldsymbol{b}$ we obtain

$$-\frac{\partial p}{\partial \boldsymbol{x}} + G\,\mathrm{tr}\left(\frac{\partial \boldsymbol{B}}{\partial \boldsymbol{x}}\right) = \rho\,\frac{d^2\boldsymbol{x}}{dt^2}. \qquad \{38\}$$

Since, the prescribed motion in Eq. {36} is in Lagrangian representation, Eq. {38} needs to be reformulated as follows

$$-\frac{\partial p}{\partial \boldsymbol{X}}\boldsymbol{F}^{-1} + G\,\mathrm{tr}\left(\frac{\partial \boldsymbol{B}}{\partial \boldsymbol{X}}\boldsymbol{F}^{-1}\right) = \rho\,\frac{d^2\boldsymbol{x}}{dt^2} \qquad \{39\}$$

Solving for p we get the following system

$$\frac{\partial p}{\partial \boldsymbol{X}} = G\,\mathrm{tr}\left(\frac{\partial \boldsymbol{B}}{\partial \boldsymbol{X}}\boldsymbol{F}^{-1}\right)\boldsymbol{F} - \rho\,\frac{d^2\boldsymbol{x}}{dt^2}\boldsymbol{F} = -\begin{bmatrix} \frac{\rho}{\sqrt{\lambda(t)}}\left(\frac{X}{\sqrt{\lambda(t)}}\right)_{tt} \\ \frac{\rho}{\sqrt{\lambda(t)}}\left(\frac{Y}{\sqrt{\lambda(t)}}\right)_{tt} \\ \rho\lambda(t)(\lambda(t)\,Z + z_0(t))_{tt} \end{bmatrix}. \qquad \{40\}$$

Integrating Eq. {40}, and after using Eq. {36} to return to the Eularian representation, results in

$$p(x,y,z,t) = -\frac{\rho}{2}\sqrt{\lambda(t)}\left(\frac{1}{\sqrt{\lambda(t)}}\right)_{tt}(x^2+y^2) - \frac{\rho}{2}\frac{\ddot{\lambda}(t)}{\lambda(t)}(z-z_0)^2 - \rho\ddot{z}_0(t)(z-z_0). \qquad \{41\}$$

Now, substituting Eq. {41} into {35} and imposing traction free condition on the lateral boundaries (because the web is in the free span), we note that the terms that are dependent on $(z-z_0)$ drop off; this is consistent with the assumption that the stretch is only time dependent in the span which implies that the traction should not depend on z. Thus, we obtain the following traction component along the machine direction.

$$\sigma_{zz}(x,y,z,t) = G\left(\lambda^2(t) - \frac{1}{\lambda(t)}\right) + \frac{\rho}{2}\sqrt{\lambda(t)}\left(\frac{1}{\sqrt{\lambda(t)}}\right)_{tt}(x^2+y^2-\frac{b^2+h^2}{4}) \qquad \{42\}$$

where b and h, respectively, are the deformed web width and web thickness. Note that in the z direction the boundary conditions are the velocities of the web on the rollers which are already accounted for in Eq. {12}. By integrating Eq. {42} with respect to the cross-sectional area and considering the engineering stress definition, we obtain the following expression for tension in a moving web:

$$t_i(t) = GA\left(\lambda_i(t) - \frac{1}{\lambda_i^2(t)}\right) - \frac{\rho}{12}\sqrt{\lambda_i(t)}\left(\frac{1}{\sqrt{\lambda_i(t)}}\right)_{tt}\frac{A(h_w^2 + b_w^2)}{\lambda_i^3(t)} \qquad \{43\}$$

where A, h_w and b_w, respectively, are the undeformed web cross-sectional area, web thickness and web width.

Application of the small strain assumption to Eq. {41} we obtain following (linearized Hookean model based) expression for tension in a moving web:

$$t_i(t) = EA\varepsilon_i(t) - \frac{\rho}{12}\sqrt{1+\varepsilon_i(t)}\left(\frac{1}{\sqrt{1+\varepsilon_i(t)}}\right)_{tt} A(h_w^2 + b_w^2). \qquad \{44\}$$